Geometrical aspects of local gauge symmetry

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Abstract

This paper is an analysis of the geometrical interpretation of local gauge symmetry for theories of the Yang-Mills type. It concludes, at least in the case of nonrelativistic quantum mechanics, that gauge symmetry denotes a flexibility in the local representation of nonlocal aspects of the theory.

Contents

1 Introduction

In the context of physics symmetry is defined as an immunity to possible change.[1] In other words, it is the possibility of making a change that leaves some aspect of the situation unchanged. Thus a symmetry is always relative to a class of changes and what is invariable under this class must be specified. In most contexts the application of this concept is not philosophically puzzling, but in a few philosophers and physicists meet difficulties of interpretation. For example in quantum mechanics, if a symmetry associates two apparently distinct states of a system, in what cases should we identify these two states? Obviously the answer to this question is of great importance when we try to identify the ontology of physical theory. Local gauge symmetry is one of these problematic cases. A gauge symmetry is defined as a certain class of changes that does not affect in a physically significant way the Lagrangian. If we want to go beyond this fact and precise exactly what does not change in the theory in order to conserve the empirical content of a physical situation, we encounter conceptual problems. Is gauge symmetry the result of a surplus of structure like it is commonly thought? If it is the result of a surplus of structure, why is this symmetry present in all of our best theories modeling fundamental interaction? This last question express maybe the more puzzling aspect about gauge symmetry. How a surplus of structure could be nonarbitrary?

We believe these questions could be clarify if we develop a sufficiently rich representation framework of gauge theories. In such framework we can hope to clearly distinguish between what depends from the representation and what depends from the the physical model, in other words we hope to identify the pertinent ontological features associated to gauge symmetry. In this paper, we will argue like most physicists[2] that a geometrical representation

[1] I borrow this elegant definition to Joe Rosen [22].

[2] For an example of the physicist point of view, see the classical paper of Daniel and

of gauge theories is an interesting framework for the philosophical analysis. As we will see this interpretation provide clues that in for the quantized Yang-Mills theories[3] the gauge symmetry is the result of a surplus of structure; a surplus that is produced by representing locally non local features of gauge theories. Of course I used the word "clues" and not "proof" because the geometrical interpretation has limits that we will expose. The geometrization program of physics is not advanced enough to assert firmly an interpretation of gauge symmetry.

1.1 What is local gauge symmetry

In many analysis of gauge theory philosophers limit themselves to the simplest case: electrodynamics. General conclusions induced from this theory could be misleading. For example in electrodynamics the gauge field $F_{\mu\nu}$ is gauge invariant. This is not generally the case. Subsequently the belief that $F_{\mu\nu}$ represents the "real field" because it is a covariant field that is gauge invariant cannot be generally defended. Since I believe that gauge theories form a "natural class", it seems clear to me that an analysis of a special case will not do. Theories that we call gauge theories are not put in the same category for arbitrary reasons. They share a basic structure that incline me to believe that a good philosophical analysis should apply to all of them.[4] Therefore I will focus my analysis on a wide class of gauge theories, often called Yang-Mills theories. This class includes among others electrodynamics and quantum chromodynamics.

Let us assume that we have a field theory where a matter field, represented by $\psi(x)$, is coupled to a gauge potential $\mathcal{A}_\mu = A^a_\mu t^a$, where t^a are generators of the gauge group forming the algebra $[t^a, t^b] = if^{abc}t^c$, where f^{abc} are structures constants. The local gauge symmetry implies that this theory is unchanged, that the Lagrangian is invariant, under the following

Viallet [6].

[3]In this paper we will put aside general relativity which is also a gauge theory. This we will do for two reasons: 1) the ontological implication of gauge symmetry is different than in other theories because it affects directly space-time; 2) the quantization of general relativity is not enough developed for discussing the case properly.

[4]Like I said general relativity could be an exception.

transformations:

$$\psi(x) \rightarrow V(x)\psi(x) \tag{1}$$

$$A^a_\mu t^a \rightarrow V(x)\left(A^a_\mu t^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x) \tag{2}$$

where matrices $V(x) = exp\,(i\alpha^a(x)t^a)$ form a group G, where $\alpha^a(x)$ are smooth functions of space-time, g is the charge associated to the gauge interaction. Note that it is because $\alpha^a(x)$ are explicitly function of space-time that we call these transformations local. In this theory the field-tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{3}$$

and transforms under an infinitesimal gauge transformation as

$$F^a_{\mu\nu} \rightarrow F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu} \tag{4}$$

As an example, let us take $V(x) = exp(i\alpha(x))$. In this case[5] the $V(x)$ form a group isomorphic to $U(1)$. The gauge potential and the field tensor transform as

$$A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu\alpha(x) \tag{5}$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \tag{6}$$

We recognize here the case of electrodynamics. Since all known observable quantities are gauge invariant, it seems plausible that local gauge symmetries are the result of a surplus of structure in the theory.[6] Face with this surplus, two strategies possible:

1. We try to formulate gauge theories in a gauge invariant way.[7]

2. We identify the surplus, continue to work with it for pragmatic reasons, and stay careful about the interpretation of the theory.

The first strategy, though philosophically very interesting, creates a mystery. If we could work without gauge freedom, why have we formulate all theories about fundamental interaction with such freedom? What are the pragmatic reasons that incline us to do so? We will see that a geometrical formulation of gauge theories could be appropriate to answer this question.

[5]Note that in the case the group generator is the unity matrix.

[6]This position was recently discussed by Michael Redhead [21], but has been already defended by Eugene P. Wigner in 1964 [26].

[7]A good example of that is the constrained Hamiltonian formulation of electrodynamics. See Gordon Belot [3].

2 Geometrical formulation of gauge theory

2.1 Why the principal fibre bundle formalism?

Even if we agree on the merits of a geometrical formulation of gauge theories, we are still faced to a number of choices on how to built such a representation. What lies at the center of these choices is how we will model geometrically properties that are associated *with* space-time but are not properties *of* space-time. In the case of classical gauge theories physicists massively believe that the principal fibre bundle formalism is the right one for the job. In this subsection I will expose three reasons for choosing this formalism; three reasons that are rarely explicit in the physics litterature.

1. In the principal fibre bundle formalism, the gauge potential is understood naturally as a connection. This is appealing because it models interaction as a topological feature.

2. Gauge theories exhibit global aspects. The principal fibre bundle formalism can represent these aspects.

3. The principal fibre bundle theory was already well developed when physicists needed it.

2.1.1 Geometrical formulation of gauge potential

The conception defended in this subsection has been influenced by Michael Atiyah work [2]. The gauge principle (also called gauge argument) requires that the coupling between a free matter field and the free interacting field is made through the application of an operator on the matter field. This operator is called a covariant derivative[8]:

$$D_\mu = \partial_\mu - ig\mathcal{A}_\mu \tag{7}$$

The commutator of this operator generates the field tensor (equation 3):

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}t^a \tag{8}$$

[8]I am adopting here the position of Lyre on the gauge argument which interpret this one as a constraint on how to couple two free field theories [13]. For more details on the gauge principle see Martin dissertation [16]. For a geometrical interpretation of the gauge argument see Teller [23].

Under a gauge transformation (equation 2), the gauge potential transforms in an inhomogeneous way, in fact like a connection. This is not the case for the field tensor:

$$F^a_{\mu\nu} t^a \to V(x) F^a_{\mu\nu} t^a V^\dagger(x) \tag{9}$$

which transforms like a vector (or tensor).

Let us take seriously that $\mathcal{A}_\mu$ could be some kind of connection. But a connection of what space? It is not apparently a connection of space-time. In fact, the attempt of Hermann Weyl (1918) to relates this connection to the topology of space-time failed.[9] To interpret $\mathcal{A}_\mu(x)$ as a connection, we have to add something to space-time (represented as a Minkowski space M^4): a charge space[10]. Each element of this internal space will be labeled by elements $g \in G$ (the gauge group). This way, gauge transformations exhaust this internal space. See Figure 1. In general we conceive the internal space

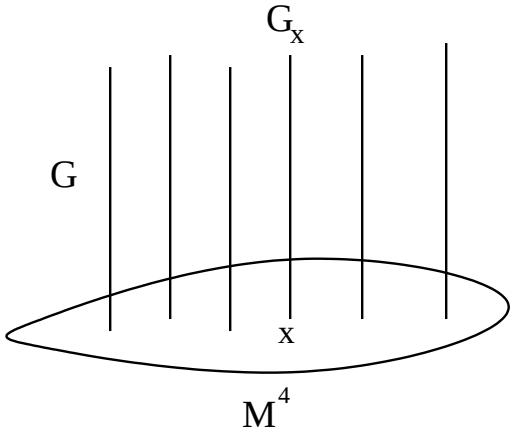

Figure 1: Total space P.

G_x and G_y for $x \neq y$, $x, y \in M^4$ as not being indentified. In fact, we have no a priori reason to believe that property space at different point in space-time are the same. The best we can do is to consider G_x and G_y are of the same type since they represent properties pertinent for the same interaction. So we can draw the total space P as a collection of fibres.

In the absence of external field (interacting field), we consider conventionally that all G_x can be identified to each other. I insist that this identification

[9] For more details see the book of Lochlainn O'Raifeartaigh [20], chapter 1.

[10] I borrow this non standard expression to Faddeev and Slavnov, page 8 of [8].

is purelly conventional. See Figure 2. In that case, we can define horizontal

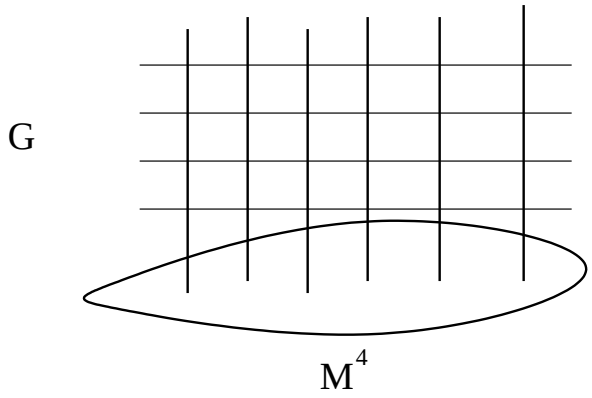

Figure 2: P without interaction.

lines (called sections) to compare points in different fibres. An interacting field has the effect of distorting the relative alignment of fibres so no coherent identification is possible between G_x at different point. However we will presume that G_x and G_y can still be compared if we choose a definite path in M^4 from x to y. This identification of fibres along the path is the familiar notion of parallel transport. Of course we have no a priori reason to believe that if we choose a different path from x to y, we will obtain the same identification. This difference or "phase shift" can be viewed as the total curvature over the region enclosed by the two paths.

If we pass to an infinitesimal description, the field that represents the infinitesimal shift in fibres that is produced by moving of an infinitesimal in any direction in M^4 is what we call the connection. The infinitesimal curvature depends of two directions at x and takes values in the Lie algebra of G_x. It is also an infinitesimal shift. If now we compare this situation to the case without interacting field, we note that $\mathcal{A}_\mu$ plays the role of the connection in the covariant derivative and $\mathcal{F}_{\mu\nu}$ the curvature. The interacting field by mean of the field tensor modifies locally the geometry of the internal space. This identification of fields with geometrical distortion is the heart of Einstein's theory of gravitation. This analogy is appealing.

In order to give a geometrical meaning to $\mathcal{A}_\mu$, we compared the situation with an interacting field ($\mathcal{F}_{\mu\nu} \neq 0$) to the situation in the absence of an external field. This was not the general case. We could have chosen a

different identification of the fibres. This coherence represents the absence

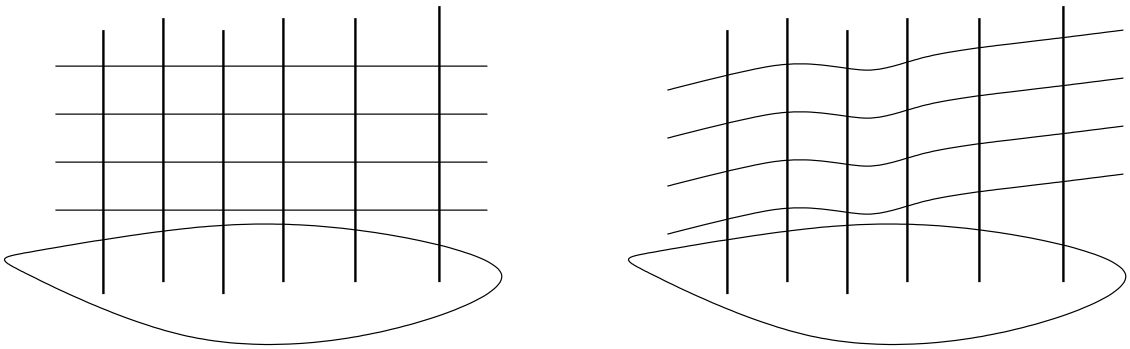

Figure 3: Two identifications of the fibres (two gauges).

of field. A particular choice is called choosing a gauge and we can change from one choice to another by a gauge transformation. In that sens a gauge transformation is a passive transformation and plays a role analogous to a coordinate transformations in Riemannian geometry. The construction that we just described is a special case of principal fibre bundle.

2.1.2 On the importance of being global

The second reason to choose principal fibre bundle formalism is that it can represent global topological features. In physics, field theories are believed to be local, but this main line of thought does not exclude that global features could not be significative. In recent litterature two subjects implying global properties have been widly discussed: monopoles and instantons. In this subsection I will discuss how that kind of examples have put constraints on possible geometrisation of Yang-Mills theories. Constraints that are easily representable in the principal fibre bundle formalism.

I will illustrate my point by discussing of a particular example the Dirac's monopole [7]. This case applies to electrodynamics but could be generalized for non-abelian gauge theories. For details see [28]. Let us imagine that we have in a certain referential a magnetic monopole. In other words in this referential:

$$\vec{E} = 0, \vec{B} = \frac{g}{\rho^2}\hat{e}_\rho \text{ for } \rho \neq 0, \tag{10}$$

where g is the magnetic charge. It can be proven that it is not possible to define a vector potential $\vec{A}$ such that $\vec{\nabla} \times \vec{A} = \vec{B}$ on $\mathbb{R}^3 - 0$. But it is possible to define such field $\vec{A}$ on $U = \mathbb{R}^3 - D$, where D is a Dirac string, a continuous curve starting at the origin and going to infinity. If we choose two strings D_- and D_+, the two vector potentials defined $\vec{A}$ on $\mathbb{R}^3 - 0$. What is interesting is that on $U_- \cap U_+$: $\vec{A}_+ - \vec{A}_- = g\vec{\nabla}\alpha$, where α is a smooth function of space. We note that going from $\vec{A}_-$ to $\vec{A}_+$ only necessitate a gauge transformation. From this example we can infer that the choice of a gauge is essentially local. It consists in choosing a certain number of charts on which are defined the local gauge potentials. If the number of charts cannot be reduced to one, it is because the total space P defined in the last subsection cannot be reduced to $M^4 \times G$. It possess a global structure. Fibre bundle are specially appropriate to represent this structure.

It seems that in few cases global features are important in gauge theories, but why should we make a requisite for the framework of philosophical analysis that it should be able to manage them? Are monopoles and instantons not marginal examples? Is it necessary to cope with them, if our ultimate goal is to discuss the interpretation of gauge symmetry? My answer to these legitimate inquiries is in the same line that the one I gave against concentrate our analysis on electrodynamics. Physicists do not isolate parts of gauge theories and so should not we. Gauge symmetry is a very general aspect of gauge theories, our analysis must be as general. In fact, special cases are traditionally illuminating in physics. For example, physicists began to reconsider the status of the gauge potential in electrodynamics because of the discovery of the Aharonov-Bohm effect [1]. They did not see this effect as marginal, but as an important case where we can see what is truly going on. In the same way, global features in gauge theories are not exceptions but illustrations of what could be important in physical interactions.

Following this spirit, it is not surprising that in 1975 the physicists Wu and Yang in [28] asserted that the Dirac's monopole model force us to adopt a more general notion of gauge; notion that will influence our understanding of local features of gauge theories. An model like the monopole makes us realize that there are two, and not only one, kinds of flexibilities in the description of a gauge potential. First, there is the choice of atlas (set of charts), and second, there is the choice of a gauge potential in a certain chart. This double flexibility has a direct consequence; a gauge does not refer to a particular gauge potential, contrary to what we would have expected. This implies that there are two forms of gauge transformation: 1) Passing from

a chart to another in a overlapping region is what we call a passive gauge transformation. 2) Passing from a gauge potential to an equivalent one in the same chart will be an active gauge transformation. It is interesting to

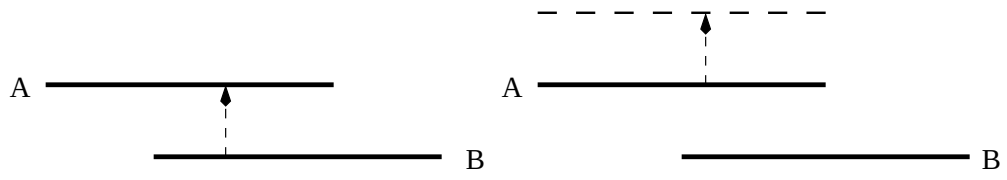

Figure 4: Respectively passive and active gauge transformation between local gauge A and B.

note that in the mathematics litterature a gauge transformation always refer to the active case. In the physics litterature it is the contrary. If these two kinds of transformations are clearly mathematically equivalent, they are not philosophically the same. Usually passive transformation are considered as changes of representation and active ones as physical changes. I will return to this point latter in this article.

2.1.3 The formalism was already there

The two reasons I have exposed are strong motivations to choose the principal fibre bundle formalism to model classical gauge theories, but alternatives are possible.[11] The Lie groupoid is an interresting example, but it is not as developed and as known as the principle fibre bundle formalism.

In 1970, the physicist A. Trautman [24] advocated for the use of fibre bundle techniques to clarify concepts often confused like relativity, symmetry, covariance, invariance and gauge or coordinate transformations. Fibre space was already a well developed field of research in mathematics. In 1930's, H. Seifert has introduced the notion of fibre space. By 1950, the notion of fibre space and fibre bundle had become central in the study of algebraic topology. The specific notion that interests us the principal fibre bundle was fully developed by the mathematician C. Ehresmann in the 50' under the influence of Cartan. It happens so often that physicists have to develop their own mathematical tools that it must have been a relief to find

[11]For the historical part of this subsection, we rely on an unpublished paper by John McCleary [19].

a mathematical theory already there when you need it. This seems to me the main reason why physicists did not produce alternative geometrical constructions of gauge theories. The mathematics already exist and it fits their purposes. As a philosopher we have to be careful because it is possible that this formalism does not fit our purpose which is to have a clear understanding of the conceptual foundations of gauge symmetry.

2.2 Presentation of the principal fibre bundle formalism

In this subsection I will briefly present some essential elements of the principal fibre bundle theory applied to Yang-Mills theories. The reader is invited to note how closely this formalism is mapping the conception I discussed in the precedents subsections.

Definition 1 (Yang-Mills principal fibre bundle) *A Yang-Mills principal fibre bundle $P(M^4, G, \pi)$ consists of the following elements:*

1. *A differentiable manifold P called the total space.*
2. *A Minkowski manifold M^4 called base space.*
3. *A Lie gauge group G called the fibre.*[12]
4. *A surjection $\pi : P \to M^4$ called projection. The inverse image $\pi^{-1}(x) \equiv G_x \cong G$ is called the fibre at x.*
5. *A Lie group G called the structure group, which acts on the fibre G on the left.*[13]
6. *A set of open covering $\{U_i\}$ of M^4 with a diffeomorphism $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ such that $\pi\phi_i(x, g) = x$. The map ϕ_i is called the local gauge or local trivialization since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times G$.*
7. *If we write $\phi_i(x, g) = \phi_{i,x}(g)$, the map $\phi_{i,x} : G \to G_x$ is a diffeomorphism. On $U_i \cap U_j \neq \emptyset$, we require that $S_{ij}(x) \equiv \phi_{i,x}^{-1}\phi_{j,x} : G \to G$ be*

[12] In the case of Yang-Mills theories $G \equiv SU(N)$.

[13] What make a fibre bundle principal is that the structure group is isomorphic to the fibre.

an element of the structure group G. The ϕ_i and ϕ_j are related by a smooth map $S_{ij} : U_i \cap U_j \to G$ such as $\phi_j(x, g) = \phi_i(x, S_{ij}(x)g)$. $\{S_{ij}\}$ are called transition functions or passive gauge transformations.

As you can see this definition express explicitly the notions discussed in the precedent subsection. Note the fact that the structure group is isomorphic to the fibre will guaranty that for every passive gauge transformation there will be an equivalent active gauge transformation. This is the geometrical implementation of the Yang and Wu conception of a gauge.

There is something that was not emphasized in the preceding discussion. Because the structure group is isomorphic to the fibre, we can define the right action of G on the fibre independently of local gauges. Let ϕ_i be a local gauge given by $\phi_i^{-1}(p) = (x, g_i)$, where $p \in \pi^{-1}(U_i)$ and $\pi p = x$. The right action of G on $\pi^{-1}(U_i)$ is defined by $\phi_i^{-1}(pg) = (x, g_i g)$, that is, $pg = \phi_i(x, g_i g)$ for any $g \in G$ and $p = \pi^{-1}(x)$. This definition is independent of the local gauges. If $x \in U_i \cap U_j$, then

$$pg = \phi_j(x, g_j g) = \phi_j(x, S_{ji}(x)g_i g) = \phi_i(x, g_i g) \tag{11}$$

This action will be important for the groupoid description of gauge theories.

In this formalism, the notion of section that we briefly used can be defined as:

Definition 2 (A section) *A section is a smooth map $s : M^4 \to P$ which satisfies $\pi s = id_{M^4}$.*

Given a section $s_i(x)$ over U_i, we can define a prefered local gauge ϕ_i as follows. For $p \in \pi^{-1}$, $x \in U_i$, there is a unique element $g_p \in G$ such that $p = s_i(x)g_p$, then we define ϕ_i by $\phi_i^{-1}(p) = (x, g_p)$. In this local gauge the section is expressed as $s_i(x) = \phi_i(x, e)$. This local gauge is called the canonical local trivialization (or gauge).

We have a well defined geometrical space where the gauge potentials live, but now we have to define the associated vector bundle appropriate for the geometrisation of the matter field $\psi(x)$. Given a principal fibre bundle $P(M^4, G, \pi)$, we may construct the associated vector bundle as follows. Let G acts on a k-dimensional vector space V on the left. Define an action of $g \in G$ on $P \times V$ by $(p, v) \to (pg, \rho(g)^{-1}v)$, where ρ is the k-dimensional unitary representation of G. Then the associated vector bundle $E(M^4, G, V, P, \pi_E)$ is an equivalence class $P \times V/G$ in which two points (p, v) and $(pg, \rho(g)^{-1}v)$

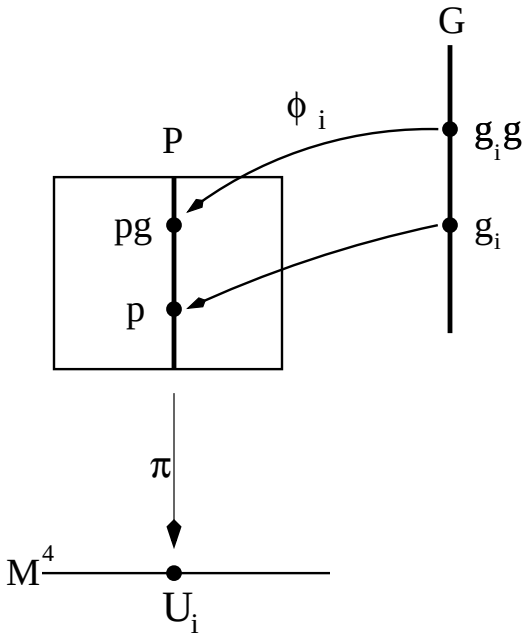

Figure 5: The right action of the gauge group.

are identified. The fibre bundle structure of $E = P \times_\rho V$ is given as follows. The projection $\pi_E : E \to M^4$ is defined by $\pi_E(p, v) = \pi(p)$. The local trivialization is given by $\Phi_i : U_i \times V \to \pi_E^{-1}(U_i)$. Transition functions of E are given by $\rho(S_{ij}(x))$ where S_{ij} are transitions functions on P.

A section s_i on P can be used to determine both local trivialization on P and the local on E:

$$\phi_{i,x}^{-1} \circ s_i(x) = Id_G \tag{12}$$

$$\Phi_{i,x}^{-1} \circ s_i(x) = Id_V \tag{13}$$

where $\Phi_{i,x}^{-1} : \pi_E^{-1}(x) \to V$, $x \in U_i$.

Definition 3 (A matter field ψ) *A matter field of type (ρ, V) is naturally defined as a section $\psi : M^4 \to E$.*

There is an alternative definition that does not depend of the gauge.

Definition 4 (A matter field $\tilde{\psi}$) *A matter field of type (ρ, V) is a map $\tilde{\psi} : P \to V$ which is equivariant under G; $\tilde{\psi}(pg) = \rho(g^{-1})\tilde{\psi}(p)$, $\forall p \in P$, $\forall g \in G$.*

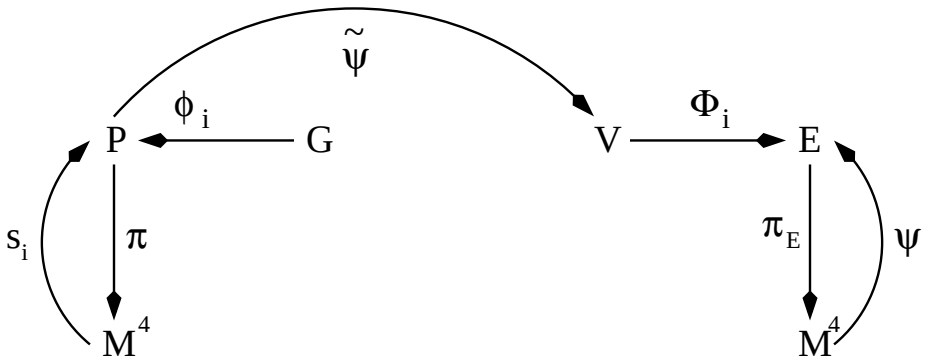

Figure 6: Definitions of the matter field.

The two definitions are related in the following way:

$$\Phi_{i,x}^{-1} \circ \psi(x) = \tilde{\psi}(s_i(x)) \tag{14}$$

where $\Phi_{i,x}^{-1}$ is the canonical local gauge associated to the section s_i.

In physics it is far more convenient to work with $\psi(x)$ because physical quantities are always implicitly defined in a certain gauge, in a certain frame of reference. On the other hand in philosophical analysis $\tilde{\psi}(x)$ is a much more interesting definition of the matter field because it refers to an intrinsic geometrical object. Contrary to a change of $\psi(x)$, a change of $\tilde{\psi}(x)$ is very probably an active transformation, one that includes a physical change.

2.3 The gauge potential as a connection

As we have seen earlier, the gauge potential does not transform like a vector, but like a connection. In this subsection, I will show in what way the gauge potential could be identified as the connection on a principal fibre bundle in a certain gauge.

First let us give a very general definition of what is a connection on a principal bundle.

Definition 5 (A connection on P) *Let $P(M^4, G, \pi)$ be a principal bundle. A connection on P is a unique separation of the tangent space T_pP into the vertical subspace V_pP and the horizontal subspace H_pP such that*

1. $T_pP = H_pP \oplus V_pP$

2. *A smooth vector field X on P is separated into smooth vector fields $X^H \in H_pP$ and $X^V \in V_pP$ as $X = X^H + X^V$.*

3. *$H_{pg}P = R_{g^*}H_pP$ for arbitrary $p \in P$ and $g \in G$.*

The vertical subspace V_pP is a subspace of T_pP which is tangent to G_x at p. The third condition states that H_pP and $H_{pg}P$ on the same fibre are related by a linear map R_{g^*} induced by the right action of the gauge group.

With this definition of the connection, we can easily define the notion of parallel transport of a vector along a curve in M^4 using the notion of horizontal lift.

Definition 6 (Horizontal lift) *Let $P(M^4, G, \pi)$ be a principal bundle and let $\gamma : [0,1] \to M^4$ be a curve in M^4. A curve $\tilde{\gamma} : [0,1] \to P$ is said to be a horizontal lift of γ if $\pi\tilde{\gamma} = \gamma$ and if the tangent vector to $\tilde{\gamma}(t)$ always belongs to $H_{\tilde{\gamma}(t)}P$.*

These definitions imply two nice theorems that I will not prove here:

Theorem 1 *Let $\gamma : [0,1] \to M^4$ be a curve in M^4 and let $p \in \pi^{-1}(\gamma(0))$, then there exists a unique horizontal lift $\tilde{\gamma}(t)$ in P such that $\tilde{\gamma}(0) = p$.*

Corollary 1 *Let $\tilde{\gamma}'$ be another horizontal lift of γ, such that $\tilde{\gamma}'(0) = \tilde{\gamma}(0)g$, then $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$ for all $t \in [0,1]$.*

We can see that this corollary implies among others things the global gauge symmetry. A global right action does not modify the connection structure of the principal fibre bundle.

For practical reasons physicists prefer to work with an equivalent definition of the connection called the Ehresmann one-form connection.

Definition 7 (A connection on P) *A connection one-form ω is a projection of T_pP onto the vertical component $V_pP \simeq \mathcal{G}$ the Lie algebra of G. The projection property is summarized by the following requirements:*

1. *$\omega(v) = \hat{v}$, where $v \in V_p$ and $\hat{v} \in \mathcal{G}$ are related by the canonical isomorphism.*[14]

[14] Let $\hat{v} \in \mathcal{G}$ correspond to $dg(s)/ds|_{s=0} \in T_e(G)$; the equation $v(p) = d(R_g p)/ds|_{s=0} \in V_p$ defines the canonical isomorphism between $\mathcal{G}$ and V_p. For more details see [4], pp 359-360.

2. $R_g^*\omega = Ad_{g^{-1}}\omega$ *that is for* $u \in T_pP$, $R_g^*\omega_{pg}(u) = \omega_{pg}(R_g^*u) = g^{-1}\omega_p(u)g$.

The connection ω is independant of local gauges but $\mathcal{A}_\mu$ are not. The relation between ω and $\mathcal{A}_\mu$ is simple. For a certain U_i and a certain local gauge ϕ_i:

$$A_i \equiv s_i^*\omega \tag{15}$$

where s_i^* is the pullback of the local section canonically related to ϕ_i. Note that A_i is defined only on U_i. If x_μ are local coordinates on U_i, then we recover the well known gauge potential

$$A_i \equiv (-igA_\mu^a t^a dx_\mu)_i \tag{16}$$

where $-i$ is the Lie algebra factor and g the charge associated to the interaction. We are now ready to attack our main question: the status of the gauge symmetry.

3 The gauge symmetry

3.1 Gauge transformations

What is a gauge transformation in our geometrical construction? Like we saw for the Dirac's monopole there is not only one answer to this question. Passive and active interpretation of gauge transformations must be studied.

Let us begin with the passive viewpoint. We are looking at the same $p \in P$, $\pi p = x$, but in two different local gauges (local coordinations) ϕ_i and ϕ_j. We can write that $\phi_i^{-1}(p) = \phi_j^{-1}(R(p))$, where R is a vertical automorphism, in other words $R : \pi^{-1}(x) \to \pi^{-1}(x)$, $R(pg) = R(p)g$, $p \in P$, $g \in G$. This is a consequence of the gauge independence of the right action of G. What happens when we pass from the gauge i to the gauge j?

$$\text{the connection: } \omega \to \omega \tag{17}$$

$$\text{the potential: } A_i = s_i^*\omega \to (R \circ s_i)^*\omega \tag{18}$$

$$\text{the matter field: } \tilde{\psi} \to \tilde{\psi} \tag{19}$$

$$\psi \to \rho(S_{ji}) \circ \psi \tag{20}$$

In coordinates:

$$\psi(x) \to V(x)\psi(x) \tag{21}$$

$$A_\mu^a t^a \to V(x)\left(A_\mu^a t^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x) \tag{22}$$

This transformation is clearly a simple change of representation. To the passive viewpoint corresponds an active one. Let us look at two points $p, R(p) \in P$ related by the same automorphism than above. This time, we stay in the same local gauge ϕ_i. The effect of this vertical automorphism of the principqal fibre bundle is

$$\text{the connection: } \omega \to R^*\omega \tag{23}$$

$$\text{the potential: } A_i = s_i^*\omega \to s_i^*(R^*\omega) \tag{24}$$

$$\text{the matter field: } \tilde{\psi} \to R^*\tilde{\psi} \tag{25}$$

It is now time to pass to our main objective.[15]

3.2 Interpretation: classical physics

In classical electrodynamics, the gauge is understood as a surplus of structure, because only $F_{\mu\nu}$ is considered as a physical field (not A_μ). A necessary but not sufficient condition to be considered a physical field is to be gauge invariant. We will see that this conclusion about gauge symmetry status can be extend to all classical gauge theories.

If the meaning of gauge transformation was restricted only to the passive case, gauge symmetry would only denote the freedom we have in choosing a representation framework (coordinations). Knowing that, we would still have to identify which intrinsic structures of the geometrical construction "correspond" to physical entities. If in electrodynamics the locality of interaction and Maxwell's equations favor that the curvature of the connection, $F_{\mu\nu}$ in coordinates, is the interaction field, in non-Abelien cases things are not that simple. In these cases $\mathcal{F}_{\mu\nu}$ is not gauge invariant. Since physicists have a strong presumption that only gauge invariant objects can represent real entities, the active field in classical non-abelien theories could not be $\mathcal{F}_{\mu\nu}$. On the other hand under a passive gauge transformation ω stays inchanged.

[15] In the conception of Yang and Wu of a gauge the difference between a global and a local transformation is blurred. In fact, a transformation $\psi \to V(x)\psi$ for a domain of space-time can be interpreted as

- a local transformation measured in a gauge using the same coordination for all the domain or
- a global transformation measured in a gauge where there is a different coordination for each point of the domain.

Apparently, the intrinsic connection could be the physical field. However, this analysis is forgetting the active partner of a gauge transformation.

If we extend the definition of gauge transformation to include active ones, the conclusion about the status of gauge symmetry does not change. Furthermore, since ω and $\tilde{\psi}$ are not gauge invariant, we cannot defend that the topology of the fibre bundle construction could represent a physical state of affairs. An infinite number of connection ω represent the same physical content and nothing in the fibre bundle construction helps us to choose one of them.[16]

But do we have to include active transformations in the set of gauge transformations? I remind you that this distinction between active and passive transformations comes directly from the analysis of models, like monopoles, with global geometrical properties. Put aside active transformations would unable us to understand the gauge structure of these examples. In fact, only active interpretation of gauge transformations allows gauge symmetry to have a possible physical meaning. Therefore, we should not exclude active gauge transformations without imperative reasons.

At this point our geometrical construction has permitted us to extend the well known interpretation of gauge symmetry in electrodynamics to non-abelien classical gauge theories. Nevertheless, we are no more advanced in our understanding of the origin of this symmetry. To go beyond we will have to examine the application of fibre bundle formalism in non-classical domains.

3.3 Interpretation: nonrelativistic quantum mechanics

We will examine in this subsection the semi-classical case, namely when quantized charges particles interact with a classical external gauge field. This category is far to be exotic. Most of problems solved in non-relativistic quantum mechanics are in this class. The very well known Aharonov-Bohm effect, much discussed in philosophy recently, falls also in this category. Thus for the rest of this subsection, we will consider that $\psi(x)$ is a wave function[17] representing appropriately charged particles which interact with a classical gauge field.

[16] Richard Healey made a similar analysis for electrodymamics in semi-classical contexts [11].

[17] This analysis is applicable to the case when ψ is a wave function because in this situation the Schrödinger equation is locally gauge invariant.

3.3.1 The gauge groupoid

Let us return to the fibre bundle construction. If $\psi(x)$ is representing a wave function then its absolute phase is not significant. This has a direct geometrical consequence. The elements $p \in G_x$ have no physical meaning, only variation between elements in different fibres have one. By defining for each $x \in M^4$ an internal space isomorphic to G, we built to much structure in the geometrical construction. Does it mean that fibre bundle formalism is useless in quantum physics? Of course not, but we have to reexamine the formalism and extract the phase invariant structure. This work has already been done by mathematicians like Ehresmann in the 1950's, following his work on the concept of principal bundle. To do so, he used the groupoid theory.[18] A groupoid is a generalization of a group that is particularly handy to express local symmetries of geometrical structure. See [25] for more details.

The particular groupoid that we need is called the Ehresmann or gauge groupoid. As we will see in this formalism above the base space we do not find internal spaces but arrows that denote difference of internal properties between points of space-time.

Definition 8 (The gauge groupoid) *Let $P(M^4, G, \pi)$ be a principal bundle. Let G act on $P \times P$ to the right by $(p_2, p_1)g = (p_2 g, p_1 g)$; denote the orbit of (p_2, p_1) by $\langle p_2, p_1 \rangle$ and the set of orbits by $\frac{P \times P}{G}$, then the gauge groupoid consists of two set $\Omega = \frac{P \times P}{G}$ and M^4, called respectively the groupoid and the base, together with two maps $\alpha : \langle p_2, p_1 \rangle \mapsto \pi(p_1)$ and $\beta : \langle p_2, p_1 \rangle \mapsto \pi(p_2)$, called respectively the source and the target, a map $\varepsilon : M^4 \to \Omega; x \mapsto \tilde{x} = \langle p, p \rangle$, called the object inclusion map, and of a partial multiplication in Ω defined on the set $\Omega \times \Omega = \{(\langle p_3, p_2' \rangle, \langle p_2, p_1 \rangle) \in \Omega \times \Omega \,|\, \alpha(\langle p_3, p_2' \rangle) = \beta(\langle p_2, p_1 \rangle)\}$*

$$\langle p_3, p_2' \rangle \langle p_2, p_1 \rangle = \langle p_3, p_1 \delta(p_2', p_2) \rangle \tag{26}$$

where $\delta : P \times P \to G$ is the map $(pg, p) \mapsto g$.

The gauge groupoid is a geometrical construction very close to the principal fibre bundle. The main difference is that information about particular points in fibres is lost in the groupoid, only relative relations between fibres is kept. Thus the gauge groupoid seems more in accordance with quantum theory.

[18] To my knowledge the first that proposed using groupoid in a physical context was Meinhard Mayer in 1988 [17].

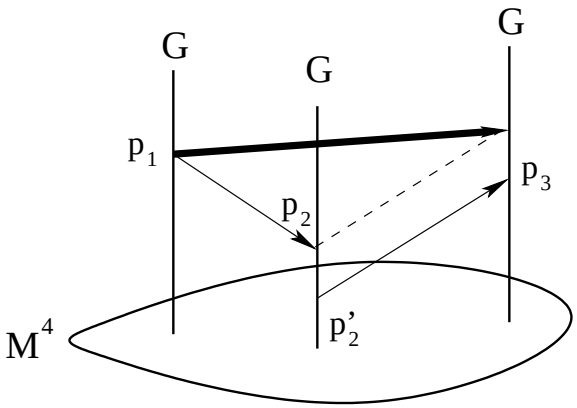

Figure 7: Multiplication in the gauge groupoid.

There is also a notion of gauge freedom in the groupoid. We can choose any representative of a groupoid element (an orbit). By example, we want to multiply $\langle p_3, p_2' \rangle$ by $\langle p_2, p_1 \rangle$. To know that $\alpha(\langle p_3, p_2' \rangle) = \beta(\langle p_2, p_1 \rangle)$ guarantees that p_2 and p_2' are on the same fibre. By means of a "gauge transformation"[19] one can always choose representatives on the orbits so that $p_2 = p_2'$. In this case, the multiplication becomes a "cancellation of the middle". This gauge freedom is clearly passive.

Like we said the principal fibre bundle has more structure than the gauge groupoid. The two constructions are not bijective, but an isomorphism between them can be defined. This isomorphism will allow us to find the groupoid morphism induced by an active gauge transformation of the principal fibre bundle.

Definition 9 (Isomorphism of the principal fibre bundle) *Let $P(M^4, G, \pi)$ be a principal fibre bundle. Choose $p_0 \in P$ (a reference point) and write $x_0 = \pi(p_0)$. Then the map*

$$P \to \left.\frac{P \times P}{G}\right|_{x_0} ; \quad p \mapsto \langle p, p_0 \rangle \tag{27}$$

is a homeomorphism. The map

$$G \to \left.\frac{P \times P}{G}\right|_{x_0}^{x_0} ; \quad g \mapsto \langle p_0 g, p_0 \rangle \tag{28}$$

[19]Expression used by M. Mayer in [18].

is an isomorphism of topological gauge groups. Together they form an isomorphism between the principal bundle and the gauge groupoid.

Thus the way to relate principal bundle to gauge groupoid is by the means of one or more reference points. This dependance made Kirill Mackenzie asserts that if a phenomenon on a principal fibre bundle is formulated in groupoid terms then it will be an intrinsic concept, independant of reference points (page 30, [14]). The gauge groupoid is describing the intrinsic relations between properties at different space-time points. This is all we need in quantum mechanics. The introduction of a charge space was inspired by the way we represent degrees of freedom in space-time. Since internal space (charge space) and space-time could be of totally different nature, it is not necessary to use the same geometrical mean to represent them.

We have seen how is important the notion of connection in principal fibre bundle, what is the equivalent concept in gauge groupoid?

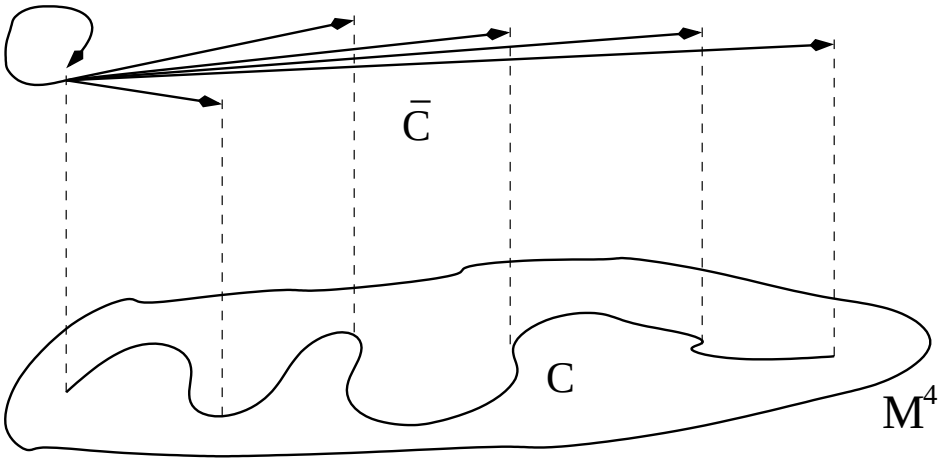

Figure 8: C^0 connection in the groupoid.

Definition 10 (A C^0 path connection) *For any groupoid Ω and any $x, y \in M^4$, write $\Omega_x = \alpha^{-1}(x)$, $\Omega^y = \beta^{-1}(y)$ and $\Omega_x^y = \Omega_x \cap \Omega^y$. Each element $\xi \in \Omega_x^y$ defines a right translation $R_\xi : \Omega_y \to \Omega_x$; $\eta \mapsto \eta\xi$. Let $P_0^\alpha(\Omega)$ be the set of paths $\gamma : I \to \Omega$, which commence at an identity of Ω, for which*

$\alpha \circ \gamma : I \to M^4$ *is constant, where* $I = [0,1]$. *A* C^0 *connection*[20] *in* Ω *is a map* $\Gamma : C(I, M^4) \to P_0^\alpha(\Omega)$; $c \mapsto \bar{c}$, *satisfying the following conditions:*

1. $\bar{c}(0) = \widetilde{c(0)}$ *and* $\beta \circ \bar{c} = c$
2. *If* $\varphi : [0,1] \to [a,b] \subseteq [0,1]$ *is a homeomorphism then* $\overline{c \circ \varphi} = R_{\bar{c}(\varphi(0))^{-1}} \circ (\bar{c} \circ \varphi)$.

This connection is the equivalent for groupoid of the horizontal lift for principal fibre bundle. Of course they are not the same because Γ is defined independently of any reference point of P. To study the effect of an active gauge transformation on the groupoid it will be clearer not to work directly with Γ, but rather with the holonomy subgroupid.

Definition 11 (The holonomy subgroupoid) *The holonomy subgroupoid is a subgroupoid of the gauge groupoid defined as*

$$\Psi = \Psi(\Gamma) = \left\{ \bar{c}(1) \,|\, c \in C(I, M^4) \right\} \tag{29}$$

It associates to each path in $C(I, M^4)$, *one element of the gauge groupoid, namely the last* $\bar{c}$ *produced by* Γ.

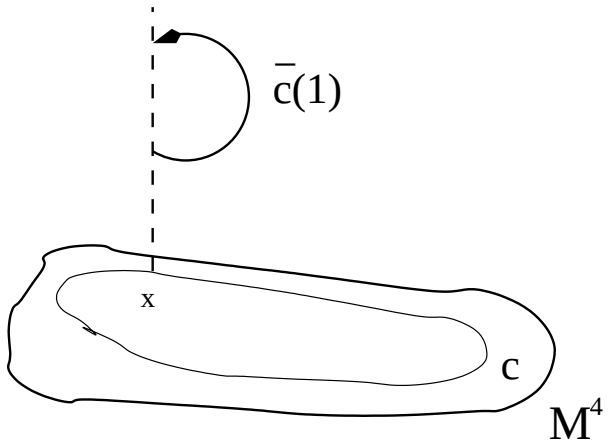

Figure 9: An element of Ψ_x^x.

[20] In this paper we will not use the concept of differential connection. Such notion can be defined in groupoid theory using algebroids. See chapter 3, [14].

Holonomy subgroupoid or holonomy groupoid is a very rich concept. If we restrict $C(I, M^4)$ only to path finishing at the same point they have begun, we obtain Ψ_x^x, the holonomy groups in principal bundle, in other words the subgroup of G that can be related by a parallel transport on a closed path. For more details see the book of Lichnerowicz [12]. Add to Ψ_{x_0} a reference point p_0 such that $\pi(p_0) = x_0$ and we otain the holonomy bundle of $P(M^4, G)$ through p_0, in other words all points that can be parallel transported from p_0. Ψ clearly represent the intrinsic topological structure generated by the Ehresmann connection ω.

We have seen that to an active gauge transformation corresponds a vertical automorphism R of the principal fibre bundle. Using the principal bundle isomorphism, it is easy to define the associated groupoid morphism $\mathcal{R}$. If we apply this morphism to the holonomy groupoid we note that in general $\Psi' = \mathcal{R}(\Psi) \neq \Psi$. However, we can also note a very interesting fact. For any vertical automorphism R and for all $x \in M^4$ then $\mathcal{R}(\Psi_x^x) = \Psi_x^x$. This fact will be very important in the next subsection.

3.3.2 The gauge symmetry in nonrelativistic quantum mechanics

One of the difficulties to apply geometrical ideas to quantum mechanics is that the phase space in such a theory is based on a noncommutative algebra. This fact seems to imply that this theory could not be represented by a geometry of points. One possibility to bypass this difficulty is to work in the frame Alain Connes noncommutative geometry [5]. This is maybe the more natural path, but it is not what we will do here. There is a way to get philosophical insights about gauge symmetry without using the full arsenal of noncommutative geometry. As quantization method, we will use the Feynman path integral (first developed in [9]). This approach of quantum mechanics and quantum field theory has rarely been used in philosophical discussions. However it is a reconstruction of quantum theory that is perfectly legitimate. It is generally admitted that this so-called space-time or functional approach is completely equivalent to the standard Hamiltonian quantization methods, but for our purpose the advantage of the Feynman formulation is that quantities involved in calculations remain classic. In this context, our geometrical construction could still be used.

Let us begin with the simplest case: quantized particles under the influence of a classical electromagnetic field. In that model, the path integral is defined as

Definition 12 (Path integral) *The probability amplitude (also called propagator), that a particle that was at the position $\vec{q}$ at time $t = 0$, is at position $\vec{p}$ at time $t = T$, is*

$$K(\vec{p}, T; \vec{q}, 0) = \int D(\vec{q}(t))\, e^{\frac{i}{\hbar} S[\vec{q}(t)]} \tag{30}$$

where $S[\vec{q}(t)]$ is the classical action of the path $\vec{q}(t)$; in other words $S[\vec{q}(t)] = \int_0^T L(\dot{\vec{q}}, \vec{q})\, dt$, where $L(\dot{\vec{q}}, \vec{q})$ is the Lagrangian of the particle. The integral $\int D(\vec{q}(t))$ is a sum over all possible trajectories between $x = (0, \vec{q})$ and $y = (T, \vec{p})$.

To calculate the propagator, we have to sum functions of the action for all possible trajectories between x and y. Note that the contributiuon of each path is a phase, an imaginary number of norm 1. To illustrate this

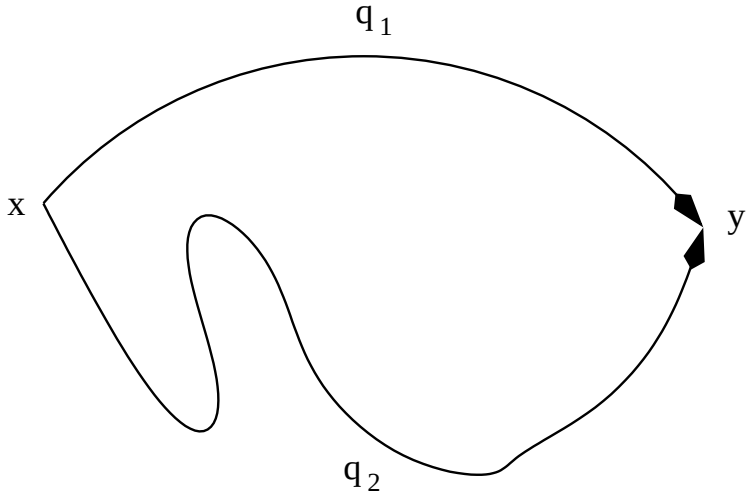

Figure 10: Two paths from x to y.

definition, let us consider the partial sum of only two paths q_1 and q_2. For this two paths, the amplitude is

$$e^{\frac{i}{\hbar} S[\vec{q_1}(t)]} + e^{\frac{i}{\hbar} S[\vec{q_2}(t)]} = e^{\frac{i}{\hbar} S[\vec{q_1}(t)]} \left(1 + e^{\frac{i}{\hbar}(S[\vec{q_2}(t)] - S[\vec{q_1}(t)])}\right) \tag{31}$$

$$= e^{\frac{i}{\hbar} S[\vec{q_1}(t)]} \left(1 + e^{\frac{i}{\hbar} \Delta_{21}}\right) \tag{32}$$

We note that the relative phase between these two contributions Δ_{21} is the action difference between the two paths. Since absolute phase is not measurable in quantum mechanics, if we had only these two paths to sum, only this relative phase would have a physical meaning.

What is the specific effect of electromagnetism on this particle? Classically we know that if we add an electromagnetic interaction, the Lagrangian of the charged particle (of charge e) becomes:

$$L(\dot{\vec{q}}, \vec{q}) \to L(\dot{\vec{q}}, \vec{q}) + e\left(\frac{\vec{v}(t)}{c} \cdot \vec{A}(\vec{q}(t)) - \phi(\vec{q}(t))\right) \tag{33}$$

where $\vec{v}(t)$ is the velocity of the particle[21]. And the contribution to the path integral is modified in the following way:

$$\exp\left[\frac{i}{\hbar}S[\vec{q}(t)]\right] \to \exp\left[\frac{i}{\hbar}\left(S[\vec{q}(t)] + e\int\left(\frac{\vec{v}(t)}{c} \cdot \vec{A}(\vec{q}(t)) - \phi(\vec{q}(t))\right)\, dt\right)\right] \tag{34}$$

$$= \exp\left[\frac{i}{\hbar}S[\vec{q}(t)]\right] \cdot \exp\left[\frac{ie}{c\hbar}\int\left(\frac{d\vec{q}}{dt} \cdot \vec{A} - c\phi\right)\, dt\right] \tag{35}$$

$$= \exp\left[\frac{i}{\hbar}S[\vec{q}(t)]\right] \cdot \exp\left[-\frac{ie}{c\hbar}\int_q A_\mu dx^\mu\right] \tag{36}$$

What the electromagnetic interaction does to the contribution of the path is to multiply it by a factor that depends only of the net parallel transport in the principal fibre bundle over $\vec{q}$ between x and y. This factor is a phase that depends in general of the particular $\vec{q}$. This model clearly points to consider electromagnetism as a nonintegrable phase factor (see this position in Yang [29]). As we have seen in the last subsection, to the $\int_q A_\mu dx^\mu$ corresponds an element of Ψ_x^y. By induction, we can see that the set Ψ_x^y contains all the information that we need to calculate the effect of an electromagnetic interaction on the propagator of a particle.

As we said, absolute phase is not measurable, only relative phase between

[21] For this discussion we will neglect the effect of the spin.

paths is. The effect of electromagnetism on Δ_{21} is

$$\Delta_{21} \rightarrow \Delta_{21} - \frac{e}{c}\left(\int_{q_2} A_\mu dx^\mu - \int_{q_1} A_\mu dx^\mu\right) \tag{37}$$

$$= \Delta_{21} - \frac{e}{c}\left(\int_{q_2} A_\mu dx^\mu + \int_{-q_1} A_\mu dx^\mu\right) \tag{38}$$

$$= \Delta_{21} - \frac{e}{c}\oint A_\mu dx^\mu \tag{39}$$

From this result we can see that what is physically pertinent[22] is not directly Ψ_x^y, but gauge groupoid elements such as $\xi\eta \in \Psi_x^x$, where $\eta \in \Psi_x^y$ and $\xi \in \Psi_y^x$. Holonomy groups are the only part of our geometrical construction (the structure express by the connection) that is physically significant in nonrelativistic quantum mechanics. This is an important point. We are not saying that the connection structure could be reformulate in terms of holonomy groups. What we are defending is stronger than this mathematical claim. In the space-time approach of quantum mechanics, if we take in account the phase symmetry, Ψ_x^x is the only needed geometrical structure to express electromagnetic interaction. This observation could lead us to believe that quantum interaction is based on nonlocal properties (in the case of Healy's interpretation [11] on nonseparable properties). This conclusion will not be defended here. Many ontological interpretations could fit the same mathematical structure. Symmetries are more stable than ontologies. Thus I will concentrate my analysis on the symmetry status, leaving the ontological problem open.

This result has an interesting effect on the interpretation of gauge symmetry. Let us define the closed loop in space-time $q = q_1 + q_2$, let us write $\Psi(q) = \{\bar{q}(1)\}$ the element of the holonomy groupoid corresponding to the path q, then

$$\Psi(q) = \Psi(q_2)\Psi(q_1) = \Psi'(q_2)\Psi'(q_1) = \Psi'(q) \tag{40}$$

where $\Psi' = \mathcal{R}(\Psi)$ is the holonomy groupoid after an active gauge transformation. In that case, the active gauge symmetry express a flexibility in the choice of local relational properties compatible with a nonlocal structure represented by holonomy groups. Gauge symmetry is a surplus of structure caused by the fact that we represent locally nonlocal relations.

[22] Remember we can factorize the contribution of q_1 from the contribution of every paths.

This interpretation of gauge symmetry solves our puzzling question. Remember the question was how gauge symmetry could be the result of a surplus of structure and in the same time not be arbitrary. This interpretation gives us an explanation. In the light of the Feynman approach of quantum mechanics, the gauge symmetry is a surplus of structure. It is a symmetry of the Lagrangian that has no empirical consequence. On the other hand, subset Ψ_x^x of the holonomy groupoid have measurable effect. They have even a direct empirical effect in interference experiments, like the Aharonov-Bohm effect.[23] We saw that Ψ_x^x are gauge invariant. It seems reasonable to interpret gauge symmetry as a flexibility in the representation of Ψ_x^x. Since this flexibility appears in an intrinsic geometrical context, it is not a case of coordinates change. It is rather a flexibility in the choice of a local connection compatible with holonomy groups. The gauge symmetry is not arbitrary. It is a consequence of the fact we have a choice of local constructions that represent nonlocal geometrical structure.

Our work can easily be extended to nonabelien gauge interaction. In these cases, the phase factor (or Wilson line) will be

$$U_c(y,x) = P\left\{e^{\frac{ig}{c\hbar}\int_0^1 ds \frac{dx^\mu}{ds} A_\mu^a(x(s))t^a}\right\}, \tag{41}$$

where $P\{\}$ is a prescription called path-ordering. This prescription takes in account the fact that $A_\mu^a t^a$ matrices do not necessarily commute at different points. For a closed path gauge invariant phase factor (called Wilson loop), we have to take the trace of $U_c(x,x)$. The interpretation of gauge symmetry is the same than in the electromagnetic case.

3.4 Interpretation: relativistic quantum mechanics

For now we do not have a solid geometrical construction that we can apply to a fully quantized Yang-Mills theory. Thus the results of the preceding subsection cannot directly be transfered to relativistic quantum mechanics. Let us examine briefly what is the main difficulty.

If we introduce a dynamical gauge potential, the path integral becomes not well defined.

$$\int D(\mathcal{A}) e^{\frac{i}{\hbar}S[\mathcal{A}]} \tag{42}$$

[23] See MacKenzie [15] for a treatment of the Aharonov-Bohm effect using path integrals.

We have to sum on all possible configurations of the gauge potential. In doing so, we will count an infinite number of equivalent $\mathcal{A}_\mu$ that differ only to a gauge transformation. If we fix the gauge to solve this problem, we loose the unitarity of the theory and create unphysical states. The standard solution is to introduce new terms in the Lagrangian; terms that include new fields, the Faddeev-Popov ghost fields. Therefore we add a surplus of structure to compensate for another surplus of structure generated by gauge freedom. It is not clear how geometry enters in this picture. Should we include a second associated vector bundle to accommodate ghost fields? What will be the gain in understanding?

Another angle would be to work in a gauge invariant formulation of Yang-Mills theory. There is a well known example in the literature. The concept of Wilson loop was invented by Kenneth Wilson [27] to discussed of a particular effect in quantum chromodynamics, a nonabelian gauge theory. In his work,

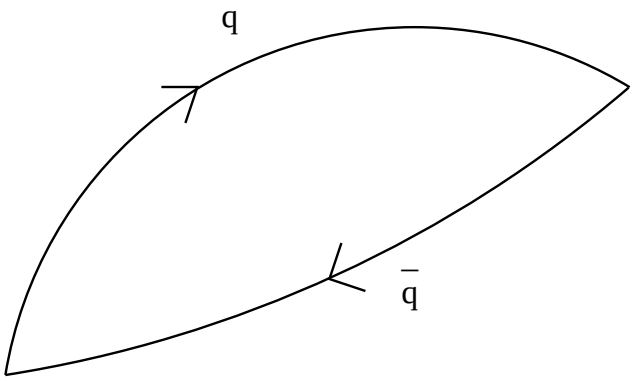

Figure 11: Example of quark q and antiquark $\bar{q}$ paths connecting.

the confinement of quarks is described approximately by Wilson loops. Since in relativistic quantum theory an antiparticle is equivalent to a particle of opposite charge moving reverse in time, the figure 11 represents paths of a quark and an antiquark connecting. The weight of this diagram in the serie of possible interaction includes a factor that is the associated Wilson loop. This loop is an approximation of the strength of the gauge interaction between the quark and the antiquak. Based on these factors Wilson proposed a confinement mechanism. In that kind of special cases, our analysis of gauge symmetry is still useful, but the range of applicability is not wide. Future developments are needed.

After Wilson more general quantum loop formulation were proposed. See Gambini and Pullin [10] on the subject. But we are far from a sufficiently developed theory to do philosophical analysis. The interpretation of gauge symmetry is still an open problem in this context.

4 Conclusion

In this paper we push the geometrical interpretation of gauge symmetry as far as we could. We obtain a result in the context of nonrelativistic quantum mechanics. The gauge symmetry denotes there a flexibility in the local representation of nonlocal features of the interaction. The question of the interpretation of gauge symmetry for quantum field theory is still open.

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