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Substitutional Validity for Modal Logic

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Abstract In the substitutional framework, validity is truth under all substitutions of the non logical vocabulary. I develop a theory where the $\Box$ is interpreted as substitutional validity. I show how to prove soundness and completeness for common modal calculi using this definition.

1 Philosophical Backbone and Context of this Work

1.1 Simple truth vs truth in a model According to the substitutional account of logical consequence, an argument is logically valid if it preserves truth under all uniform substitutions of the non logical vocabulary. The substitutional account has a longstanding pedigree.[1] Quine is probably the main modern proponent and the target of most of the contemporary discussion, especially in his 'Philosophy of Logic'. As it is not exactly clear what formal theory Quine had in mind for his account—if he had any—the discussion of his account of logical validity is complicated and entangled with sophisticated historical points. Many have attacked him [15, 4, 8, 13], and some have tried to defend him, or at least to soothe the critics [2, 19, 7].

It is common to view the substitutional account as an alternative to the model theoretic account, and yet when it comes to make it precise, people have often used model theory. For example, [2] developed one of the best versions of the substitutional account, which however is framed in model theory. First, they argue that Quine's definition of logical truth is incomplete 'since "truth" must be relative to an *interpreted* language'. Then, they consider one specific model—what they call a 'Quine model'—and define a sentence to be logically true when true *in that model*, under all substitutions. Whilst their account answers many questions others have raised, it neglects the fact that the substitutional account does not need the scaffolding of model theory to work: it is really a genuine alternative. In their system, on the other hand, the substitutional account is not very different from model theory: all the semantic work is done exactly in the same way, what changes is only the definition of validity.

What is the point of developing an alternative to model theory to frame substitutional validity? In model theory we define the concept of truth in a model. However,

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the substitutional definition does not need the concept of truth in a model to work, but only a concept of simple truth: validity is preservation of truth—simple truth—under all substitutions of the non logical vocabulary. It follows that all the models are superfluous, except maybe for the intended model where truth in a model collapses into simple truth for the language. However, such an intended model is often nowhere to be found. For example, in no model of the language of ZFC truth in that model equates to truth *simpliciter*. All models have a set sized domain, and a set sized extension of membership.[2] Yet, the intended extension of membership is not set sized, nor is it the intended domain of quantification of the meta theory, where we quantify over all models and thus over all sets. Consequently, for such languages truth in a model cannot recapture simple truth. We require truth for a language to satisfy a suitable T schema, maybe restricted to truth free formulae. Yet, no model satisfies a T schema even for atomic formulae of the language of ZFC: there is no model where, for any x and y, membership holds between them in the model exactly when x is in y. We also expect truth to give a homophonic reading of some expressions, like the quantifiers, yet there is no model where $\ulcorner$Everything is $\phi\urcorner$ is true in the model iff *everything* satisfies ϕ in the model: quantification is always restricted to a set.

1.2 Advantages of the axiomatic approach If we wish to apply the substitutional account to the language of set theory, or to any language that shares a similar level of generality and strength, the concept of truth in a model looks superfluous, and even unsuitable to capture the intuitive meaning of validity in substitutional terms. If all we need for the substitutional account is substitutions and simple truth, it seems reasonable to just build a theory of truth for the object language directly, rather than hoping—often unsuccessfully—that the model theoretic machinery will provide us with a suitable substitute. Recently, Halbach followed this idea [2019, 2020]. He builds a truth theory for ZFC using an axiomatically defined notion of simple satisfaction; validity is preservation of satisfaction under all substitutions and assignments of values to the variables: no concept of truth in a model is needed.

The axiomatic approach has many advantages over the model theoretic account. I list a few. Firstly, there is a substitution—the identity function—that provides a homophonic translation of all formulae, and under which truth satisfies the compositional axioms for connectives and quantifiers, and a T schema for all formulae of set theory. There exists no model that matches the homophonic substitution, however, nor is there a model equivalent to any substitution which maps membership to some formula that fails to define a set, like $x \notin x$. In contrast, for any model there exists a substitution where truth in that model is equivalent to (simple) truth under that substitution [12].[3]

A second advantage is the ability to model absolute generality. Many philosophers have argued that quantifiers are often absolutely general, especially when we do logic and we state logical truths: when we say that everything is self identical, we really mean that everything whatsoever is self identical 'without restriction'. [34].[4] Many have struggled to make sense of the restriction to a set in the interpretation of quantifiers, which is otherwise essential in model theory to ensure that truth in a model is definable in set theory. In the axiomatic approach, the absolute generality of quantifiers is ensured by the homophonic reading and by the (unrestricted) clause for the satisfaction of quantified sentences. There is no need to relativize truth to a set, because truth is primively axiomatized.

Many have tried the higher order route to provide interpretations where there is no restrictions to a set, often appealing to plural resources [3, 27, 25, 24]. For the strategy to be successfull one needs to bear a lot of ideological baggage: Quinean skepticism about higher order languages needs to be set aside; higher order quantification needs to be *sui generis*, irreducible to quantification over singular objects, like sets or other kinds of collections. In the substitutional approach, there is no need to appeal to higher order resources to model absolute generality: everything can be done in a first order language. This is a great advantage because all that ideological baggage is not needed. Also, in this way we can provide a universal theory of validity that applies to possibly any language, not just to languages of some strictly lower order, as it is the case in the higher order theories of validity just mentioned.

Lastly, in the axiomatic approach one can show that logical truths are true and that validity preserves truth simply by looking at the definitions. For example, since logical truths are true under all substitutions, they are true under the homophonic one: they are true. The situation is quite different in model theory, where there is no intended model, so no model where truth in it really means simple truth. Hence, the relation between validity and truth is far from obvious: one will need to appeal to reflection principles to achieve analogous results, if and when such principles are available.[5]

1.3 Intensional languages All the substitutional accounts out there have been focusing on first order logic. As far as I know, no one has tried to extend the account to intensional languages. Quine surely was not interested in this endeavour, for he was skeptical of modal logic to begin with. The substitutional tradition thus far has followed him in disregarding intensional languages. Yet, much of the success of model theory resides in its adaptability to any language, and in particular in its ability to interpret intensional languages through possible worlds semantics. If the substitutional account cannot work for intensional languages at all, it would be less interesting compared to model theory.

I propose a substitutional account of validity for modal languages where the box is a way of talking about what is (substitutionally) valid. If what we are interested in is not this kind of modality, but something else like epistemic modality, or deontic modality, then the substitutional account is not really an alternative to the Kripke style semantics, where it seems we have more flexibility on what we take the possible worlds of the theory to be. Yet, when they are compatible, the two accounts are genuine alternatives, and I would argue that the substitutional account has many advantages over the usual Kripkean account, spelled out in set theory. I will name a few.

First, similar reasons in favour of the substitutional approach carry over into intensional languages. The lack of an intended interpretation puts into jeopardy the relation between validity and truth: we cannot show that logical truths are true, nor that validity is truth preserving just by looking at the definitions. Kripke semantics is spelled out in ZFC where there are no modal operators. It follows that reflection principles are of no help: it cannot be proven that $\Box\phi$ holds only if it is satisfiable, and that therefore if $\Box\phi$ is true in all models, it is true. So, as far as we know, it might be that $\Box\phi$ is true in all models but false, or false in all models but true.

Suppose that $\Box$ means 'it is valid that'. If we try to model 'it is valid that' in Kripke semantics we run into problems. According to the model theoretic account, validity is preservation of truth under all interpretations. Therefore is natural to think of points in the model as interpretations of the language. 'It is valid that ϕ' is true at some interpretation if true at all interpretations. It should be clear from the definition of $\Box$ that $\Box\phi$ implies that ϕ is true because validity implies truth in all interpretations, and in particular truth in the intended interpretation. Of course, we can designate a point in the Kripke model that plays the role of the intended interpretation, but, for the same reasons above, truth at the designated point cannot be truth. The membership relation under the intended interpretation has the extension it actually has, and the quantifiers range over what they actually range. Since all points of any Kripke model receive a set sized domain, the actual meanings of the quantifiers and membership cannot be matched by any of them. So, even though the schema $\Box\phi \rightarrow \phi$ might be valid, it cannot be shown that the truth of $\Box\phi$ implies the truth of ϕ, even though it should be obvious from the definitions. Other, obvious links between validity, actuality and truth are missing. It should be obvious from the definition of validity that if ϕ is logically true then its necessitation is true, and that if ψ follows from ϕ then the strict implication between ϕ and ψ is satisfied. Yet none of this can be shown in standard Kripke semantics.

1.3.1 A proposal Here is the approach I will follow. The substitutional account comes with a concept of simple truth, so the pivotal relations between actuality, necessity and truth just mentioned plainly hold, if we extend the substitutional account to modal logic. My account will be reductive: the language of necessity can be interpreted in a non modal language. However, I do not wish to add a concept of truth in a world, for the same reasons I do not want a concept of truth in a model: substitutional validity is preservation of simple truth under all substitutions. It is neither preservation of truth in a model nor preservation of truth in a world: the possible world machinery is only superfluous structure, from the substitutional point of view. All that it is needed is a primitive concept of truth and a definition of substitution.

Our approach will be based on the following, reductive analysis of modality: the truth of $\Box\phi$ is reduced to (possibly weaker versions of) logical truth. The broadest notion of necessity is logical validity: $\Box\phi$ is true exactly when ϕ is logically true. To be logically true, according to the substitutional account, is to be true under all substitutions of the non logical vocabulary. Thus, $\Box\phi$ is true exactly when ϕ is true under all substitutions of the non logical vocabulary.[6] In virtue of this definition, we can read $\Box$ roughly as 'it is valid that'.

Valid:: Where $\Box$ means 'It is valid that', $\Box\phi$ is true iff ϕ is true under all substitutions.

In the resulting system, the links highlighted above between truth, validity and necessity are met: if ϕ is logically true then it is true under all substitutions. In particular, under the identity substitution, so ϕ is true. Also, because the truth of ϕ is preserved under all substitutions, $\Box\phi$ is true, as well. If ψ follows from ϕ then the implication and the strict implication between ϕ and ψ is satisfied. Since there exists a homophonic substitution, if $\Box\phi$ is true then ϕ is true.

It would be wrong and unintended to suggest that this definition of $\Box$ works in general, given any suitable use of modal logic. $\Box$ in the system does not talk about truth in different possible worlds, and it is unsuitable to model such a reading of modality. Our semantics is not neutral: it presupposes a specific understanding of the box as a way of expressing logical validity, or suitable expansions of logical validity.

If this understanding is unfeasible, the semantics is unfeasible. The aim of this paper is to show that there is an interesting way of framing the substitutional understanding of validity for intensional languages, I do not wish to replace model theory in all of its applications.

Different truth theories can specify different interpretations of the modal operators, by restricting the range of substitutions relevant to the truth of $\Box\phi$, with the limit cases being the trivial and universal restriction. This nicely tracks the fact that, usually, if something is valid then its necessitation is true, even though the reverse direction might not hold. Given any element i of the powerset of the set of substitution functions, there is a truth theory where the following holds:

Valid (extension):: $\ulcorner\Box\phi\urcorner$ is true exactly when ϕ is true under all substitutions in i.

Thus, in effect I obtain (countably) many different theories of modal truth, that reflect different extensions of 'it is (substitutionally) valid'.

It is an open question whether, for some given calculus, there is a truth theory where validity coincides with provability in the calculus. In such a case, I call the truth theory 'canonical' for the modal calculus. In the rest of the paper, I will show how to develop a canonical theory when possible, and I prove soundness and completeness for the most common (normal) modal logics.

1.3.2 Why I don't choose a non reductive account In the axiomatic, substitutional approach we define truth axiomatically and we read logical expressions homophonically, via the recursive axioms of truth. One might wonder why I do not try the same approach with modal operators, by reading necessity homophonically, and by stipulating the commutativity of $\Box$ and satisfaction axiomatically, treating $\Box$ pretty much like negation. Yet, whether this axiom is feasible depends on what $\Box$ is taken to mean. In fact, many doubted this axiom under fairly common readings of $\Box$: even though 'Socrates is necessarily human' is true, it is not clear that necessarily 'Socrates is human' is true, since the sentence might have meant something else completely [14, 32]. Some suggested different ways of getting out of the problem, for example by fixing a particular interpretation of the language [22, 10]. However, [6] doubts even this solution because, even if we relativized truth to a particular language (and thus to a particular interpretation), since languages themselves are contingent objects, in some world they would hold no relation to the sentence. Crucially, commutativity would not be sound if we read $\Box$ as 'It is valid that', either, as we are trying to do here: 'It is valid that it rains if it rains' is true, yet it is not valid that 'It rains if it rains' is true.

2 Technical Preliminaries

2.1 Language I start with the language and theory of ZFC. The only relation symbols are $\in$ and $=$.[7] The only connectives and quantifiers are: $\forall$, $\neg$, $\wedge$. The other common quantifiers and connectives are to be defined as meta abbreviations. To this language I add the modal operator $\Box$, to be read informally as 'it is valid that', and I extend the definition of well formed formula, adding $\Box\phi$ as well formed. I use $\Diamond\phi$ as an abbreviation for $\neg\Box\neg\phi$. I finally add a binary satisfaction predicate 'Sat', and define $\mathrm{Sat}(v, w)$ as well formed, where '$\mathrm{Sat}(v, w)$' is to be read informally as 'v is satisfied under w'. I call the resulting language L_{SM}. I use the following notation:

i. L_{ZFC}: the language of ZFC;
ii. $L_{\Box}$: the result of adding $\Box$ to L_{ZFC}, and $\Box\phi$ among the wel formed formulae;

iii. L_{Sat}: the result of adding Sat to L_{ZFC} and updating the definition of well formed formula accordingly (the language has no $\Box$);
iv. L_{SM}: the result of merging L_{Sat} and $L_{\Box}$.

I code all formulae of L_{SM} in a natural way in set theory. There are no function symbols in set theory, but functions can be represented through suitable formulae. 'Form' is the recursively definable condition of being (the code of) a formula of L_{SM}. I use Greek letters ϕ, ψ etc as meta variables for things that satisfy 'Form'. For readability, I do not distinguish between formulae and codes when I use Greek meta variables, hoping that it does not create confusion. That is, where ψ is some condition and $\ulcorner\phi\urcorner$ the code of ϕ, I do not write $\psi(\ulcorner\phi\urcorner)$ but just $\psi(\phi)$ to say that ϕ satisfies ψ. I do need some coding notation when I need to show the structure of the formula: in that case I either use semi quotes $\ulcorner\urcorner$ or coding functions. A dot under a symbol represents its correspondent coding function: $\underset{\cdot}{\neg}$ is a function from the code of ϕ to the code of its negation, which I write either as $\ulcorner\neg\phi\urcorner$ or as $\underset{\cdot}{\neg}\phi$; $\underset{\cdot}{\in}$ is a function from the code of variables v and w to $\ulcorner w \in v\urcorner$ or, which is the same, $w\underset{\cdot}{\in}v$. I seldom abuse notation, using coding functions for connectives that do not belong to the language: $\phi\underset{\cdot}{\rightarrow}\psi$ is an abbreviation for $\underset{\cdot}{\neg}(\phi\underset{\cdot}{\wedge}\underset{\cdot}{\neg}\psi)$. When I need to talk about the numeral of the code of ϕ, I use $\overline{\phi}$. To save space I sometimes write $\forall x, y\phi$ instead of $\forall x\forall y\phi$. For functions, instead of $f(g(x))$ I often write $fg(x)$. $|\phi, v_1/w_1, .., v_n/w_n|$ is the result of uniformly substituting v_1 for w_1, .., v_n for w_n in ϕ, and it is recursively definable. ϕ_{at} is a meta variable for (the code of) atomic formulae; that is, $\underset{\cdot}{\text{S}}\text{at}(v_n, v_m)$, $v_n\underset{\cdot}{\in}v_m$ or $v_n\underset{\cdot}{=}v_m$.

2.2 Substitution function A substitution function $\mathcal{I}$ that reinterprets the non logical expressions is defined recursively on formulae of L_{SM}. I do not write $\mathcal{I}(\ulcorner\phi\urcorner)$, much less $\psi(\ulcorner\mathcal{I}(\ulcorner\phi\urcorner)\urcorner)$, because it makes the text confusing and aesthetically unpleasant.

Substitution functions ought to treat logical expressions homophonically. In the following, I consider logical expressions $\neg$, $\wedge$, $\forall$ and $\Box$. I take a Quinean stance on identity: it is not logical, but this can be changed if we wished to. A substitution function can treat quantifiers in two different ways: it can relativize them to some condition or not. I call the first 'relativized', the second 'unrelativized'. If it adds a condition, one might say that quantifiers are not treated as logical constants, after all. A similar objection holds for model theory, where in different models quantifiers range over different domains. Throughout the article I will assume that substitutions are unrelativized.

I do not allow a substitution function to add modal operators: they respect the modal depth of formulae. The modal depth md can be defined recursively:

$$\begin{aligned}
&\text{md}(\phi_{at}) = 0\\
&\text{md}(\underset{\cdot}{\neg}\phi) = \text{md}(\phi)\\
&\text{md}(\phi\underset{\cdot}{\wedge}\psi) = \max(\text{md}(\phi), \text{md}(\psi))\\
&\text{md}(\underset{\cdot}{\forall}v\phi) = \text{md}(\phi)\\
&\text{md}(\underset{\cdot}{\Box}\phi) = \text{md}(\phi) + 1
\end{aligned}$$

I call a substitution $\mathcal{I}$ 'modally bounded' whenever the following holds:

$$\forall\phi(\text{md}(\phi) = \text{md}(\mathcal{I}(\phi)))$$

I stipulate that every $\mathcal{I}$ is modally bounded. We will see later that this requirement ensures relative consistency and the validity of substitution of identicals.

Usually a substitution function will rename free variables. Variable renaming is natural, because a renaming of variables looks like a correct substitution of the formula. Also, it is used to avoid unintended variable bindings. Yet, we cannot let substitutions rename free variables, because variable renaming disrupts the relation between modality and quantification. Consider $\exists x \Box Fx$. We would like to say that the sentence is true if some thing y satisfies every modally relevant substitution of Fx. To do so, we need to talk about that specific y across different substitutions, and yet it might be that the variable x gets substituted arbitrarily, making us 'lose track' of the thing we were considering through the existential quantifier when we started to evaluate the formula. There is a lack of coordination between the way we rename variables and the unravelling of the formula via the compositional axioms of truth. The lack of coordination also makes substitution of identicals invalid.

For the above reasons I do not let substitutions rename free variables. To avoid unintended variable bindings, substitutions must uniformly rename only bound variables. I define a formula in 'signed form' as follows:

Definition 2.1 (Signed form) A formula ϕ is in signed form if and only if every free variable in ϕ is even, and every bound variable in ϕ is odd.

I stipulate that $\mathcal{I}$ maps $\in$, Sat and $=$ to three formulae which are all in signed form. I stipulate that in each of these formulae we ordered every free and bound variable: I use α_1 for the first first free variable, α_2 for the second, etc, and β_1 for the first bound variable, β_2 for the second etc. I show how to define $\mathcal{I}$ for the case $\ulcorner v_n \in v_m \urcorner$. Similar definitions hold for $\ulcorner v_n = v_m \urcorner$ and $\ulcorner \text{Sat}(v_n, v_m) \urcorner$. First, where $\beta_1, .., \beta_s$ are the bound variables in $\mathcal{I}(\in)$, and k is the maximum between the indices in $\beta_1, .., \beta_s$, define the following transformation τ:

$$\tau(r, \mathcal{I}(\in)) := |\mathcal{I}(\in), v_{2(k+r+1)+1}/\beta_1, .., v_{2(k+r+s)+1}/\beta_s|$$

Now, define $\mathcal{I}(v_n \underset{\cdot}{\in} v_m)$ as follows:

$$\mathcal{I}(v_n \underset{\cdot}{\in} v_m) := \begin{cases} |\tau(n, \mathcal{I}(\in)), v_n/\alpha_1, v_m/\alpha_2| & \text{if } \max(n,m) = n \\ |\tau(m, \mathcal{I}(\in)), v_m/\alpha_2, v_n/\alpha_1| & \text{if } \max(n,m) = m \end{cases}$$

$\mathcal{I}$ always chooses a bigger odd subscript for bound variables, so the free variables it substitutes do not get bound, and the old (even) free variables $\alpha_1, .. ,\alpha_n$ do not get bound, either. It then substitutes the variable with the biggest subscript first; otherwise, where $\mathcal{I}(\in) = \phi(\alpha_1, \alpha_2)$, $\mathcal{I}(\alpha_2 \underset{\cdot}{\in} \alpha_3) = |\phi(\alpha_1, \alpha_2), \alpha_2/\alpha_1, \alpha_3/\alpha_2|$, which is the unintended $\phi(\alpha_2, \alpha_2)$.

We extend the definition of $\mathcal{I}$ recursively to all formulae of L_{SM}. By construction, where k is the maximum of all variables in $\mathcal{I}(\psi)$, $u = 2k + 1$.

$$\mathcal{I}(\phi) := \begin{cases} \underset{\cdot}{\neg}\mathcal{I}(\psi) & \text{if } \phi \text{ is } \underset{\cdot}{\neg}\psi \\ \mathcal{I}(\psi) \underset{\cdot}{\wedge} \mathcal{I}(\chi) & \text{if } \phi \text{ is } \psi \underset{\cdot}{\wedge} \chi \\ \underset{\cdot}{\forall} u |\mathcal{I}(\psi), u/v| & \text{if } \phi \text{ is } \underset{\cdot}{\forall} v \psi \\ \underset{\cdot}{\Box}\mathcal{I}(\psi) & \text{if } \phi \text{ is } \underset{\cdot}{\Box}\psi. \end{cases}$$

In the proofs below I seldom omit variables renaming because it makes the formula unreadable, hoping that the reader trusts us with the fact that variable renaming works and we can avoid unintended variable bindings.

2.3 Satisfaction I do not use a concept of simple truth, but rather a concept of satisfaction under an assignment. Simple truth is truth under all assignments. Satisfaction under an assignment of values to the variables is still an acceptable concept, from the perspective of a truth conditional semantics and a substitutional understanding of validity: satisfaction is what [29] uses to define truth for a language, and to deal with quantified formulae. A variable assignment is a function from the domain of variables. I use Asg as an abbreviation for the condition of being a variable assignment, which is specifiable in set theory in the usual way. I use Gothic letters $\mathfrak{a}$, $\mathfrak{b}$ etc as meta variables for things that satisfy Asg. By '$\mathfrak{a}^x_v$' I mean the result of changing the value of v to x in $\mathfrak{a}$, keeping everything else the same:

$$\mathfrak{a}^v_x(u) = \begin{cases} x & \text{if } u = v \\ \mathfrak{a}(u) & \text{otherwise.} \end{cases}$$

In the following I will discuss different truth theories, which differ on how they interpret modal formulae. However, all the truth theories I consider share the following Tarskian axioms, for all (codes of) formulae of L_{SM}:

$$\forall v, w, \mathfrak{a}(\mathrm{Sat}(v \underset{\cdot}{\in} w, \mathfrak{a}) \leftrightarrow \mathfrak{a}(v) \in \mathfrak{a}(w)) \tag{SM1}$$

$$\forall v, w, \mathfrak{a}(\mathrm{Sat}(v \underset{\cdot}{=} w, \mathfrak{a}) \leftrightarrow \mathfrak{a}(v) = \mathfrak{a}(w)) \tag{SM2}$$

$$\forall \phi, \mathfrak{a}\big(\mathrm{Sat}(\underset{\cdot}{\neg}\phi, \mathfrak{a}) \leftrightarrow \neg\mathrm{Sat}(\phi, \mathfrak{a})\big) \tag{SM3}$$

$$\forall \phi, \psi, \mathfrak{a}\big(\mathrm{Sat}(\phi \underset{\cdot}{\wedge} \psi, \mathfrak{a}) \leftrightarrow (\mathrm{Sat}(\phi, \mathfrak{a}) \wedge \mathrm{Sat}(\psi, \mathfrak{a}))\big) \tag{SM4}$$

$$\forall \mathfrak{a}, \phi(\mathrm{Sat}(\underset{\cdot}{\forall} v \phi, \mathfrak{a}) \leftrightarrow \forall y \mathrm{Sat}(\phi, \mathfrak{a}^y_v)) \tag{SM5}$$

$$\forall v, w, \mathfrak{a}, \phi_{at}\big(\mathfrak{a}(v) = \mathfrak{a}(w) \to \mathrm{Sat}(\phi_{at} \leftrightarrow |\phi_{at}, v/w|, \mathfrak{a})\big) \tag{SM6}$$

$$\forall \mathfrak{a}, \phi\big(\phi \in L_{\mathsf{ZFC}} \to (\mathrm{Sat}(\phi, a(u)) \leftrightarrow \mathrm{Sat}(\underset{\cdot}{\mathrm{Sat}}(v, u), a^\phi_v))\big) \tag{SM7}$$

$$\forall x, y(\neg(\mathrm{Form}(x) \vee \mathrm{Asg}(y)) \to \neg\mathrm{Sat}(x, y)) \tag{SM8}$$

By extending ZFC with axioms SM1–SM8 and extending the axioms and rules of ZFC to L_{SM}, we obtain the theory SM. The compositional axioms for connectives and quantifiers also hold for modal formulae. Axioms SM1–SM5 provide a weak type free theory of satisfaction: they are but the extension to the modal language of the theory Halbach [11] labels S. Since we lack a T schema for formulae with the satisfaction predicate, the scope of Sat is opaque: SM6 ensures that substitution of identicals holds regardless, for all atomic formulae. SM6 for $\phi_{at} = v \underset{\cdot}{\in} u$ or $v \underset{\cdot}{=} u$ is redundant. We will see how the axiom is enough to ensure that substitution of identicals holds inside the scope of the modal operator, as well.

We can prove a T schema for formulae of L_{ZFC} (the language without Sat and $\Box$):

Lemma 2.2 *For all formulae ϕ of L_{ZFC} with free variables $x_1, .., x_n$, the following holds:*

$$\mathsf{SM} \vdash \forall \mathfrak{a}\big(\mathrm{Sat}(\phi(x_1, .., x_n), \mathfrak{a}) \leftrightarrow \phi(\mathfrak{a}(x_1), .., \mathfrak{a}(x_n))\big)$$

Proof The proof is by induction on the complexity of ϕ using axioms SM1–SM5. $\Box$

SM7 allows us to iterate satisfaction twice, if we start from a formula of set theory. It will prove useful for the adequacy theorem. The restriction of formulae of set theory only ensures consistency. The axioms of SM are debatable and can be changed if we wish to. With some modifications, what we prove can probably be done in another truth theory of similar strength. If we choose a truth theory like that of [16], which was axiomatized by [9], the resulting theory of validity will probably be non classical and para complete. A classic, typed Tarskian satisfaction may work too, even though the adequacy proof will have to be changed.[8] In a typed system, however, logical consequence cannot be defined for all first order languages, but only for weaker object languages. We lose self applicability: validity cannot be defined in a language for that language.

2.3.1 Modal truth theories We can provide different interpretations of the box through different truth theories, which result from adding specific axioms that cover the clause for $\Box\phi$. The simplest equates its satisfaction to the satisfaction of ϕ under all substitutions:

$$\forall\mathfrak{a},\phi\big(\mathrm{Sat}(\underset{\cdot}{\Box}\phi,\mathfrak{a})\leftrightarrow\forall\mathcal{I}\ \mathrm{Sat}(\mathcal{I}(\phi),\mathfrak{a})\big)$$

We call a formula 'simply true' when satisfied under all assignments. Via this axiom, $\ulcorner\Box\phi\urcorner$ is simply true exactly when it is simply true under all substitutions: when it is logically true.[9]

Different interpretations of modality can be given through different modifications of the above truth clause. In general, for any collection of substitutions, there is a truth theory that makes it the set of modally relevant substitutions. Where $\Im$ is the set of substitution functions defined as above over L_{SM}, for any element i of $\wp\Im$, there is a theory of truth $\mathsf{SM}+\mathsf{SMi}$, which is the result of adding to SM the following axiom:

$$\forall\mathfrak{a},\phi\big(\mathrm{Sat}(\underset{\cdot}{\Box}\phi,\mathfrak{a})\leftrightarrow\forall\mathcal{I}(\mathcal{I}\in i\rightarrow\mathrm{Sat}(\mathcal{I}(\phi),\mathfrak{a}))\big) \qquad \text{(SMi)}$$

There are countably many $\mathcal{I}$, so there are uncountably many interpretations of the box available. The first one I considered equated i with the class of $\mathcal{I}$. It is important to point out that, even though we have many ways of interpreting $\Box$, when we specify validity we always need to choose one first, and then work in that truth theory only: we never define validity by quantifying over different ways of interpreting $\Box$. If we did, truth could not possibly be simple truth, when we define validity.

If substitutions are not modally bounded the system might be inconsistent. For example, consider $i=\{\mathcal{I}:\mathcal{I}(F)=\underset{\cdot}{\neg}\Box F\}$, where F is some designated atomic formula of L_{SM}. Via $\mathsf{SM}+\mathsf{SMi}$, $\mathrm{Sat}(\underset{\cdot}{\Box}F,\mathfrak{a})$ iff $\mathrm{Sat}(\mathcal{I}(F),\mathfrak{a})$, for any $\mathcal{I}$ in i. That is, iff for any $\mathcal{I}$ such that $\mathcal{I}(F)=\underset{\cdot}{\neg}\Box F$, $\mathrm{Sat}(\mathcal{I}(F),\mathfrak{a})$; that is, iff $\neg\mathrm{Sat}(\underset{\cdot}{\Box}F,\mathfrak{a})$. Contradiction. However, when substitutions are modally bounded, we have reasons to believe that the system is consistent. I call S the restriction of the axioms of Sat to the non modal language: to L_{Sat}. S is a very weak type free theory, and it is what is true at level 0 of modal depth. We obtain the following lemma:

Lemma 2.3 *Give any* $\mathsf{SM}+\mathsf{SMi}$*, given any level of modal depth* n*, what is true at level* n *in* $\mathsf{SM}+\mathsf{SMi}$ *is inconsistent only if what is true at level* 0 *is inconsistent.*

Proof Take any SMi, and any ϕ, and say that $\mathrm{md}(\phi)=n$. Via SMi, $\mathrm{Sat}(\underset{\cdot}{\Box}\phi,\mathfrak{a})$ exactly when $\mathrm{Sat}(\mathcal{I}(\phi),\mathfrak{a})$, for any $\mathcal{I}\in i$. Since $\mathcal{I}$ is modally bounded, $\mathrm{md}(\phi)$ is $\mathrm{md}(\mathcal{I}(\phi))$, and thus $\mathrm{md}(\mathcal{I}(\phi))<\mathrm{md}(\underset{\cdot}{\Box}\phi)$.

Now, if $\mathrm{md}(\phi)$ is 0, S would be inconsistent, so the lemma holds. Assume that for any ϕ of modal depth $m<n$, for any x, $\neg\mathrm{Sat}(\phi\underset{\cdot}{\wedge}\underset{\cdot}{\neg}\phi,x)$ (IH1). Suppose the lemma does not hold for formulae of modal complexity n. We apply another induction, this time on the complexity of the formula. Suppose that, for any $m<k$, for any x, the lemma holds for any ψ of complexity m and modal complexity n; that is, $\neg\mathrm{Sat}(\psi\underset{\cdot}{\wedge}\underset{\cdot}{\neg}\psi,x)$ (IH2). ψ of complexity k and modal depth n is either (a) $\underset{\cdot}{\neg}\chi$, (b) $\chi\underset{\cdot}{\wedge}\gamma$, (c) $\underset{\cdot}{\forall}v\chi$ or (d) $\underset{\cdot}{\Box}\chi$.

a. Assume Sat$(\underset{.}{\neg}\chi\underset{.}{\wedge}\underset{.}{\neg}\underset{.}{\neg}\chi, x)$. Via the axioms for satisfactions SM3 and SM4, Sat$(\underset{.}{\neg}\chi\underset{.}{\wedge}\chi, x)$, contrary to IH2.
b. Assume Sat$(\chi\underset{.}{\wedge}\gamma\underset{.}{\wedge}\underset{.}{\neg}(\chi\underset{.}{\wedge}\gamma), x)$. Via the axioms for satisfaction SM3 and SM4, from Sat$(\underset{.}{\neg}(\chi\underset{.}{\wedge}\gamma), x)$ either Sat$(\underset{.}{\neg}\chi, x)$ or Sat$(\underset{.}{\neg}\gamma, x)$, and so in any case either Sat$(\chi\underset{.}{\wedge}\underset{.}{\neg}\chi, x)$ or Sat$(\gamma\underset{.}{\wedge}\underset{.}{\neg}\gamma, x)$, contrary to IH2.
c. Assume Sat$(\underset{.}{\forall}v\chi\underset{.}{\wedge}\underset{.}{\neg}\underset{.}{\forall}v\chi, x)$. Via the axioms for satisfaction SM3, SM4 and SM5, there is a y such that Sat$(\chi \wedge \neg\chi, y)$, contrary to IH2.
d. Assume Sat$(\underset{.}{\Box}\chi\underset{.}{\wedge}\underset{.}{\neg}\underset{.}{\Box}\chi, x)$. Then, by axioms SMi, SM3 and SM4, there is a substitution $\mathcal{I} \in i$ such that Sat$(\mathcal{I}(\chi)\underset{.}{\wedge}\underset{.}{\neg}\mathcal{I}(\chi), x)$. Since $\mathcal{I}$ is modally bounded, md$(\mathcal{I}(\chi)) < n$, so we obtain a contradiction via IH1.

We conclude by induction on the complexity of formulae, that for any ψ of modal depth n, the lemma holds. We therefore conclude by induction on the modal depth of formulae that for any ϕ of any modal depth, the lemma holds. □

3 Substitutional Logical Consequence

Substitutional logical consequence is defined in the language of L_{SM}: I label it '$\models_S$'.

Definition 3.1 (Substitutional Validity) Where Δ and ϕ are sets of formulae and a formula of L_{SM}, respectively:

$$\forall\Delta, \phi\Big(\Delta \models_S \phi := \forall\mathfrak{a}\forall\mathcal{I}\big(\forall\psi(\psi \in \Delta \to \mathrm{Sat}(\mathcal{I}(\psi), \mathfrak{a})) \to \mathrm{Sat}(\mathcal{I}(\phi), \mathfrak{a})\big)\Big)$$

For the proofs in the next sections, it is useful to highlight some properties of logical validity. A quite obvious but useful lemma is the following:

Lemma 3.2 *Given any* $\mathsf{SM} + \mathsf{SMi}$*:*

$$\mathsf{SM} + \mathsf{SMi} \vdash \forall\phi\psi(\phi) \to \forall\phi, \mathcal{I}\psi(\mathcal{I}(\phi))$$

$\mathcal{I}(\phi)$ is always a formula. Therefore, if some condition ψ holds of all formulae, it also holds of all their substitutions.

Theorem 3.3 *Given any* $\mathsf{SM} + \mathsf{SMi}$, *where FOL is first order logic without identity, and ϕ and Δ are meta variables for formulae and sets of formulae of L_{SM}, the following holds:*

$$\mathsf{SM} + \mathsf{SMi} \vdash \forall\phi, \Delta(\Delta \vdash_{FOL} \phi \to \Delta \models_S \phi)$$

Proof The proof is by induction on the complexity of formulae: the axioms hold for all formulae and thus for all substitutions, via lemma 3.2. In particular, we can safely prove soundness for universal instantiation and generalization even for *de re* modal formulae, since $\mathcal{I}$ does not change free variables.[10] □

Granted that ϕ is logically necessary exactly when it is logically true, as expected, where i is the class of all $\mathcal{I}$, necessity is logical necessity, and $\models_S \phi$ is equivalent to $\forall\mathfrak{a}$Sat$(\underset{.}{\Box}\phi, \mathfrak{a})$: the logical truth of a formula is equivalent to the truth of its logical necessitation. For arguments, we can prove that, if ϕ follows from a conjunction $\psi_1, .., \psi_n$, then the strict conditional from $\psi_1 \wedge .. \wedge \psi_n$ to ϕ is satisfied. We cannot show that being logically true is equivalent to being logically necessary, because there is no T schema for modal sentences, nor in general for all sentences of L_{Sat}, on pain of contradiction. For similar reasons, in any truth theory we consider we cannot show that logical truths are necessary, but only that their necessitation is true.

Substitution of identicals is valid. Note that the result relies on substitutions being modally bounded.

Lemma 3.4 *Given any* $\mathsf{SM} + \mathsf{SMi}$*:*

$$\mathsf{SM} + \mathsf{SMi} \vdash \forall v, w, \mathfrak{a}, \phi\big(\mathfrak{a}(v) = \mathfrak{a}(w) \to (\mathrm{Sat}(\phi, \mathfrak{a}) \leftrightarrow \mathrm{Sat}(|\phi, v/w|, \mathfrak{a}))\big)$$

Proof The proof is by induction on the modal depth of formulae. When the modal depth is 0, we prove the lemma for atomic formulae via SM6, and for the other non modal formulae via SM3–5 (via induction on the complexity of ϕ). Assume that the lemma holds for all ψ whose md is less than m. Then whenever ϕ has $md = m$, we have added a $\Box$ to some subformula χ of ϕ. Assume $\mathrm{Sat}(\underset{\cdot}{\Box}\chi, \mathfrak{a})$ but $\neg\mathrm{Sat}(|\underset{\cdot}{\Box}\chi, v/u|, \mathfrak{a})$, even though $\mathfrak{a}(v) = \mathfrak{a}(u)$. Then for any $\mathcal{H}$ in i, $\mathrm{Sat}(\mathcal{H}(\chi), \mathfrak{a})$. Since by construction, $\mathcal{H}$ does not rename variables, $|\mathcal{H}(\chi), v/u|$= $\mathcal{H}(|\chi, v/u|)$. So, for some $\mathcal{H}$ in i, $\neg\mathrm{Sat}(\mathcal{H}(|\chi, v/u|), \mathfrak{a})$. By construction of $\mathcal{H}$, $\mathcal{H}$ is modally bounded, so for any ψ $\mathcal{H}(\psi)$ has the same modal depth of ψ. We can therefore apply the induction hypothesis, and conclude that for any $\mathcal{H}$ in i, $\mathrm{Sat}(\mathcal{H}(\chi), \mathfrak{a}) \leftrightarrow \mathrm{Sat}(\mathcal{H}(|\chi, v/u|), \mathfrak{a})$, which contradicts our assumption that $\neg\mathrm{Sat}(\mathcal{H}(|\chi, v/u|), \mathfrak{a})$ for some $\mathcal{H}$ in i. Since the lemma holds for $\underset{\cdot}{\Box}\chi$, we can show that it holds for ϕ, via the definition of modal depth and well formed formula. □

Lemma 3.5 *Given any* $\mathsf{SM} + \mathsf{SMi}$,

$$\mathsf{SM} + \mathsf{SMi} \vdash \forall\phi(\models_S \phi \to \forall\mathfrak{a}\mathrm{Sat}(\phi, \mathfrak{a}))$$

Proof If $\models_S \phi$ then ϕ is true under all $\mathfrak{a}$ and $\mathcal{I}$. In particular, it is true under the identity function, so $\mathrm{Sat}(\phi, \mathfrak{a})$, for any $\mathfrak{a}$. □

Lemma 3.6 (Additivity) *For any $\mathcal{I}$ and $\mathcal{I}'$, there exists a $\mathcal{H}^{\mathcal{I}+\mathcal{I}'}$ such that, for all ϕ, $\mathcal{I}'\mathcal{I}(\phi) = \mathcal{H}^{\mathcal{I}+\mathcal{I}'}(\phi)$*

Proof I construct the relevant $\mathcal{H}$ as follows:

$$\mathcal{H}(\in) = \mathcal{I}'\mathcal{I}(\in)$$
$$\mathcal{H}(\mathrm{Sat}) = \mathcal{I}'\mathcal{I}(\mathrm{Sat})$$
$$\mathcal{H}(=) = \mathcal{I}'\mathcal{I}(=)$$

The proof is a simple induction on the complexity of formulae.[11] □

We can read the definition of logical truth and necessity through an accessibility relation between formulae. A formula ψ is *logically accessible* to ϕ if ψ is a substitution of ϕ. So, ϕ is logically true exactly when all the formulae logically accessible to it are true. Say that i is the set of modally relevant substitutions: ψ is *modally accessible* to ϕ if there is a substitution function in i from ϕ to ψ. $\underset{\cdot}{\Box}\phi$ is true iff every modally accessible ψ from ϕ is true. The process can be iterated: $\underset{\cdot}{\Box}\underset{\cdot}{\Box}\phi$ is true iff every ψ modally accessible from ϕ is such that every χ modally accessible from ψ is true.

Definition 3.7 (Accessibility relations)

$$\mathcal{L}(\phi, \psi) := \exists\mathcal{I}(\mathcal{I}(\phi) = \psi)$$
$$\mathcal{N}(\phi, \psi) := \exists\mathcal{I}(\mathcal{I} \in i \wedge \mathcal{I}(\phi) = \psi)$$

Lemma 3.8 *Given any* $\mathsf{SM} + \mathsf{SMi}$ *the following holds:*

$$\textit{(L7.1)}\ \forall\phi(\models_S \phi \leftrightarrow \forall\psi, \mathfrak{a}(\mathcal{L}(\phi,\psi) \to \mathrm{Sat}(\psi,\mathfrak{a})))$$

$$\textit{(L7.2)}\ \forall\phi\forall\mathfrak{a}\big(\mathrm{Sat}(\underset{\cdot}{\Box}\phi,\mathfrak{a}) \leftrightarrow \forall\psi(\mathcal{N}(\phi,\psi) \to \mathrm{Sat}(\psi,\mathfrak{a}))\big)$$

Proof The proofs are trivial given the definitions. □

4 Soundness

In this section I discuss soundness. I first consider a modal logic with a constant domain: where Barcan and Converse are provable.

4.1 Soundness for K

K is first order logic plus the following:

Nec: $\vdash_K \phi \Rightarrow \vdash_K \Box\phi$
K: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

All the modal logics I consider for now have the following axioms (of which some might be redundant in some of them):

BF: $\forall v\Box\phi \to \Box\forall v\phi$
CBF: $\Box\forall v\phi \to \forall v\Box\phi$

K is sound with any theory of truth we consider, for the following holds:

Lemma 4.1 *Given any* $\mathsf{SM} + \mathsf{SMi}$*:*

$$\mathsf{SM} + \mathsf{SMi} \vdash \forall\phi(\models_S \phi \to \models_S \underset{\cdot}{\Box}\phi)$$

Proof Given any $\mathsf{SM} + \mathsf{SMi}$:

$$\begin{aligned}
\models_S \phi \quad & \to \\
& \to \forall\mathcal{I},\mathfrak{a}\,\mathrm{Sat}(\mathcal{I}(\phi),\mathfrak{a}) \\
& \to \forall\mathcal{I},\mathcal{I}',\mathfrak{a}\,\mathrm{Sat}(\mathcal{H}^{\mathcal{I}+\mathcal{I}'}(\phi),\mathfrak{a}) \\
& \to \forall\mathcal{I},\mathcal{I}',\mathfrak{a}\,\mathrm{Sat}(\mathcal{I}\mathcal{I}'(\phi),\mathfrak{a}) \\
& \to \forall\mathcal{I},\mathcal{I}',\mathfrak{a}\,(\mathcal{I} \in i \to \mathrm{Sat}(\mathcal{I}\mathcal{I}'(\phi),\mathfrak{a})) \\
& \to \forall\mathcal{I}',\mathfrak{a}\,\mathrm{Sat}(\underset{\cdot}{\Box}\mathcal{I}'(\phi),\mathfrak{a}) \\
& \to \forall\mathcal{I}',\mathfrak{a}\,\mathrm{Sat}(\mathcal{I}'(\underset{\cdot}{\Box}\phi),\mathfrak{a}) \\
& \to \models_S \underset{\cdot}{\Box}\phi
\end{aligned}$$

□

Lemma 4.2 *Given any* $\mathsf{SM} + \mathsf{SMi}$*:*

$$\mathsf{SM} + \mathsf{SMi} \vdash \forall\phi,\psi(\models_S \underset{\cdot}{\Box}(\phi \underset{\cdot}{\to} \psi) \underset{\cdot}{\to} (\underset{\cdot}{\Box}\phi \underset{\cdot}{\to} \underset{\cdot}{\Box}\psi))$$

Proof Via the axioms for truth, $\mathrm{Sat}(\underset{\cdot}{\Box}(\phi \underset{\cdot}{\to} \psi),\mathfrak{a})$ is equivalent to the following:

$$\forall\mathcal{I}(\mathcal{I} \in i \to (\mathrm{Sat}(\mathcal{I}(\phi),\mathfrak{a}) \to \mathrm{Sat}(\mathcal{I}(\psi),\mathfrak{a})))$$

Also, $\mathrm{Sat}(\underset{\cdot}{\Box}\phi,\mathfrak{a})$ is equivalent to $\forall\mathcal{I}(\mathcal{I} \in i \to \mathrm{Sat}(\mathcal{I}(\phi),\mathfrak{a}))$. The proof easily follows by logic. □

Lemma 4.3 *Given any* $\mathsf{SM} + \mathsf{SMi}$*:* $\mathsf{SM} + \mathsf{SMi} \vdash \forall\phi \models_S \underset{\cdot}{\Box}\underset{\cdot}{\forall} v\phi \underset{\cdot}{\leftrightarrow} \underset{\cdot}{\forall} v\underset{\cdot}{\Box}\phi$

Proof Note that $|\mathcal{I}(\phi), u/v| = \mathcal{I}(|\phi, u/v|)$ where u and v are free. We show the equivalence as follows, where $\mathcal{H}$ are the substitutions in i:

$$\begin{array}{ll}
\quad \mathrm{Sat}(\underset{.}{\Box}\underset{.}{\forall} v\phi, \mathfrak{a}) & \quad \mathrm{Sat}(\underset{.}{\forall} v\underset{.}{\Box}\phi, \mathfrak{a}) \\
\leftrightarrow \forall\mathcal{H}\mathrm{Sat}(\mathcal{H}(\underset{.}{\forall} v\phi), \mathfrak{a}) & \leftrightarrow \mathrm{Sat}(\underset{.}{\forall} u\underset{.}{\Box}|\phi, u/v|, \mathfrak{a}) \\
\leftrightarrow \forall\mathcal{H}\mathrm{Sat}(\underset{.}{\forall} u\mathcal{H}(|\phi, u/v|), \mathfrak{a}) & \leftrightarrow \forall y\mathrm{Sat}(\underset{.}{\Box}|\phi, u/v|, \mathfrak{a}^y_u) \\
\leftrightarrow \forall\mathcal{H}\forall y\mathrm{Sat}(\mathcal{H}(|\phi, u/v|), \mathfrak{a}^y_u) & \leftrightarrow \forall\mathcal{H}\forall y\mathrm{Sat}(\mathcal{H}(|\phi, u/v|), \mathfrak{a}^y_u)
\end{array}$$

The last two lines are the same. □

Theorem 4.4 *K is sound in any* SM + SMi

Proof The proof is by induction on the length of proofs of K. By theorem 3.3, lemma 4.2 and 4.3, each proof of length 0 is valid. The induction step follows by theorem 3.3 and lemmas 4.1, 4.2 and 4.3. □

Since no matter which i I consider **K** is sound, the minimal logic for which I can define validity in the present substitutional system is K: the weakest normal modal logic.

4.2 Soundness for other common modal calculi I give now some examples of relations between features of $\mathcal{N}$ and modal axioms, as well as a more general way of checking soundness.

M: $\Box\phi \rightarrow \phi$
B: $\phi \rightarrow \Box\Diamond\phi$
4: $\Box\phi \rightarrow \Box\Box\phi$
5: $\Diamond\phi \rightarrow \Box\Diamond\phi$

We offer now a resolution strategy to check soundness.

Definition 4.5 (Path) .

i. $\phi \approx \psi : \forall\mathfrak{a}\mathrm{Sat}(\phi\underset{.}{\leftrightarrow}\psi, \mathfrak{a})$;
ii. There is a direct path from ϕ to ψ: $\mathcal{N}(\phi, \psi)$;
iii. There is an indirect path from ϕ to ψ: $\exists\chi(\mathcal{N}(\phi, \chi) \wedge \chi \approx \psi)$;
iv. Path reflexivity: for any ϕ, there is always a path from ϕ to ϕ;
v. Path symmetry: for any ϕ and ψ, if there is a direct path from ϕ to ψ, there is an indirect path from ψ to ϕ;
vi. Path transitivity: for any ϕ, ψ, χ, if there is a direct path from ϕ to ψ and ψ to χ, then there is an indirect path from ϕ to χ;
vii. Path Euclideanicity: for any ϕ, ψ, χ, if there is a direct path from ϕ to ψ and ϕ to χ, then there is an indirect path from ψ to χ.

Clearly, since anything is self equivalent, if there is a direct path between x and y there is also an indirect one. We offer the following lemma:

Lemma 4.6 *Validity conditions:*

*i. **M** is valid if path reflexivity holds;*
*ii. **B** is valid if path symmetry holds;*
*iii. **4** is valid if path transitivity holds;*
*iv. **5** is valid if path Euclideanicity holds.*

Proof I prove lemma 4.6.i–ii. Similar proofs work for 4.6.iii–iv.

Suppose $\mathrm{Sat}(\underset{.}{\Box}\phi, \mathfrak{a})$. By lemma 3.8.ii, $\forall\psi(\mathcal{N}(\phi,\psi) \rightarrow \mathrm{Sat}(\psi,\mathfrak{a}))$. Via path reflexivity, some $\mathcal{I}(\phi)$ is equivalent to ϕ, so $\mathrm{Sat}(\phi,\mathfrak{a})$. ϕ and $\mathfrak{a}$ were arbitrary, so I can generalize to any ϕ and $\mathfrak{a}$ and, via lemma 3.2, to any $\mathcal{I}$ and $\mathfrak{a}$.

For lemma 4.6.ii, assume that $\mathrm{Sat}(\phi,\mathfrak{a})$ and yet $\neg\mathrm{Sat}(\underset{.}{\Box}\underset{.}{\Diamond}\phi,\mathfrak{a})$. Where $\mathcal{H}$ and $\mathcal{I}$ are all in i, via the axioms for truth, this second formula is equivalent to: $\exists\mathcal{H}\forall\mathcal{I}\neg\mathrm{Sat}(\mathcal{I}\mathcal{H}(\phi),\mathfrak{a})$. By definition of $\mathcal{N}$, trivially $\mathcal{N}(\phi,\mathcal{H}(\phi))$. Via path symmetry, since $\mathcal{N}(\phi,\mathcal{H}(\phi))$, for some $\mathcal{I} \in i$ $\mathcal{N}(\mathcal{H}(\phi),\mathcal{I}\mathcal{H}(\phi))$, where $\mathcal{I}\mathcal{H}(\phi) \approx \phi$. But $\mathrm{Sat}(\phi,\mathfrak{a})$, so $\mathrm{Sat}(\mathcal{I}\mathcal{H}(\phi),\mathfrak{a})$, contrary to our assumption that, for all $\mathcal{I} \in i$, $\neg\mathrm{Sat}(\mathcal{I}\mathcal{H}(\phi),\mathfrak{a})$. □

System T is K plus axiom **M**. System S4 is K plus **4** and **M**. S5 is K plus **M** plus **5**.

Theorem 4.7 (Soundness for T, S4, S5) *The following holds:*

i. Calculus T is sound for validity in any $\mathsf{SM}+\mathsf{SMi}$ *where path reflexivity holds;*
ii. Calculus S4 is sound for validity in any $\mathsf{SM}+\mathsf{SMi}$ *where path reflexivity and path transitivity holds;*
iii. Calculus S5 is sound for validity in any $\mathsf{SM}+\mathsf{SMi}$ *where path reflexivity and path Euclideanicity holds.*

Proof The proofs are by induction on the length of proofs of T, S4 and S5. □

5 Completeness

In this section I explore some completeness results. Completeness in modal logic is a notoriously complex matter: some techniques work for some logics but not for others and, in any case, there are continuum many Kripke incomplete normal modal logics.[12]

5.1 Canonical model and adequacy For canonical modal logics, we can construct a canonical truth theory. A truth theory X is canonical for a modal calculus Y whenever Y is adequate for validity defined in X. That is, when X proves the following: $\forall\phi(Y \vdash \phi \leftrightarrow \models_S \phi)$. I call a choice i of $\mathcal{I}$ canonical for Y if the canonical truth theory for Y is $\mathsf{SM}+\mathsf{SMi}$. I will use a Henkin style completeness proof for the calculus K, by specifying the canonical i Similar proofs can be run for other canonical modal calculi. To prove the adequacy theorem, we need to prove the following 'substitutional truth lemma':

Lemma 5.1 (Substitutional Truth lemma) *There is a truth theory* $\mathsf{SM}+\mathsf{SM}\xi$ *where the following holds, where* Γ *is a meta variable for Henkin extensions of K:*

$$\exists\mathfrak{c}\forall\Gamma\exists\mathcal{I}\forall\phi\big(\phi \in \Gamma \leftrightarrow \mathrm{Sat}(\mathcal{I}(\phi),\mathfrak{c})\big)$$

The rest of the section contains the proof of the lemma, and finally the adequacy theorem that follows from it.

The canonical Kripke model is usually built by taking as points of the model all complete and consistent extensions of the modal calculus with certain properties, which we may call 'Henkin extensions'. It is then shown that, with a suitable choice of accessibility relation, membership to the extension is equivalent to truth at that point. I will do something similar: the points in my system are substitutions, and the accessibility relation is $\mathcal{N}$.

A first issue is that the cardinality of the canonical model for K is the continuum, yet we have but denumerably many substitution functions. We have two options: one is to add uncountably many variables. Alternatively, we can appeal to a theorem similar to the downward Löwenheim Skolem:

Theorem 5.2 *Given any consistent extension of K, for any of its characteristic Kripke model there is an equivalent countable model; that is, a model with at most countably many worlds.*[13]

Proof For a proof, see Chagrov and Zakharyaschev 5, theorem 6.29. □

By theorem 5.2, there is a countable equivalent to the uncountable canonical model: its countable filtration; it follows that our denumerable substitutions are enough.

A Henkin extension of K for the language of L_{SM} is a set of sentences Γ with the following properties:

$$\forall\phi(\mathsf{K}\vdash\phi\rightarrow\phi\in\Gamma) \qquad \text{(K conservativity)}$$

$$\forall\phi(\underset{\cdot}{\neg}\phi\in\Gamma\rightarrow\phi\notin\Gamma) \qquad \text{(Consistency)}$$

$$\forall\phi(\phi\notin\Gamma\rightarrow\underset{\cdot}{\neg}\phi\in\Gamma) \qquad \text{(Maximality)}$$

$$\forall\phi,v\exists v'(\ulcorner\exists v\phi\rightarrow|\phi,v'/v|\urcorner\in\Gamma) \qquad \text{(Henkin Witness)}$$

Where v and v' are meta variables for variables. The Henkin witness property (Henkin property, for short) is usually achieved by adding constants to the language; here we use variables as witnesses, as in Halbach 12. We will also use variables as proxies for points of the Kripke model. Because of this two fold use, and because substitutions cannot switch free variables, variables are to be handled carefully to avoid variable clashes. Out of notational convenience, in the following I will assume that variables in the language have a subscript and a superscript, the latter being either 0 or 1. It is routine logic homework to show that everything that can be done using superscripts and subscripts can be done using only subscripts: we can always get rid of one superscript $m>0$ in the following way: first we set $v_n^{m-1}=v_{2n+1}^{m-1}$, and then we set $v_n^m=v_{2n}^{m-1}$. By reiterated applications of the process, we end up with only one superscript: 0, and then we just set $v_n^0=v_n$. When we translate v_n^1 as an even variable and v_n^0 as an odd variable, in the new notation to be in signed form means that every free variable has superscript 1 and every bound variable has superscript 0.

Given a consistent set of formulae, we build Henkin extensions in a somewhat unorthodox way: first, in each formula we get rid of the superscript 1 in the way just described: now each variable of each formula has superscript 0. Then, we use variables with superscript 1 as witnesses, which are guaranteed to be fresh by construction. We then again get rid of the superscript 1: we are left with formulae whose variables have all superscript 0. There is no loss of generality: as just shown, superscripts do not add any expressive strength to the language. All formulae of the Henkin extensions have variables with superscript 0; that is, odd subscripts, if we had no superscripts.

We assume we have well ordered the countable subset of Henkin extensions we need to match the countable filtration of the canonical model in a suitable way. Given such a well order, define $g(\Gamma)=n$ iff Γ is the nth Henkin extension. We will use the following two variable assignments $\mathfrak{c}$ and $\mathfrak{b}$:

$$\mathfrak{c}(v^0_n) = n \text{ for all } n \in \omega \qquad \mathfrak{b}(v^1_{g(\Gamma)}) = \Gamma \text{ for all } \Gamma$$
$$\mathfrak{c}(v^1_3) = \mathfrak{b}$$

The values not listed are set to 0 and do not matter.

For any Γ, to define its characteristic substitution, called $\mathcal{I}^\Gamma$, we only need to set the substitution of the atomic formulae. We do so for '$\in$', similar definitions hold for 'Sat' and '$=$'. Henkin extensions have only formulae with variables with superscript 0, by construction. Thus, our aim is to show that $\text{Sat}(\mathcal{I}^\Gamma(v^0_n \underset{\cdot}{\in} v^0_m), \mathfrak{c}) \leftrightarrow v^0_n \underset{\cdot}{\in} v^0_m \in \Gamma$. We define the following recursive function f:

$$f(x) = \begin{cases} x & \text{if } x \in \omega \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{I}^\Gamma(\in)$ is the following formula, where $g(\Gamma)$ is k:

$$\forall x, y, t, z\big((x = f(v^1_1) \land y = f(v^1_2) \land t = \ulcorner v^0_x \in v^0_y \urcorner \land z = \bar{t} \underset{\cdot}{\in} v^1_k) \to \text{Sat}(z, v^1_3)\big)$$

I abbreviate it to:

$$\forall x(x = \overline{v^0_{f(v^1_1)} \underset{\cdot}{\in} v^0_{f(v^1_2)}} \underset{\cdot}{\in} v^1_{g(\Gamma)} \to \text{Sat}(x, v^1_3))$$

$\mathcal{I}^\Gamma(\in)$ is in signed form (we assume x, y, z and t have superscript 0). The formula has only v^1_1, v^1_2 and v^1_3 free. All these variables have superscript 1, so they will not be inadvertently substituted when we construct $\mathcal{I}^\Gamma(\phi)$, where ϕ is in some Henkin extension, since Henkin extensions have only variables with superscript 0. Via the definition of $\mathcal{I}$, if we ignore the renaming of the bound variables, we obtain the following:

$$\mathcal{I}^\Gamma(v^0_n \underset{\cdot}{\in} v^0_m) = \forall x(x = \overline{v^0_{f(v^0_n)} \underset{\cdot}{\in} v^0_{f(v^0_m)}} \underset{\cdot}{\in} v^1_{g(\Gamma)} \to \text{Sat}(x, v^1_3))$$

Under $\mathfrak{c}$, v^1_3 is $\mathfrak{b}$ and v_n is n; under $\mathfrak{b}$, $v^1_{g(\Gamma)}$ is Γ, so the formula is saying that the formula that says that $\ulcorner v^0_n \in v^0_m \urcorner$ is in Γ is true. Since the formula talks about a formula of set theory, SM7 applies, so if it is true that such a formula is true, then the formula is true. The equivalence between $\mathcal{I}^\Gamma$ and Γ is proven below (every line is equivalent to the next):

$$\begin{array}{ll}
\text{Sat}(\mathcal{I}^\Gamma(v^0_n \underset{\cdot}{\in} v^0_m), \mathfrak{c}) & \\
\forall x(x = \overline{v^0_{f(\mathfrak{c}(v^0_n))} \underset{\cdot}{\in} v^0_{f(\mathfrak{c}(v^0_m))}} \underset{\cdot}{\in} v^1_{g(\Gamma)} \to \text{Sat}(\underset{\cdot}{\text{Sat}}(v, v^1_3), \mathfrak{c}^x_v)) & \text{lm.2.2, df. } \mathcal{I}^\Gamma \\
\forall x(x = \overline{v^0_n \underset{\cdot}{\in} v^0_m} \underset{\cdot}{\in} v^1_{g(\Gamma)} \to \text{Sat}(\underset{\cdot}{\text{Sat}}(v, v^1_3), \mathfrak{c}^x_v)) & \text{df. } f, \mathfrak{c} \\
\text{Sat}(\ulcorner v^0_n \in v^0_m \urcorner \underset{\cdot}{\in} v^1_{g(\Gamma)}, \mathfrak{c}(v^1_3)) & \text{SM7} \\
\ulcorner v^0_n \in v^0_m \urcorner \in \mathfrak{b}(v^1_{g(\Gamma)}) & \text{df. } \mathfrak{c}, \text{lm.2.2} \\
\ulcorner v^0_n \in v^0_m \urcorner \in \Gamma & \text{def. } \mathfrak{b}
\end{array}$$

This takes care of the induction base. We now proceed to the induction step. The cases for negation and conjunction are trivial. The case for the quantifier is handled as in Halbach 12, 325. For the modal case, first we need to define the set ξ of modally relevant $\mathcal{I}$ that is canonical for K. The canonical accessibility relation C between Henkin extensions is defined as follows:

$$C(x, y) := \forall \phi(\underset{\cdot}{\Box} \phi \in x \to \phi \in y)$$

We now need to define ξ via C, so that, when we apply a modally relevant substitution to a $\mathcal{I}^{\Gamma}(\phi)$, we 'end up' in a formula that is equivalent to some $\mathcal{I}^{\Omega}$ such that $C(\Gamma, \Omega)$ (and, conversely, whenever $C(\Gamma, \Omega)$, there is a modally relevant substitution that makes us jump from $\mathcal{I}^{\Gamma}$ to $\mathcal{I}^{\Omega}$). Now, if we could switch free variables, this would be easy, but substitutions cannot switch free variables. So, instead, I chose to 'double quote' the formula, and then change the clause for $\underset{\cdot}{\mathrm{Sat}}(v, u)$, so that instead of v being the formula that says that x is in Γ, it is the formula that says that x is in Ω.[14] Here are the details:

Definition 5.3 (Canonical condition) $\mathcal{I}$ is in ξ if and only if it satisfies the following condition: there is a Γ and Ω such that $C(\Gamma, \Omega)$ and, for all ϕ_{at}, $\mathcal{I}$ is as follows:

$$\mathcal{I}(\phi_{at}) = \begin{cases} \phi_{at} & \text{if } \phi_{at} \neq \ulcorner \mathrm{Sat}(v, u) \urcorner \\ \ulcorner \mathrm{Sat}(|v, v^1_{g(\Omega)}/v^1_{g(\Gamma)}|, u) \urcorner & \text{otherwise} \end{cases}$$

We make use of the following observation:

Lemma 5.4 (Composition lemma) *For any two Γ and Ω, $C(\Gamma, \Omega)$ exactly when there is a substitution function $\mathcal{H}$ is in ξ such that, for any ϕ, truth under $\mathcal{H}\mathcal{I}^{\Gamma}$ is equivalent to truth under $\mathcal{I}^{\Omega}$. That is:*

$$\forall \Gamma, \Omega \Big(C(\Gamma, \Omega) \leftrightarrow \exists \mathcal{H} \big(\mathcal{H} \in \xi \wedge \forall \phi, \mathfrak{a} (\mathrm{Sat}(\mathcal{H}\mathcal{I}^{\Gamma}(\phi), \mathfrak{a}) \leftrightarrow \mathrm{Sat}(\mathcal{I}^{\Omega}(\phi), \mathfrak{a})) \big) \Big)$$

Proof The right to left direction follows from the definition of ξ. For the other direction, by construction of ξ, the relevant $\mathcal{H}$ is clearly the one that swaps $\underset{\cdot}{\mathrm{Sat}}(v, u)$ with $\underset{\cdot}{\mathrm{Sat}}(|v, v^1_{g(\Omega)}/v^1_{g(\Gamma)}|, u)$. That truth under $\mathcal{H}\mathcal{I}^{\Gamma}$ is equivalent to truth under $\mathcal{I}^{\Omega}$ is shown by induction on the complexity of ϕ. We prove the base, using $v^r_n \underset{\cdot}{\in} v^s_m$ as an example. First, in $\mathcal{I}^{\Gamma}(\in)$ there is only one occurrence of Sat. So, the following holds:

$$\mathcal{H}\mathcal{I}^{\Gamma}(v^r_n \underset{\cdot}{\in} v^s_m) = \forall x (x = \overline{v^0_{f(v^r_n)} \underset{\cdot}{\in} v^0_{f(v^s_m)}} \underset{\cdot}{\in} v^1_{g(\Gamma)} \rightarrow \mathrm{Sat}(|x, v^1_{g(\Omega)}/v^1_{g(\Gamma)}|, v^1_3))$$

The proof continues as follows (every line is equivalent to the next):

$$\mathrm{Sat}(\mathcal{H}\mathcal{I}^{\Gamma}(v^r_n \underset{\cdot}{\in} v^s_m), \mathfrak{a}) \tag{1}$$

$$\mathrm{Sat}(\underset{\cdot}{\forall} x (x \underset{\cdot}{=} \overline{v^0_{f(v^r_n)} \underset{\cdot}{\in} v^0_{f(v^s_m)}} \underset{\cdot}{\in} v^1_{g(\Gamma)} \underset{\cdot}{\rightarrow} \underset{\cdot}{\mathrm{Sat}}(|x, v^1_{g(\Omega)}/v^1_{g(\Gamma)}|, v^1_3)), \mathfrak{a}) \tag{2}$$

$$\forall x (x = \overline{v^0_{f(\mathfrak{a}(v^r_n))} \underset{\cdot}{\in} v^0_{f(\mathfrak{a}(v^s_m))}} \underset{\cdot}{\in} v^1_{g(\Gamma)} \rightarrow \mathrm{Sat}(\underset{\cdot}{\mathrm{Sat}}(|v, v^1_{g(\Omega)}/v^1_{g(\Gamma)}|, v^1_3), \mathfrak{a}^x_v)) \tag{3}$$

$$\forall x (x = \overline{v^0_{f(\mathfrak{a}(v^r_n))} \underset{\cdot}{\in} v^0_{f(\mathfrak{a}(v^s_m))}} \underset{\cdot}{\in} v^1_{g(\Gamma)} \rightarrow \mathrm{Sat}(|x, v^1_{g(\Omega)}/v^1_{g(\Gamma)}|, \mathfrak{a}(v^1_3))) \tag{4}$$

$$\ulcorner v^0_{f(\mathfrak{a}(v^r_n))} \in v^0_{f(\mathfrak{a}(v^s_m))} \urcorner \in \mathfrak{a}(v^1_3)(v^1_{g(\Omega)}) \tag{5}$$

$$\forall x (x = \overline{v^0_{f(\mathfrak{a}(v^r_n))} \underset{\cdot}{\in} v^0_{f(\mathfrak{a}(v^s_m))}} \underset{\cdot}{\in} v^1_{g(\Omega)} \rightarrow \mathrm{Sat}(x, \mathfrak{a}(v^1_3))) \tag{6}$$

$$\mathrm{Sat}(\mathcal{I}^{\Omega}(v^r_n \underset{\cdot}{\in} v^s_m), \mathfrak{a}) \tag{7}$$

In the step from line 3 to line 4 we are applying SM7, since the formula belongs to L_{ZFC}. The step from 4 to 5 is proven as follows: $\mathfrak{a}(v^1_3)$ must be an assignment. If it was not, then $\forall x \neg \mathrm{Sat}(x, \mathfrak{a}(v^1_3))$ via SM8, in particular when $x = \overline{v^0_{f(\mathfrak{a}(v^r_n))} \underset{\cdot}{\in} v^0_{f(\mathfrak{a}(v^s_m))}} \underset{\cdot}{\in} v^1_{g(\Gamma)}$, so line 4 would not hold, contrary to our assumption. Since $\mathfrak{a}(v^1_3)$ is an assignment, and x is a formula of set theory, we can apply lemma 2.2, and proceed to line 5.

The induction step follows by definition of $\mathcal{I}$ and the compositional axioms of Sat. □

The lemma shows that truth under $\mathcal{H}\mathcal{I}^{\Gamma}(\phi)$ is equivalent to truth under $\mathcal{I}^{\Omega}(\phi)$. For this reason, we call $\mathcal{H}$ the composition from $\mathcal{I}^{\Omega}$ to $\mathcal{I}^{\Gamma}$, written: $\mathcal{H}^{\mathcal{I}^{\Gamma}\mapsto\mathcal{I}^{\Omega}}$.

We are now ready to prove the induction step of the substitutional truth lemma for the modal case. First, suppose $\underset{\cdot}{\Box}\phi \in \Gamma$ and yet $\neg\text{Sat}(\mathcal{I}^{\Gamma}(\underset{\cdot}{\Box}\phi), \mathfrak{c})$. Via commutation of $\mathcal{I}$ with $\Box$, $\neg\text{Sat}(\underset{\cdot}{\Box}\mathcal{I}^{\Gamma}(\phi), \mathfrak{a})$. By the axioms for truth, there exists a $\mathcal{H}$ in ξ such that $\neg\text{Sat}(\mathcal{H}\mathcal{I}^{\Gamma}(\phi), \mathfrak{c})$. By construction of ξ, there is a $\mathcal{H}^{\mathcal{I}^{\Gamma}\mapsto\mathcal{I}^{\Omega}}$ such that $C(\Gamma, \Omega)$ and $\neg\text{Sat}(\mathcal{H}^{\mathcal{I}^{\Gamma}\mapsto\mathcal{I}^{\Omega}}\mathcal{I}^{\Gamma}(\phi), \mathfrak{c})$. Via the composition lemma, $\neg\text{Sat}(\mathcal{I}^{\Omega}(\phi), \mathfrak{c})$. By induction hypothesis, $\phi \notin \Omega$. By construction of C, since $\underset{\cdot}{\Box}\phi \in \Gamma$, $\phi \in \Omega$. Contradiction.

For the other direction, assume that $\underset{\cdot}{\Box}\phi \notin \Gamma$ and yet $\text{Sat}(\mathcal{I}^{\Gamma}(\underset{\cdot}{\Box}\phi), \mathfrak{c})$. We claim that the set $S = \{\psi : \underset{\cdot}{\Box}\psi \in \Gamma\} \cup \{\underset{\cdot}{\neg}\phi\}$ is consistent. Suppose it was not: then a finite subset $\psi_1, .., \psi_n$ in Γ is such that $(\psi_1 \wedge .. \wedge \psi_n) \to \bot$. By logic, $(\psi_1 \wedge .. \wedge \psi_n) \to \phi$. But Γ is closed under K rules, so by necessitation and **K** $(\Box\psi_1 \wedge .. \wedge \Box\psi_n) \to \Box\phi$. Thus, $\underset{\cdot}{\Box}\phi \in \Gamma$, contrary to our assumption. Since S is consistent, by Lindenbaum's lemma we can extend it to a maximal and consistent set, to which we can add the Henkin Witness property via the renaming of variables explained above. We obtain a Henkin extension Ω where, by construction, $C(\Gamma, \Omega)$, and $\phi \notin \Omega$. We construct the substitution that maps $\text{Sat}(v, u)$ to $\text{Sat}(|v, v^1_{g(\Omega)}/v^1_{g(\Gamma)}|, u)$, and reads the other atomic formulae homophonically. By construction of ξ, the substitution is in ξ: it is $\mathcal{H}^{\mathcal{I}^{\Gamma}\mapsto\mathcal{I}^{\Omega}}$. By assumption, $\text{Sat}(\underset{\cdot}{\Box}\mathcal{I}^{\Gamma}(\phi), \mathfrak{c})$, so $\text{Sat}(\mathcal{I}\mathcal{I}^{\Gamma}(\phi), \mathfrak{c})$, for any $\mathcal{I} \in \xi$, in particular $\text{Sat}(\mathcal{H}^{\mathcal{I}^{\Gamma}\mapsto\mathcal{I}^{\Omega}}\mathcal{I}^{\Gamma}(\phi), \mathfrak{c})$. Via the composition lemma $\text{Sat}(\mathcal{I}^{\Omega}(\phi), \mathfrak{c})$, and $\phi \in \Omega$, by induction hypothesis, contrary to our assumption. This concludes the induction and the proof of the substitutional truth lemma.

Theorem 5.5 (K adequacy) *There is a canonical truth theory for K. That is, a truth theory* $\mathsf{SM} + \mathsf{SMi}$ *such that:*

$$\mathsf{SM} + \mathsf{SMi} \vdash \forall\phi(K \vdash \phi \leftrightarrow \models_S \phi)$$

Proof The theory in question is of course $\mathsf{SM} + \mathsf{SM}\xi$. Theorem 4.4 ensures soundness. For the right to left direction, suppose that ϕ is not provable in K. Then $\underset{\cdot}{\neg}\phi$ is in some Henkin extension of K. Via lemma 5.1, $\neg\text{Sat}(\mathcal{I}^{\Gamma}(\phi), \mathfrak{a})$, and thus $\not\models_S \phi$. $\Box$

The proofs used in this section can be generalized to the usual, familiar canonical modal logics like S4, S5, B etc. By construction, ξ will ensure the relevant features for soundness (that is, path reflexivity, path transitivity etc). We show this for T. The canonical condition for T is reflexive, so it follows that there is, for any Γ, a $\mathcal{H}$ in ξ such that $\mathcal{H}(\text{Sat}(v, u) = \text{Sat}(|v, v^1_{g(\Gamma)}/v^1_{g(\Gamma)}, u)$, leaving the other atomic formulae the same. We show that, for any ϕ, $\phi \approx \mathcal{H}(\phi)$. The base is trivial for $v \in u$ and $v = u$. For $\text{Sat}(v, u)$, we reason as follows. $|x, y/y| = x$ and $\text{Sat}(|x, y/y'|, z)$ is actually a short for $\forall t(t = |x, y/y'| \to \text{Sat}(t, z))$.

$$\begin{aligned}
\text{Sat}(\underset{\cdot}{\text{Sat}}(v, u), \mathfrak{a}) &\leftrightarrow \forall x(x = \mathfrak{a}(v) \to \text{Sat}(\underset{\cdot}{\text{Sat}}(v_n, u), \mathfrak{a}^x_{v_n})) \\
&\leftrightarrow \forall x(x = |\mathfrak{a}(v), v^1_{g(\Gamma)}/v^1_{g(\Gamma)}| \to \text{Sat}(\underset{\cdot}{\text{Sat}}(v_n, u), \mathfrak{a}^x_{v_n})) \\
&\leftrightarrow \text{Sat}(\underset{\cdot}{\forall} v_n(v_n \underset{\cdot}{=} |v, v^1_{g(\Gamma)}/v^1_{g(\Gamma)}| \underset{\cdot}{\to} \underset{\cdot}{\text{Sat}}(v_n, u)), \mathfrak{a}) \\
&\leftrightarrow \text{Sat}(\mathcal{I}^{\Gamma}(\text{Sat}(v, u)), \mathfrak{a})
\end{aligned}$$

The induction step follows because $\mathcal{I}$ always reads connectives and quantifiers homophonically. Thus, since $\mathcal{H}(\phi) \approx \phi$, there is always a path from ϕ to ϕ, and the system is path reflexive.

The adequacy proof cannot be extended to modal logics for which there is no canonical model, even when such logics can be shown to be Kripke complete via alternative methods. For example, the logic of provability GL is K4 plus Löb's axiom $\Box(\Box\phi \to \phi) \to \phi$. GL is not canonical (and not strongly complete) but admits selective filtration, thus it has the finite model property, being characterized by the class of finite strict partial orders [5, 150].[15] Filtrations do not work in the substitutional framework, because each truth theory and choice of i can copy at most one filtration, for some specific ϕ. Of course, if we wanted we could change the definition of validity, by quantifying over different truth theories, which differ in the choice of i. An argument is valid iff it is substitutionally valid in all these truth theories. Then, we can pick as relevant all and only the modal theories that correspond to each filtration, and adequacy would follow. However, this choice would betray the substitutional understanding of validity: validity is preservation of truth—simple truth—under all substitutions. It is not preservation of different models of truth under all substitutions. The correct philosophical conclusion is simply that substitution theoretic semantics is not as strong as Kripke semantics.

6 Identity and Quantification

The state of identity in the substitutional account is a tricky subject. There is a trade off between three different conditions:

a. Quantifiers are logical;
b. Identity is logical;
c. Validity for a first order, non modal language without identity should coincide with classical first order logic (without identity).

So far, I chose to give up (b), and keep (a) and (c). If we keep (a) and (b), we need to give up (c), because the system will prove 'There are at least n things', for any n, which is not a classical validity. This condition can be stated using only identity, quantifiers and connectives. Since we are embracing (a) and (b), all these expressions should be read homophonically, so the sentence is logically true if true; it is true, so it is logically true.[16]

As [12] discusses, if we use the 'relativized substitutions' mentioned above, the definition of a substitution function $\mathcal{I}$ is changed so that quantifiers are treated 'quasi homophonically', and can get restricted by a fixed condition δ; that is, $\mathcal{I}(\underset{\cdot}{\forall} v\phi)$ might be $\underset{\cdot}{\forall} v(\delta \to \phi))$. The new system can handle domain variation. In this case, we are partially giving up (a); however, we can keep (b). The logic one naturally obtains is free and inclusive, since whenever $\mathcal{I}$ chooses $\delta = \bot$, $\mathcal{I}(\exists v\phi)$ is never true.[17] So, we are giving up (c) as well, this time by under generating classical validities.[18]

In the remainder of the paper I sketch a semantics for modal logic where identity is logical and quantifiers are read quasi homophonically. First, we redefine the clause for $\underset{\cdot}{\forall} v\phi$ and $v \underset{\cdot}{=} u$ in the definition of $\mathcal{I}$ as follows:

$$\mathcal{I}(v \underset{\cdot}{=} u) := v \underset{\cdot}{=} u$$
$$\mathcal{I}(\underset{\cdot}{\forall} v\phi) := \underset{\cdot}{\forall} u(|\delta, u/\alpha_1| \underset{\cdot}{\to} |\mathcal{I}(\phi), u/v|)$$

δ is some fixed condition. We assume that δ is in signed form, and the usual care needs to be applied to avoid unintended variable bindings. Restricting quantifiers to a condition δ is optional: we are considering both unrelativized and relativized substitutions, so the new system is a proper extension of the old one.

In the new system we can falsify BF and CBF. A substitution $\mathcal{I}$ is now such that $\mathcal{I}(\underset{\cdot}{\forall} v \underset{\cdot}{\Box}\phi \underset{\cdot}{\rightarrow} \underset{\cdot}{\Box}\underset{\cdot}{\forall} v\phi)$ is $\underset{\cdot}{\forall} v(\delta \underset{\cdot}{\rightarrow} \underset{\cdot}{\Box}\mathcal{I}(\phi)) \underset{\cdot}{\rightarrow} \underset{\cdot}{\Box}\underset{\cdot}{\forall} v(\delta \underset{\cdot}{\rightarrow} \mathcal{I}(\phi))$. We can easily falsify this formula in some truth theory. For example, consider the following instance of BF, where ϕ is $\neg x = x$ and δ is $x \in x$:

$$\forall x(x \in x \rightarrow \Box\neg x = x) \rightarrow \Box\forall x(x \in x \rightarrow \neg x = x)$$

This instance of BF is false in a truth theory where, for all $\mathcal{I}$ in i, $\mathcal{I}(v \underset{\cdot}{\in} v) = \underset{\cdot}{\neg} v \underset{\cdot}{\in} v$, for any v. The same truth-theory falsifies CBF; just consider the following instance of it where ϕ is $\neg x = x$ and δ is $\neg x \in x$:

$$\Box\forall x(\neg x \in x \rightarrow \neg x = x) \rightarrow \forall x(\neg x \in x \rightarrow \Box\neg x = x)$$

I will not analyze further the conditions for the validity of BF and CBF. Rather, in the rest of the section, I show soundness and completeness for FKI: the result of adding to free logic the axioms of identity, the modal axiom **K** and the rule **Nec** (BF and CBF are not valid).

6.1 Soundness and completeness for FKI We rewrite the definition of $\models_S$ using the new definition of $\mathcal{I}$. One can easily check the following lemma:

Lemma 6.1 *For any* $\mathsf{SM}+\mathsf{SMi}$*, where* ϕ *and* Δ *are formulae and sets of formulae of* L_{SM}*, the following holds:*

$$\mathsf{SM}+\mathsf{SMi} \vdash \forall\phi, \Delta(\Delta \vdash_{\mathsf{FK}=} \phi \rightarrow \Delta \models_S \phi)$$

Proof The proof is by induction on the length of proofs of FKI. Substitution of identicals is valid by lemma 3.4 and because $\mathfrak{a}(v) = \mathfrak{a}(u)$ exactly when under all $\mathcal{I}$, $\mathrm{Sat}(\mathcal{I}(v \underset{\cdot}{=} u), \mathfrak{a})$. $\Box$

The resolution strategies to check soundness for different modal calculi remain the same. Statements of identity are all 'path equivalent'. Theorem 4.4 and 4.7 still hold.

For completeness, we show that if something is substitutionally valid, it is true in the canonical Kripke model of FKI, so it is provable in FKI. A Kripke model $\mathfrak{M} = \langle W, D, R, V, Q\rangle$ for the language of L_{SM} is defined in the usual way, where W is the set of worlds, D the domain, R the accessibility relation, V the valuation function and Q a function from W to the power set of D. $\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \phi$ is defined as usual:

$$\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \underset{\cdot}{\mathrm{Sat}}(v, u) \leftrightarrow \langle\mathfrak{a}(v), \mathfrak{a}(u), w\rangle \in V(\mathrm{Sat})$$
$$\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models v \underset{\cdot}{\in} u \leftrightarrow \langle\mathfrak{a}(v), \mathfrak{a}(u), w\rangle \in V(\in)$$
$$\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models v \underset{\cdot}{=} u \leftrightarrow \mathfrak{a}(v) = \mathfrak{a}(u)$$
$$\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \underset{\cdot}{\neg}\phi \leftrightarrow \neg\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \phi$$
$$\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \psi \underset{\cdot}{\wedge} \phi \leftrightarrow \langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \psi \wedge \langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \phi$$
$$\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \underset{\cdot}{\forall} v\phi \leftrightarrow \forall y(y \in Q(w) \rightarrow \langle\mathfrak{M}, w, \mathfrak{a}^y_v\rangle \models \phi)$$
$$\langle\mathfrak{M}, w, \mathfrak{a}\rangle \models \underset{\cdot}{\Box}\phi \leftrightarrow \forall x(R(w, x) \rightarrow \langle\mathfrak{M}, x, \mathfrak{a}\rangle \models \phi)$$

ϕ is true in $\mathfrak{M}$ iff true in $\mathfrak{M}$ under all $\mathfrak{a}$ and $w \in W$.

Define π as the result of getting rid of superscript 1:

$$\pi(v^0_n) = v^0_{2n+1}$$
$$\pi(v^1_n) = v^0_{2n}$$

$\pi(\phi) =$ the result of uniformly applying π to each variable in ϕ.

We prove the following lemma:

Lemma 6.2 *For any countable Kripke model* $\mathfrak{M}$ *of FKI there is a truth theory* $\mathsf{SM}+\mathsf{SM}\xi$ *where the following holds, where* w *are worlds of* $\mathfrak{M}$ *and* $\mathcal{H}$ *substitutions in* ξ*:*

$$\forall\mathfrak{a}\exists\mathfrak{c}\forall w\exists\mathcal{H}\forall\phi\big(\langle\mathfrak{M},w,\mathfrak{a}\rangle\models\pi(\phi)\leftrightarrow\mathrm{Sat}(\mathcal{H}\pi(\phi),\mathfrak{c})\big)$$

The rest of the section is the proof of lemma 6.2 by induction on the complexity of the formulae, and the adequacy theorem that follows from it.

For any ϕ, $\pi(\phi)$ has only variables with superscript 0. Consider any countable Kripke model $\mathfrak{M}=\langle W,D,R,V,Q\rangle$. We match the elements of W with the natural numbers. We order the elements of the set $W\cup\{V,Q\}$. $g(x)=n$ iff x is the xth thing in the order. Consider any $\mathfrak{a}$. Define the following assignments:

$$\mathfrak{c}(v_n^0)=\mathfrak{a}(v_n^0)\qquad\qquad \mathfrak{b}(v^1_{g(x)})=x$$
$$\mathfrak{c}(v_3^1)=\mathfrak{b}$$

The values not mentioned are set to zero. Given a w of $\mathfrak{M}$, we build its characteristic $\mathcal{I}^w$ inductively from the substitution for atomic formulae and the condition for the quantifiers. We define $\mathcal{I}^w(\in)$. A similar definition works for $\mathcal{I}^w(\mathrm{Sat})$. In the definition, $\forall v_n^1$ ranges over variables with superscript 1.

$$\begin{aligned}
x\simeq y &:= \forall v_n^1 x(v_n^1)=y(v_n^1)\\
\alpha &:= \langle v_1^0,v_2^0,v^1_{g(w)}\rangle\in v^1_{g(V)}(\in)\\
\beta(x,y,z,t) &:= \mathrm{Asg}(x)\land x(v_1^0)=y\land x(v_2^0)=z\land x\simeq t\\
\mathcal{I}(\in) &:= \forall x,y((x=\ulcorner\alpha\urcorner\land\beta(y,v_1^1,v_2^1,v_3^1))\rightarrow\mathrm{Sat}(x,y))
\end{aligned}$$

The induction base for $v_n^0\underset{\cdot}{\in}v_m^0$ is proven as follows (every line is equivalent to the next):

$$\mathrm{Sat}(\mathcal{I}^w(v_n^0\underset{\cdot}{\in}v_m^0),\mathfrak{c}) \tag{1}$$
$$\mathrm{Sat}(\underset{\cdot}{\forall}v,u((v\underset{\cdot}{=}\ulcorner\alpha\urcorner\underset{\cdot}{\land}\beta(u,v_n^0,v_m^0,v_3^1))\underset{\cdot}{\rightarrow}\underset{\cdot}{\mathrm{Sat}}(v,u)),\mathfrak{c}) \tag{2}$$
$$\forall x,y((x=\ulcorner\alpha\urcorner\land\beta(y,\mathfrak{c}(v_n^0),\mathfrak{c}(v_m^0),\mathfrak{c}(v_3^1)))\rightarrow\mathrm{Sat}(\underset{\cdot}{\mathrm{Sat}}(v,u),\mathfrak{c}^{x,y}_{v,u})) \tag{3}$$
$$\forall x,y((x=\ulcorner\alpha\urcorner\land\beta(y,\mathfrak{a}(v_n^0),\mathfrak{a}(v_m^0),\mathfrak{b}))\rightarrow\mathrm{Sat}(\underset{\cdot}{\mathrm{Sat}}(v,u),\mathfrak{c}^{x,y}_{v,u})) \tag{4}$$
$$\forall\mathfrak{d}\big((\mathfrak{d}(v_1^0)=\mathfrak{a}(v_n^0)\land\mathfrak{d}(v_2^0)=\mathfrak{a}(v_n^0)\land\mathfrak{d}\simeq\mathfrak{b})\rightarrow \rightarrow\mathrm{Sat}(\langle v_1^0,v_2^0,v^1_{g(w)}\rangle\underset{\cdot}{\in}v^1_{g(V)}(\in),\mathfrak{d})\big) \tag{5}$$
$$\forall\mathfrak{d}(\mathfrak{d}\simeq\beta\rightarrow\langle\mathfrak{a}(v_n^0),\mathfrak{a}(v_m^0),\mathfrak{d}(v^1_{g(w)})\rangle\in\mathfrak{d}(v^1_{g(V)})(\in)) \tag{6}$$
$$\langle\mathfrak{a}(v_n^0),\mathfrak{a}(v_m^0),w\rangle\in V(\in) \tag{7}$$
$$\langle\mathfrak{M},w,\mathfrak{a}\rangle\models v_n^0\underset{\cdot}{\in}v_m^0 \tag{8}$$

A similar reasoning works for $\underset{\cdot}{\mathrm{Sat}}(v_n^0,v_m^0)$. The induction base is straightforward for identity: $\mathcal{I}^w(v\underset{\cdot}{=}u)=v\underset{\cdot}{=}u$, so $\mathrm{Sat}(\mathcal{I}^w(v_n^0\underset{\cdot}{=}v_m^0),\mathfrak{c})$ iff $\mathfrak{c}(v_n^0)=\mathfrak{c}(v_m^0)$ iff $\langle\mathfrak{M},w,\mathfrak{a}\rangle\models v_n^0\underset{\cdot}{=}v_m^0$

For the induction step of lemma 6.2, the axioms for satisfaction trivially take care of negation and conjunction. The quantified case is handled as follows. First, we define δ ($x \simeq z$ is defined as above):

$$\alpha' := v_1^0 \in v^1_{g(Q)}(v^1_{g(w)})$$
$$\beta'(x,y,z) := \text{Asg}(x) \wedge x(v_1^0) = y \wedge x \simeq z$$
$$\delta := \forall x, y((x = \ulcorner \alpha' \urcorner \wedge \beta'(y, v_1^1, v_3^1)) \to \text{Sat}(x,y))$$

We prove $\mathfrak{a}(v_n^0) \in Q(w) \leftrightarrow \text{Sat}(\delta(v_n^0), \mathfrak{c})$ with a proof very similar to the one just given for the induction base. Then, the proof continues as follows:

$$\langle \mathfrak{M}, w, \mathfrak{a} \rangle \models \dot{\forall} v_n^0 \phi$$
$$\forall y (y \in Q(w) \to \langle \mathfrak{M}, w, \mathfrak{a}^y_{v_n^0} \rangle \models \phi)$$
$$\forall y (\mathfrak{a}^y_{v_n^0}(v_n^0) \in Q(w) \to \langle \mathfrak{M}, w, \mathfrak{a}^y_{v_n^0} \rangle \models \phi)$$
$$\forall y (\text{Sat}(\delta(v_n^0), \mathfrak{c}^y_{v_n^0}) \to \text{Sat}(\mathcal{I}^w(\phi), \mathfrak{c}^y_{v_n^0}))$$
$$\text{Sat}(\forall u |\delta, u/v_n^0| \to |\phi, u/v_n^0|, \mathfrak{c})$$
$$\text{Sat}(\mathcal{I}^w(\forall v_n^0 \phi), \mathfrak{c})$$

For the modal case, we set ξ like before, however we focus on the R of the Kripke model at hand:

Definition 6.3 (ξ for $\mathfrak{M}$) Given a countable $\mathfrak{M}$ and its R, the correspondent ξ is defined as follows: $\mathcal{I}$ is in ξ if and only there is a w and w' in W of $\mathfrak{M}$ such that $R(w,w')$ and, for all ϕ_{at}, $\mathcal{I}$ is as follows:

$$\mathcal{I}(\phi_{at}) = \begin{cases} \phi_{at} & \text{if } \phi_{at} \neq \ulcorner \text{Sat}(v,u) \urcorner \\ \ulcorner \text{Sat}(|v, v^1_{g(w')}/v^1_{g(w)}|, u) \urcorner & \text{otherwise} \end{cases}$$

Modal substitutions are unrelativized. By construction of ξ and via the definition of $\mathcal{I}^w$, we can check that a result parallel to lemma 5.4 holds:

Lemma 6.4

$$\forall w, w' \Big(R(w,w') \leftrightarrow \exists \mathcal{H} \big(\mathcal{H} \in \xi \wedge \forall \phi, \mathfrak{a} (\text{Sat}(\mathcal{H}\mathcal{I}^w(\phi), \mathfrak{a}) \leftrightarrow \text{Sat}(\mathcal{I}^{w'}(\phi), \mathfrak{a})) \big) \Big)$$

Proof The only new cases are for $v \dot{=} u$ in the induction base and $\mathcal{I}^w(\dot{\forall} v \phi)$ in the induction step. The case for $v \dot{=} u$ is trivial. For the quantified case, the proof works very similarly to the atomic cases, by exploiting the fact that the only difference between δ of $\mathcal{I}^w$ and δ of $\mathcal{I}^{w'}$ is the occurrence of $v^1_{g(w')}$ instead of $v^1_{g(w)}$ in α'. □

The modal case for the induction for lemma 6.2 proceeds as follows: $\langle \mathfrak{M}, w, \mathfrak{a} \rangle \models \dot{\Box} \phi$ iff for all w' such that $R(w,w')$, $\langle \mathfrak{M}, w', \mathfrak{a} \rangle \models \phi$. Via induction hypothesis, for all w' such that $R(w,w')$, $\text{Sat}(\mathcal{I}^{w'}(\phi), \mathfrak{c})$. Via lemma 6.4, this is so iff $\forall \mathcal{H}(\mathcal{H} \in \xi \to \text{Sat}(\mathcal{H}\mathcal{I}^w(\phi), \mathfrak{c}))$, iff $\text{Sat}(\dot{\Box}\mathcal{I}^w(\phi), \mathfrak{c})$ iff $\text{Sat}(\mathcal{I}^w(\Box\phi), \mathfrak{c})$. This concludes the induction and the proof of lemma 6.2.

Theorem 6.5 *There is a canonical truth theory for FKI.*

Proof We show that, for any ϕ, $\vdash_{FKI} \phi \leftrightarrow \models_S \phi$. For the left to right direction, we use lemma 6.1. For the other direction, pick the Kripke canonical model of FKI. By theorem 5.2, there is an equivalent countable model: its 'countable filtration' $\mathfrak{M}$. A canonical truth theory for FKI is the characteristic truth theory modelled from R of $\mathfrak{M}$ via definition 6. Assume $\nvdash_{FKI} \phi$. Then, in some world w of the countable filtration, under some $\mathfrak{a}$, $\langle \mathfrak{M}, w, \mathfrak{a} \rangle \not\models \phi$. Construct $\mathfrak{a}'$ as follows: $\mathfrak{a}'(\pi(v_n^s)) = \mathfrak{a}(v_n^s)$. Clearly,

$\langle \mathfrak{M}, w, \mathfrak{a} \rangle \models \phi$ exactly when $\langle \mathfrak{M}, w, \mathfrak{a}' \rangle \models \pi(\phi)$, so $\langle \mathfrak{M}, w, \mathfrak{a}' \rangle \not\models \pi(\phi)$. By lemma 6.2, $\neg\mathrm{Sat}(\mathcal{I}^w(\pi(\phi)), \mathfrak{c})$, therefore $\pi(\phi)$ is not substitutionally valid, and thus ϕ is not substitutionally valid either, since validity is preserved under uniform substitutions of free variables. $\square$

Theorem 6.5 can be extended to other familiar canonical extensions of FKI without BF nor CBF. We can always apply lemma 6.2 to the countable filtration of their canonical model. Soundness will still hold by construction of ξ, as before.

Notes

1. For a discussion, see Halbach 11, note 2. The substitutional account arguably traces back at least to Buridan.

2. By ‘set sized’ I mean the size of something that can be collected in a set.

3. Quine thinks this cannot be the case due to a variation of what he calls ‘Grelling’s paradox’. He considers the set of all sentences of the object language that do not satisfy themselves. In a type free system this set is unproblematic: the satisfaction predicate is in the object language, the paradox is stopped because a T schema is not provable for sentences like ‘not x satisfies x’.

4. For a contemporary and in depth discussion on absolute generality, see Rayo and Uzquiano 26.

5. Again, I point to Halbach 11 and Halbach 12 for a discussion. Reflection principles roughly show that any formula of set theory is satisfiable if what it says holds in the universe of sets. So if it is true in all models, it holds; that is, it is true. They have been proved for ZFC by [17] and [21]. They are not provable in higher order languages, though [28].

6. If ϕ is a formula and not a sentence, it is simply true iff satisfied under all assignments, and thus $\Box\phi$ is true iff true under all assignments and substitutions.

7. [11] builds the theory from ZFC, as well. Substitutional validity for first order logic can be defined using Peano arithmetic as base. Arguably, similar results hold for modal logic.

8. My guess is that one will need either to add some syntax to the language, thus strengthening the substitution functions, or have two typed truth predicates, and work in a meta meta language to mimic axiom SM7.

9. Note that $\underset{\cdot}{\Box}\phi$ might be satisfied but not logically true, like for $\phi(v)\underset{\cdot}{\rightarrow}\phi(u)$, where $\mathfrak{a}(v) = \mathfrak{a}(u)$ (since substitutions do not rename free variables). To be true is not to be satisfied under some assignment: to be true is to be satisfied under all assignments.

10. Here is what I mean. Suppose each $\mathcal{I}$ in i did rename variables via some recursive mapping $'$, in particular $v_n' = v_n$, $v_m' = v_n$ and $v_k' \neq v_n$. Suppose further that $\mathcal{I}(x \underset{\cdot}{\in} y) = x' \underset{\cdot}{=} y'$ and that $\mathfrak{a}(v_n) = n$. Then Sat$(\underset{\cdot}{\forall} v_n \underset{\cdot}{\Box} v_n \underset{\cdot}{\in} v_m, \mathfrak{a})$: by the axioms for satisfaction, $\forall y, \mathcal{I}(\mathcal{I} \in i \rightarrow$ Sat$(\mathcal{I}(v_n \underset{\cdot}{\in} v_m), \mathfrak{a}^y_{v_n}))$ and, by construction of i, $\forall y$Sat$(v_n \underset{\cdot}{=} v_n, \mathfrak{a}^y_{v_n})$. Yet, Sat$(|\underset{\cdot}{\Box} v_n \underset{\cdot}{\in} v_m, v_k/v_m|, \mathfrak{a})$ fails, because it is equivalent to Sat$(v_n \underset{\cdot}{=} v_k', \mathfrak{a})$, with $r \neq k$, where $v_k' = v_r$. Universal instantiation fails!

11. The reader might worry that lemma 3.6 relies on substitutions being unrelativized. However, this is not the case: if we allow them to add a condition δ to the quantifiers, we just need to set $\delta^{\mathcal{H}} = \delta^{\mathcal{I}} \underset{\cdot}{\wedge} \delta^{\mathcal{I}'}$. For the induction step in the quantified case, we use the fact that $(\phi \wedge \psi) \rightarrow \chi$ is equivalent to $\phi \rightarrow (\psi \rightarrow \chi)$ under satisfaction, via axioms SM3–SM4 and the fact that $\phi \rightarrow \psi$ iff $\neg(\phi \wedge \neg\psi)$.

12. See for example theorem 6.22 in Chagrov and Zakharyaschev 5, 173 and Miyazaki 20.

13. The theorem is from Makinson 18. It does not hold for frames, see for example Chagrov and Zakharyaschev 5, 189.

14. I could have added a name for every Henkin extension, and then let the substitutions swap names, yet I do not like the solution for two reasons: first, it makes the system rely on syntactic resources too much, which is one of the main criticisms of the substitutional account [8, 15]. Second, if we wished to consider identity logical, we would probably want $a = b \to \Box a = b$ to be logically true, which it would not be if names could be swapped.

15. For some other examples, see Chagrov and Zakharyaschev 5, 5.5.

16. This is often seen as an objection to the substitutional account [8]. Some endorse the result as independently plausible if we read the quantifiers as absolutely general [33, 25].

17. For a discussion of free and inclusive logic, see Bencivenga 1.

18. For a related discussion about the status of quantifiers in the substitutional system see Etchemendy 8.

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