Trivalent Conditionals: Stalnaker's Thesis and Bayesian Inference

Paul Égré   Lorenzo Rossi   Jan Sprenger

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Abstract

This paper develops a trivalent semantics for indicative conditionals and extends it to a probabilistic theory of valid inference and inductive learning with conditionals. On this account, (i) all complex conditionals can be rephrased as simple conditionals, connecting our account to Adams's theory of p-valid inference; (ii) we obtain Stalnaker's Thesis as a theorem while avoiding the well-known triviality results; (iii) we generalize Bayesian conditionalization to an updating principle for conditional sentences. The final result is a unified semantic and probabilistic theory of conditionals with attractive results and predictions.

1   Introduction

The logical properties of a conditional connective $\rightarrow$ representing the natural language indicative conditional (e.g., "if Mary went to bed, she will be sleeping now") prompt several challenging and stimulating questions. Do conditional sentences have truth conditions? What is the probability of conditionals? What does valid inference with conditionals amount to? Finally, how do we *learn* conditionals and how does conditional inference relate to Bayesian inference?

This paper argues that these questions hang together and should be answered jointly. It then develops a unified answer, based on trivalent truth conditions and non-classical probability functions, generalizing Bayesian inference in a natural way to a language with a conditional.

There is a well-known obstacle to such a project—at least if it wants to recover the old and widely acknowlewdged idea that evaluating a

conditional amounts to evaluating B on the *supposition* that A (Ramsey 1929/1990). On such accounts, the probability of a simple conditional $A \rightarrow B$ is, for any probability distribution p, naturally explicated as the conditional probability of B given A:

For conditional-free A and B: $p(A \rightarrow B) = p(B|A)$ (Adams's Thesis)

This claim, known as "Adams's Thesis", has been defended in a large variety of semantic approaches, and independently of whether or not conditionals are analyzed as propositions with truth conditions.[1] It is more controversial whether the above equality should also hold for *arbitrary* sentences A and B (e.g., allowing that A and B themselves are conditionals):

For arbitrary A and B: $p(A \rightarrow B) = p(B|A)$ (Stalnaker's Thesis)

This thesis takes its name from Stalnaker 1970, and it is also referred to as "The Equation" due to its ubiquitous occurrence in Bayesian analyses of conditional reasoning (e.g., Evans and Over 2004). However, David Lewis (1976) showed that treating $A \rightarrow B$ as a proposition that is subject to the standard laws of probability will lead to an unacceptable trivialization of Stalnaker's Thesis. Triviality proofs for indicative conditionals, including versions involving epistemic modals, come in several variants (e.g., Bradley 2000; Milne 2003; Hájek 2011; Fitelson 2015; Goldstein and Santorio 2021), but their takeaway message is simple: treating indicative conditionals as propositions with truth conditions is incompatible with respecting Stalnaker's Thesis in reasonable generality.

The strategies for avoiding triviality results roughly fall into three categories: (i) to construct a truth-conditional semantics without a systematic connection to probabilistic inference, (ii) to amend a truth-conditional semantics with a non-classical account of probability where (a reasonable approximation to) Stalnaker's Thesis is obtained, but the triviality results are avoided, or (iii) to reject the thesis that conditionals have truth conditions.

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[1]The first prominent occurrence of this claim in the literature goes back to Adams (1965, 1975), but it has also been identified as a plausible desideratum by Lewis (1976) and Edgington (1995). It is viewed as a plausible desideratum in a number of more recent accounts, including Bradley (2002), Égré and Cozic (2011), Goldstein and Santorio (2021) and Ciardelli and Ommundsen (forthcoming). For experimental support of Adams's Thesis, see Evans and Over (2004), Over and Cruz (2023) and the references cited in these works. Criticism of Adams's Thesis is mainly articulated by inferentialist accounts of conditionals, such as Douven (2016), Douven, Elqayam, and Krzyżanowska (2023), and Crupi and Iacona (2022).

Strategy (i) is popular among philosophers of language, especially in accounts based on premise semantics, dynamic semantics, and information states (e.g., Gillies 2009; Bledin 2015; Ciardelli 2020; Punčochár and Gauker 2020; Santorio 2022). While such accounts do not offer a probabilistic theory of uncertain reasoning with conditionals, they provide consequence relations which explicate reasoning with conditionals when premises are *certain*. Strategy (ii) is followed by McDermott (1996) in a trivalent setting, but also by Goldstein and Santorio (2021) using informational state semantics. The probability of conditionals is then obtained indirectly by a theory of updating such states. However, their formalisms do not integrate smoothly with standard Bayesian inference. Strategy (iii) amounts to defending a non-propositional ("suppositional") account of conditionals, giving up on truth conditions, and focusing exclusively on probabilistic reasoning with conditionals. The most well-known defenders of this view are Adams (1975) and Edgington (1995); recent arguments in Ciardelli and Ommundsen (forthcoming) also support this option.

Let us look at option (iii) in more detail. Suppose we have a propositional language $\mathcal{L}^{\rightarrow}$ with the usual Boolean operators $\wedge$, $\vee$ and $\neg$, and a conditional connective $\rightarrow$. Its Boolean fragment, excluding conditional expressions, is $\mathcal{L}$ and its flat fragment, allowing at most simple, non-nested conditionals, and no compounds of conditionals, is $\mathcal{L}_1^{\rightarrow}$. Adams then proposes the following criterion for valid probabilistic inference:

p-**valid inference (Adams 1975)** Suppose $\Gamma \subset \mathcal{L}_1^{\rightarrow}$, $B \in \mathcal{L}_1^{\rightarrow}$. Then $\Gamma \models_p B$ if and only if for all probability functions $p : \mathcal{L}_1^{\rightarrow} \mapsto [0,1]$, the uncertainty of the conclusion does not exceed the cumulative uncertainty of the premises:

$$U(B) \leq \sum_{A \in \Gamma} U(A) \qquad \qquad (\text{p-valid inference})$$

where $U(X) := 1 - p(X)$ for any $X \in \mathcal{L}_1^{\rightarrow}$.

McGee (1989, p. 485) gives a catchy description of the merits and limitations of Adams's account:

> The theory [of *p*-valid inference] describes what English speakers assert and accept with unfailing accuracy, yet the theory has won only limited acceptance. A principal reason for this has been that the theory is so limited in its scope. While the theory does a marvelous job of accounting for how we use simple conditionals, it tells us nothing

about compound conditionals or about Boolean combinations of conditionals. [. . .] This limitation has been thought to be insuperable, so that Adams's theory has appeared to be a dead end, highly accurate in a narrowly specialized domain, but isolated from the rest of logical theory and unable to overcome that isolation.

To overcome the isolation McGee talks about, Adams's theory needs to be extended from a theory on $\mathcal{L}_1^{\rightarrow}$ to a general theory covering nested conditionals, compounds of conditionals, and in general, all sentences of the full language $\mathcal{L}^{\rightarrow}$.

This paper provides such an extension using a trivalent semantics. More specifically, we define factual—and even truth-functional—truth conditions for conditionals, and a (non-classical) probability function on $\mathcal{L}^{\rightarrow}$ where the probability of a sentence depends in the standard way on its truth conditions. We show that this account generalizes p-valid inference and Bayesian conditionalization to arbitrary formulas of the full language $\mathcal{L}^{\rightarrow}$. Our account follows strategy (ii): we provide a unified truth-conditional and probabilistic semantics that yields Adams's and even Stalnaker's Thesis, *generalizing* Bayesian reasoning to a more expressive language, instead of developing an *alternative* account of updating that agrees with Bayesian inference in their common domain. We argue that the features required for obtaining these results—non-classical behavior of conjunction, and failure of the ratio analysis for conditional probability—can be motivated on independent grounds. In our opinion, strategy (iii)—denying that conditionals are propositions with truth conditions—makes an unnecessary philosophical sacrifice.

We proceed as follows: Section 2 explains our basic idea for trivalent truth conditions and the probability of indicative conditionals. Section 3 shows why nested conditionals and compound conditionals are, on this semantics, equivalent to simple conditionals. Section 4 defines trivalent logical consequence relations that generalize Adams's logics for certain and uncertain reasoning to the full language $\mathcal{L}^{\rightarrow}$. Due to the equivalence result shown in the previous section, this result implies that the valid inferences with sentences of the flat fragment $\mathcal{L}_1^{\rightarrow}$ (i.e., p-valid inferences) determine all valid inferences in $\mathcal{L}^{\rightarrow}$, rebutting McGee's objections to Adams's logic. We then discuss how these results relate to an impossibility result established by Schulz (2009), inspired from McGee (1981), for three-valued representations of p-valid inference involving compounds of conditionals.

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Two philosophically significant applications to Baysesian follow. Section 5 introduces a suitable notion of conditional probability for $\mathcal{L}^{\rightarrow}$-sentences that avoids Lewis-style triviality results and yields Stalnaker's Thesis in full generality. Section 6 generalizes Bayesian Conditionalization to a language with a conditional connective, and shows that updating on a simple indicative conditional amounts to updating on the corresponding material conditional. The final Section 7 concludes.

2   Outline of Trivalent Semantics

It is controversial whether indicative conditionals have factual truth conditions and whether they express propositions in the same way in which conditional-free and modal-free sentences do (e.g., see the dialogue in Jeffrey and Edgington 1991). However, even a defender of a non-truth-conditional view such as Adams (1965, p. 187) admits that we feel compelled to say that a conditional "if A, then B" has been *verified* if we observe both A and B, and *falsified* if we observe A and $\neg B$". For example, take the sentence "if it rains, the match will be cancelled"; it seems to be true if it rains and the match is in fact cancelled, and false if the match takes place in spite of rain. No similarly strong intuitions apply to the case where the antecedent is false (i.e., in case it does not rain).

This observation motivates the treatment of the indicative conditional "if A, then B" as a *conditional assertion*—i.e., as an assertion about B upon the supposition that A is true. On this account, when the antecedent is false, the speaker is committed to neither truth nor falsity of the consequent (e.g., de Finetti 1936; Quine 1950; Belnap 1973). There is simply no factual basis for evaluating the assertion. Therefore the conditional assertion is classified as neither true nor false. See Table 1.

Truth value of $A \rightarrow B$	B true	B false
A true	true	false
A false	neither	neither

Table 1: Partial truth table for a conditional $A \rightarrow B$ analyzed as a conditional assertion.

The question is whether "neither" should be understood as a truth-value gap, i.e., whether the valuation function for $A \rightarrow B$ should be a *partial function*, or whether we should treat "neither" as a third semantic value ("indeterminate", "nonassertive") that can freely interact with the

two standard semantic values. In the remainder, we will pursue the second option and develop it systematically.

Suppose that we have a set of possible worlds W, where each possible world corresponds to a complete trivalent valuation function over all sentences of our language $\mathcal{L}^{\rightarrow}$. (Of course, we have not yet specified *which* valuation functions we select for $\mathcal{L}^{\rightarrow}$; we will do this shortly, but the general shape of our approach can be presented without committing ourselves to specific valuation functions). Sentences of $\mathcal{L}^{\rightarrow}$ are interpreted as propositions, i.e., as functions from W to a set of truth values $\{1, 0, 1/2\}$ (=true, false, or neither). In addition, we assume that atomic sentences, and in general all conditional-free sentences, only take classical truth values (equivalently, that valuations are atom-classical). The guiding idea, going back to Cooper (1968), is that $\mathcal{L}^{\rightarrow}$ is an *extension* of the Boolean propositional language $\mathcal{L}$ to a language with a conditional connective, and so valuations of sentences of $\mathcal{L}$ should assign them classical truth values.

Suppose further that we have a credence function $c : \mathcal{A} \longmapsto [0, 1]$ on the measurable space of possible worlds $(W, \mathcal{A})$, where $\mathcal{A}$ is an algebra defined on subsets of W, representing the subjective plausibility of a particular set of possible worlds. Moreover, we assume that any algebra $\mathcal{A}$ includes the singletons of worlds, i.e., for every $w \in W$, $\{w\} \in \mathcal{A}$. We assume that the credence function c is a finitely additive probability function, i.e., for all singleton worlds $\{w\} \in \mathcal{A}$, we have $c(\{w\}) \geq 0$, whereas $c(\varnothing) = 0$, $c(W) = 1$, and $c(X \cup Y) = c(X) + c(Y)$ whenever $X \cap Y = \varnothing$.

We can then define a (non-classical) probability function $p : \mathcal{L}^{\rightarrow} \longmapsto [0, 1]$ on the language $\mathcal{L}^{\rightarrow}$, taking into account that sentences of $\mathcal{L}^{\rightarrow}$ can receive three values: true, false, or neither ("indeterminate").[2] For convenience, define

$$A_T = \{w \in W \mid v_w(A) = 1\} \qquad A_I = W \backslash (A_F \cup A_T)$$
$$A_F = \{w \in W \mid v_w(A) = 0\}$$

where v_w is the valuation function associated with the world w (recall, worlds essentially are valuation functions). In other words, A_T, A_F, and A_I are the sets of possible worlds where A is true, false and neither true nor false, respectively. What should then be the probability of A? Traditionally, the probability of a sentence A is simply the credence assigned to all possible worlds where A is true: $p(A) = c(A_T)$. But when A involves

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[2]For a survey of non-classical probability functions, see Williams (2016).

a conditional, it is possible that $c(A_F \cup A_T) < 1$—and we may end up assigning a small probability to A, even though A is much more likely to be true than false.

Therefore, for trivalent valuations, the relative weight of truth and falsity is a better indicator of the probability of a sentence than the credence that it is true (the same point is made by Bradley 2002, p. 363). Apart from its intuitive appeal, this idea is supported by Bayesian accounts that explicate probability by means of fair *betting odds* (Ramsey 1929/1990; de Finetti 1974). In particular, the subjective probability of A is the inverse of the decimal betting odds on A: $p(A) = 1/O(A)$, e.g., a probability of 1/3 corresponds to 3:1 betting odds. These odds specify the factor by which the bettor's stake is multiplied in case A occurs and she wins the bet. When A is a conditional assertion, bets are naturally generalized as follows: they are settled in the ordinary way if A takes classical truth value, and they are *called off* otherwise (i.e., the original stakes are returned). Indeed, it is hard to imagine how we should declare a bet on a conditional assertion like "if it rains on Saturday, the match will be cancelled" as won or lost unless it actually rains on Saturday.

The relation between probabilities and fair betting odds helps us to show why only the relative weight of $c(A_T)$ and $c(A_F)$ should affect the probability of A. Suppose that the bettor is betting on A at stake $S > 0$ and odds $O(A) > 0$ and that we are sampling possible worlds at random, according to a credence function c. Since the bet on A will be called off in case A does not take a classical truth value, with the stakes returned to the bettor, her long-run net gain will in the limit approach the following quantity:

$$
\begin{aligned}
G &= -S + c(A_T) \times S \times O(A) + c(A_F) \times 0 + c(A_I) \times S \\
&= S \times (-1 + c(A_T) \times O(A) + c(A_I))
\end{aligned}
$$

If bettor and bookie agree on credence function c, the bet is fair if and only if $G = 0$, i.e., neither side is supposed to have a long-run advantage. This can be shown to be equal to

$$
c(A_F)/c(A_T) = O(A) - 1
$$

or, even simpler,

$$
O(A) = \frac{c(A_T) + c(A_F)}{c(A_T)},
$$

which agrees with the definition of decimal betting odds in the standard bivalent case. Moving back from conditional bets to probabilities for sentences with trivalent valuations, and defining probability as the inverse of fair betting odds (as in the bivalent case), we obtain:

$$p(A) := \frac{c(A_T)}{c(A_T) + c(A_F)} \qquad \text{if } c(A_T) + c(A_F) > 0 \qquad \text{(Probability)}$$

(If $c(A_T) + c(A_F) > 0$, then $p(A) = 1$ by convention.) For a Boolean, conditional-free sentence A, this will collapse to the standard definition of probability as the credence in the set of possible worlds where A is true (since $A_T \cup A_F = W$ and hence $c(A_T) + c(A_F) = 1$). Whereas for a simple conditional "if A, then B", with conditional-free sentences A and B, we obtain, given Table 1,

$$p(A \to B) = \frac{c(A_T \cap B_T)}{c(A_T)} = \frac{p(A \wedge B)}{p(A)} = p(B|A) \qquad \text{(Adams's Thesis)}$$

where the probability function in the last two expressions refers to standard bivalent probability on $\mathcal{L}$. In other words, Adams's Thesis—the equality of probability of conditionals and conditional probability for simple conditionals—follows as a *corollary* of the trivalent semantics and need not be stipulated as a definition, as in Adams's own account.

These considerations do not yet show how we can assign a probability to Boolean compounds of conditionals, nested conditionals, and other complex sentences of $\mathcal{L}^\to$. We need to extend Table 1 and the truth tables for the regular Boolean connectives to cases where both the antecedent and the consequent can take the third semantic value, which we call "indeterminate". For the conditional, there are two main options: the de Finetti conditional $\to_{DF}$ where indeterminate antecedents are grouped with false ones (e.g., de Finetti 1936; Belnap 1970; McDermott 1996; Dubois and Prade 1994; Rothschild 2014) and the Cooper conditional $\to_C$ where they are grouped with true ones (e.g., Cooper 1968; Belnap 1973; Cantwell 2008). See Table 2.[3] Semantic values are represented by numbers, with 1, 1/2 and 0 standing for true, indeterminate and false.

Reasons for preferring the Cooper conditional to the de Finetti conditional, i.e., treating an indeterminate antecedent like a true one, are dis-

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[3]Negation is Strong Kleene negation and conjunction and disjunction happen to coincide with those of Sobociński (1952). The combination of these Boolean connectives and the connective $\to_C$ was first advocated by Cooper (1968).

$\to_C$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

$\to_{DF}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	1/2
0	1/2	1/2	1/2

	$\neg$
1	0
1/2	1/2
0	1

$\wedge$	1	1/2	0
1	1	1	0
1/2	1	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	0
0	1	0	0

Table 2: Truth tables for the trivalent conditional according to Cooper ($\to_C$) and to de Finetti ($\to_{DF}$), and the Boolean connectives in our trivalent logic.

cussed in detail in Égré, Rossi, and Sprenger (2021), as are those for deviating from the Strong Kleene connectives for conjunction and disjunction. We just mention one important motivation: as argued by Bradley (2002), we want to be able to consider "partitioning" sentences like

(1)   If the sun shines tomorrow, John goes to the beach; and if it doesn't, he goes to the museum.

as true when one of two conditional predictions is verified (e.g., the sun shines and John goes to the beach). But if we used Strong Kleene connectives for conjunction, (1) would always be indeterminate or false. This means that as long as John's actions are uncertain, (1) will have probability zero. This would be an unwelcome prediction since (1) is certainly more assertable than "If it rains tomorrow, John goes to the beach; and if not, he goes to the museum.". Therefore, we consider the conjunction of a true and an indeterminate sentence to be true. In other words, the third truth value acts as a chameleon: the conjunct or disjunct with classical truth value determines the truth value of the compound.[4] In spite of their non-standard nature, the truth tables for conjunction and disjunction interact with each other, and with Strong Kleene negation, in the usual way, respecting the de Morgan rules and the distributivity laws. They also define a particularly important family of valuation functions:

Definition 1 (Cooper valuations). *A valuation function $v : \mathcal{L}^{\to} \longmapsto \{1, 1/2, 0\}$ that assigns semantic values to all formula of $\mathcal{L}^{\to}$ is called a* Cooper valuation

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[4]Another unwelcome consequence of the Strong Kleene tables is that regardless of whether we follow Cooper or de Finetti in the interpretation of the conditional, $v((A \to B) \vee (B \to A)) \geq 1/2$ for all valuations v. In other words, such sentences—encoding the so-called *linearity* principle—cannot be false.

if and only if (i) it assigns classical values to all atomic formulas and (ii) respects all constraints in the truth tables of Table 2 with the $\to_C$ interpretation of the conditional connective.

In other words, Cooper valuations combine the "ideal" configuration of truth tables with the aforementioned idea that the conditional connective extends a language with classical semantic valuations. Specifically, presupposing Cooper valuations, our trivalent probability function obtains more structure and satisfies the following three principles:

(1) $p(\top) = 1$ and $p(\bot) = 0$.

(2) $p(A) = 1 - p(\neg A)$ (unless $c(A_F \cup A_T) = 0$, in which case $p(A) = p(\neg A)$).

(3) If $A_T \cap B_T = \varnothing$ (i.e., A and B cannot both be true) and $A_T \cup A_F = B_T \cup B_F$ (i.e., they take classical truth values in the same conditions), then $p(A \vee B) = p(A) + p(B)$.

These axioms are similar to the standard axioms of probability, where the additional requirement $A_T \cup A_F = B_T \cup B_F$ for the Additivity axiom reflects that the probability of a disjunction reduces to the probability of its disjunctions only if the disjunctions take classical truth values in the same worlds.

Notably, our conjunction is non-classical and allows for violations of the standard probability law $p(X \wedge Y) \leq p(X)$. This behavior can be rationalized as follows: suppose that A, B and C are conditional-free sentences, with A and B false and C true. Then the bet on $(A \to B) \wedge C$ yields a positive return (because the expressed proposition is true for Cooper valuations), while the bet on $A \to B$ is called off. Betting on X is therefore not always safer than betting on X and Y. Since these properties of betting odds transfer to probabilities, some probability functions will have the feature $p((A \to B) \wedge C) > p(A \to B)$, in notable difference to standard bivalent probability. Of course, as long as X and Y are conditional-free, the standard law $p(X \wedge Y) \leq p(X)$ will continue to hold. Section 4 will get back to this point in more detail.

3   Simplifying Complex Conditionals

This trivalent account deals with nested and compound conditionals in a particularly pleasant way. It is a well-known drawback of Adams's account

that complex conditionals (and compounds of conditionals) such as $A \rightarrow (B \rightarrow C)$ or $(A \rightarrow B) \wedge (C \rightarrow D)$ with $A, B, C, D \in \mathcal{L}$, cannot be directly analyzed; instead they have to be *rephrased* by means of a formula in $\mathcal{L}_1^{\rightarrow}$ that approximates their meaning. For instance, Adams *defines* the meaning of a conjunction of conditionals $(A \rightarrow B) \wedge (C \rightarrow D)$ as $(A \vee C) \rightarrow ((A \supset B) \wedge (C \supset D))$ (Adams 1975, 1998, pp. 164-165). This move, however, is devoid of independent philosophical justification.[5]

The advantage of a fully compositional trivalent semantics is that Adams's "translations" from $\mathcal{L}^{\rightarrow}$ to its flat fragment $\mathcal{L}_1^{\rightarrow}$ cease to be *definitions*. Instead, they are *statements* whose truth or falsity can be decided in our semantics, in line with the following definition.

Definition 2. *For two formulas $\varphi, \psi \in \mathcal{L}^{\rightarrow}$, we write $\varphi \approx \psi$ if and only if for all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{1, 1/2, 0\}$, $v(\phi) = v(\psi)$.*

We can now state some equivalence results for nested conditionals and Boolean compounds of conditionals.

Proposition 1. *For $A, B, C \in \mathcal{L}^{\rightarrow}$ and a Cooper valuation $v : \mathcal{L}^{\rightarrow} \longmapsto \{1, 1/2, 0\}$:*

$$A \rightarrow (B \rightarrow C) \approx (A \wedge B) \rightarrow C \qquad \text{(Import-Export)}$$
$$\neg(A \rightarrow B) \approx A \rightarrow \neg B \qquad \text{(Negation Commutation)}$$

In addition, for all Boolean sentences A, B, C, and $D \in \mathcal{L}$:

$$(A \rightarrow B) \rightarrow C \approx (A \supset B) \rightarrow C \qquad \text{(Left-Nesting)}$$
$$(A \rightarrow B) \wedge (C \rightarrow D) \approx (A \vee C) \rightarrow ((A \supset B) \wedge (C \supset D)) \qquad \text{(Conjunction)}$$
$$(A \rightarrow B) \vee (C \rightarrow D) \approx (A \vee C) \rightarrow ((A \wedge B) \vee (C \wedge D)) \qquad \text{(Disjunction)}$$

Proof: By inspection of the (trivalent) truth tables. □

The results for negation, conjunction and disjunction of conditionals are intuitive and correspond to Adams's translation procedures (e.g., his "quasi-conjunction" of conditionals). Right-Nesting is the Import-Export Principle: the nested conditional $A \rightarrow (B \rightarrow C)$ expresses the simple conditional $(A \wedge B) \rightarrow C$. Import-Export enjoys strong theoretical support (for

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[5]Actually, Adams assigns "ersatz truth values" to the conjunction of conditionals $P_1 \wedge \ldots \wedge P_n$, classifying it as false if any of the P_i is false, as true if none of the P_i is false and at least one of them is true, and as being neither true nor false otherwise. This description agrees exactly with Cooper's conjunction as described in Table 2.

recent defenses, see Ciardelli 2020; Ciardelli and Ommundsen forthcoming) and it is also supported by empirical work in cognitive psychology (van Wijnbergen-Huitink, Elqayam, and Over 2015). Even accounts that reject Import-Export, like Khoo and Mandelkern (2019) and Mandelkern (2020), concede that these two expressions have the same meaning in many contexts, and specifically when A, B and C themselves do not contain conditionals.

By contrast, left-nested conditionals like $(A \rightarrow B) \rightarrow C$ express, on our account, an assertion of the consequent C conditional on the material conditional $(A \supset B)$. This analysis is supported by the observation that $A \rightarrow B$ and $A \supset B$ have (for $A, B \in \mathcal{L}$) the same falsity conditions, i.e., both of them are false if and only if A is true and B is false. They may not have the same *meaning*, but they express the same *supposition* in the antecedent of a conditional. This observation will be important in Section 6, where we develop a Bayesian theory of learning conditionals.[6]

Finally, note the important restriction to Boolean sentences for Left-Nesting, Conjunction, and Disjunction: these equivalences do not hold for arbitrary sentences. For example, for Left-Nesting, counterexamples to the scheme $(F \rightarrow G) \rightarrow H \approx (F \supset G) \rightarrow H$ require $v(G) = 1/2$ (and $v(F) = 1$ and $v(H) \neq 1/2$), i.e., G is a conditional with false antecedent. Still, the equivalences are practically useful since natural language conditionals will rarely be more complex than the ones displayed in Proposition 1.

The above observations allow us to derive a general and important result:

Theorem 1 (Reduction Theorem). *For every $X \in \mathcal{L}^{\rightarrow}$ there is an $X_1 \in \mathcal{L}_1^{\rightarrow}$, i.e., the fragment of $\mathcal{L}^{\rightarrow}$ containing at most simple conditionals, such that $X \approx X_1$.*

While the formal proof by induction is in the appendix, we can summarize the argument informally: Suppose $A \rightarrow B$ and $C \rightarrow D$ are simple, non-nested conditionals, i.e., $A, B, C, D \in \mathcal{L}$. Then each compound of these sentences is semantically equivalent to another simple conditional.

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[6]Actually, all combinations of truth tables—de Finetti or Cooper conditional, Strong Kleene connectives or quasi-connectives for conjunction and disjunction—validate Import-Export and Commutation with Negation (Égré, Rossi, and Sprenger 2021). However, only Cooper valuations validate the entire set of equivalences. Specifically, Strong Kleene connectives fail Conjunction and Disjunction of Conditionals (e.g., $v(A) = 0$, $v(C) = c(D) = 1$), while de Finetti's conditional fails Left-Nesting: it satisfies $(A \rightarrow B) \rightarrow C \approx (A \wedge B) \rightarrow C \approx A \rightarrow (B \rightarrow C)$, i.e., it makes no difference between left-nested and right-nested conditionals.

For conjunction, disjunction, and negation, this is immediate from Proposition 1, while the $\mathcal{L}_1^{\rightarrow}$-sentence corresponding to $(A \rightarrow B) \rightarrow (C \rightarrow D)$ is $((A \supset B) \wedge C) \rightarrow D$. Hence, compounds of $\mathcal{L}_1^{\rightarrow}$-sentences are equivalent to $\mathcal{L}_1^{\rightarrow}$-sentences, regardless of the chosen connective. Now we gradually simplify any complex $\mathcal{L}^{\rightarrow}$-sentence X, starting with the innermost elements, until X can be rewritten as a simple conditional.

This means that Adams was right on an important point: restricting the language of interest and the logic of conditionals to $\mathcal{L}_1^{\rightarrow}$, i.e., the flat fragment of $\mathcal{L}^{\rightarrow}$, does not lose anything logically nor semantically essential because complex conditionals are *equivalent* to simple conditionals. We can consider them with Adams, if we want, as convenient linguistic abbreviations. The next section explores the consequences of the Reduction Theorem for a theory of valid inference with conditionals.

4 Inferences with Conditionals

So far, we have dealt with the semantics and probability of trivalent conditionals; we have not said much about inference, except indirectly for the equivalences stated in Proposition 1. Following Adams, we characterize a consequence relation $\models_{\mathsf{U}}$ for valid inference in $\mathcal{L}^{\rightarrow}$ in terms of non-increasing uncertainty: the conclusion must not be less probable than the conjunction of the premises.

Definition 3 (Valid Inference in U). *For a set Γ of formulas of $\mathcal{L}^{\rightarrow}$ and a formula $B \in \mathcal{L}^{\rightarrow}$: $\Gamma \models_{\mathsf{U}} B$ if and only if there is a finite subset of the premises $\Delta \subseteq \Gamma$ such that for all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$:*

$$p(\bigwedge_{A \in \Delta} A) \leq p(B)$$

As a limiting case of U, we can define a logic of *reasoning with certain premises* where probability 1 is preserved:

Definition 4 (Valid Inference in C). *For a set Γ of formulas of $\mathcal{L}^{\rightarrow}$ and a formula $B \in \mathcal{L}^{\rightarrow}$: $\Gamma \models_{\mathsf{C}} B$ if and only if for all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$:*

if for all $A \in \Gamma, p(A) = 1, then\ p(B) = 1.$

Both notions of probabilistic validity can be characterized semantically in a trivalent setting (see Égré, Rossi, and Sprenger forthcoming for proofs, which we omit here, and further analysis of the properties of C and U):

Proposition 2 (Trivalent Characterization of C). *For a set Γ of formulas of $\mathcal{L}^{\rightarrow}$ and a formula $B \in \mathcal{L}^{\rightarrow}$: $\Gamma \models_C B$ if and only if for all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$:*

if for all $A \in \Gamma, v(A) \geq 1/2$, then $v(B) \geq 1/2$.

In other words, preservation of probability 1 is equivalent to preservation of designated values $D = \{1, 1/2\}$ in passing from the premises to the conclusion. For uncertain reasoning in U, we can derive a similar result:

Definition 5 (Consistent and Inconsistent Sets). *A set of formulas Γ is consistent if it has no subset $\Delta \subseteq \Gamma$ such that $p\left(\bigwedge_{A \in \Delta} A\right) = 0$ in all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0, 1]$. It is inconsistent if such a subset exists.*

Proposition 3 (Trivalent Characterization of U). *For a consistent set of formulas Γ of $\mathcal{L}^{\rightarrow}$ and a formula $B \in \mathcal{L}^{\rightarrow}$ with $\not\models_C B$, $\Gamma \models_U B$ if and only if there is a finite subset $\Delta \subseteq \Gamma$ such that for all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$:*

$$v\left(\bigwedge_{A_i \in \Delta} A_i\right) \leq v(B).$$

This means that an inference in U is valid if and only if both sets of designated values $D = \{1, 1/2\}$ and $D' = \{1\}$ are preserved in passing from the premise from the conclusion. Equivalently, the semantic value of the conclusion must never drop below the semantic value of the conjunction of (a subset of) the premises.[7] These results connect the (trivalent) truth conditions of conditionals to probabilistic reasoning and answer one of the questions raised above at the opening of this paper: valid *uncertain* reasoning preserves both truth and non-falsity, and valid *certain* reasoning preserves non-falsity.

Crucially, inference in U generalizes Adams's *p*-valid inference to all sentences of $\mathcal{L}^{\rightarrow}$:

Proposition 4. *Suppose $\Gamma \subset \mathcal{L}_1^{\rightarrow}$, $B \in \mathcal{L}_1^{\rightarrow}$ and $\Gamma \models_p B$. Then, for all Cooper valuations, we also have $\Gamma \models_U B$.*

Proof. According to Adams (1986, p. 264), the premises $\Gamma = \{A_1, \ldots A_n\}$ *yield* a conclusion B (where $A_1, \ldots A_n$ and B are sentences of $\mathcal{L}_1^{\rightarrow}$) if and only if (1) any atom-classical valuation that falsifies none of the A_i, and

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[7]The restriction to consistent premise sets is required to deal with some degenerate cases, but imposes no substantial limitation.

verifies at least one of them, verifies B, too; (2) any atom-classical valuation that falsifies B also falsifies at least one of the A_i. He then shows ("Meta-Metatheorem 3") that $\Gamma \models_p B$ if and only if a subset of Γ yields the conclusion B. It is not difficult to see that Adams's conditions on yielding are equivalent to conditions for valid inference in our system U: for a subset $\Delta \subseteq \Gamma$, if $v(\bigwedge_{A \in \Delta} A) = 1$ then $v(B) = 1$, and if $v(B) = 0$, then $v(\bigwedge_{A \in \Delta} A) = 0$, where the logical vocabulary satisfies the Cooper truth tables (Table 2), as the Cooper connectives behave classically on classical inputs. $\square$

The relation between U und p-valid inference is, however, more complex than generalization. Taking into account the Reduction Theorem, we can establish an equivalence between the two logics, in the following sense.

Theorem 2 (Representation Theorem for U). *Suppose $\Gamma \subset \mathcal{L}^{\rightarrow}$ and $B \in \mathcal{L}^{\rightarrow}$. Then there is a function $f : \mathcal{L}^{\rightarrow} \longmapsto \mathcal{L}_1^{\rightarrow}$ such that*

$$\Gamma \models_{\mathsf{U}} B \qquad \textit{if and only if} \qquad f(\Gamma) \models_p f(B).$$

Proof. Immediate from Theorem 1—the function f is the translation of $\mathcal{L}^{\rightarrow}$-formulas into $\mathcal{L}_1^{\rightarrow}$-formulas in the proof of the theorem—and from Adams's yielding criterion cited in the proof of Proposition 4. $\square$

In other words, all valid or invalid conditional inferences can be expressed by means of valid or invalid inferences with *simple* conditionals. The Representation Theorem thus answers McGee's challenge to Adams from the introduction. It shows why Adams's restriction of the logic of probability conditionals to $\mathcal{L}_1^{\rightarrow}$ is not harmful: p-valid inference can be characterized as inference in trivalent semantics (and vice versa), covering also nested conditionals and Boolean compounds of conditionals.

The Representation Theorem raises interesting puzzles and questions. For instance, Modus Ponens is invalid in U, but p-valid. How can this happen if valid inferences in both logics can be mapped to each other? The reason is that the translation procedure from $\mathcal{L}^{\rightarrow}$ to $\mathcal{L}_1^{\rightarrow}$ changes the logical form of the premises. For conditional-free sentences $A, B, C \in \mathcal{L}$, Modus Ponens fails in U for inferences of the form "$A \rightarrow (B \rightarrow C)$ and A, therefore $B \rightarrow C$" (compare Égré, Rossi, and Sprenger forthcoming, Section 8). However, when we project these formulas to their semantic equivalents in $\mathcal{L}_1^{\rightarrow}$, we obtain a different inference scheme, namely "$(A \wedge B) \rightarrow C$ and A, therefore $B \rightarrow C$". This inference is p-invalid, as predicted by Theorem

2. For $\mathcal{L}_1^{\rightarrow}$-premises, however, we retain that $A \rightarrow B, A \models_p B$, and similarly in U.

Another interesting question prompted by the Representation Theorem concerns impossibility results for representations of p-validity in many-valued logic. McGee (1981) showed that no consequence relation in many-valued logic defined as preservation of a set of designated values agrees with p-validity on Adams's restricted language. Adams (1995) proved that such a characterization is possible if validity is defined more liberally, basically as we established above for U (namely not in terms of the preservation of a fixed set of designated values, but by quantifying over such sets). However, Schulz (2009, Corollary 3.3) shows that no such characterization is possible for a language admitting compounds of conditionals when the conjunction is supposed to be "classical" with respect to the logical consequence relation, i.e., if the following two conditions are satisfied:

(i) If $\Gamma \models \phi$ and $\Gamma \models \psi$, then $\Gamma \models \phi \wedge \psi$.

(ii) If $\Gamma \models \phi \wedge \psi$, then $\Gamma \models \phi$ and $\Gamma \models \psi$.

Schulz concludes:

> This poses a dilemma for proponents of a three-valued account of compounds of conditionals which is supposed to conform to the thesis that conditionals are evaluated by conditional probabilities. Either they will not succeed in defining a classical conjunction, or their conception of validity will disagree with p-validity on the restricted language. (Schulz 2009, p. 516)

For our consequence relation $\models_U$, condition (i) holds, but condition (ii) doesn't.[8] This means that conjunction in U is indeed not classical, and so we choose the first horn of Schulz's dilemma. However, while a non-classical conjunction may look non-standard at first, there are good independent reasons to adopt it.

First, as argued in Section 2 (especially p. 9), the chameleon-like behavior of the third truth value (i.e., $1 \wedge 1/2 = 1$, etc.) is essential for showing that specific conjunctions of conditionals, which we often employ in natural language, can be true. Second, there is strong evidence that when conditionals are involved, the probability of conjunctions need not be smaller than the probability of a conjunct. Consider the following example from Santorio and Wellwood (2023)—a similar case is made by Ciardelli and Ommundsen (forthcoming):

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[8]Counterexample: $v(\Gamma) = v(\phi) = 1$, but $v(\psi) = 1/2$. In this case $\Gamma \models \phi \wedge \psi$, but $\Gamma \nvDash \phi$.

(2)   If the outcome of the die roll was even, it was two; and if it was odd, it was one.

While "the die landed 2 if it was even" and "the die landed 1 if it was odd" both have probability 1/3, their conjunction seems to be true exactly when the die landed one or two, and false in all other cases. So (2) should have probability 1/3, too. This prediction is borne out in our framework, but it is at odds with what we would expect from a classical conjunction. Santorio and Wellwood (2023) back up their argument with experimental data, and so it seems that the non-classical behavior of conjunction in uncertain reasoning has both normative and empirical support.[9]

Hence, we propose to interpret Schulz's result as follows: a non-classical conjunction is the only way of (i) providing a many-valued logic that agrees with Adams's widely accepted p-validity in their common domain, while (ii) yielding an adequate account of the truth values of compounds of conditionals. Some properties of classical conjunction simply do not generalize to more complex languages.

5 Conditional Probability and Stalnaker's Thesis

A workable definition of conditional probability is essential for Bayesian reasoning and for updating on incoming evidence in particular. Unfortunately, the standard analysis of conditional probability

$$p(B|A) := \frac{p(A \wedge B)}{p(A)}, \qquad \text{if } p(A) > 0, \qquad \text{(Ratio Analysis)}$$

fails in the trivalent case. Since the non-classical behavior of conjunction allows for cases where $p(A \wedge B) > p(A)$, we could end up with $p(B|A) > 1$, which is unacceptable for any conditional probability function.

In this section, we develop a surrogate notion of conditional probability, which coincides with standard conditional probability for conditional-free sentences, but extends to sentences which can take all three truth values. Our definition is simple: for all $A, B \in \mathcal{L}^{\rightarrow}$, the conditional probability of B given A is the probability of the conditional $A \rightarrow B$.

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[9]We view such examples as providing an answer to Schulz's challenge as set in Schulz (2009, p. 513): "If one designs a semantic theory for the unrestricted conditional language, can there be any doubt that the conjunction should obey the standard introduction and elimination rules? The burden of proof would be on the side of those who think that it should not. Counterexamples would have to be produced".

Definition 6. *For all $A, B \in \mathcal{L}^{\rightarrow}$, the* trivalent conditional probability *of B given A, in symbols $p_A(B)$, is defined as follows:*

$$p_A(B) := p(A \rightarrow B) = \frac{c(A_{TI} \cap B_T)}{c(A_{TI} \cap (B_T \cup B_F))} \qquad \text{if } c(A_{TI}) > 0$$

<div align="right">(Trivalent Conditional Probability)</div>

where $A_{TI} := A_T \cup A_I$ (i.e., the set of worlds where A is not false).

It is clear by the truth-table for Cooper's conditional that this definition respects the operational definition of probability: it is the ratio of the weight of worlds where the conditional is true to the weight of worlds where it is defined, i.e. it has a classical value. It is also clear that this definition agrees with standard axioms of conditional probability if A and B are conditional-free: $A_{TI} = A_T$ because $A_I = \varnothing$, and the numerator will thus be equal to $p(A \wedge B)$ and the denominator to $p(A)$ (since $B_T \cup B_F = W$). In this special case, $p_A(B)$ also satisfies the standard axioms for conditional probability.[10] We need to check, however, whether the behavior of $p_A(B)$ agrees with what we expect from a conditional probability function if A and B can contain conditionals. As before, we assume that propositional atoms receive

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[10]Using $p(A \rightarrow B)$ as a surrogate notion for $p(B|A)$ has been suggested first by McGee (1989) as a means of introducing conditional probability into a language with a conditional. However, McGee does not provide a full truth-conditional semantics for his language. So he cannot interpret $p(X)$ directly as the credence in the worlds where X is true, or the credence ratio between worlds where X is true or false, etc. Instead, McGee (1989, p. 504) provides an axiomatic characterization of the function $p(A \rightarrow B)$, and of its interaction with the probability of conditional-free sentences. The main pillar in his edifice is the

(Simple) Independence Principle (McGee 1989, p. 499). For conditional-free sentences $A, B, C \in \mathcal{L}$, and assuming that A and C are logically incompatible and $p(A) > 0$, then

$$p(C \wedge (A \rightarrow B)) = p(C) \cdot p(A \rightarrow B). \tag{C1}$$

This principle actually amounts to a *definition*. After all, McGee does not have an account that relates the probability of sentences containing Boolean and/or conditional operators to their truth conditions. So he needs to *stipulate* (C1). But on our semantics, (C1) is *invalid*: it is immediate from $C \models_{\mathsf{CL}} \neg A$ that the truth value of $C \wedge (A \rightarrow B)$ is identical to the truth value of C. (If C is true, then $A \rightarrow B$ is indeterminate and hence the conjunction is true.) Hence $p(C \wedge (A \rightarrow B)) = p(C)$, which is almost always larger than $p(C) \cdot p(A \rightarrow B)$. This is actually the only divergence between our account and McGee's: our trivalent probability function satisfies the adequacy conditions C2–C8 that McGee imposes, together with the Independence Principle, as necessary and sufficient conditions for a (conditional) probability distribution on a language with a conditional.

only classical values, and therefore that conditionals are the only source of the third truth value.

Here below we list Popper's axioms for a conditional probability functions $p(\cdot|\cdot)$ for classical propositional logic (taken from Hawthorne 2016):

1. $0 \leq p(B|A) \leq 1$.

2. If $\models_{CL} \neg B$ and $\models_{CL} A$, then $p(B|A) = 0$.

3. If $A \models_{CL} B$, then $p(B|A) = 1$.

4. Left Logical Equivalence: If $A \models_{CL} B$ and $B \models_{CL} A$, then $p(C|A) = p(C|B)$.

5. Additivity: If $C \models_{CL} \neg(A \wedge B)$, then either $p(A \vee B|C) = p(A|C) + p(B|C)$ or $p(D|C) = 1$ for any D.

6. The Product Rule: $p(A \wedge B|C) = p(A|B \wedge C) \times p(B|C)$.

Before evaluating these axioms with respect to $p_A(B)$, we need to reformulate them, replacing $p(B|A)$ with $p_A(B)$ and classical entailment with its generalization to the language $\mathcal{L}^{\rightarrow}$, i.e., entailment in C (=preservation of certainty).[11] This means that the first four axioms read:

1'. $0 \leq p_A(B) \leq 1$.

2'. If $\models_C \neg B$ and $\models_C A$, then $p_A(B) = 0$.

3'. If $A \models_C B$, then $p_A(B) = 1$.

4'. Left Logical Equivalence: If $A \models_C B$ and $B \models_C A$, then $p_A(C) = p_B(C)$.

The reader is invited to verify that trivalent conditional probability satisfies (1')–(4')—the proofs are simple.[12]

The Additivity axiom (5) has to be modified more substantially, similar to the case of unconditional probability:

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[11]Note that C generalizes certain reasoning from a classical Boolean setting to a language with a conditional connective. Therefore C has the same role for trivalent probability as classical logic has for classical, bivalent probability.

[12]1' is immediate. As to 2', notice that $B_T = \varnothing$. As to 3', note that $A_{TI} \cap B_F = \varnothing$, and so numerator and denominator of $p_A(B)$ in (Trivalent Conditional Probability) are equal. As to 4', notice that given $A \models_C B$ and $B \models_C A$, the conditionals $A \rightarrow C$ and $B \rightarrow C$ take the same truth values in all Cooper valuations.

5′. Trivalent Additivity: Suppose (i) $c(C_{TI}) > 0$ (ii) $C \models_c \neg(A \wedge B)$ and (iii) if $v(C) \geq 1/2$, then $|v(A) - v(B)| \neq 1/2$. Then:

$$p_C(A \vee B) = p_C(A) + p_C(B).$$

To see that Trivalent Additivity is satisfied by our definition of conditional probability, consider the division of the credence attached to sets of possible worlds in C_{TI} according to the following table:

$W \cap C_{TI}$	B_T	B_I	B_F
A_T	✗	✗	α
A_I	✗	δ	✗
A_F	β	✗	γ

Conditions (ii) ensures that the upper left corner is empty and condition (iii) ensures that all direct neighbors to the central square are empty. Hence $\alpha + \beta + \gamma + \delta = 1$. We can now calculate the conditional probabilities of A, B and $A \vee B$ according to Definition 6 and obtain

$$p_C(A) = \frac{\alpha}{\alpha + \beta + \gamma} \quad p_C(B) = \frac{\beta}{\alpha + \beta + \gamma} \quad p_C(A \vee B) = \frac{\alpha + \beta}{\alpha + \beta + \gamma}$$

showing that Trivalent Additivity holds.

Notably, trivalent conditional probability does *not* satisfy axiom 6, the Product Rule. The reason for the failure of the Product Rule is inherited from unconditional probability and the non-classical behavior of conjunction in particular: the term $p(A \wedge B)/p(B)$ can be greater than 1, and so we cannot define conditional probability via the familiar Ratio Analysis (i.e., $p(B|A) = p(A \wedge B)/p(B)$). This feature is inherited from the non-classical behavior of conjunction in our semantics, but it does not undermine the status of $p_A(B)$ as the appropriate conditional probability function. Of course, the Product Rule is satisfied if we restrict ourselves to conditional-free sentences.[13] (We will say more about the Product Rule in a minute.)

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[13]Trivalent conditional probability does *not* satisfy the trivalent version of the Law of Total Probability

$$p(B) = p(A \rightarrow B) \times p(A) + p(\neg A \rightarrow B) \times p(\neg A), \tag{LTP}$$

in full generality (it holds, of course, for $A, B \in \mathcal{L}$). Actually, this is good news for avoiding Lewis-style triviality results: as shown by Lassiter (2020), these arguments presuppose features of probability that are typical of classical, bivalent probability functions, such as

Let us now look at a famous bone of contention for theories of conditionals: Stalnaker's Thesis $p(B \rightarrow C) = p(C|B)$ and its generalization $p(B \rightarrow C|A) = p(C|A \wedge B)$. Lewis (1976) has shown that any bivalent semantics of conditionals where Stalnaker's Thesis holds will trivialize the probability function, as long as we assume it to be closed under conditionalization. In the wake of this and successor results (such as Bradley 2000 and Fitelson 2015), many theorists opted either for (i) saving Stalnaker's Thesis at the price of abandoning (full) truth conditions for conditionals (e.g., Adams 1975; McGee 1989; Edgington 1995; Ciardelli and Ommundsen forthcoming), or (ii) declaring Stalnaker's Thesis to be false (e.g., Fitelson 2015; Khoo and Mandelkern 2019; Goldstein and Santorio 2021). As shown by Lassiter (2020), the triviality proofs rely on *classical* features of probability functions, such as the Product Rule and the Law of Total Probability, which do not generally apply to probability functions over trivalent valuations. Indeed, our trivalent definition of conditional probability yields Stalnaker's Thesis as a mathematical fact:

Theorem 3 (Stalnaker's Thesis, general form). *For any $A, B, C \in \mathcal{L}^{\rightarrow}$ with $c(A_{TI}) > 0$ and $c((A \wedge B)_{TI}) > 0$, and any trivalent probability function $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$:*

$$p_A(B \rightarrow C) = p_{A \wedge B}(C) \qquad \text{(Stalnaker's Thesis)}$$

The special case $A = \top$ yields the familiar-looking $p(B \rightarrow C) = p_B(C)$.

Proof. The result follows immediately from Import-Export and Definition 6:

$$p_A(B \rightarrow C) = p(A \rightarrow (B \rightarrow C)) = p((A \wedge B) \rightarrow C) = p_{A \wedge B}(C).$$

$\square$

Recently, Fitelson (2022) has shown that we can obtain Stalnaker's Thesis only at the price of giving up the Product Rule. Whenever conjunction is supposed to behave classically, the failure of the Product Rule looks like

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(LTP), and will be blocked in trivalent semantics. Lewis's derivation goes through, however, when B carries no information regarding C. But then, Lewis's paradoxical conclusion $p(B \rightarrow C) = p(C)$ actually makes sense and is compatible with $p(B \rightarrow C) = p(C|B)$: conditioning on B does not change our credences. In other words, our account circumvents the general triviality result *and* it explains why the paradoxical result is acceptable whenever the derivation is not blocked.

an unacceptable price for obtaining Stalnaker's Thesis. But in our framework, we have independent reasons for rejecting classical conjunction, and therefore also for giving up the Product Rule (when conditional sentences are involved). Since our account of conditional probability satisfies axioms 1'-5' and violates only the Product Rule, it is therefore as similar to standard conditional probability as one can hope for if one wants to obtain Stalnaker's Thesis. For us, the triviality results simply show that no probability of conditionals can be a *fully classical* conditional probability function—but we can reject this requirement on independent grounds.

Goldstein and Santorio (2021) argue, however, that Stalnaker's Thesis should be invalid. It is instructive to study their counterexample. They consider a fair die and the sentences

A: If the die landed even, then if it didn't land on two or four, it landed on six.

B: If the die did not land on two or four, it landed on six.

C: The die landed even.

We agree with Goldstein and Santorio's premise that an adequate theory of the probability of conditionals should assign the values $p(A) = 1$, $p(B) = 1/4$ and $p(C) = 1/2$. Then they note that C ("the die landed even") is just another way of expressing the *material* conditional corresponding to B ("either the die landed 2 or 4, or it landed on 6"). Since indicative conditionals are, on their account, logically stronger than the corresponding material conditionals, B must entail C. On the other hand, $B \wedge C$ entails each of its conjuncts, and so $B = \models B \wedge C$ and therefore also $p(B \wedge C) = p(B)$. Moreover, by Stalnaker's Thesis and the identity $A = C \rightarrow B$, we infer that $p(A) = p(C \rightarrow B) = p(B|C)$. But then we obtain a contradiction:

$$1 = p(A) = p(C \rightarrow B) = p(B|C) = \frac{p(B \wedge C)}{p(C)} = \frac{p(B)}{p(C)} = 1/2,$$

From our viewpoint, it is tempting to attack the equality $p(B|C) = p(B \wedge C)/p(C)$ since we know that the Product Rule fails in our semantics. But in this concrete case, $p(B \wedge C) = 1/2$ and the above equality actually holds. Rather, we should reject the equality $p(B \wedge C) = p(B)$, which is false on our account (since $p(B) = 1/4$). Specifically, due to the non-classical behavior of conjunction, $B \wedge C$ can have greater probability than B (note that B is itself a conditional). This suffices to block the above argument.

Relatedly, our account declares simple material and indicative conditionals to be logically equivalent in C, the logic of certain reasoning (see Égré, Rossi, and Sprenger forthcoming for details). This explains why Or-To-If inferences are so compelling when premises are certain. ("Either the butler or the gardener dit it. Therefore, if the butler did not do it, the gardener did."—see Stalnaker 1975 and more recently, Boylan and Schultheis 2022.) However, in *uncertain* reasoning $A \vee B \not\models_U \neg A \rightarrow B$. For conditional-free A and B, even $\neg A \rightarrow B \models_U A \vee B$ holds due to the well-known theorem $p(\neg A \rightarrow B) = p(B|\neg A) \leq p(A \vee B)$. This means that the simple indicative conditional is more demanding to assert than the corresponding material conditional (as desired). In particular, Or-to-If fails in U.

Accounts that declare Stalnaker's Thesis invalid rarely do so because they consider it fundamentally mistaken. Goldstein and Santorio agree with the basic intuition about the probability of conditionals as conditional probabilities and they declare Adams's Thesis—the restriction of Stalnaker's Thesis to conditional-free statements—valid. They reject Stalnaker's Thesis because it clashes with having a standard probability function over conditional statements together with a specific view on the logical relationship between the indicative and the material conditional (e.g., Gibbard 1981; Gillies 2009)—requirements that we consider mistaken. Similarly, in the light of the central role of Import-Export both in Gibbard's 1981 collapse result and in the trivialization of Stalnaker's Thesis, Khoo and Mandelkern (2019) give up Import-Export as a logical principle while retaining that two sentences with the logical forms $A \rightarrow (B \rightarrow C)$ and $(A \wedge B) \rightarrow C$ express the same proposition in any context. Such solutions are feasible, but they look to us like workarounds: it is more attractive, and certainly more straightforward, to preserve Import-Export and to align Stalnaker's and Adams's Thesis. Our account declares both of them valid and blocks the triviality results in an independently motivated way.[14]

6   Bayesian Learning in a Trivalent Setting

In the previous section, we have mapped the *statics* of uncertain reasoning with conditionals: we have defined conditional probability in a trivalent

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[14]Regarding Import-Export, Ciardelli and Ommundsen (forthcoming) argue, in our opinion convincingly, that the use of Import-Export in the trivialization of Stalnaker's Thesis is innocent from a normative point of view. For the role of Import-Export in Gibbard's triviality proof, see Égré, Rossi, and Sprenger (2023).

setting via the equation $p_A(B) := p(A \rightarrow B)$ and shown how this account yields Stalnaker's Thesis. We now move on to the *dynamics* of probabilistic reasoning with conditionals, i.e., updating a prior probability distribution to a posterior probability distribution in the light of incoming evidence. Bayesian learning, the standard theory of changing one's credences in the light of new information, defines updates by means of conditional probability (for evidence E and hypothesis H):

$$p^E(H) := p(H|E) \qquad \text{(Bayesian Conditionalization)}$$

In other words, the rational credence in H after learning E should be the rational credence in H conditional on E. The natural generalization of Bayesian Conditionalization to the trivalent setting is therefore based on our definition of conditional trivalent probability. In this way, we can be sure that the behavior of trivalent conditionalization mimicks standard Bayesian Conditionalization as much as possible.

Definition 7 (Trivalent Conditionalization). *Suppose we learn $E \in \mathcal{L}^\rightarrow$ with $c(E_{TI}) > 0$. Then the rational credence in $H \in \mathcal{L}^\rightarrow$ is the trivalent conditional probability of H, given E:*

$$p^E(H) := p_E(H) = p(E \rightarrow H) \qquad \text{(Trivalent Conditionalization)}$$

The first thing to note is the posterior probability function p^E is itself a trivalent probability function (proof omitted, but straightforward). Second, C-equivalent sentences produce the same update under trivalent conditionalization:

Proposition 5. *For any formulas $E, E', H \in \mathcal{L}^\rightarrow$, if $E =\!|\!|=_\mathsf{C} E'$, then $p^E(H) = p^{E'}(H)$.*

Proof. Follows immediately from the observation that trivalent conditional probability satisfies Left Logical Equivalence with respect to C (Property 4', p. 19). $\qquad \square$

When A and B are conditional-free, we also have $A \supset B =\!|\!|=_\mathsf{C} A \rightarrow B$, and the above proposition implies the following corollary:

Proposition 6. *For any $H \in \mathcal{L}^\rightarrow$, and $A, B \in \mathcal{L}$:*

$$p^{A \rightarrow B}(H) = p^{A \supset B}(H) \qquad \text{(Updating on Simple Conditionals)}$$

In other words, *learning a simple conditional $A \rightarrow B$ is the same as learning the material conditional $A \supset B$.* This prediction is also endorsed by Goldstein and Santorio (2021) and Santorio (2022, Section 7): it is required for explaining why, upon learning a disjunction such as "either the butler or the gardener did it", we also fully accept the sentence "if the butler did not do it, the gardener did it". While Santorio's account does not treat these sentences as logically equivalent, but only as update-equivalent, our account explains the equivalence between learning a simple conditional and learning the corresponding material conditional in terms of their semantic and logical properties in C.

These results are helpful for tackling open problems in Bayesian epistemology. Note that Bayesian conditionalization is a principle for *propositional* learning. When indicative conditionals are not treated as standard propositions, it is unclear how Bayesian should update on $A \rightarrow B$. The various proposals in the literature, such as adding the constraint $p'(B|A) = 1$ for the posterior distribution p', or conditionalizing on $A \supset B$, are thus not buttressed by a theory of the semantics of conditionals (for discussion, see Douven and Romeijn 2011; Eva, Hartmann, and Rafiee Rad 2020).

By contrast, trivalent semantics naturally extends the scope of Bayesian updating to conditional sentences. Specifically, we have justified why learning a simple conditional should amount to learning the corresponding material conditional. In fact, we can also show that trivalent conditionalization generalizes Bayesian conditionalization in another important way. One way of motivating conditionalization is to consider it as a special case of a more general updating rule: *minimizing the divergence between prior distribution p and posterior distribution p'.* Indeed, results by Cziszár (1967, 1975) and Diaconis and Zabell (1982) show that for a certain class of divergence functions—the so-called f-divergences—the two following updating policies are equivalent:

(1) Bayesian conditionalization on the event E;

(2) minimizing the f-divergence between the distributions p and p', subject to the constraint that $p'(E) = 1$.

In general, for discrete probability spaces $\Omega = \{\omega_1, \ldots, \omega_n\}$, f-divergences have the form

$$D_f(p, p') = \sum_{i=1}^{N} p(\omega_i) f\left(\frac{p'(\omega_i)}{p(\omega_i)}\right),$$

where $f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ is a convex and differentiable function satisfying $f(1) = 0$. A well-known f-divergence is the Kullback-Leibler divergence or relative entropy, which is obtained by choosing $f(x) = x \log x$.

The divergence minimization approach not only agrees with Bayesian Conditionalization for propositional learning, but it is also independently motivated as a conservative method of belief revision. Incoming evidence should change our beliefs only to the extent that this is strictly required; if possible, we would like to stay close to the original prior distribution. Moreover, in comparison to standard Bayesian conditionalization, it has the advantage of being applicable to a wider variety constraints on the posterior probability distribution (e.g., constraints which we cannot express in the object language).

We can now show that also trivalent conditionalization minimizes divergence between prior and posterior distribution:

Theorem 4 (Updating Theorem). *The following two procedures for updating credences from a prior probability distribution $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$ to a posterior probability distribution $p' : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$ are equivalent:*

  (1) *trivalent conditionalization on the proposition $E \in \mathcal{L}^{\rightarrow}$;*

  (2) *minimizing the f-divergence between p and p', subject to the constraint that $p'(E) = 1$.*

Proof. Suppose we learn $E \in \mathcal{L}^{\rightarrow}$, which, by the Reduction Theorem, can be written as $E \approx A \rightarrow B$, with $A, B \in \mathcal{L}$. Trivalent conditionalization and Proposition 6 yield $p^E = p_{A \rightarrow B} = p_{A \supset B}$, i.e., learning E amounts to conditionalizing on the corresponding material conditional $A \supset B$. Sprenger and Hartmann (2019, Theorem 4.3) show that updating p on the *material* conditional $A \supset B$ is equivalent to minimizing the f-divergence $D_f(p, p')$ subject to the constraint that $p'(B|A) = 1$ (this expression denotes standard conditional probability in $\mathcal{L}$). However, by Adams's Thesis, this constraint on the posterior distribution is equivalent to $p'(E) = p'(A \rightarrow B) = p'(B|A) = 1$. The converse direction makes use of the same identities. $\square$

Since trivalent conditionalization agrees with the minimization of f-divergence, it can be defended as a conservative method of belief revision in the Bayesian spirit. And it is much more powerful than standard Bayesian conditionalization: it applies to learning conditionals, and to compounds of conditionals of arbitary complexity. The Updating Theorem complements the account of rational credence and static inference in

$\mathcal{L}^{\rightarrow}$ outlined in the previous sections with a theory of dynamic inference in $\mathcal{L}^{\rightarrow}$, and with an epistemological defense of Trivalent Conditionalization.[15] Summing up, we have shown that (i) trivalent conditionalization naturally generalizes Bayesian conditionalization to a language with a conditional connective; (ii) C-equivalent sentences generate the same updates, (iii) learning a simple indicative conditional corresponds to learning the corresponding material conditional; and (iv) trivalent Bayesian conditionalization can be represented as minimizing divergence between prior and posterior distribution.

7   Conclusions

The present paper has proposed a trivalent approach to both the truth conditions and the probability of indicative conditionals, based on the idea of assigning a third truth value when the antecedent is false. We have shown that this semantics, when paired with an appropriate consequence relation for uncertain reasoning, generalizes Adams's logic of p-valid inference to arbitrary compounds and nestings of conditionals. Moreover, this account (i) shows that all complex conditionals can be rephrased equivalently as simple conditionals, (ii) validates Stalnaker's Thesis for the probability of conditionals in its most general form; (iii) models the learning of (conditional) information by means of generalizing Bayesian conditionalization to updating on conditionals.

The Reduction and Representation Theorem in particular vindicate Adams's conjecture that complex conditionals represent "linguistic shortcuts"—we show them to be extensionally equivalent to simple conditionals. In particular, p-validity is a sufficient criterion for evaluating all probabilistically valid inferences in the unrestricted language $\mathcal{L}^{\rightarrow}$, too. An important corollary is that one does not have to choose between a truth-conditional and a suppositional, Adams-style analysis of indicative conditionals: we can have both. Trivalent semantics provides one with fully truth-functional, compositional truth conditions *and* with an Adams-style

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[15]The Updating Theorem does not address the paradoxical examples of updating on conditionals where learning the material conditional seems to deliver the wrong answer (e.g., Douven and Dietz 2011; Douven and Romeijn 2011). However, it provides the semantic and epistemological foundations for solution proposals, e.g. that the posterior distribution should not only be close to the prior, but also preserve the causal structure of the example, and the probabilistic independence constraints implied by this structure (Eva, Hartmann, and Rafiee Rad 2020).

theory of probabilistic reasoning. Committing to the view that indicative conditionals do not have truth conditions is unnecessary philosophical baggage for proponents of an Adams-style approach: it bars the road to important insights into how we use complex conditionals, how we learn them, and how valid inference relates to truth preservation.

The second part of the paper was devoted to conditional probability and to an integration of our account with Bayesian inference. Conditional probability $p_A(B)$ is *defined* as the probability of the conditional $A \to B$ (for all $A, B \in \mathcal{L}^{\to}$). We have argued that this is a natural generalization of conditional probability to the trivalent case. Stalnaker's Thesis in its general form, i.e., $p_{A \wedge B}(C) = p_A(B \to C)$, follows, for all sentences of $\mathcal{L}^{\to}$, as an immediate corollary, without falling prey to Lewis-style triviality results or having to restrict the scope of plausible principles such as Import-Export. Instead, triviality is avoided because trivalent probability does not satisfy the Product Rule. The reason behind this is the (independently motivated) non-classical behavior of conjunction in our semantics. This allows us not only to obtain Stalnaker's Thesis, but also to dodge the impossibility result presented by Schulz (2009) for representing uncertain inference with compounds of conditionals in trivalent logic.

Finally, the Updating Theorem establishes that trivalent conditionalization is a form of divergence minimization, thereby showing that our trivalent conditionalization is a natural generalization of Bayesian conditionalization to the trivalent case, preserving its epistemological motivations. Specifically, we obtain that updating on a simple, non-nested indicative conditional is equivalent to updating on the corresponding material conditional—a prediction that agrees with the results by Goldstein and Santorio (2021) and Santorio (2022). Despite our agreement with these and other authors on crucial predictions, we consider our account simpler, more unified, and more attractive in its results.

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A Proof of the Reduction Theorem

We reason by induction on the depth of formulae, assigning formulae of $\mathcal{L}_1^{\rightarrow}$ a depth of 0, and positive depth to compounds of $\mathcal{L}_1^{\rightarrow}$-formulae containing at least one occurrence of $\rightarrow$. This depth $d(X)$ is defined recursively as follows:

$$
d(X) = \begin{cases} 0 & \text{if } X \in \mathcal{L}_1^{\rightarrow} \\ d(Y)+1 & \text{if } X \in \mathcal{L}^{\rightarrow} \setminus \mathcal{L}_1^{\rightarrow} \text{ has the form } \neg Y \\ \max[d(Y), d(Z)] + 1 & \text{if } X \in \mathcal{L}^{\rightarrow} \setminus \mathcal{L}_1^{\rightarrow} \text{ has the form } Y \circ Z \\ & \quad \text{and } \circ \in \{\wedge, \vee, \rightarrow\} \end{cases}
$$

For the base case $d(X) = 0$, i.e., $X \in \mathcal{L}_1^{\rightarrow}$, the theorem holds trivially. We now consider more complex formulae $X \notin \mathcal{L}_1^{\rightarrow}$ case by case, according to the formula's main connective.

Negation Suppose $X = \neg X'$ for some $X' \in \mathcal{L}^{\rightarrow}$. We observe $d(X) = d(X') + 1$, and so, by the induction hypothesis, there are $A', B' \in \mathcal{L}$

such that $X' \approx A' \to B'$. So $X \approx \neg(A' \to B')$. The Negation Commutation property stated in Proposition 1 yields $X \approx A' \to \neg B' \in \mathcal{L}_1^{\to}$.

Binary Connectives Suppose $X = Y \circ Z$, with $\circ \in \{\wedge, \vee, \to\}$, and $Y, Z \in \mathcal{L}^{\to}$. Obviously, $d(Y) < d(X)$ and $d(Z) < d(X)$. Thus we apply the inductive hypothesis to Y and Z and infer $Y \approx A \to B$ and $Z \approx C \to D$ with $A, B, C, D \in \mathcal{L}$. We have to consider three cases:

$$
\begin{aligned}
X &= Y \to Z \\
&\approx (A \to B) \to (C \to D) && \text{(by compositionality)} \\
&\approx (A \supset B) \to (C \to D) && \text{(by Prop. 1, Left-Nesting)} \\
&\approx ((A \supset B) \wedge C) \to D && \text{(by Prop. 1, Import-Export)}
\end{aligned}
$$

$$
\begin{aligned}
X &= Y \wedge Z \\
&\approx (A \to B) \wedge (C \to D) && \text{(by compositionality)} \\
&\approx (A \vee C) \to [(A \supset B) \wedge (C \supset D)] && \text{(by Prop. 1, Conjunction)}
\end{aligned}
$$

$$
\begin{aligned}
X &= Y \vee Z \\
&\approx (A \to B) \vee (C \to D) && \text{(by compositionality)} \\
&\approx (A \vee C) \to ((A \wedge B) \vee (C \wedge D)) && \text{(by Prop. 1, Disjunction)}
\end{aligned}
$$

In all three cases we have shown that X is semantically equivalent to a $\mathcal{L}_1^{\to}$-formula, completing the proof.