The Arbitrariness of Local Gauge Symmetry

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Abstract

This paper shows how the study of surpluses of structure is an interesting philosophical task. In particular I explore how local gauge symmetry in quantized Yang-Mills theories is the by-product of the specific dynamical structure of interaction. It is shown how in non relativistic quantum mechanics gauge symmetry corresponds to the freedom to locally define global features of gauge potentials. Also discussed is how in quantum field theory local gauge symmetry is replaced by BRST symmetry. This last symmetry is apparently the result of the fact that we do not know how to define quantum Yang-Mills theories without unphysical gauge boson states. Since Yang-Mills theories describe successfully three of the four fundamental interactions the elucidation of this symmetry is a pressing philosophical question.

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1 Introduction

In the context of physics symmetry is defined as an immunity to possible change.[1] In other words, it is the possibility of making a change that leaves some aspect of the situation unchanged. Thus a symmetry is always relative to a class of changes and what is invariant under this class must be specified. In most contexts the application of this concept is not philosophically puzzling, but in a few we meet difficulties of interpretation. Local gauge symmetry in quantum Yang-Mills (YM) theories is one of these problematic cases. A local gauge symmetry is defined as a certain class of local changes of fields that do not affect the empirical outcome of a particular theory. For example it could be a class of transformations that leave the Lagrangian unchanged, or change it at most by a total derivative. But this criterion is not a necessary condition. If we want to go beyond this fact and specify exactly what does not change in order to conserve the empirical content of a physical theory, we encounter conceptual problems. Since no observable quantity allows us to distinguish between situations related by a gauge transformation, how can we even know if there is an active interpretation[2] of that change? At first sight local gauge symmetry seems to be a redundancy of the theory. On the other hand we often encounter in physics opinions like:

> Gauge arbitrariness of electrodynamics may appear sometimes annoying and sometimes a deep and far-reaching principle. C. Itzykson and J.-B. Zuber, (page 10, [12]).

Since three of the four fundamental interactions are modeled by quantum Yang-Mills theories, these opinions seem well-founded. But how can redundancy be profound? The idea seems absurd, since profundity in the context

[1] I borrow this elegant definition from Joe Rosen [21].

[2] An active interpretation implies that transformations involved are actual changes of states.

of physics is associated with capturing the structure of the world through models and theories, or more modestly to produce a minimal theory. At best, redundancy refers to a surplus of structure in the theory but does not refer in anyway to the world. Thus, it could not be profound. In a recent work [16], C. Martin suggested that

> the local gauge symmetry is rather a *by-product* or accompaniment of the specific dynamical interaction field(s) in question. [Emphasis in original].

He draws an interesting analogy with the place of general covariance in the foundation of the general relativity (GR), which is a formal principle that seems not to be arbitrary. Unfortunately GR and YM theories are sufficiently different to make it difficult to push this analogy very far. In this paper I will explore Martin's thesis and show how local gauge symmetry is a by-product. This discussion will defend that certain surpluses of structure are in a certain sense unavoidable. They are the product of our way of theorizing. This is a new task for philosophy of science: to identify these surpluses and understand their role in theoretical structure. In section 2, I will expose the different philosophical positions we can adopt towards the status of gauge symmetry. This will give us a framework for the discussion of section 3, where local gauge symmetry will be discussed in non relativistic and relativistic quantum mechanics. For the latter I will show how local gauge symmetry is replaced by BRST symmetry and how new questions arise from this change.

1.1 What is local gauge symmetry?

Most of the philosophical analyses of YM theories limit themselves to the simplest case: electrodynamics.[3] General conclusions induced from this theory could be misleading. For example in electrodynamics the gauge field $F_{\mu\nu}$ is gauge invariant. This is not generally the case. Subsequently the belief that $F_{\mu\nu}$ represents a physical field because it is a covariant field that is gauge invariant cannot be generally defended. Since I believe that YM theories form a natural class, it seems clear to me that an analysis of a special case will not do. YM theories are not put in the same category for arbitrary reasons.

[3]Note that, contrary to certains authors, I extend the category of Yang-Mills theories to electrodynamics, an Abelien gauge theory.

They share a basic structure that inclines me to believe that a good philosophical analysis should apply to all of them.[4] Therefore I will focus on the entire class of Yang-Mills theories, and not merely on electrodynamics. This class includes among others quantum electrodynamics and quantum chromodynamics. I will concentrate my analysis on quantized YM theories and say very little about local gauge symmetry in classical YM theories. This is for reasons of space but also because I believe that the need for a philosophical discussion is far greater in quantum physics; furthermore, I do not believe that classical interpretations apply in quantum context.

Let us assume that we have a field theory where a matter content[5] is represented by a multiplet $\psi(x)$, where ψ belongs to an irreducible representation r of the gauge group G. This field is coupled to a gauge potential $\mathcal{A}_\mu = A^a_\mu t^a_r$, where t^a_r are generators of G forming the algebra $[t^a_r, t^b_r] = i f^{abc} t^c_r$, where f^{abc} are structure constants. Note that the repeated indices $a, b \cdots$ are summed over the generators of G. The local gauge symmetry implies that there are no empirical consequences to the following transformations:

$$\psi(x) \to V(x)\psi(x) \tag{1}$$

$$\mathcal{A}_\mu \to V(x)\left(\mathcal{A}_\mu + \frac{i}{g}\partial_\mu\right)V^\dagger(x) \tag{2}$$

where every matrix $V(x) = exp\,(i\alpha^a(x)t^a_r) \in G$, where $\alpha^a(x)$ are smooth real functions of space-time, and g is the charge associated to the gauge interaction. The corresponding infinitesimal transformation laws are

$$\psi(x) \to (1 + i\alpha^a t^a_r)\psi(x) \tag{3}$$

$$A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c \tag{4}$$

Note that it is because $\alpha^a(x)$ are explicitly functions of space-time that we call these transformations local. In this theory the field tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{5}$$

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[4] Gauge transformations in general relativity are the set of diffeomorphisms of the space-time manifold. These transformations are not limited to an internal property like Yang-Mills transformations, they affect the space-time manifold itself. Furthermore general relativity is a classical theory. Yang-Mills theories are not. For these reasons I am not expecting my analysis to apply to general relativity.

[5] In this paper I will work in "natural" units, where $\hbar = c = 1$.

and transforms under an infinitesimal gauge transformation as

$$F^a_{\mu\nu} \rightarrow F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu} \tag{6}$$

As an example let us take $G = U(1)$. In this case[6] $V(x) = exp(i\alpha(x))$. The gauge transformations consist of

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \tag{7}$$

$$A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu\alpha(x) \tag{8}$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \tag{9}$$

where $-g$ is the electron charge $e = -|e|$. We recognize here the case of quantum electrodynamics or in the context of non relativistic quantum mechanics, the case of a quantized particle in interaction with a classical electromagnetic field.

2 Philosophical positions about local gauge symmetry

The issue I want to explore in this paper is the conflict between redundancy and profundity of local gauge symmetry. First, we can put aside discussions using the gauge principle (often called the gauge argument) that relate local gauge symmetry to a kind of logic of nature. Authors following that vein may have been tempted to justify the deepness of the local gauge symmetry based on the logical order of nature exposed in the gauge principle. In his Ph.D. dissertation [15], C. Martin clearly showed that the gauge argument is at best heuristic, and that we cannot clarify the status of gauge symmetry using only this principle. There are too many holes in the argument to justify anything about the status of gauge symmetry. In most formulations of the argument, local gauge symmetry is postulated without a solid justification. For example see the fundamental paper of C.N. Yang and R.L. Mills [24].

If the gauge argument is of no help, I propose to approach gauge symmetry directly. On the status of gauge symmetry, there are three mutually exclusive philosophical positions[7]: 1) the local gauge symmetry is a physical symmetry,

[6]Note that in the case the group generator is the unit matrix.

[7]Note that these positions map the trilemma proposed by M. Redhead [20].

2) the local gauge symmetry is not physical and we should formulate our theories in a gauge invariant way, and 3) the local gauge symmetry is not physical and we should keep the gauge dependence of our theories. Note that these positions go beyond the ontological status of gauge symmetry. They have an epistemological scope. In the philosophy of physics, we often discuss independently ontological and epistemological questions. In the case of local gauge symmetry this would be unwise. This symmetry is at the core of these considerations.

2.1 The gauge symmetry is physical

This interpretation implies that a gauge transformation relates distinct physical states. This possibility was evoked by Y. Aharonov and D. Bohm [1] as a possible consequence of the local interpretation of the effect that bears their names. This is also what R. Feynman probably had in mind when he defended that gauge potentials are physical actors in quantum mechanics (chapter 15, [8]). If we adopt this interpretation the gauge symmetry is of course not redundant because it is a symmetry of the physical model. Let me explain this in more detail.

In a realist perspective a good physical model corresponds at least approximately to some aspects of the external world. In the best scenario distinct physical states in the physical model[8] represent distinct states of affairs in the world. Since gauge symmetry relates distinct states, identifying this symmetry was a scientific discovery and is not an arbitrary construction. From the antirealist point of view things are more complex. In this context arbitrariness is a question of degree. Asserting that the local gauge symmetry is physical implies that this symmetry is less arbitrary than a purely formal symmetry of the model. It is affirming that the gauge symmetry is not just a symmetry of the mathematical representational machinery but also a symmetry of the physical model itself.

This said, this position is far from without flaws. Two features make this option very costly. First, this position allows that there are distinct states of affairs which no possible observation could distinguish. Remember gauge symmetry implies complete empirical equivalence. The question is not that these undistinguishable states could not exist, but in what measure they should appear in physical theory. In the spirit of simplicity, such states should

[8] A physical model is a collection of idealized physical entities and their relations.

be identified unless we can find at least indirect evidence of their existence. This point is of course not fatal to the interpretation but combined with the next point it may be. If we index the states of affairs by gauge potentials, in other words if each distinct gauge potential refers to a different state, the theory is indeterministic. This was clearly exposed by G. Belot [3]. In this condition, the geometry of the phase space of classical electromagnetism is determined by a presymplectic form. This structure is too weak to determine the evolution of states. There are infinitely many trajectories through each point of phase space. In other words, the theory is not strong or rich enough to predict the evolution of physical states indexed by $\mathcal{A}_\mu$. To my knowledge there is no equivalent proof for the quantum case, but as I will show in section 3.1, the local value of $\mathcal{A}_\mu$ is not the physically meaningful entity. Let us return to the classical case. Since these evolutions are not distinguishable, by observation we find ourselves in a very uncomfortable situation. The reader, I am sure, recognizes the premisses of the hole argument [5]. While some forms of indeterminism are acceptable in physics, the incapacity of the theory to choose between possible evolutions is not. As a response we could argue that states of affairs should not be represented by different gauge potentials. But in that case what does it mean for the gauge symmetry to be physical? What is changed under a gauge transformation?

Of course these points do not completely rule out that local gauge symmetry could be physical. But for that we will need new experimental data. If we discover an empirical phenomenon that exhibits in a certain way a gauge dependence, we will have to reconsider the question. Until then, we have to interpret the local gauge symmetry as a redundancy of the theory.

2.2 Theories should be gauge invariant

If we accept the conclusion of the last subsection we have to consider local gauge symmetry as the result of a superfluous theoretical structure. Thus this symmetry is formal. There are two possible epistemological attitudes towards this surplus. One of them would be to defend that all theories should be formulated in a gauge invariant way. Gauge freedom should be eliminated. Ideally this reduction of the theory should be done before quantization, at the classical level. This is the position of J. Ismael and B. C. van Fraassen:

> Formalisms with little superfluous structure are nice, of course, because they reflect cleanly the structure of what they represent;

> they have fewer extra mathematical hooks on which to hang the mental structure that we project onto phenomena. (page 390, [11])

This position makes sense but it should not be taken literally. Even if surpluses of structure are mental projections onto the world that do not refer to a possible world structure, this does not imply that these surpluses are always arbitrary. A surplus of structure could be an unavoidable by-product of our way of constructing a theoretical model. This possibility has to be explored. It is only after the analysis of the role of surpluses of structure in the theory that we can eliminate them without a second thought. In the cases that occupy us another question arises. When do we eliminate or reduce the gauge surplus? Before or after quantization? This is an important question. There is no obvious reason why gauge reduction and quantization should be commuting operations. Because of this the second position is more complex than it originally appeared. It is not enough to defend that surplus of structure should be eliminated. You have to specify how and when. As we will show in section 3, this is not a trivial task.

Before moving on to the last position about local gauge symmetry, let us examine in more detail how we could build classical YM theories without this symmetry. One of the more prominent research programs in that direction is to formulate gauge interaction in terms of Wilson loops.[9] K. Wilson introduced this notion to study the behavior of the interactions strength in quantum chromodynamics [22], an $SU(3)$ Yang-Mills theory. The concept of a Wilson loop is based on the notion of a Wilson line:

$$U_q(y,x) = P\left\{e^{ig\int_0^1 ds \frac{dx^\mu}{ds} A^a_\mu(x(s))t^a}\right\}, \tag{10}$$

where $P\{\}$ is a prescription called path-ordering that takes into account the fact that $\mathcal{A}_\mu$ matrices do not necessarily commute at different points x. The integration is over a path q in space-time parameterized by a real variable s beginning at $x(0) = x$ and finishing at $x(1) = y$. A Wilson loop is simply the trace of a Wilson line $U_q(x,x)$ beginning and ending at some space-time point.[10]

[9] The main goal of this research program is not to formulate a gauge invariant theory, but to develop a new theory of quantum gravity.

[10] Wilson loops are closely related to the geometrical concept of holonomy in a principal fibre bundle. For a philosophical introduction to the concept of holonomy in gauge theories, see R. Healey [10]. For a physicist's approach, see R. Gambini and J. Pullin [9].

Wilson loops are gauge invariant entities, but gauge potentials still appear explicitly in their definition. This is reminiscent of a formulation of the theory with gauge freedom. It is not mandatory to define Wilson loops using gauge potentials. By analogy with geometry, where holonomy groups could be defined independently of the connection, Wilson loops could be conceived as primary entities. In fact, Gambini and Pullin assert that Wilson loops possess the reconstruction property:

> Given the Wilson loop functions evaluated for all possible loops we can reconstruct all the gauge invariant information present in the gauge connection. (page 63, [9])

In other words, Wilson loops contain all the physically significant information included in the gauge potential (gauge connection). Therefore, they form a legitimate reduction of classical YM theories.

As far as Wilson loops are concerned, it is easy to convince ourselves that the local gauge symmetry expresses a flexibility in the choice of a locally defined gauge potential $\mathcal{A}_\mu$ compatible with the global structure represented by the values of Wilson loops. Therefore local gauge symmetry is clearly the by-product of our preference to define physical fields locally. This interpretation is limited, for now, to classical physics and is dependent on the type of reduction we choose. I will show in section 3 there is a natural way to extend this interpretation to non relativistic quantum mechanics.

2.3 We should keep the gauge dependence

The third philosophical option is to keep the gauge dependence in the formulation of the theory as long as we can, even if we know that nothing in the world corresponds to the surplus of structure that is responsible for the symmetry. This position may seem paradoxical, but in fact it is more in tune with the practice of physicists. Since the rising of quantum physics the relation between theories and the world is much more subtle than before. Like the famous example of Dirac's equation, the apparent surplus of today could be the empirical part of tomorrow. Of course, sometimes a surplus should be eliminated but not before we understand what it was doing in the theory in the first place. Consider the following example. In 1979, the physicist L. O'Raifeartaigh wrote:

> The reason that the gauge structure of the weak interactions lay

> undiscovered for so long is that the symmetry of the gauge group is in this case *broken*. [Emphasis in original], (page 162, [18]).

If we believe that the gauge surplus is an epistemological mistake and that it should have been put aside long ago, O'Raifeartaigh's remark has no pertinent content. This is of course not the right attitude to take toward the writing of physicists. The challenge for the philosopher of physics is not just to identify where the physicist is careless about foundations, but also to explain how to interpret physical discourse in a meaningful way. How otherwise can we understand spontaneously broken gauge symmetry? Physicists speak casually of the gauge structure of theories. What is this structure and what is its role in the theory? This is a question philosophers should answer. It could even be the beginning of new field of investigation in philosophy of physics: the study of the role of superfluous structure in theory. This research would probably give us new answers to old philosophical questions about the nature of scientific representation. But my immediate goal is not so ambitious. In the next section I will concentrate on local gauge symmetry in quantum YM theories.

3 The non arbitrariness of local gauge symmetry

In this section I will work within the framework of the third philosophical position about local gauge symmetry. So I will keep the gauge dependence as long as possible. I will show how this symmetry is a by-product and what we can learn from it.

3.1 Nonrelativistic quantum Yang-Mills theories

In this subsection I will show that the interpretation about local gauge symmetry that resulted from the Wilson loops formulation of YM theories is right in the non relativistic quantum context. Let us begin by studying local gauge symmetry in the simplest quantum case, when a classical electromagnetic field interacts with a non relativistic quantized particle.[11] I will analyze this case using the space-time approach to quantum mechanics, often called

[11] In this discussion I will always neglect the spin.

the Feynman path integral approach (see [7]). This quantum formalism is rarely used in philosophical analysis so I will explain it in some detail. This pedagogical effort will be rewarded. In this formalism we can show explicitly how the gauge interaction enters in the theory which will clarify greatly how supplementary degrees of freedom appear in the theory.

First, let me define the basic quantity of the Feynman path integral formalism: the propagator.

Definition 1 (The propagator) *The probability amplitude (also called propagator) that a particle that was at the position $\vec{r}_1$ at time $t = 0$ is at position $\vec{r}_2$ at time $t = T$ is*

$$K(T, \vec{r}_2; 0, \vec{r}_1) = \int \mathcal{D}(\vec{q}(t))\, e^{iS[\vec{q}(t)]} \tag{11}$$

where $S[\vec{q}(t)]$ is the classical action of the path $\vec{q}(t)$; in other words $S[\vec{q}(t)] = \int_0^T L(\dot{\vec{q}}, \vec{q})\, dt$, where $L(\dot{\vec{q}}, \vec{q})$ is the Lagrangian of the particle. The integral $\int \mathcal{D}(\vec{q}(t))$ is a sum over all possible trajectories between $x = (0, \vec{r}_1)$ and $y = (T, \vec{r}_2)$.

To calculate the propagator, we have to sum functions of the action for all possible trajectories between x and y. The presence of this sum is the main difference between quantum and classical physics. In classical physics the main objective is to find *the* trajectory followed by the particle. In quantum physics, all trajectories contribute to the propagator. It is the relative phase between these contributions that generates intrinsically quantum phenomena.[12]

Now let us imagine that our system is a quantized charged particle of Lagrangian L. What is the effect of adding an electromagnetic interaction to this system? Classically we know that the Langrangian of a charged particle (charge e) will be modified in the following way:

$$L(\dot{\vec{q}}, \vec{q}) \to L(\dot{\vec{q}}, \vec{q}) + e\left(\vec{v}(t) \cdot \vec{A}(\vec{q}(t)) - \phi(\vec{q}(t))\right), \tag{12}$$

where $\vec{v}(t)$ is the velocity of the particle, $\vec{A}$ the vector potential and ϕ the scalar potential. The contribution of each path will be modified in the fol-

[12] Of course, I am not at all saying that the particle is following all the paths at the same time. We are not talking about physical trajectories here.

lowing way:

$$\exp\left(iS[\vec{q}(t)]\right) \rightarrow \exp\left(iS[\vec{q}(t)] + ie\int\left(\vec{v}(t)\cdot\vec{A}(\vec{q}(t)) - \phi(\vec{q}(t))\right)\,dt\right) \quad (13)$$

$$= \exp\left(iS[\vec{q}(t)]\right)\cdot\exp\left(ie\int\left(\frac{d\vec{q}}{dt}\cdot\vec{A} - \phi\right)\,dt\right) \quad (14)$$

$$= \exp\left(iS[\vec{q}(t)]\right)\cdot\exp\left(-ie\int_q A_\mu dx^\mu\right) \quad (15)$$

Due to the effect of electromagnetic interaction the contribution of each path is modified by a nonintegrable phase factor, which is the Wilson line for this path. This is an important result. In 1974 [23], C.N. Yang argued that the concept of nonintegrable phase factor could be taken as the basis of a gauge interaction. In our example this means that the fact that the interaction is in the form of a Wilson line is the signature of a gauge interaction.[13] Let me rephrase this point. We recognize that we are faced with an electromagnetic interaction *because* the interaction takes the form of a Wilson line in the path integral.[14] Identifying the gauge structure of interaction is just that. If the interaction was of another form, it would not be a YM interaction.

Having established this point, let us discuss the symmetry itself. The propagator consists in the sum of contributions of all paths. If we look at the relative phase caused by the interaction between two particular paths q_1 and q_2:

$$e^{iS[\vec{q}_1(t)]}e^{-ie\int_{q_1} A_\mu dx^\mu} + e^{iS[\vec{q}_2(t)]}e^{-ie\int_{q_2} A_\mu dx^\mu} \quad (16)$$

$$= e^{-ie\int_{q_1} A_\mu dx^\mu}\left(e^{iS[\vec{q}_1(t)]} + e^{iS[\vec{q}_2(t)]}e^{-ie\oint_{q_{21}} A_\mu dx^\mu}\right) \quad (17)$$

where $q_{21} = q_2 - q_1$. Examination of the term in parentheses reveals that it is gauge invariant since the last factor - the only one that could be gauge dependent - is a Wilson loop, which as has already been noted is gauge invariant. This is a noteworthy point. Although individual phase factors are gauge dependent, the relative phase between paths is not. In other words, in any gauge the relative phase between contributions of two paths stays unchanged. Under a gauge transformation $A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha$ a phase

[13] Note that it is ultimately the reason why it is justified to interpret $\mathcal{A}_\mu$ as a connection in a principal fibre bundle.

[14] A Wilson line associated to the gauge group $U(1)$.

factor associated with a path q beginning at x and ending at y becomes:

$$\exp\left(-ie\int_q A_\mu dx^\mu\right) \rightarrow \exp\left(-ie\int_q A_\mu dx^\mu\right)\exp\left(i\int_x^y \partial_\mu\alpha(x)dx^\mu\right) \quad (18)$$

$$= V(y)\exp\left(-ie\int_q A_\mu dx^\mu\right)V^\dagger(x), \quad (19)$$

where $V(x) = \exp(i\alpha(x))$. As we can see, the effect of a gauge transformation is integrable. It only depends on the value of $\alpha(x)$ at x and y. Knowing this, we can infer that under a gauge transformation the propagator becomes:

$$K(y;x) \rightarrow K'(y;x) = V(y)K(y;x)V^\dagger(x) \quad (20)$$

Since in $K(y;x)$, y is the variable and x an initial condition, the factor $V^\dagger(x)$ is just a global phase change with no physical significance. Local gauge symmetry postulates that K and K' are physically equivalent because the propagator is a representation of a wave function. This is not trivial since $V(y)$ depends smoothly of space-time. Returning to the path integral formalism, we can see that the physical equivalence between K and K' corresponds to the absence of relative phase change between path contributions due to electromagnetic interaction. The action of $V(y)$ corresponds to changing the standard phase against which the contribution of each path is compared. Local gauge symmetry is asserting that any change of gauge potential that does not modify relative phase between path contributions is physically meaningless. In other words, it is a passive local gauge recalibration. From that I conclude that quantum mechanics is not sensible to a local change of A_μ if it does not change global properties expressed by Wilson loops. In our example, local gauge symmetry is the by-product of the way the electromagnetic interaction is coupled to a quantized particle, namely by nonintegrable phase factors. The heart of this gauge theory is the way these phase factors are defined and the fact that we have to take into account the contribution of all possible trajectories. Local gauge symmetry is the unavoidable product of this definition.[15]

Even if local gauge symmetry has no physical content it is not arbitrary. It depends on the structure of gauge interaction. It is the liberty we have

[15] This example could be generalized to non-abelian YM theories. In that case phase factors would be traces of potential matrices. But I do not know any applications of such theories in the non relativistic regime.

in defining locally distinct A_μ that are physically equivalent. The gauge structure of interaction is of course the real discovery.

If we return to the philosophical positions of section 2, positions 2 and 3 are still available to us. We can keep the gauge dependence of the theory and continue to describe the interaction in terms of the action of $\mathcal{A}_\mu$. For example, we could interpret the gauge potential as a connexion in a principal fibre bundle $P(M^4, G, \pi)$, without forgetting that it is only the global features of $\mathcal{A}_\mu$ that are physically significant.[16] Or we could argue for a more economical description in terms of Wilson loops. At this point both positions are perfectly justifiable. In this case we do not have to worry about the order between quantization and reduction since the gauge potential is not quantized.

3.2 Relativistic quantum mechanics

It is now time to study the status of local gauge symmetry in relativistic quantum mechanics. I will concentrate my study on perturbative quantum field theory. After all, it is in this theoretical framework that YM theories have been most successful.

Even in the simplest case, the quantization of a free electromagnetic field, local gauge symmetry causes trouble. Part of the quantization procedure using Feynman path integrals is to solve functional integrals of the sort

$$\int \mathcal{D}(A)\, e^{iS[A]} \tag{21}$$

where S is the gauge invariant action of a free field $S = \int -\frac{1}{4}(F_{\mu\nu})^2 d^4x$. The sum represented by the integral $\int \mathcal{D}(A)$ is over all possible potential $\mathcal{A}_\mu$ configurations. It is a direct generalization of the notion of path integral.

Due to the gauge invariance, for any potential of the form $A_\mu = -\frac{1}{e}\partial_\mu \alpha$ the action is zero. The sum over these potentials makes the integral explode. More generally, the integral is badly defined because we are redundantly integrating over a continuous infinity of physically equivalent field configurations. Gauge symmetry has become a problem. We must reduce the degrees of freedom to make the theory finite.[17] We have seen one way to do this

[16]Knowing that the gauge groupoid is probably a better geometrical construction.

[17]Note that this problem is distinct from renormalization. Here infinity is produced at the tree level of Feynman diagrams.

with Wilson loops. But it is difficult to build a computable perturbative quantum theory of loops. Physicists have devised another approach: fixing the gauge. For example, we could add a covariant constraint of the type $\partial^\mu A_\mu = 0$ (Lorentz gauge). In the Lagrangian formalism implementing this constraint corresponds to adding to L a gauge fixing term: $L_{gf} = \frac{1}{2\xi}(\partial^\mu A_\mu)^2$, where ξ can be any finite constant.[18] As you can see the reduction of the gauge surplus takes place before quantization.

At this point we might worry about the fact that the Lagrangian density is not gauge invariant anymore. By adding the gauge fixing term, did we not modify deeply the structure of the gauge theory to the point that it is not the same theory at all? Fortunately this is not the case. It can be proven that every physical prediction is in fact independent of ξ and thus, of the gauge fixing term. All other nice properties of gauge theory are conserved and especially renormalizability. If we compute empirical results based on a gauge-fixed L where photons interact with fermions, we obtain a remarkable accord with empirical data.

If we stop our analysis here we have to conclude that local gauge symmetry is essentially a classical feature. It appears in nonrelativistic quantum mechanics because gauge potentials are not quantized, but in a fully quantized theory it is an annoying nuisance that must be eliminated before quantization and therefore has no new interesting interpretation.

On the other hand, if we believe that quantum electrodynamics is just a theory in the class of Yang-Mills theories, we have to check if this pessimistic conclusion applies to non-Abelian YM theories. As I will show the answer is not simple. We have to be much more careful in our analysis. Let us study a non-Abelian theory of fermions interacting with gauge bosons represented by gauge potentials:

$$L = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \qquad (22)$$

Even if the gauge fixing term makes the path integral finite another problem arises: the unitarity of this theory is lost.[19] There is a nonzero probability for a physical process to produce external gauge bosons that are not transversely polarized. These are unphysical states because non-transverse polarized bosons would be massive, making it impossible for the theory to be gauge

[18] We add ξ to have a more general gauge fixing term.
[19] See chapter 16.1 of [19].

invariant even at the classical level. I put aside the case of spontaneously broken gauge symmetry. Moreover these states are not experimentally observed. A solution to this problem was devised in 1967 by L.D. Faddeev and V.N. Popov [6]. They decided to look at the question of the reduction of gauge theories with a new angle. They proposed to reduce directly the gauge surplus in the quantum context. To do so they had to modify the measure of the Feynman path integral to count each physical configuration only once. This way they could isolate the interesting part of the functional integral. If we use the Faddeev-Popov procedure on the free photon integral already discussed, we find that this technique is equivalent to adding a gauge fixing term. We regain the solution discussed before. The surprise is in non-Abelian cases. There the Lagrangian is transformed by the addition of two terms

$$L_{gf} + L_g = \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}^a \left(-\partial^\mu D^{ac}_\mu\right) c^c \tag{23}$$

where $D^{ac}_\mu = \partial_\mu \delta^{ac} + g f^{abc} A^b_\mu$. The new fictitious anticommuting fields are called Faddeev-Popov ghosts. These fields appear only inside Feynman diagrams as virtual particles. If they are included in computations they restore the unitarity of quantum non-Abelien YM theories. Renormalizability is also apparently the result of this inclusion.

Let us review what we have done so far. Local gauge symmetry is the result of a surplus of structure. This surplus causes problems when we quantized the gauge bosons represented by gauge potentials. To eliminate this surplus and conserve unitarity of the theory we must *add* a structure: the ghosts fields. To understand what happened to the local gauge symmetry we will have to study a new symmetry of the gauge-fixed Lagrangian, which involves the ghost in an essential way.

3.2.1 The BRST symmetry

The BRST symmetry[20] was introduced By Becchi, Rouet, Stora [2] and indepedently by Tyutin. To express this symmetry in its simplest form, let us rewrite the Faddeev-Popov Lagrangian by introducing a new commutative scalar field B^a:

$$L = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + -\frac{\xi}{2}(B^a)^2 + B^a \partial^\mu A^a_\mu + \bar{c}^a \left(-\partial^\mu D^{ac}_\mu\right) c^c \tag{24}$$

[20] Note that this subsection relies heavily on [19].

The new field has a quadratic term without derivatives so it is not a normal propagating field. It has no independent dynamics, and is known as an auxiliary field. Integrating B^a by completing the square, we recover precisely the gauge fixing term. This Lagrangian density is thus equivalent to the Faddeev-Popov Lagrangian.

Now let ϵ be an infinitesimal anticommuting parameter. The following infinitesimal BRST transformation of the fields is a symmetry of L:

$$A_\mu^a \to A_\mu^a + \epsilon \left(\partial_\mu c^a + g f^{abc} A_\mu^b c^c\right) \tag{25}$$

$$\psi \to (1 + ig\epsilon c^a t^a)\psi \tag{26}$$

$$c^a \to c^a - \frac{1}{2} g\epsilon f^{abc} c^b c^c \tag{27}$$

$$\bar{c}^a \to \bar{c}^a + \epsilon B^a \tag{28}$$

$$B^a \to B^a \tag{29}$$

The BRST transformation is a global symmetry of the gauge-fixed Lagrangian for any value of the gauge parameter ξ.[21] Note that the transformation of the fields A_μ^a and ψ is a local gauge transformation whose parameter is $\alpha^a(x) = g\epsilon c^a(x)$. Even if we are studying an apparently gauge-fixed Lagrangian the local gauge symmetry has returned, but in a new guise, since ϵ is an anticommuting (Grassmann) number. Since BRST symmetry guarantees the good behavior of the theory, it is natural to consider BRST symmetry as a fundamental symmetry of quantum YM theories, as local gauge symmetry was the foundation of classical YM theories. If this is the case the status of ghost fields is the new question. This question has been extensively discussed by T. Kugo and I. Ojima in [13]. I will review their result.

Let $Q\phi$ be the BRST transformation of the field ϕ: $\delta\phi = \epsilon Q\phi$. For example, $QA_\mu^a = D_\mu^{ac} c^c$. Then for any field $Q^2\phi = 0$. This is a consequence of the anticommuting nature of the transformation. If we change our perspective and now consider our YM theory in the Hamiltonian formalism after canonical quantization, we will find an interesting result. Because the Lagrangian has a continuous symmetry there will be a conserved current with a conserved charge Q that commutes with the Hamiltonian. What we have shown implies that the operator Q is nilpotent: $Q^2 = 0$. The action of Q

[21] In the BRST transformation the role of ghosts and antighosts seem different. This is only an appearance, it possible to define the equivalent anti-BRST transformation. See [17].

divides the eigenstates of H into three subspaces. $\mathcal{H}_1$ is the subspace of states that are not annihilated by Q. $\mathcal{H}_2$ is the subspace of states of the form $|\psi_2\rangle = Q|\psi_1\rangle$ where $|\psi_1\rangle \in \mathcal{H}_1$. Finally $\mathcal{H}_0$ is the subspace of states that satisfy $Q|\psi_0\rangle = 0$ but are not in $\mathcal{H}_2$. T. Kugo and I. Ojima argued that among the single-particle states, forward polarized gauge bosons and antighosts belong to $\mathcal{H}_1$, ghosts and backward gauge bosons belong to $\mathcal{H}_2$, and transverse gauge bosons belong to $\mathcal{H}_0$. Thus the physical Hilbert space does not contain ghosts or antighosts. Unphysical gauge boson polarization adds supplementary degrees of freedom to the theory, ghosts and antighosts subtract them. Their input could be understood as *negative* degrees of freedom. The BRST transformation Q gives us the relation between these states.

From this discussion we conclude that the reduction of the gauge surplus in classical and quantum YM theories seems different. In classical physics we could fix the gauge; this is not possible in the same sense here. Fixing the gauge implies adding ghosts and antighosts to the theory. This is equivalent to imposing BRST symmetry, which implies a new kind of gauge dependence. This apparent circularity led researchers to look for an interpretation of BRST symmetry in the context of geometry. Since in the classical case geometrical interpretation of Yang-Mills theory is considered enlightening. Indeed, classical YM theories have a very natural representation in the principal fibre bundle formalism. If we define the base space as a Minkowski space M^4 and the fibre as the gauge group G, there is a natural way to interpret the gauge potential as a connection defined on M^4. The previous analysis would suggest that the ghost field corresponds to a nonphysical part of the differential connection. This approach is appealing but unfortunately not realizable. J.M. Leinaas and K. Olaussen [14] proved that it is impossible to model a Grasmann field, like the ghost, of path integration by differential forms on a suitable finite dimensional manifold. This study blocks any simple classical interpretation of the ghosts fields. It does not mean that any geometrical representation of BRST transformation is impossible, but it must be constructed in a different geometrical framework, for example in the space of connections [4]. The anticommuting nature of the BRST symmetry gives a new flavour to local gauge symmetry making the quantum case irreducible to the classical one.

We have seen that in the non relativistic case local gauge symmetry is the result of the liberty we have to define locally non local physical features of fields. In the relativistic version, local gauge symmetry is replaced by a quantum equivalent of the BRST symmetry. This new symmetry must be imposed to conserve the unitarity and renormalizability of non-Abelien YM theories. By analogy with the non relativistic case, we could say that BRST symmetry is the result of our incapacity to define a satisfying quantum theory without unphysical bosons states. More questions need to be solved. For example, how could we relate the interpretations of local gauge symmetry in relativistic and non relativistic quantum YM theories? For now, I have no answer.

Our list of philosophical positions about local gauge symmetry seemed to fall short. Of course the gauge surplus should be eliminated (position 2). We cannot have a useful theory without this reduction. But to do it properly we have to fix the gauge *and* to add a new surplus. The result of this process is that the gauge surplus is replaced by the BRST surplus. In a certain way we seem to be forced to defend a version of position 3. This is new. Until now the choice between keeping or reducing a surplus of structure has been a philosophical one. The BRST surplus seems to be a theoretical necessity. Perhaps it is not so and this is a failure of our imagination. Maybe we did not find the right way to reduce this surplus. To quantize directly Wilson loops could be a possible answer. To my knowledge this program has not yet demonstrated that it can reproduce the empirical results obtained from the Fadeev-Popov Lagrangian.

4 Conclusion

In this paper, I have shown that the study of surpluses of structure is a fertile topic for philosophy. This important subject has often been neglected in philosophical research. In particular, the study of local gauge symmetry gives us elements about the structure of interaction and about limits and constraints that result from the mathematical framework we use to represent physical structure. These kinds of questions at the frontier between ontology and epistemology are a new domain to explore. It is in quantum physics, where theories are mathematically rich that we should begin this work. This paper provides clues but many questions about field representations, especially in relativistic quantum mechanics, remain unanswered. In particular, the tension between local and global description of fields needs to be clarified. In this context BRST symmetry should be studied in much greater detail. If, as physicists say, local gauge symmetry (BRST symmetry) is the heart of our best theories that describe fundamental interactions, it is the role of philosophers to explain how surpluses of structure could play such an important role. Surpluses of structure should be taken seriously.

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