The role of time in relational quantum theories

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Abstract

We propose a solution to the problem of time for systems with a single global Hamiltonian constraint. Our solution stems from the observation that, for these theories, conventional gauge theory methods fail to capture the classical dynamics of the full system. We propose a new strategy for consistently quantizing systems with a relational notion of time that captures the full classical dynamics of the system and allows for evolution parametrized by an *equitable* internal clock. This proposal contains the minimal temporal structure necessary to retain the ordering of events required to describe classical evolution. In the context of *shape dynamics*, an equivalent formulation of general relativity that is locally scale invariant and free of the local problem of time, our proposal constitutes a natural methodology for describing dynamical evolution in quantum gravity.

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Contents

1 Introduction **2**
1.1 Main arguments . . . 3

2 Review of Hamiltonian gauge theory **4**

3 The problem of reduction **6**
3.1 Gauge invariance versus dynamics: quantum . . . 6
3.2 Gauge invariance versus dynamics: classical . . . 7

4 The problem of relational time **9**
4.1 Relational clocks . . . 9
4.2 Internal clocks . . . 10

5 Toy models **11**
5.1 Example: double pendulum . . . 12
5.2 Example: relational free particle . . . 14

6 Relational quantization **15**
6.1 Formal procedure . . . 15
6.2 Proposed solution . . . 16

7 Remarks **17**

8 Acknowledgements **18**

1 Introduction

"Come what come may
Time and the hour runs through the roughest day."

–Macbeth, Act 1 Scene 3

As you read these words, is something happening? As your eyes scan across the page and your brain registers the meaning of the symbols, has something changed? Is there something physically different between this moment and the last? Every intuition we have about the nature of reality forces us to assert that *something* must be happening. If this is so, then how can it have become common practice in canonical quantum gravity to assert that the whole history of the universe is nothing but the unfolding of a gauge transformation? How can we reconcile ourselves to a physics locked in a static universe when our reality abounds with change?

There are two ways out of this conceptual *cul–de–sac*. First, we could accept the fundamental timelessness of quantum gravity as described by the Wheeler–DeWitt equation (or something like it) and focus our energy on recovering the impression of dynamics from the frozen formalism. Much work has been devoted to this first path over the last twenty years: for example, the time capsules approach of Barbour [1, 2], the complete and partial observables scheme developed by Rovelli [3, 4], and approaches that rely on decoherence to regain time in a semi–classical approximation [5, 6].[1]

[1] For further approaches broadly constituted along these lines, we refer the reader to the recent review of Anderson [7].

The second escape route is to insist that it is our formalism rather than our intuition that has mislead us.[2] We will present an argument demonstrating that conventional gauge theory methods *can not* provide a quantization of the appropriate classical system when they are rigidly applied to theories where the dynamical trajectories are generated by a Hamiltonian constraint. Then, we will present an alternative that *does* provide the appropriate quantization. In our relational quantization procedure, minimal temporal structure is retained through the existence of a monotonically increasing parameter that labels the ordering of successive states. In contrast to a quantum theory that retains a Newtonian notion of absolute duration, in our proposal, duration is based upon relative change within the universe *as a whole*. Our minimal notion of time is neither a pre–relativistic background parameter nor a quantity derived from an isolated subsystem. Central to our argument is the realization that it is physically unreasonable to identify all points along a dynamical solution as a gauge orbit containing only physically equivalent states. Identifying all such points using a standard gauge fixing destroys the physical information about the temporal ordering of events and is inconsistent with the classical interpretation of reparametrization invariant theories.

Many of the problems with regard to time and the quantization of general relativity originate in the curiously multifaceted – part dynamical, part symmetry generating – role that the local Hamiltonian constraints play within the classical canonical formalism. Individually, these constraints are responsible for foliation invariance and the associated local problem of time (see [9] for discussion of the problem of interpreting local Hamiltonian constraints quantum mechanically). Collectively, they are responsible for reparametrization invariance and the associated global aspects of the problem of time. There exists, however, an equivalent formulation of general relativity, called *shape dynamics*, that is free of the local problem of time. In this approach, developed in [10, 11], there is a single Hamiltonian constraint generating the dynamics of conformal 3–geometry. The resulting global reparametrization invariance is the only remaining trace of the problem of time. In this paper, we will address this global problem through the consideration of analogue models. Using shape dynamics, our solution is therefore relevant for addressing the full problem of time in quantum gravity.

1.1 Main arguments

In his influential *Lectures on Quantum Mechanics* [12], Dirac claimed that all primary first class constraints are generators of infinitesimal transformations that do not change the physical state (p. 21). Despite his proof, it has been argued [8, 13] that this assertion is invalid for first class constraints generating dynamical evolution. As shown by Barbour and Foster [14], Dirac's proof operates under the assumption that the theory in question contains a fixed time parameter. Reparametrization invariant theories violate this assumption. Therefore, for this class of theories, Dirac's proof is not applicable and our interpretation of the constraints must be dictated by their physical origin.

The logical starting point of this paper is that the Hamiltonian constraints that feature in globally reparametrization invariant theories *must* be treated as generating physical transformations associated with time evolution. In this context, usual gauge theory methodologies become inapplicable: since the integral curves of the null directions associated with Hamiltonian constraints are solutions rather than equivalence classes of identical instantaneous states, what we would normally call 'gauge orbits' are sequences of dynamically ordered physically distinct states. The ordering information is encoded in the positivity of the *lapse* multiplier associated with the constraint. Contrary to standard gauge theory, the passage to a reduced phase space, where the null directions of the Hamiltonian constraints are quotiented out, will lead to an initial data space lacking sufficient

[2] Defense of this largely unexplored second option is, to our knowledge, only found within Kuchař [8].

structure to reconstruct dynamics. This is because, to construct a solution from an initial data point, it is essential to retain the ordering information projected out by the reduction procedure. Formally, this is due to the fact that the pull back of the projection to the reduced space does not tell us how to temporally label the corresponding curves in the unreduced space.[3] We call this the *problem of reduction*. As a result of this problem, the reduced phase space quantization, and equivalent methodologies, do not correctly parametrize the *full* structure of the classical dynamics. This will be shown in detail in Section 3 using path integrals for the quantum theory (Section 3.1) and the Hamilton–Jacobi formalism for the classical theory (Section 3.2).

In addition to these arguments stemming from the problem of reduction, there is good cause to re–evaluate standard quantization techniques on the grounds of an additional *problem of relational time*. Classically, reparametrization invariant theories can be equipped with an internal *and* equitable measure of duration that constitutes a fundamentally relational notion of time. Given strong conceptual and epistemological arguments in favour of physical models with relational time (see Section 4.1), we would like to construct a quantum theory with relational dynamics. However, as shown in Section 4.2, standard gauge theory techniques exclude the use of clocks that are both internal *and* equitable (i.e. they are not relational by our definition). Thus, these techniques prohibit us from constructing a relation notion of time at the quantum level. Specific examples illustrating the problems of reduction and relation time are given in Section 5.

In Section 6.1, we will detail a formal procedure for retaining the essential dynamical ordering information through the introduction of an auxiliary field, and its momenta, that label the classical trajectories and define a relational time. The introduction of these variables is achieved via the extension of the phase space of the original theory in a precise geometrical manner.[4] We then show that the application of standard quantization techniques to the extended theory will lead to a quantum theory that correctly captures the full classical dynamics of the original theory we started with. Furthermore, as shown in Section 6.2, the quantization of the extended theory is such that it leads to a quantum dynamics with respect to a relational time. This time is relational in the sense that it is a reparametrization invariant label for sequences of states and reduces to an equitable internal measure of duration in the classical limit. Our solution is applicable to all theories with global Hamiltonian constraints. As a precursor to our analysis of the problem of time, in the following section we review the key elements of Hamiltonian gauge theory that are necessary to frame our argument in precise terminology.

2 Review of Hamiltonian gauge theory

In this section we review the essential structures of Hamiltonian gauge theory. Further details can be found in [15, 16].

A Hamiltonian theory consists of at least three structures: 1) an even dimensional phase space $\Gamma(q_\mu, p^\mu)$ coordinatized by an equal number of configuration variables q_μ and their conjugate momenta p^μ, where μ runs from 1 to the dimension d of the system, 2) a closed, non–degenerate symplectic 2–form $\Omega = dq \wedge dp$, and 3) a Hamiltonian $H(q, p)$, which is a function on Γ. The non–degeneracy of Ω implies that its inverse, with coordinates Ω^{ab} in some chart (where a, b range from 1 to $2d$), exists such that

$$\Omega^{ac}\Omega_{cb} = \delta^a_b. \tag{1}$$

[3]We are indebted to Tim Koslowski for helping to clarify this key point.

[4]We are again indebted to Tim Koslowski for his valuable insight in regards to this extension procedure.

Using Ω^{ab}, we can define the Poison bracket between functions f and g on Γ through the relation

$$\{f, g\} = \Omega^{ab} \partial_a f \partial_b g \tag{2}$$

where the derivatives are taken with respect to the coordinates on Γ. We can also define the Hamilton vector fields, v_f, of f as

$$v_f = \{f, \cdot\} . \tag{3}$$

These define the tangent space of Γ. It is conventional to choose a representation for Ω^{ab} such that

$$\Omega^{ab} = \begin{pmatrix} 0 & \delta^{\mu\nu} \\ -\delta^{\mu\nu} & 0 \end{pmatrix} . \tag{4}$$

All structures are then defined up to symplectomorphisms, or *canonical transformations*, which are diffeomorphisms of Γ that preserve Ω. The classical solutions of the theory are the integral curves of the Hamilton vector fields, v_H, of the Hamiltonian $H(q, p)$.

A Hamiltonian gauge theory has additional structure due the presence of first class constraints $\chi_i \approx 0$, where $i = 1, .., n$, that must be satisfied by the classical and quantum solutions.[5] These must form a first class algebra with themselves and H under the action of the Poison bracket:

$$\{\chi_i, \chi_j\} = c_{ij}^k \chi_k \tag{5}$$

$$\{H, \chi_i\} = k_i^j \chi_j , \tag{6}$$

for some structure constants c_{ij}^k and k_i^j.

One immediately encounters a problem when trying to define the classical dynamics on the constraint surface, Σ, defined by $\chi_i \approx 0$: the pullback, $\omega \equiv \Omega|_\Sigma$, of the symplectic 2–form, Ω, onto Σ is degenerate and closed (and, therefore, pre–symplectic). It is straightforward to show that ω will have n null directions given by the Hamilton vector fields, v_{χ_i}, of the constraints χ_i. Therefore, the Hamilton vector fields of the Hamiltonian on the constraint surface are not unique and the classical evolution is non–deterministic.

In conventional gauge theory, the solution to this problem is implied by the physical interpretation of the integral curves of v_{χ_i} that foliate Σ. Every point along an integral curve is identified as a *physically indistinguishable* state of the system. Consequently, the leaves of Σ are interpreted as equivalence classes of physical states and are named *gauge orbits*. One way to restore determinism would be to select a particular member of each gauge orbit and therefore select a unique representative for each physically distinguishable state. This surface constitutes a gauge–fixed, $2(d - n)$ dimensional submanifold, Σ_{gf}, of Γ. Furthermore, Σ_{gf} is symplectic because the pullback, $\Omega_{\text{gf}} \equiv \Omega|_{\Sigma_{\text{gf}}}$, of Ω on Σ_{gf} is, by construction, non–degenerate. The symplectic manifold $(\Sigma_{\text{gf}}, \Omega_{\text{gf}})$ is equivalent to a reduced phase space constructed from the quotient of the constraint surface Σ by the null space of ω. Crucial to this reduced formalism is the assumption that the map $\pi : \Sigma \to \Sigma_{\text{gf}}$ encodes *no* dynamical information. We will see in Section 3.2 that this assumption breaks down in theories with global Hamiltonian constraints.

The specification of Σ_{gf} is achieved in practice by imposing gauge fixing conditions $\rho_i \approx 0$ such that the Poison bracket

$$\{\chi_i(t), \rho_j(t')\} \tag{7}$$

seen as a matrix in t, t', i, and j is invertible. Equivalently,

$$\det \left| \{\chi_i(t), \rho_j(t')\} \right| \neq 0. \tag{8}$$

[5]Second class constraints can always be made first class by a suitable redefinition of Ω following Dirac [12].

Σ_{gf} is then defined as the non–degenerate intersection of $\chi_i \approx 0$ and $\rho_i \approx 0$. The condition (8) ensures that the gauge fixing only selects a single member of each gauge orbit. In other words, Σ_{gf} must be nowhere parallel to v_{χ_i}.[6] This geometric requirement of the gauge fixing will be important for our argument later.

Given a set of gauge fixing conditions $\rho_i \approx 0$ satisfying (8), it is possible to define the classical and quantum theories. The classical solutions are the integral curves of the Hamiltonian on Σ_{gf}. The quantum theory is defined by integrating over all paths on the gauge fixed surface restricted by the endpoints q_{in} and q_{fin}. The integration measure over the gauge fixed surface is given by the only non–trivial structure available to us from the formalism:

$$\det \left| \{ \chi_i(t), \rho_j(t') \} \right| . \tag{9}$$

This measure is precisely the Jacobian obtained from a canonical transformation from Γ to the reduced phase space. Thus, the path integral

$$I(q_{\text{in}}, q_{\text{fin}}) = \int \mathcal{D}q_\mu|_{q_{\text{in}}, q_{\text{fin}}} \mathcal{D}p^\mu \delta(\chi_i)\delta(\rho_i) \det \left| \{ \chi_i(t), \rho_j(t') \} \right| \exp \left\{ i \int dt \, [\dot{q}_\mu p^\mu - H] \right\} \tag{10}$$

is equal to a Feynman path integral over the reduced phase space. For details, see [17]. Additionally, it's been conjectured [18], and proven in limited cases (e.g., [19]), that quantization of the reduced phase space is equivalent Dirac quantization of constrained systems [12].

3 The problem of reduction

3.1 Gauge invariance versus dynamics: quantum

Globally reparametrization invariant theories feature action functionals in which the integration is performed with respect to an *arbitrary change parameter* λ rather than a fixed Newtonian background time. The invariance of these theories under re–scalings of this parameter leads to the defining feature of their canonical representation: that the Hamiltonian is replaced with a constraint $\mathcal{H}$, often called the *Hamiltonian constraint*. Assuming that all other first class constraints have been gauge–fixed using the method described above,[7] the remaining structure is a phase space $\Gamma(q, p)$ (possibly corresponding to a gauge fixed surface in a larger phase space), a symplectic 2–form, Ω, and a Hamiltonian constraint, $\mathcal{H}$.

We now state the fundamental difference between reparametrization invariant theories and standard gauge theories: the classical solutions are defined as the integral curves of the Hamilton vector field of the constraint $\mathcal{H}$. Because they are the dynamical solutions of the classical theory, the elements of the integral curves of $v_{\mathcal{H}}$ are no more *physically indistinguishable* from each other than this moment is from the big bang. Thus, the leaves of the foliations of the constraint surface, Σ, defined by $\mathcal{H} \approx 0$ can no longer be reasonably identified as gauge orbits - rather they are dynamical solutions. We will now show that, if one turns a blind eye to this fact, one is led to a quantum theory that, in general, cannot contain the appropriate classical limit.

Performing the gauge fixing procedure outlined in Section 2, we treat each classical history on the constraint surface as an equivalence class of physically indistinguishable states. We then seek a gauge fixing condition $\rho \approx 0$ satisfying

$$\det |\{\mathcal{H}, \rho\}| \neq 0 \tag{11}$$

[6]There is also a global requirement that Σ_{gf} only intersects the gauge orbits once. We will assume that this requirement can be satisfied.

[7]Crucially, this is the step that can be performed in shape dynamics that is highly non–trivial in the ADM formulation of general relativity.

that selects a single element of each of these foliations. However, because the Hamiltonian is just given by the constraint $\mathcal{H}$, its associated flow is everywhere parallel to the gauge orbits. Thus, this procedure completely trivializes the dynamics since there is no way to flow in any direction on the gauge fixed surface. In addition, the interpretation of the gauge fixed surface is now completely different from the case described in Section 2. $\Sigma_{\rm gf}$ constitutes a single element of each integral curve of $v_{\mathcal{H}}$. As such, it represents a space of initial data for all possible classical evolutions on the constraint surface. However, this space, by construction, necessarily excludes any set of points on the constraint surface representing a classical history. Thus, the path integral

$$I = \int \mathcal{D}q_\mu \mathcal{D}p^\mu \delta(\mathcal{H})\delta(\rho) \det|\{\mathcal{H},\rho\}| \exp\left\{ i \int d\lambda \left[\dot{q}_\mu p^\mu\right] \right\} \tag{12}$$

restricted to $\Sigma_{\rm gf}$ cannot contain any particular solution to the classical evolution problem. It, therefore, *can not* be a quantization of the original classical theory. This is equivalent to the statement that the Feynman path integral on the reduced space, which is canonically isomorphic to $\Sigma_{\rm gf}$, fails to capture the classical evolution. In Section 5.1, we give an explicit example illustrating this point.

One key problem we will solve in this paper will be establishing a consistent quantization procedure for globally reparametrization invariant theories that *does* contain the appropriate classical limit. We are faced with a dilemma: on one hand, we need to restrict our path integral to a proper symplectic manifold where the Hamilton vector field of $\mathcal{H}$ is well defined on the constraint surface; but, on the other hand, such a restriction must be such that the constraint $\rho \approx 0$ runs parallel to the foliations of $\mathcal{H}$. Unfortunately, this would imply

$$\det|\{\mathcal{H},\rho\}| = 0 \tag{13}$$

and we no longer have a natural candidate for the measure of the path integral. The solution that we will propose in Section 6 involves extending the phase space in a trivial way so that the desired classical solutions are indeed contained in the initial value problem of the extended theory. Thus, a standard gauge fixing on this extended theory corresponds to a consistent quantization of the original theory. Before describing this procedure in detail, we will show how the argument presented above is paralleled in the classical theory.

3.2 Gauge invariance versus dynamics: classical

In the semi–classical approximation, the wavefunction of a system is given by the WKB ansatz

$$\psi = e^{iS}, \tag{14}$$

where S solves the Hamilton–Jacobi (HJ) equation. When the dynamics is generated by a constraint, the HJ equation takes the form

$$H(q_\mu, \frac{\partial S}{\partial q_\mu}) = 0. \tag{15}$$

Hamilton's principal function $S = S(q_\mu, P^a)$ is a function of the configuration variables, q_μ, and the separation constants, P^a. These separation constants are obtained by solving the partial differential equation (15). In general, there will be one for each $\frac{\partial S}{\partial q_\mu}$ but these will not all be independent because (15) acts as a constraint. This is the reason for labeling P with the index a, which runs from 1 to $d-1$.

The equations of motion are obtained by treating S as a generating function for a canonical transformation from $(q_\mu, p^\mu) \to (Q_a, P^a)$ that trivializes the evolution. The canonical transformation can be determined by computing

$$Q_a = \frac{\partial S(q, P)}{\partial P^a} \tag{16}$$

$$p^\mu = \frac{\partial S(q, P)}{\partial q_\mu}. \tag{17}$$

S is defined such that the relations (17) simply reproduce the Hamiltonian constraint through (15). If the system of equations (16) can be inverted for q_μ then the equations of motion for q_μ can be determined by using the fact that

$$\dot{Q}_a = 0 \qquad\qquad \dot{P}^a = 0. \tag{18}$$

There is, however, an immediate obstruction to this procedure since the system of equations (16) has, in general, a one dimensional kernel and, thus, no unique solution. This obstruction can be overcome in two ways:

1. A *gauge* can be fixed by imposing a gauge fixing condition of the form

$$f(q_\mu, Q_a, P^a, \lambda) = 0. \tag{19}$$

   f must be chosen such that, when the condition $f = 0$ is imposed, the system of equations (16) is invertible.

2. The solution space can be parametrized by one of the q's, chosen arbitrarily. This allows us to write

$$q_a = F_a(Q_a, P^a, q_0). \tag{20}$$

The first method is the one exclusively employed in conventional gauge theory. The gauge fixing (19) reduces the dimension of the system. This is natural in standard gauge theory because the map between the original and reduced phase spaces contains no physical information. It is, therefore, reasonable to make the equations of motion invertible by quotienting away the information contained in this map. This does not kill the dynamical information because a non–trivial Hamiltonian survives the quotienting. However, for globally reparametrization invariant theories, the information contained in the kernel of (16) contains *all* the dynamical information. Thus, we must use the second method for reproducing the classical solutions. This is natural, because the relations (20) are precisely the integral curves of null directions of the presymplectic form on the constraint surface $H = 0$. Indeed, this method is consistent with how HJ theory is used to reproduce the ADM equations of motion in general relativity [20, 21]. In Section 5.2, we will apply methods 1 and 2 to a simple model to illustrate how to implement the formal procedure presented here.

A powerful argument can be made in favor of method 2 over method 1, when dealing with reparametrization invariant theories. In method 1, the pullback under the projection doesn't contain the complete dynamical information. Only in method 2 is it possible to retain information about the temporal ordering of events along the gauge orbits. This a necessary requirement for theories with Hamiltonian constraints because there must be a way to distinguish between the past and the future.[8] This is already implicit in requiring that the lapse, N, should be positive. The

[8] This distinction constitutes a temporal orientation rather than a temporal direction, which would imply an arrow of time.

fact that *only* Hamiltonian constraints have this requirement is an indication that they should be treated differently from the constraints arising in standard gauge theories.

We see that there is a substantive difference between the way the HJ formalism is used in conventional gauge theory and in globally reparametrization invariant theories. This difference is exactly mirrored in the quantum theory. The arguments given in Section 3.1 reflect what happens in the classical theory when method 1 is used: the information about the dynamics is lost by quotienting with respect to the null directions of the Hamiltonian flow. A requirement for consistency for the classical and quantum theories is that the method used in the classical theory is reflected in that used in the quantum theory. In light of this requirement and the necessity of using method 2 classically for reparametrization invariant theories, we will present a relational quantization procedure in Section 6.

4 The problem of relational time

4.1 Relational clocks

There is, without doubt, practical utility in the use of a time parameter disconnected from the dynamics of a physical system. Such an external notion of time is an essential element of both Newtonian systems and conventional approaches to quantum theory. Yet, the existence of such a temporal background is inconsistent without the structure of the physical theory that most accurately describes the behaviour of clocks: general relativity. Within this theory, time is an inherently internal notion, parasitic upon the dynamics. Thus, there is empirical motivation to search for a general procedure for consistently constructing an internal notion of time that can be used in both classical and quantum theories.

In addition, there are strong conceptual arguments against external time – many of which predate general relativity. Ernst Mach, in particular, criticized external notions of time on epistemic grounds. In the most general system, we only have access to the internal dynamical degrees of freedom. Thus, it is "utterly beyond our power to measure the changes of things by time" [22]. Rather, according to Mach, any consistent notion of time must be abstracted from change such that the inherently interconnected nature of every possible internal measure of time is accounted for. According to the Mittelstaedt–Barbour [23, 24] interpretation of Mach, we can understand this *second Mach's principle* as motivating a relational notion of time that is not merely internal but also equitable; in that it can be derived uniquely from the motions of the entire system taken together. Thus, any isolated system – and, in fact, the universe as a whole – would have its own natural clock emergent from the dynamics. Significantly, for a notion of time to be relational in this sense, it is not enough to be merely internal – it must also be unique and equitable. We cannot, therefore, merely identify an isolated subsystem as our relational clock, since to do so is not only non–unique but would also lead to an inequitable measure, insensitive to the dynamics of the clock system itself.

Within classical non–relativistic theory, relational clocks of exactly the desired type have already been constructed and utilized. As has been pointed out by Barbour, the astronomical measure of *ephemeris time*, based upon the collective motions of the solar system, has precisely the properties discussed above. In Section 5, we give an explicit expression for the ephemeris time for a large class of physically relevant finite dimensional models. Quantum mechanically, we run into a problem when attempting to construct a suitably relational notion of time. As we shall discuss in the next section, it is precisely the relational sub–set of internal clocks that are excluded under conventional quantization techniques. The logic of the next section is as follows: first we establish a general theory for describing evolution in timeless systems in terms of an internal clock as constituted by an isolated subsystem; then, we show that such clocks can never be fully adequate precisely because they are not fully relational.

4.2 Internal clocks

We will now detail a method for expressing the path integral (12) in terms of evolution with respect to an internal clock constructed from an isolated subsystem. For *any* splitting of the Hamiltonian constraint of the form[9]

$$\mathcal{H} = \mathcal{H}_{\|} + \mathcal{H}_{\perp} \tag{21}$$

we can perform a canonical transformation Π

$$\Pi : (q_\mu, p^\mu) \to (Q_i, P^i, \tau_{\text{int}}, \mathcal{H}_{\|}) \tag{22}$$

generated by the type–2 generating functional $F(q_\mu, P^a, \mathcal{H}_{\|})$

$$F(q_\mu, P^a, \mathcal{H}_{\|}) = \int dq_\mu \, p^\mu(q_\mu, P^a, \mathcal{H}_{\|}). \tag{23}$$

The index i runs from $1, .., d-1$. The functions $p^\mu(q_\mu, P^a, \mathcal{H}_{\|})$ are obtained by inverting the relations

$$\begin{aligned} P^a &= P^a(q_\mu, p^\mu) \\ \mathcal{H}_{\|}(q_\mu, p^\mu) &= \mathcal{H}(q_\mu, p^\mu) - \mathcal{H}_{\perp}(q_\mu, p^\mu). \end{aligned} \tag{24}$$

The functions $P^a(q_\mu, p^\mu)$ are arbitrary provided the above equations are invertible for p^μ. Because $\mathcal{H}_{\|}$ is fixed by the splitting (21), the canonical transformation Π has a $(d-1)$–parameter freedom parametrized by the functions $P^a(q_\mu, p^\mu)$. Up to this freedom, Π singles out an *internal time* variable τ_{int} which can be obtained from

$$\tau_{\text{int}}(q_\mu, p^\mu) = \left. \frac{\partial F}{\partial \mathcal{H}_{\|}} \right|_{P^a = P^a(q_\mu, p^\mu), \mathcal{H}_{\|} = \mathcal{H}_{\|}(q_\mu, p^\mu)}. \tag{25}$$

We say that $\mathcal{H}_{\|}$ singles out an isolated subsystem of the universe whose motion is used as an internal clock parametrizing the motion of the rest of the system. The remaining configuration variables are given by

$$Q_i(q_\mu, p^\mu) = \left. \frac{\partial F}{\partial P^i} \right|_{P^a = P^a(q_\mu, p^\mu), \mathcal{H}_{\|} = \mathcal{H}_{\|}(q_\mu, p^\mu)}. \tag{26}$$

If we chose to label curves in Γ by the arbitrary parameter λ then, in terms of the transformed coordinates, the natural gauge fixing condition

$$\rho = \tau_{\text{int}} - f(\lambda) \approx 0 \tag{27}$$

splits Γ into the two symplectic submanifolds $\Gamma = \Sigma_{\text{gf}}(Q_a, P^a) \times \Phi(\tau_{\text{int}}, \mathcal{H}_{\|})$. From this, it is clear that the freedom in defining τ_{int} through Π corresponds to the unavoidable freedom in making arbitrary canonical transformations on Σ_{gf}. Using the simple choice $f(\lambda) = \lambda$, the measure is 1. A short calculation shows that the path integral, I, of (12) becomes

$$I = \int \mathcal{D}Q_i \mathcal{D}P^i \exp\left\{ i \int d\tau_{\text{int}} \left[\dot{Q}_a P^a - \mathcal{H}_{\perp}(Q_a, P^a) \right] \right\} \tag{28}$$

[9] In practice, this split need only be approximate to some desired order of accuracy.

after integrating over $\tau_{\rm int}$ and $\mathcal{H}_\|$. As expected, it is equivalent to a Feynman path integral on the reduced phase space coordinatized by (Q_a, P^a). This path integral also corresponds to the quantization of a standard time–dependent Hamiltonian theory in terms of the variables Q^a and momenta P^a with the Hamiltonian $H = \mathcal{H}_\perp(Q_a, P^a)$. It leads to a wavefunction satisfying the time–dependent Schrödinger equation

$$i\frac{\partial \Phi}{\partial \tau_{\rm int}} = \hat{\mathcal{H}_\perp}\Psi. \tag{29}$$

Furthermore, this is equivalent to applying Dirac quantization to the Hamiltonian constraint (21) after applying Π.[10]

The quantum theory given by (28) has a well known classical limit. It is the Hamiltonian theory described by the integral curves of $\mathcal{H}_\perp$ parametrized by $\tau_{\rm int}$. Different choices of $f(\lambda)$ correspond to different parametrizations of these integral curves. Although this freedom to reparametrize the classical solutions is a feature we require, the classical solutions we obtain are *not* equal (or equivalent) to the integral curves of $\mathcal{H}$. Instead they are the integral curves of the part projected out of $v_{\mathcal{H}}$ along $\Phi(\tau_{\rm int}, \mathcal{H}_\|)$. Thus, the only way to obtain the desired classical limit is to impose $\mathcal{H}_\| = 0$. However, with this the choice, the above method fails since the relations (24) become non–invertible. This is another way of understanding the problem of reduction: gauge fixing such that we follow the integral curves of $v_{\mathcal{H}}$ leads to a zero measure in the path integral. In addition to the problem of excluding classical trajectories, this restriction on internal clocks is such that it specifically excludes relational clocks of the kind considered in the previous section. In phase space, the classical ephemeris time is precisely the variable canonically conjugate to the full Hamiltonian of the system. In the following section we demonstrate this explicitly in a simple model.

5 Toy models

We will now make our formal arguments concrete by applying them to specific models. These models will also help to motivate our new quantization procedure. We will consider models of the form

$$S = \int d\lambda \sqrt{g^{\mu\nu}(q)\dot{q}_\mu \dot{q}_\nu}, \tag{30}$$

where g is some specified metric on configuration space. The variation of this action with respect to q implies that it is a geodesic principle on configuration space. Thus, (30) is invariant under $\lambda \to f(\lambda)$, which is the reparametrization invariance we require. These models are useful gravitational models because they include the mini–superspace approximation and contain many key features of general relativity and shape dynamics. Furthermore, because they are also equivalent to non–relativistic particle models, they have considerable heuristic value.

We will treat the case where g_{ab} is conformally flat. Identifying the conformal factor with $2(E - V(q))$, the Hamiltonian constraint is

$$\mathcal{H} = \frac{\delta_{\mu\nu}}{2} p^\mu p^\nu + V(q) - E \approx 0. \tag{31}$$

It is important to note that the origin of this constraint can be traced back to the reparametrization invariance of the action (30). As such, its interpretation is crucially different from that of the gauge generating constraints discussed in Section 2.

[10] See [25] p.280 for a analogous case.

The classical theory corresponding to the Hamiltonian (31) is just that of non–relativistic particles under the influence of a potential $V(q)$ with total energy E and mass $m = 1$. The classical equations of motion are easily seen to lead to

$$\sqrt{\frac{E-V}{T}}\frac{d}{d\lambda}\left(\sqrt{\frac{E-V}{T}}\frac{dq_\mu}{d\lambda}\right) = -\frac{\partial V}{\partial q^\mu}, \tag{32}$$

where $T = \frac{\delta_{\mu\nu}}{2}p^\mu p^\nu$ is the kinetic energy. If we define the reparametrization invariant quantity

$$\tau_{\text{eph}} \equiv \int d\lambda \sqrt{\frac{T}{E-V}} \tag{33}$$

then (32) becomes

$$\frac{d^2 q_\mu}{d\tau^2} = -\frac{\partial V}{\partial q^\mu}, \tag{34}$$

which are Newton's equations with τ_{eph} playing the role of absolute time. Newton's theory is then given by the integral curves of (31) parametrized by the ephemeris time label τ_{eph}.

5.1 Example: double pendulum

Consider the double pendulum consisting of 2 particles q_1 and q_2 in 1 dimension under the influence of a potential

$$V(q) = \frac{1}{2}\left(q_1^2 + q_2^2\right) \tag{35}$$

corresponding to 2 uncoupled harmonic oscillators whose spring constants k have been set to 1. The Hamiltonian constraint is[11]

$$\mathcal{H}_{\text{HO}} = \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) - E. \tag{36}$$

Its Hamilton vector field $v_{\mathcal{H}}$ is

$$v_{\mathcal{H}} = p^\mu \frac{\partial}{\partial q_\mu} - q_\mu \frac{\partial}{\partial p^\mu} \tag{37}$$

and the constraint surface is the S^3 boundary of the 4–sphere of radius E. The integral curves on $\mathcal{H} = 0$ are circles when projected into the (q_μ, p^μ)–planes as is familiar from the usual harmonic oscillator.

Performing a standard gauge fixing, as described in Section 2, and following the procedure described in Section 4.2, we split the Hamiltonian constraint into the pieces

$$\mathcal{H}_\parallel = \frac{1}{2}\left(p_1^2 + q_1^2\right) \qquad\qquad \mathcal{H}_\perp = \frac{1}{2}\left(p_2^2 + q_2^2\right) - E. \tag{38}$$

Using this splitting, we can single out particle 1 as an internal clock for the system. We perform a canonical transformation that takes us to the internal clock variables for particle 1 and leaves particle 2 unchanged. The relations (24) become

$$p_2 = P_2 \tag{39}$$

$$\mathcal{H}_\parallel = \frac{1}{2}\left(p_1^2 + q_1^2\right). \tag{40}$$

[11] In this section we will sometimes write the coordinates of p using lower case indices for convenience.

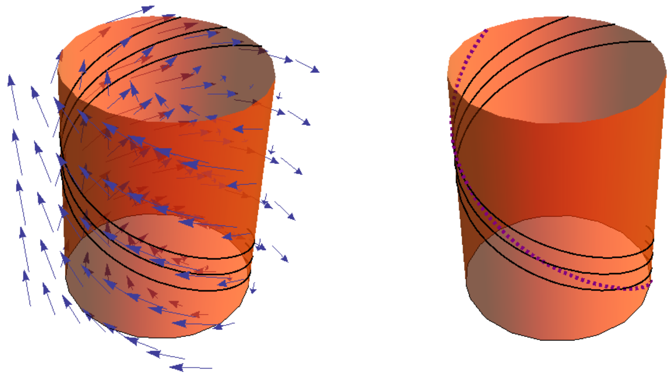

Figure 1: The left hand graphic shows the constraint surface $\mathcal{H}_{\parallel} = 0$, the vector field $v_{\mathcal{H}}$ and three examples of classical solutions (integral curves of $v_{\mathcal{H}}$). The right hand graphic shows, as a dashed line, a sample path that is included in the integral (45), but, by definition, is nowhere parallel to the classical solutions.

Inverting these, we are led to the generating functional

$$F = \int dq_1 \sqrt{2\mathcal{H}_{\parallel} - q_1^2} + q_2 P_2. \tag{41}$$

The transformed Q_2 coordinate is q_2 as expected and the internal time variable canonically conjugate to $\mathcal{H}_{\parallel}$ is

$$\tau_{\text{int}} = \left.\frac{\partial F}{\partial \mathcal{H}_{\parallel}}\right|_{\mathcal{H}_{\parallel}=\frac{1}{2}\left(p_1^2+q_1^2\right)} = \arctan\left(\frac{q_1}{p_1}\right). \tag{42}$$

As can be seen from the definitions of τ_{int} and $\mathcal{H}_{\parallel}$ in terms of q_1 and p_1, this canonical transformation takes us to polar coordinates on the (q_1, p_1)–plane of phase space.

The transformed Hamiltonian is

$$\mathcal{H} = \mathcal{H}_{\parallel} + \frac{1}{2}\left(P_2^2 + Q_2^2\right) - E. \tag{43}$$

Its Hamilton vector field is

$$v_{\mathcal{H}} = P_2 \frac{\partial}{\partial Q_2} - Q_2 \frac{\partial}{\partial P_2} + \frac{\partial}{\partial \tau_{\text{int}}}. \tag{44}$$

The constraint surface is a cylinder along the τ_{int} direction about the (q_2, p_2)–plane of radius $E - \mathcal{H}_{\parallel}$. The integral curves of $v_{\mathcal{H}}$ are helices along the τ_{int}–direction and wrap around the $\mathcal{H}_{\parallel}$–direction implying that $\mathcal{H}_{\parallel}$ is a classical constant of motion (see Figure (1)).

If we impose the gauge fixing condition $\tau_{\text{int}} = \lambda$, the path integral (28) takes the form

$$I_{\text{HO}} = \int \mathcal{D}Q_2 \mathcal{D}P_2 \exp\left\{ i \int d\tau_{\text{int}} \left[\dot{Q}_2 P_2 - \frac{1}{2}\left(P_2^2 + Q_2^2\right) + E \right] \right\}, \tag{45}$$

which leads to the time–dependent Schrödinger equation

$$i \frac{\partial \Psi}{\partial \tau_{\text{int}}} = \left(-\frac{1}{2} \frac{\partial^2}{\partial Q_2^2} + \frac{1}{2} Q_2^2 - E \right) \Psi. \tag{46}$$

This is the same theory we would have obtained had we quantized the 1D harmonic oscillator in the usual way. However, the freedom in redefining $\tau_{\text{int}} = f(\lambda)$ allows us the freedom to reparametrize the paths in phase space arbitrarily. Although this is the reparametrization freedom we want, it doesn't give the freedom to reparametrize the *full* set of classical solutions.

An easy way to see that these paths will be excluded is to realize that these paths will contribute to the path integral with zero measure because the determinant $\det|\{\mathcal{H}, \tau_{\text{int}}\}|$ is zero for these paths. On the other hand, the paths that *are* captured in the integration are the ones corresponding to the 1D harmonic oscillator when projected down to the (q_2, p_2)–plane. This fact is reflected in our final result. In other words, this gauge fixing has effectively quantized the reparametrization invariant 1D harmonic oscillator, *not* the 2D oscillator we started with.

5.2 Example: relational free particle

In this section, we will solve for the classical trajectories of the relational free particle using the HJ formalism. We will compare methods 1 and 2, presented in Section 3.2, to show why conventional gauge theory methods should not be used in this case.

For the free particle, the Hamiltonian constraint (31) takes the form

$$\mathcal{H} = \frac{\delta_{\mu\nu}}{2} p^\mu p^\nu - E \approx 0. \tag{47}$$

Thus, the HJ equation reads

$$\frac{\delta_{\mu\nu}}{2} \frac{\partial S}{\partial q_\mu} \frac{\partial S}{\partial q_\nu} - E = 0. \tag{48}$$

This equation can be explicitly solved by introducing the $d-1$ separation constants P^a. The solution is

$$S(q_\mu, P^a) = q_a P^a \pm \sqrt{2E - P^2}\, q_0, \tag{49}$$

where $P^2 = \delta_{ab} P^a P^b$. We can solve the classical equations of motion by solving for Q_a and p^μ, then by inverting these relations in terms of q_μ. Differentiating S gives

$$Q_a = \frac{\partial S}{\partial P^a} = q_a \mp \frac{\delta_{ab} P^b}{\sqrt{2E - P^2}} q_0 \tag{50}$$

$$p^a = \frac{\partial S}{\partial q_a} = P^a \tag{51}$$

$$p^0 = \frac{\partial S}{\partial q_0} = \pm\sqrt{2E - P^2}. \tag{52}$$

We recover the Hamiltonian constraint, (47), immediately from the last two relations for p^μ. As expected, (50) is non–invertible for q_μ.

There are two possible ways to deal with the non–invertibility of (50):

- **Method 1:** Impose the gauge fixing condition

  $$q_0 = \lambda. \tag{53}$$

  Then,

  $$q_a(\lambda) = Q_a \pm \frac{\delta_{ab} P^b}{\sqrt{2E - P^2}} \lambda. \tag{54}$$

  This does not represent the full set of classical solutions. The reason for this is that, when a gauge fixing is performed, the information about the gauge fixing condition itself is lost. This must be the case, otherwise the theory would not be gauge invariant. Thus, these solutions give curves in the space of q_a's, *not* the space of q_μ's. What is lost is the dynamical information of the gauge fixed variable q_0.

- **Method 2:** We can parametrize the solutions for q_a in terms of q_0, giving

$$q_a(q_0) = Q_a \pm \frac{\delta_{ab} P^b}{\sqrt{2E - P^2}} q_0. \tag{55}$$

These are indeed the correct classical solutions as they represent straight lines on configuration space with parameters specified by initial conditions. The two branches of the solution represent the ambiguity of specifying an arrow of time, since our formalism is indifferent to the direction in which time is increasing.

We can straightforwardly see that method 2 is the correct way of reproducing the classical trajectories. However, this method is *not* compatible with standard techniques used for dealing with gauge systems. This is because gauge invariance requires that the gauge fixed theory is ignorant to the details of the gauge fixing itself. This information, however, is necessary for determining the classical trajectories. Thus, it can not be the case that applying standard gauge theory methods to reparametrization invariant theories will lead to the appropriate quantum theory.

6 Relational quantization

In the preceding discussion, we have shown how standard gauge theory techniques fail to deliver the appropriate quantum theory when applied to theories with global Hamiltonian constraints. Our proposed solution to the problem is not to abandon these techniques altogether – to do so would be to deny ourselves access to number of important mathematical results. Rather, we will outline a formal procedure for modifying an arbitrary globally reparametrization invariant theory such that existing gauge methods *can* be applied and lead to the appropriate quantum theory. In doing so, we will also allow for the construction of a class of quantum theories featuring dynamics with respect to a relational time.

6.1 Formal procedure

Consider the general reparametrization invariant theory $\mathcal{T}$ on the phase space, $\Gamma(q,p)$, with symplectic 2–form, $\Omega = dq \wedge dp$, and Hamiltonian constraint, $\mathcal{H}$. We assume that all other first class constraints have been gauge fixed according to the procedure outline in Section 2. We define the central element, ϵ, of the Poisson algebra as an observable that commutes with all functions on Γ. As such, ϵ is a constant of motion. Thus, provided we fix its value by observation, it can be added to the Hamiltonian without affecting the theory: $\mathcal{H} \to \mathcal{H} + \epsilon$.

Now, consider the two dimensional symplectic manifold (F, Ω_F) coordinatized by ϵ and its conjugate momentum τ (i.e. $\Omega_F = d\epsilon \wedge d\tau$). We can construct the fibre bundle $(\Gamma_\mathrm{e}, \Gamma, \pi_\mathrm{e}, F)$ where F is the fibre, Γ is the base space, Γ_e is the fibre bundle itself, and π_e is a continuous surjection $\pi_e : \Gamma_\mathrm{e} \to \Gamma$. This structure is illustrated in Figure (2). We propose that the fibre bundle Γ_e is the phase space, $\Gamma_\mathrm{e}(q,p,\tau,\epsilon)$, of an extended theory $\mathcal{T}_\mathrm{e}$ that, when quantized with conventional gauge theory methods, leads to a quantum theory that: 1) correctly describes the classical solutions of $\mathcal{T}$ in the semi–classical limit without any additional assumptions, and 2) describes quantum dynamics with respect to a relational notion of time.

We can establish 1) as follows. First, consider $\mathcal{T}_\mathrm{e} = \{\Gamma_\mathrm{e}, \Omega_\mathrm{e}, \mathcal{H}_\mathrm{e}\}$ where $\mathcal{H}_\mathrm{e}$ is the extended Hamiltonian constraint

$$\mathcal{H}_\mathrm{e} = \mathcal{H}' + \epsilon, \tag{56}$$

where $\mathcal{H}'(q,p,\tau,\epsilon)$ is the pullback of $\mathcal{H}(p,q)$ under the bundle projection. Next, consider the constraint surface Σ defined by $H_\mathrm{e} = 0$ and the closed degenerate two form $\omega_\mathrm{e} \equiv \Omega_\mathrm{e}|_\Sigma$. Crucially, the

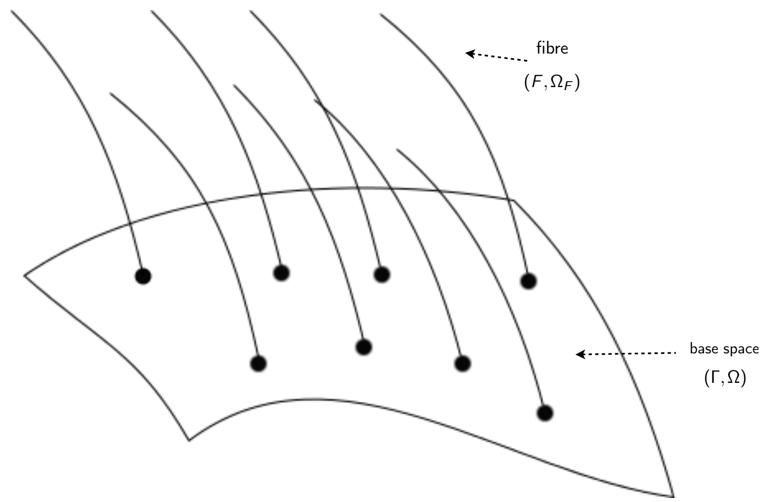

Figure 2: This picture shows the fibre bundle structure of the extended theory. The base space is the phase space of the original theory, $\Gamma(q,p)$, and the fibres are two dimensional symplectic manifolds coordinatized by ϵ and τ.

kernel of ω_{e} does not, by definition, contain any dynamical information because the null directions it encodes are associated with the physically trivial extension procedure. Thus, a gauge fixing on Σ corresponds simply to a section on the bundle. Since the reduced phase space is isomorphic to a section on the bundle, it is also isomorphic to the base space. The classical solutions are contained, therefore, in the reduced phase space because the projection π_{e} maps

$$\pi_{\mathrm{e}} : v_{\mathcal{H}_{\mathrm{e}}} \to v_{\mathcal{H}}, \tag{57}$$

which is the Hamilton vector field of the original Hamiltonian. In Section 6.2, we show how this achieved in practice, with an explicit example.

6.2 Proposed solution

We can establish 2) as follows. The path integral for this theory is defined using the methods outlined in Section 2 and using boundary conditions for τ that are consistent with the value of ϵ determined observationally. Note that $\mathcal{H}_{\mathrm{e}}$ is already in the form $\mathcal{H} = \mathcal{H}_{\parallel} + \mathcal{H}_{\perp}$. Using $\mathcal{H}_{\parallel} = \epsilon$ and $\mathcal{H}_{\perp} = \mathcal{H}'$, we can treat τ as an internal clock by imposing the gauge fixing condition $\tau = \lambda$. Thus, τ is an ephemeris clock for the theory $\mathcal{T}$ as it is canonically conjugate to $\mathcal{H}$ under the bundle projection. The gauge fixed path integral is

$$I = \int \mathcal{D}q\mathcal{D}\tau\mathcal{D}p\mathcal{D}\epsilon\delta(\mathcal{H}_{\mathrm{e}})\delta(\tau - \lambda)\det|\{\mathcal{H}_{\mathrm{e}}, \tau - \lambda\}|\exp\left\{i\int d\lambda\,(\dot{q}p + \dot{\tau}\epsilon)\right\} \tag{58}$$

Integration over τ and ϵ leads to:

$$I_{\mathcal{T}} = \int \mathcal{D}q\mathcal{D}p\exp\left\{i\int d\tau\left[\frac{dq}{d\tau}p - \mathcal{H}'\right]\right\}, \tag{59}$$

which obeys the differential equation

$$i\frac{\partial\Psi}{\partial\tau} = \hat{\mathcal{H}}'\Psi. \tag{60}$$

Thus, we obtain a time–dependent Schrödinger equation where the Hamiltonian is the Hamiltonian of the original theory and the time variable τ has a classical analogue corresponding to the total change of the system. We have, therefore, passed to a quantum theory where evolution is with respect to a relational notion of time. Although a convenient gauge choice has been made to write this result there is still freedom to use an arbitrary reparametrization $\tau = f(\lambda)$ as the path integral is invariant under the choice of gauge fixing functions. This implies that the fundamental symmetry of the classical theory is still respected quantum mechanically.

As an example, we can apply this quantization procedure to the toy model of Section 5. The central element ϵ is identified with the negative of the total energy E of the system. We now extend the phase space to include ϵ and its conjugate momentum τ and $E \to -\epsilon$ in the Hamiltonian,

$$\mathcal{H}_{\mathrm{e}} = \frac{\delta_{\mu\nu}}{2} p^{\mu} p^{\nu} + V(q) + \epsilon. \tag{61}$$

Using the gauge fixing $\tau = \lambda$, the quantum theory is given by the path integral

$$I = \mathcal{D}q_{\mu} \mathcal{D}p^{\mu} \exp \left\{ i \int d\tau \left[\frac{dq_{\mu}}{d\tau} p^{\mu} - \left(\frac{\delta_{\mu\nu}}{2} p^{\mu} p^{\nu} + V(q) \right) \right] \right\}. \tag{62}$$

This corresponds to the time–dependent Schrödinger theory

$$i \frac{\partial \Psi}{\partial \tau} = \left[-\frac{\partial^2}{\partial q_{\mu}^2} + V(q) \right] \Psi = \hat{\mathcal{H}} \Psi. \tag{63}$$

In the semi–classical limit, (63) reduces to the HJ equation for the phase, S, of the wavefunction

$$\frac{\delta_{\mu\nu}}{2} \frac{\partial S}{\partial q_{\mu}} \frac{\partial S}{\partial q_{\nu}} + \frac{\partial S}{\partial \tau} + V(q) = 0. \tag{64}$$

We can do a separation ansatz of the form

$$S(q, \tau; P, \epsilon) = W(q, P) - E\tau, \tag{65}$$

where $W(q, P)$ solves the equation

$$\frac{\delta_{\mu\nu}}{2} \frac{\partial W}{\partial q_{\mu}} \frac{\partial W}{\partial q_{\nu}} + V(q) = E. \tag{66}$$

We, thus, recover the usual HJ formalism. This procedure, however, is invariant under $\lambda \to f(\lambda)$ so that we maintain the required reparametrization invariance.

7 Remarks

Standard quantum mechanics inherits from Newtonian theory a bipartite absolute temporal structure: both the ordering of events (i.e. topological structure) and the duration (i.e. metric structure) are externally fixed. Quantum systems based on a Wheeler–DeWitt–type equation completely dispense of both these aspects of time. Our relational quantization procedure allows us to dispense of the absolute notion of duration whilst retaining the topological structure necessary to recover the classical dynamics. Although it may be possible to recover temporal phenomenology from a timeless formalism, since “the Wheeler-DeWitt equation does not know about...ordering parameter[s]” [2] the recovery of the full classical notion of time is not possible without some additional principle (e.g. Barbour's time capsules [1]). Our proposal does not require any additional principle since the minimum temporal structure is retained throughout.

The quantization procedure presented here constitutes a novel solution to the problem of time for globally reparametrization invariant theories. It provides a fundamental insight into the treatment of the Hamiltonian constraint in quantum gravity, the definition of observables, and the recovery of dynamics. It has immediate applicability in the context of shape dynamics which, like the models considered above, has dynamics driven by a global Hamiltonian constraint. Additionally, given that equitable duration emerges as a purely classical notion, this may provide insight into the interpretation of classical observations in quantum theory.

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References

[1] J. Barbour, "The end of time: The next revolution in physics,". Oxford University Press, UK.

[2] J. J. Halliwell, "Trajectories for the wave function of the universe from a simple detector model," *Phys. Rev.* **D64** (2001) 044008, arXiv:gr-qc/0008046.

[3] C. Rovelli, "Partial observables," *Phys. Rev. D* **65** (2002) 124013.

[4] B. Dittrich, "Partial and complete observables for Hamiltonian constrained systems," *General Relativity and Gravitation* **39** (2007) 1891.

[5] C. Kiefer, "Wave packets in minisuperspace," *Phys. Rev. D* **38** no. 6, (Sep, 1988) 1761–1772.

[6] J. J. Halliwell and J. Thorwart, "Life in an energy eigenstate: Decoherent histories analysis of a model timeless universe," *Phys. Rev.* **D65** (2002) 104009, arXiv:gr-qc/0201070.

[7] E. Anderson, "The Problem of Time in Quantum Gravity," arXiv:1009.2157 [gr-qc].

[8] K. Kuchař, *The problem of time in quantum geometrodynamics*, pp. 169–195. Oxford University Press, New York, 1999.

[9] K. P. Y. Thébault, "Quantisation, Representation and Reduction; How Should We Interpret the Quantum Hamiltonian Constraints of Canonical Gravity?," *Symmetry* **3** (2011) 134.

[10] H. Gomes, S. Gryb, and T. Koslowski, "Einstein gravity as a 3D conformally invariant theory," *Class. Quant. Grav.* **28** (2011) 045005, arXiv:1010.2481 [gr-qc].

[11] H. Gomes and T. Koslowski, "The Link between General Relativity and Shape Dynamics," arXiv:1101.5974 [gr-qc].

[12] P. A. M. Dirac, *Lectures on Quantum Mechanics.* Dover Publications, Yeshivea University, New York, 1964.

[13] J. B. Barbour, "The Timelessness of quantum gravity. 1: The Evidence from the classical theory," *Class. Quant. Grav.* **11** (1994) 2853–2873.

[14] J. Barbour and B. Z. Foster, "Constraints and gauge transformations: Dirac's theorem is not always valid," arXiv:0808.1223 [gr-qc].

[15] V. I. Arnold, V. V. Kozlov, and A. N. A.I., *Mathematical aspects of celestial mechanics.* Springer-Verlag, 1988.

[16] J. Souriau, *Structure of dynamical systems: a symplectic view of physics.* Birkhauser, 1997.

[17] L. D. Faddeev, "Feynman integral for singular Lagrangians," *Theor. Math. Phys.* **1** (1969) 1–13.

[18] V. Guillemin and S. Sternberg, "Geometric quantization and multiplicities of group representations," *Inventiones Mathematicae* **67** no. 3, (1982) 515–538.

[19] M. J. Gotay, "Constraints, reduction, and quantization," *Journal of Mathematical Physics* **27** (Aug., 1986) 2051–2066.

[20] U. H. Gerlach, "Derivation of the ten einstein field equations from the semiclassical approximation to quantum geometrodynamics," *Phys. Rev.* **177** (1969) 1929–1941.

[21] C. Rovelli, *Quantum Gravity.* Cambridge University Press, Cambridge, 2004.

[22] E. Mach, *Die Mechanik in ihrer Entwicklung Historisch-Kritsch Dargestellt.* Barth, Leipzig, 1883. English translation: Mach, E 1960 *The Science of Mechanics*, Open Court, Chicago (translation of 1912 German edition).

[23] P. Mittelstaedt, *Der Zeitbegriff in der Physik.* B.I.-Wissenschaftsverlag, Mannheim, Germany, 1976.

[24] J. B. (ed.) Barbour and H. (ed.) Pfister, "Mach's principle: From Newton's bucket to quantum gravity. Proceedings, Conference, Tuebingen, Germany, July 26-30, 1993,". Boston, USA: Birkhaeuser (1995) 536 p. (Einstein studies. 6).

[25] M. Henneaux and C. Teitelboim, *Quantization of gauge systems.* Univ. Pr., Princeton, USA, 1992. 520 p.