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Research Article

Study of Ion-Acoustic Solitary Waves in a Magnetized Plasma Using the Three-Dimensional Time-Space Fractional Schamel-KdV Equation

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The study of ion-acoustic solitary waves in a magnetized plasma has long been considered to be an important research subject and plays an increasingly important role in scientific research. Previous studies have focused on the integer-order models of ion-acoustic solitary waves. With the development of theory and advancement of scientific research, fractional calculus has begun to be considered as a method for the study of physical systems. The study of fractional calculus has opened a new window for understanding the features of ion-acoustic solitary waves and can be a potentially valuable approach for investigations of magnetized plasma. In this paper, based on the basic system of equations for ion-acoustic solitary waves and using multi-scale analysis and the perturbation method, we have obtained a new model called the three-dimensional(3D) Schamel-KdV equation. Then, the integer-order 3D Schamel-KdV equation is transformed into the time-space fractional Schamel-KdV (TSF-Schamel-KdV) equation by using the semi-inverse method and the fractional variational principle. To study the properties of ion-acoustic solitary waves, we discuss the conservation laws of the new time-space fractional equation by applying Lie symmetry analysis and the Riemann-Liouville fractional derivative. Furthermore, the multi-soliton solutions of the 3D TSF-Schamel-KdV equation are derived using the Hirota bilinear method. Finally, with the help of the multi-soliton solutions, we explore the characteristics of motion of ion-acoustic solitary waves.

1. Introduction

Ion-acoustic solitary waves are well-known to be an important example of nonlinear phenomena in modern plasma research [1–3]. Many researchers have studied ion-acoustic solitary waves in different plasma systems such as thermal, magnetized, and unmagnetized plasmas. Among the different plasma systems, magnetized plasma systems have attracted intense interest. Many authors have studied ion-acoustic solitary waves in magnetized plasma based on the quantum hydrodynamic (QHD) model [4, 5]. The QHD model is derived from the basic system of equations of ion-acoustic solitary waves and is one of the macroscopic mathematical models used to describe the hydrodynamic behavior of quantum plasmas.

For simplicity, 1D and 2D nonlinear partial differential equations have been used to describe the evolution of nonlinear ion-acoustic solitary waves. For the simplest 1D geometry where the ion-acoustic solitary waves become solitons, Washimi and Taniuti [6] derived the KdV equation by using the reductive perturbation method. Kako and Rowlands [7] derived the 2D KP equation based on the results of Washimi and Taniuti. However, in the real magnetized plasma environment, 1D and 2D models cannot solve some of the problems encountered in the motion of ion-acoustic solitary waves. Thus, it is necessary to introduce higher-dimensional theories for the nonlinear ion-acoustic solitary waves. Therefore, in this paper, we discuss a new 3D model for nonlinear ion-acoustic solitary waves.

Most of the QHD models, such as the KdV model, mKdV model, and KP model, are integer-order models. Fractional order models have rarely been considered. Fractional calculus is a generalization of integer calculus. Many of the physical processes that have been explored to date are nonconservative. It is important to be able to apply the power of fractional differentiation [8–10]. However, because of its nonlocal character, fractional calculus has not been used in physics and engineering. With the development of nonlinear science, fractional calculus theory has been continuously developed to date. Researchers have discovered that the derivatives and integrals of fractional order models are suitable for describing various physical phenomena. In recent years, the application of fractional differential equations has attracted increasing attention in plasma physics [11]. Thus, research on fractional order models is necessary.

The solution of the integer equation is a research hot spot in the field of research and development of various models [12–14], and similarly, the solution of fractional models has been a focus of our research [15, 16]. Thus, many solution methods have been found and used to solve the fractional order equation. For instance, the iterative method [17–19], Hirota bilinear method [20, 21], trial function method [22], Homotopy perturbation [23], and other methods have all been developed in the recent decades. In the past, researchers solved integer-order models by using the Hirota bilinear method. Recently, the Hirota bilinear method has been used to solve fractional models. In this paper, using the Hirota bilinear method, we obtain soliton solutions for the new model. Various phenomena can be explained via the application of the solutions given by the above methods [24–26]. Additionally, the use of these methods enables a better understanding of various magnetized plasma phenomena. Therefore, based on the solutions derived by the abovementioned methods, we seek to determine the properties of ion-acoustic solitary waves. The properties of the model include conservation laws [27, 28], boundary value problems [29, 30], and integrable systems [31, 32].

The research on conservation laws plays an important role in the study of the physical phenomena in nonlinear magnetized plasma. Conservation laws are a mathematical formulation, and they indicate that the total amount of a certain physical quantity remains the same during the evolution of a physical system [33, 34]. In 1918, Noether [35] proved that each conservation law is associated with an appropriate symmetry and can be derived from the Lagrangian function and the invariance principle. In 1996, Riewe [36] introduced the Lagrangian function for the fractional derivative. In the past two decades, many different types of fractional Euler-Lagrangian equations have been generalized. Based on these conclusion, some fractional generalizations of Noether's theorem were proved [37], and many fractional conservation laws were obtained [38]. To study the conservation laws of the fractional differential equations, we use Lie symmetry analysis to construct the conserved vectors [39, 40]

In this paper, applying the basic system of equations of ion-acoustic solitary waves [41], we develop a new 3D model. Using the new model, we study the conservation laws and the solution of ion-acoustic solitary waves. The rest of the paper is structured as follows: In Section 2, based on the basic system of equations of ion-acoustic solitary waves, we obtain a new 3D Schamel-KdV equation by using multi-scale analysis and the perturbation method [42]. A new 3D TSF-Schamel-KdV equation is obtained in Section 3 according to the new integer-order model and by using the semi-inverse method and the fractional variational principle [43, 44]. In Section 4, applying the Riemann-Liouville fractional derivative [39, 40], we discuss the conservation laws of the new fractional model. In Section 5, according to the Hirota bilinear method, we obtain the soliton solutions of the 3D TSF-Schamel-KdV equation. The propagation of solitary waves is important because it describes the characteristic nature of the interaction of the waves and the plasmas. Therefore, using soliton solutions [17, 18], we study the characteristics of motion of ion-acoustic solitary waves.

2. Derivation of the 3D Schamel-KdV Equation

We use the basic system of equations of ion-acoustic solitary waves given by

$$\begin{gathered}\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}+\frac{\partial (nv)}{\partial y}+\frac{\partial (nw)}{\partial z}=0,\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial \phi}{\partial x}+u^{\Lambda}\Omega e_x,\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial \phi}{\partial y}+v^{\Lambda}\Omega e_x,\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial \phi}{\partial z}+w^{\Lambda}\Omega e_x,\\ \frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=n_e-n,\end{gathered}\tag{1}$$

where n is the ion number density, and u, v, w are the ion fluid velocities in the x-, y-, and z-directions, respectively. ϕ is the electric field potential, n_e is the electron number density, and Ω is the uniform external magnetic field. Ion-acoustic solitary waves are assumed to propagate in the x-direction, and the direction is specified by the unit vector e_x.

We consider the propagation of ion-acoustic solitary waves in 3D space (x, y, z) and introduce the following independent stretched variables:

$$\begin{aligned}T&=\epsilon^{3/4}t,\\ X&=\epsilon^{1/4}(x-t),\\ Y&=\epsilon^{1/4}y,\\ Z&=\epsilon^{1/4}z,\end{aligned}\tag{2}$$

where ϵ is a small parameter characterizing the strength of the nonlinearity. Thus, we can obtain

$$\begin{aligned}
\frac{\partial}{\partial t} &= \epsilon^{3/4}\frac{\partial}{\partial T} - \epsilon^{1/4}\frac{\partial}{\partial X},\\
\frac{\partial}{\partial x} &= \epsilon^{1/4}\frac{\partial}{\partial X},\\
\frac{\partial}{\partial y} &= \epsilon^{1/4}\frac{\partial}{\partial Y},\\
\frac{\partial}{\partial z} &= \epsilon^{1/4}\frac{\partial}{\partial Z}.
\end{aligned} \tag{3}$$

The dependent variables are expanded in the following form:

$$\begin{aligned}
n(X,Y,Z,T) &= 1 + \epsilon n_1(X,Y,Z,T) + \epsilon^{3/2} n_2(X,Y,Z,T) + \cdots,\\
\phi(X,Y,Z,T) &= \epsilon\phi_1(X,Y,Z,T) + \epsilon^{3/2}\phi_2(X,Y,Z,T) + \cdots,\\
u(X,Y,Z,T) &= \epsilon u_1(X,Y,Z,T) + \epsilon^{3/2}u_2(X,Y,Z,T) + \cdots,\\
v(X,Y,Z,T) &= \epsilon^{5/4} v_1(X,Y,Z,T) + \epsilon^{3/2} v_2(X,Y,Z,T) + \cdots,\\
w(X,Y,Z,T) &= \epsilon^{5/4} w_1(X,Y,Z,T) + \epsilon^{3/2} w_2(X,Y,Z,T) + \cdots,\\
n_e(X,Y,Z,T) &= 1 + \phi(X,Y,Z,T) - \frac{4}{3}b\phi^{3/2}(X,Y,Z,T) + \frac{1}{2}\phi^2(X,Y,Z,T),
\end{aligned} \tag{4}$$

and the boundary conditions are given by

$$\begin{aligned}
n &= n_e = 1,\\
u &= v = w = \phi = 0,\\
&\text{as } X \longrightarrow \infty.
\end{aligned} \tag{5}$$

Substituting (3) and (4) into (1), we can obtain the approximate equations for ϵ in the following form:

$$\epsilon : n_1 = \phi_1, \tag{6}$$

$$\epsilon^{5/4} : \begin{cases}
\dfrac{\partial\phi_1}{\partial X} = \dfrac{\partial u_1}{\partial X} = \dfrac{\partial n_1}{\partial X},\\
\dfrac{\partial\phi_1}{\partial Y} = v_1^{\Lambda}\Omega e_x = \theta w_1,\\
\dfrac{\partial\phi_1}{\partial Z} = w_1^{\Lambda}\Omega e_x = -\theta v_1,
\end{cases} \tag{7}$$

$$\epsilon^{3/2} : \begin{cases}
\dfrac{\partial v_1}{\partial Y} = -\dfrac{\partial w_1}{\partial Z},\\
\dfrac{\partial v_1}{\partial Y} = -v_2^{\Lambda}\Omega e_x = -\theta w_2,\\
\dfrac{\partial w_1}{\partial X} = -w_2^{\Lambda}\Omega e_x = \theta v_2,
\end{cases} \tag{8}$$

$$\epsilon^{7/4} : \begin{cases}
\dfrac{\partial n_1}{\partial T} - \dfrac{\partial n_2}{\partial X} + \dfrac{\partial u_2}{\partial X} + \dfrac{\partial v_2}{\partial Y} + \dfrac{\partial w_2}{\partial Z} = 0,\\
\dfrac{\partial u_1}{\partial T} - \dfrac{\partial u_2}{\partial X} = \dfrac{\partial\phi_2}{\partial X},\\
\dfrac{\partial v_2}{\partial X} = \dfrac{\partial\phi_2}{\partial Y},\\
\dfrac{\partial w_2}{\partial X} = \dfrac{\partial\phi_2}{\partial Z}.
\end{cases} \tag{9}$$

According to (6) and (7), we can obtain

$$\begin{aligned}
\phi_1 &= u_1 = n_1,\\
w_1 &= \frac{1}{\theta}\frac{\partial\phi_1}{\partial Y},\\
v_1 &= -\frac{1}{\theta}\frac{\partial\phi_1}{\partial Z}.
\end{aligned} \tag{10}$$

Substituting (10) into (8) and (9) and eliminating ϕ_2, u_2, v_2, w_2 and n_2, we can obtain

$$\frac{\partial^3\phi_1}{\partial X^3} + \frac{\partial^3\phi_1}{\partial X\partial Y^2} + \frac{\partial^3\phi_1}{\partial X\partial Z^2} + 2b\phi_1^{1/2}\frac{\partial\phi_1}{\partial X} + 2\frac{\partial\phi_1}{\partial T} = 0. \tag{11}$$

Letting $\phi_1(X,Y,Z,T) = A(X,Y,Z,T)$, (11) can be rewritten as

$$A_T + a_1\sqrt{A}A_X + a_2 A_{XXX} + a_3 A_{XYY} + a_4 A_{XZZ} = 0, \tag{12}$$

where $a_1 = b$, $a_2 = 1/2$ and $a_3 = a_4 = (1/2)(1 + 1/\theta^2)$.

Remark 1. Because of the nonlinear term $\sqrt{A}A_X$, when $a_1 \neq 0$ and $a_3 = a_4 = 0$, (12) can be reduced to the 1D Schamel-KdV equation. When $a_3 = a_4 \neq 0$, (12) is a 3D equation. Therefore, (12) is called the 3D Schamel-KdV equation. Compared to the KdV and mKdV models [6], the nonlinearity of the 3D Schamel-KdV equation is relatively weak. Therefore, the 3D Schamel-KdV equation presents a new research direction for the study of ion-acoustic solitary waves.

3. Derivation of the 3D TSF-Schamel-KdV Equation

In Section 2, we have obtained a new 3D integer-order Schamel-KdV equation. To learn more about ion-acoustic solitary waves, we seek to obtain the 3D TSF-Schamel-KdV equation by using the semi-inverse method and the fractional variational principle. First, we introduce some definitions as follows.

Definition 2 (see [44]). The left Riemann-Liouville fractional derivation of a function $A(X,Y,Z,T)$ is defined as

$$\begin{aligned}
{}_0D_T^{\omega}A &= \frac{1}{\Gamma(n-\omega)}\frac{d^n}{dT^n}\int_0^T (T-t)^{n-\omega-1}A\,dt,\\
&\qquad n-1 \leq \omega < n.
\end{aligned} \tag{13}$$

Definition 3 (see [45]). The Riemann-Liouville fractional derivation of a function $A(X,Y,Z,T)$ is defined as

$$D_T^{\omega}A = \frac{\partial^{\omega}A}{\partial T^{\omega}} = \begin{cases} \dfrac{1}{\Gamma(n-\omega)}\dfrac{\partial^n}{\partial T^n}\displaystyle\int_0^{T_0}(T-s)^{n-\omega-1}A\,ds, & n-1\le\omega<n, \\ \dfrac{\partial^n A}{\partial T^n}, & \omega = n. \end{cases} \quad (14)$$

According to the integer-order 3D Schamel-KdV equation,

$$A_T + a_1\sqrt{A}A_X + a_2A_{XXX} + a_3A_{XYY} + a_4A_{XZZ} = 0, \quad (15)$$

assuming $A(X,Y,Z,T) = B_X(X,Y,Z,T)$, where $B(X,Y,Z,T)$ is a potential function, and therefore, the potential equation of the 3D Schamel-KdV equation can be written in the following form:

$$B_{XT} + a_1\sqrt{B_X}B_{XX} + a_2B_{XXXX} + a_3B_{XXYY} + a_4B_{XXZZ} = 0. \quad (16)$$

Then, the function of the potential equation (16) can be described as

$$J(B) = \iiint_V dX\,dY\,dZ\int_{T^*}dT\Big[B(X,Y,Z,T)\Big(b_1B_{XT} + b_2a_1\sqrt{B_X}B_{XX} + b_3a_2B_{XXXX} + b_4a_3B_{XXYY} + b_5a_4B_{XXZZ}\Big)\Big], \quad (17)$$

where $b_i, i = 1,2,3,4,5$, are Lagrangian multipliers which can be obtained later.

Using integration by parts for (17) and taking $B_X|_R = B_Y|_R = B_Z|_R = B_T|_{T^*} = B_{XX}|_R = B_{XY}|_R = B_{XZ}|_R = 0$, we obtain

$$J(B) = \iiint_V dX\,dY\,dZ\int_{T^*}dT\Big[-b_1B_TB_X - \frac{2}{3}b_2a_1B_X^{5/2} + b_3a_2(B_{XX})^2 + b_4a_3(B_{XY})^2 + b_5a_4(B_{XZ})^2\Big]. \quad (18)$$

Using the variation of the above function, integrating each term by parts and applying the variation optimum condition, we obtain

$$\begin{aligned} &F(X,Y,Z,T,B,B_T,B_X,B_{XX},B_{XY},B_{XZ}) \\ &= \frac{\partial F}{\partial B} - \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial B_T}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial B_X}\right) + \frac{\partial^2}{\partial X^2}\left(\frac{\partial F}{\partial B_{XX}}\right) + \frac{\partial}{\partial X\partial Y}\left(\frac{\partial F}{\partial B_{XY}}\right) + \frac{\partial}{\partial X\partial Z}\left(\frac{\partial F}{\partial B_{XZ}}\right) \\ &= 2c_1B_{XT} + \frac{5}{2}b_2a_1B_X^{1/2}B_{XX} + 2b_3a_2B_{XXXX} + 2b_4a_3B_{XXYY} + 2b_5a_4B_{XXZZ} = 0. \end{aligned} \quad (19)$$

Comparing (19) with (16), we obtain the following Lagrangian multipliers:

$$b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{5}, \quad b_3 = \frac{1}{2}, \quad b_4 = \frac{1}{2}, \quad b_5 = \frac{1}{2}. \quad (20)$$

Therefore, the Lagrangian form of the integer-order 3D Schamel-KdV equation is given by

$$L(B_T,B_X,B_{XX},B_{XY},B_{XZ}) = -\frac{1}{2}B_TB_X - \frac{4}{15}a_1(B_X)^{5/2} + \frac{1}{2}a_2(B_{XX})^2 + \frac{1}{2}a_3(B_{XY})^2 + \frac{1}{2}a_4(B_{XZ})^2. \quad (21)$$

Similarly, the Lagrangian form of the 3D TSF-Schamel-KdV equation is given by

$$L\left(D_T^{\omega}B, D_X^{\alpha}B, D_X^{\alpha\alpha}B, D_X^{\alpha}D_Y^{\beta}B, D_X^{\alpha}D_Z^{\gamma}B\right) = -\frac{1}{2}D_T^{\omega}BD_X^{\alpha}B - \frac{4}{15}a_1(D_X^{\alpha}B)^{5/2} + \frac{1}{2}a_2(D_X^{\alpha\alpha}B)^2 + \frac{1}{2}a_3\left(D_X^{\alpha}D_Y^{\beta}B\right)^2 + \frac{1}{2}a_4(D_X^{\alpha}D_Z^{\gamma}B)^2, \quad (22)$$

where $D_X^{\alpha\alpha}B = D_X^{\alpha}(D_X^{\alpha}B)$. Thus, the function of the 3D TSF-Schamel-KdV equation can be obtained as

$$J_F(B) = \int_R(dX)^{\alpha}\int_R(dY)^{\alpha}\int_R(dZ)^{\gamma}\int_{T^*}(dT)^{\omega}\cdot F\left(D_T^{\omega}B, D_X^{\alpha}B, D_X^{\alpha\alpha}B, D_X^{\alpha}D_Y^{\beta}B, D_X^{\alpha}D_Z^{\gamma}B\right). \quad (23)$$

According to the Agrawal's method [46, 47], the variation of functional Eq. (23) can be written as

$$\delta J_F(B) = \int_R(dX)^{\alpha}\int_R(dY)^{\alpha}\int_R(dZ)^{\gamma}\int_{T^*}(dT)^{\omega}\cdot\left[\left(\frac{\partial F}{\partial D_T^{\omega}B}\right)\delta D_T^{\omega}B + \left(\frac{\partial F}{\partial D_X^{\alpha}B}\right)\delta D_X^{\alpha}B\right.$$

$$+\left(\frac{\partial F}{\partial D_X^{\alpha\alpha}B}\right)\delta D_X^{\alpha\alpha}B+\left(\frac{\partial F}{\partial D_X^{\alpha}D_Y^{\beta}B}\right)\delta D_X^{\alpha}D_Y^{\beta}B$$
$$+\left(\frac{\partial F}{\partial D_X^{\alpha}D_Z^{\gamma}B}\right)\delta D_X^{\alpha}D_Z^{\gamma}B\Bigg], \quad (24)$$

where

$$\int_a^T (d\tau)^j f(\tau) = j\int_a^T dx\,(T-\tau)^j f(\tau). \quad (25)$$

Using the fractional integration by parts,

$$\int_a^b (d\tau)^j f(z)\, D_z^j g(z) = \Gamma(1+j)\left[g(z) f(z)\big|_a^b - \int_a^b (dz)^j g(z) D_z^j f(z)\right], \quad f(z), g(z) \in [a,b], \quad (26)$$

we can obtain

$$\delta J_F(B) = \int_R (dX)^\alpha \int_R (dY)^\alpha \int_R (dZ)^\gamma \int_{T^*} (dT)^\omega \cdot \left[-D_T^{\omega}\left(\frac{\partial F}{\partial D_T^{\omega}B}\right) - D_X^{\alpha}\left(\frac{\partial F}{\partial D_X^{\alpha}B}\right) + D_X^{\alpha\alpha}\left(\frac{\partial F}{\partial D_X^{\alpha\alpha}B}\right) + D_X^{\alpha}D_Y^{\beta}\left(\frac{\partial F}{\partial D_X^{\alpha}D_Y^{\beta}B}\right) + D_X^{\alpha}D_Z^{\gamma}\left(\frac{\partial F}{\partial D_X^{\alpha}D_Z^{\gamma}B}\right)\right]. \quad (27)$$

Optimizing the variation Eq. (24), $\delta J_F(B) = 0$, we can obtain the Euler-Lagrange equation of the 3D TSF-Schamel-KdV equation as

$$-D_T^{\omega}\left(\frac{\partial F}{\partial D_T^{\omega}B}\right) - D_X^{\alpha}\left(\frac{\partial F}{\partial D_X^{\alpha}B}\right) + D_X^{\alpha\alpha}\left(\frac{\partial F}{\partial D_X^{\alpha\alpha}B}\right) + D_X^{\alpha}D_Y^{\beta}\left(\frac{\partial F}{\partial D_X^{\alpha}D_Y^{\beta}B}\right) + D_X^{\alpha}D_Z^{\gamma}\left(\frac{\partial F}{\partial D_X^{\alpha}D_Z^{\gamma}B}\right) = 0. \quad (28)$$

Substituting (22) into (28), we obtain

$$D_T^{\omega}D_X^{\alpha}B + a_1\left(D_X^{\alpha}B\right)^{1/2}D_X^{\alpha\alpha}B + a_2 D_X^{\alpha\alpha\alpha\alpha}B + a_3 D_X^{\alpha\alpha}D_Y^{\beta\beta}B + a_4 D_X^{\alpha\alpha}D_Z^{\gamma\gamma}B = 0. \quad (29)$$

Letting $D_X^{\alpha}B(X,Y,Z,T) = A(X,Y,Z,T)$ and substituting $D_X^{\alpha}B(X,Y,Z,T)$ into (29), we can obtain

$$D_T^{\omega}A + a_1\sqrt{A}D_X^{\alpha}A + a_2 D_X^{\alpha\alpha\alpha}A + a_3 D_X^{\alpha}D_Y^{\beta\beta}A + a_4 D_X^{\alpha}D_Z^{\gamma\gamma}A = 0. \quad (30)$$

Eq. (30) is the 3D TSF-Schamel-KdV equation.

4. Conservation Laws of the 3D TSF-Schamel-KdV Equation

4.1. Lie Symmetry Analysis. In the previous section, we have obtained the 3D TSF-Schamel-KdV equation. To learn about the properties of the new model, we study the conservation laws [48, 49]. First, we convert (30) to the following fractional partial differential equation form:

$$D_T^{\gamma}A = Q\left(X,Y,Z,T,Am,D_T^{\omega}A,D_X^{\alpha}A,D_X^{\alpha\alpha\alpha}A, D_X^{\alpha}D_Y^{\beta\beta}A, D_X^{\alpha}D_Z^{\gamma\gamma}A,\ldots\right), \quad \omega,\alpha,\beta,\gamma > 0. \quad (31)$$

We assume that (31) is invariant under a one parameter Lie group of point transformations in the following form:

$$\begin{aligned}
X' &= x + \epsilon\xi(X,Y,Z,T,A) + O\left(\epsilon^2\right),\\
Y' &= Y + \epsilon\zeta(X,Y,Z,T,A) + O\left(\epsilon^2\right),\\
Z' &= Z + \epsilon\psi(X,Y,Z,T,A) + O\left(\epsilon^2\right),\\
T' &= T + \epsilon\tau(X,Y,Z,T,A) + O\left(\epsilon^2\right),\\
A' &= x + \epsilon\eta(X,Y,Z,T,A) + O\left(\epsilon^2\right),\\
D_T^{\omega}A' &\longrightarrow D_T^{\omega}A + \epsilon\eta_{\omega}^{T} + O\left(\epsilon^2\right),\\
D_X^{\alpha}A' &\longrightarrow D_X^{\alpha}A + \epsilon\eta_{\alpha}^{X} + O\left(\epsilon^2\right),\\
D_X^{\alpha\alpha\alpha}A' &\longrightarrow D_X^{\alpha\alpha\alpha}A + \epsilon\eta_{\alpha}^{XXX} + O\left(\epsilon^2\right),\\
D_X^{\alpha}D_Y^{\beta\beta}A' &\longrightarrow D_X^{\alpha}D_Y^{\beta\beta}A + \epsilon\eta_{\alpha,\beta}^{XYY} + O\left(\epsilon^2\right),\\
D_X^{\alpha}D_Z^{\gamma\gamma}A' &\longrightarrow D_X^{\alpha}D_Z^{\gamma\gamma}A + \epsilon\eta_{\alpha,\gamma}^{XZZ} + O\left(\epsilon^2\right),
\end{aligned} \quad (32)$$

where ξ, ζ, ψ, τ, and η are infinitesimal functions, and η_{ω}^{T}, η_{α}^{X}, η_{α}^{XXX}, $\eta_{\alpha,\gamma}^{XYY}$, and $\eta_{\alpha,\gamma}^{XZZ}$ are the prolongations of infinitesimal functions defined as

$$\begin{aligned}
\eta_{\gamma}^{T} &= D_T^{\omega}(\eta) + \xi D_T^{\omega}(A_X) - D_T^{\omega}(\xi A_X) + \zeta D_T^{\omega}(A_Y) - D_T^{\omega}(\zeta A_Y) + \psi D_T^{\omega}(A_Z) - D_T^{\omega}(\psi A_Z)\\
&\quad + D_T^{\omega}(D_T(\tau)A) - D_T^{\gamma+1}(\tau A) + \tau D_T^{\gamma+1}(A),\\
\eta_{\alpha}^{X} &= D_X^{\alpha}(\eta) + D_X^{\alpha}(A)D_X(\xi) - D_Y^{\beta}(A)D_X(\zeta) - D_Z^{\gamma}AD_Z(\psi),\\
\eta_{\alpha}^{XXX} &= D_X^{\alpha}\left(\eta_{\alpha}^{XX}\right) - A_{XXX}D_X^{\alpha}(\xi) - A_{XXY}D_X^{\alpha}(\zeta) - A_{XXZ}D_Z(\psi) - A_{XXT}D_X^{\alpha}(\tau),\\
\eta_{\alpha,\beta}^{XYY} &= D_X^{\alpha}\left(\eta_{\beta}^{YY}\right) - A_{XXY}D_X^{\alpha}(\xi) - A_{XYY}D_X^{\alpha}(\zeta) - A_{XYZ}D_X^{\alpha}(\psi) - A_{XYT}D_X^{\alpha}(\tau),
\end{aligned}$$

$$\eta_{\alpha,\beta}^{XZZ} = D_X^\alpha\left(\eta_\gamma^{ZZ}\right) - A_{XXZ}D_X^\alpha(\xi) - A_{XYZ}D_X^\alpha(\zeta) - A_{XZZ}D_X^\alpha(\psi) - A_{XZT}D_X^\alpha(\tau), \tag{33}$$

where D_T and D_X are the total derivative operators given by

$$\begin{aligned} D_T &= \frac{\partial}{\partial T} + A_T\frac{\partial}{\partial T} + A_{TT}\frac{\partial}{\partial A_T} + A_{XT}\frac{\partial}{\partial A_X} + A_{YT}\frac{\partial}{\partial A_Y} + A_{ZT}\frac{\partial}{\partial A_Z} + \cdots, \\ D_X &= \frac{\partial}{\partial X} + A_X\frac{\partial}{\partial A} + A_{XX}\frac{\partial}{\partial A_X} + A_{TX}\frac{\partial}{\partial A_T} + A_{YX}\frac{\partial}{\partial A_Y} + A_{ZX}\frac{\partial}{\partial A_Z} + \cdots. \end{aligned} \tag{34}$$

Applying the generalized Leibnitz rule as given by

$$D_T^\omega(f(t)g(t)) = \sum_{n=0}^{\infty}\binom{\omega}{n}D_t^{\omega-n}f(t)D_t^n g(t), \quad \omega > 0, \tag{35}$$

where

$$\binom{\omega}{n} = \frac{(-1)^{n-1}\omega\Gamma(n-\omega)}{\Gamma(1-\omega)\Gamma(n+1)}, \tag{36}$$

and the chain rule for a compound function defined as

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}\left[-g(t)^{k-r}\right]\frac{d^k f(g(t))}{dt^k}, \tag{37}$$

we can obtain the following equation:

$$\begin{aligned} \eta_\omega^T &= D_T^\omega(\eta) - \omega D_T^\omega(\tau)\frac{\partial^\omega A}{\partial T^\omega} - \sum_{n=1}^{\infty}\binom{\omega}{n}D_T^n(\xi)D_T^{\omega-n}A_X - \sum_{n=1}^{\infty}\binom{\omega}{n}D_T^n(\zeta) \\ &\cdot D_T^{\omega-n}A_Y - \sum_{n=1}^{\infty}\binom{\omega}{n}D_T^n(\psi)D_T^{\omega-n}A_Z - \sum_{n=1}^{\infty}\binom{\omega}{n+1}D_T^{n+1}(\tau)D_T^{\omega-n}A. \end{aligned} \tag{38}$$

For the chain rule given by (37), when $f(t) = 1$, we obtain

$$D_T^\omega = \frac{\partial^\omega \eta}{\partial T^\omega} + \eta_A\frac{\partial^\omega A}{\partial T^\omega} - A\frac{\partial^\omega \eta_A}{\partial T^\omega} + \sum_{n=1}^{\infty}\binom{\omega}{n}\frac{\partial^n \eta_A}{\partial T^n}D_T^{\omega-n}A + Ra, \tag{39}$$

where

$$Ra = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\omega}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{T^{n-\omega}}{\Gamma(n+1-\omega)}(-A)^r\frac{\partial^A}{\partial T^A}(A)^{k-r}\frac{\partial^{n-m+k}\eta}{\partial T^{n-m}\partial A^k}\right]. \tag{40}$$

Therefore, (38) can be rewritten as

$$\begin{aligned} \eta_\omega^T &= \frac{\partial^\omega \eta}{\partial T^\omega} + (\eta_A - \omega D_T(\tau))\frac{\partial^\omega A}{\partial T^\omega} - A\frac{\partial^\omega \eta_A}{\partial T^\omega} + \sum_{n=1}^{\infty}\left[\binom{\omega}{n}\frac{\partial^\omega \eta_A}{\partial T^\omega} - \binom{\omega}{n+1}D_T^{n+1}(\tau)\right]D_T^{\omega-n} \\ &- \sum_{n=1}^{\infty}\binom{\omega}{n}\left[D_T^n(\xi)D_T^{\omega-n}(A_X) + D_T^n(\zeta)D_T^{\omega-n}(A_Y) + D_T^n(\psi)D_T^{\omega-n}A_Z\right] + Ra. \end{aligned} \tag{41}$$

Similarly, using the generalized Leibnitz rule and the chain rule for a compound function, we also obtain the following equation:

$$\begin{aligned} \eta_\alpha^X &= \frac{\partial^\alpha \eta}{\partial X^\alpha} + (\eta_A - \alpha D_X(\xi))\frac{\partial^\alpha A}{\partial X^\alpha} - A\frac{\partial^\alpha \eta_A}{\partial X^\alpha} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_A}{\partial X^\alpha} - \binom{\alpha}{n+1}D_T^{n+1}(\xi)\right]D_X^{\alpha-n} \\ &- \sum_{n=1}^{\infty}\binom{\alpha}{n}\left[D_T^n(\xi)D_T^{\alpha-n}A_Y + D_X^n(\psi)D_X^{\alpha-n}(A_Z) + D_T^n\tau D_T^{\alpha-n}(A_T)\right] + Rb, \end{aligned} \tag{42}$$

where

$$Rb = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{X^{n-\alpha}}{\Gamma(n+1-\alpha)}(-A)^r\frac{\partial^m}{\partial X^m}\left[(A)^{k-r}\right]\frac{\partial^{n-m+k}\eta}{\partial X^{n-m}\partial A^k}\right]. \tag{43}$$

The infinitesimal generator M can be defined as follows:

$$M = \xi\frac{\partial}{\partial X} + \zeta\frac{\partial}{\partial Y} + \psi\frac{\partial}{\partial Z} + \tau\frac{\partial}{\partial T} + \eta\frac{\partial}{\partial A}. \quad (44)$$

Under the infinitesimal transformations, the invariance of the system (31) leads to the following invariance condition:

$$\begin{aligned} &Pr^{(n)}M(\Delta)\big|_{\Delta=0} = 0, \quad n = 1,2,3,\ldots, \\ &\Delta = D_T^{\omega}A + a_1\sqrt{A}D_X^{\alpha}A + a_2D_X^{\alpha\alpha\alpha}A \\ &\quad + a_3D_X^{\alpha}D_Y^{\beta\beta}A + a_4D_X^{\alpha}D_Z^{\gamma\gamma}A. \end{aligned} \quad (45)$$

According to (42) and (43), we can obtain

$$\begin{aligned} Pr^{(\omega,\alpha,\beta,\gamma,4)}M(\Delta) &= \tau\frac{\partial^{\omega}}{\partial T^{\omega}} + \xi\frac{\partial^{\alpha}}{\partial X^{\alpha}} + \zeta\frac{\partial^{\beta}}{\partial Y^{\beta}} + \psi\frac{\partial^{\gamma}}{\partial Z^{\gamma}} \\ &\quad + \eta\frac{\partial}{\partial A} + \eta_{\gamma}^{T}\frac{\partial}{\partial D_T^{\gamma}A} + \eta_{\alpha}^{X}\frac{\partial}{\partial D_X^{\alpha}A} \\ &\quad + \eta_{\alpha}^{XXX}\frac{\partial}{\partial D_X^{\alpha\alpha\alpha}A} \\ &\quad + \eta_{\alpha,\beta}^{XYY}\frac{\partial}{\partial D_X^{\alpha}D_Y^{\beta\beta}A} \\ &\quad + \eta_{\alpha,\gamma}^{XZZ}\frac{\partial}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}. \end{aligned} \quad (46)$$

Then, we can obtain the following invariance criterion:

$$\begin{aligned} &\eta_{\omega}^{T} + \frac{1}{2}a_1\frac{1}{\sqrt{A}}\eta D_X^{\alpha}A + a_1\sqrt{A}\eta_{\alpha}^{X} + a_2\eta_{\alpha}^{XXX} + a_3\eta_{\alpha,\beta}^{XYY} \\ &\quad + a_4\eta_{\alpha,\gamma}^{XZZ} = 0. \end{aligned} \quad (47)$$

Substituting (33), (34), (41), and (42) into (47) and equating the coefficients of alike partial derivatives, fractional derivatives and powers of A, the set of determining equations can be obtained as

$$\begin{aligned} &\binom{\omega}{n}\frac{\partial^{\gamma}\eta_A}{\partial T^{\gamma}} - \binom{\omega}{n+1}D_T^{n+1}(\tau) = 0, \\ &\binom{\alpha}{n}\frac{\partial^{\alpha}\eta_A}{\partial X^{\alpha}} - \binom{\alpha}{n+1}D_X^{n+1}(\xi) = 0, \\ &\xi_A = \xi_T = 0, \\ &\zeta_A = \zeta_T = 0, \\ &\tau_X = \tau_Y = 0, \\ &\eta = A(3\xi_X - \omega\tau_T), \\ &\eta_{AA} = 0, \\ &\alpha\xi_X - \omega\tau_T = 0, \\ &\frac{\partial_{\omega}\eta}{\partial T_{\omega}} - A\frac{\partial_{\omega}\eta_A}{\partial T_{\omega}} + a_1\sqrt{A}\left(\frac{\partial_{\alpha}\eta}{\partial X^{\alpha}} - A\frac{\partial_{\alpha}\eta_A}{\partial X^{\alpha}}\right) \\ &\quad + a_2D_X^{\alpha\alpha\alpha}(\eta) + a_3D_X^{\alpha}D_Y^{\beta\beta}(\eta) + a_4D_X^{\alpha}D_Z^{\gamma\gamma}(\eta) = 0. \end{aligned} \quad (48)$$

By solving the above equations, we can obtain a series of Lie algebra of point symmetries as

$$\begin{aligned} \tau &= c_2\alpha T + c_1, \\ \xi &= c_2\omega X + c_3, \\ \zeta &= c_2\gamma Y + c_4Z + c_5, \\ \psi &= c_2\beta Z - c_4Y + c_6, \\ \eta &= -c_2\alpha(3-\omega)A. \end{aligned} \quad (49)$$

Hence, a series of Lie algebra of point symmetries can be written as

$$\begin{aligned} M_1 &= \frac{\partial}{\partial T}, \\ M_2 &= \frac{\partial}{\partial X}, \\ M_3 &= \frac{\partial}{\partial Y}, \\ M_4 &= \frac{\partial}{\partial Z}, \\ M_5 &= \omega X\frac{\partial}{\partial X} + \gamma Y\frac{\partial}{\partial Y} + \beta Z\frac{\partial}{\partial Z} + \alpha T\frac{\partial}{\partial T} \\ &\quad - \alpha(3-\omega)A\frac{\partial}{\partial A}. \end{aligned} \quad (50)$$

4.2. Conservation Laws. We have obtained the Lie symmetry generator in Section 4.2. In this section, we will discuss conservation laws of the 3D TSF-Schamel-KdV equation based on the obtained Lie symmetry generator. We know that the conservation laws of (30) satisfy the following equation:

$$D_T(C^T) + D_X(C^X) + D_Y(C^Y) + D_Z(C^Z) = 0, \quad (51)$$

where C^T, C^X, C^Y and C^Z are the conserved vectors.

A formal Lagrangian for the 3D TSF-Schamel-KdV equation can be presented as

$$\begin{aligned} \mathscr{L} &= s(X,Y,Z,T)\left(D_T^{\omega}A + a_1\sqrt{A}D_X^{\alpha}A + a_2D_X^{\alpha\alpha\alpha}A\right. \\ &\quad \left. + a_3D_X^{\alpha}D_Y^{\beta\beta}A + a_4D_X^{\alpha}D_Z^{\gamma\gamma}A\right), \end{aligned} \quad (52)$$

where $s(X,Y,T)$ is a new dependent variable. According to the formal Lagrangian, an action integral is defined as

$$\int_R\int_R\int_R\int_{T^*}\mathscr{L}\left(X,Y,Z,T,A,s,D_T^{\omega}A,D_X^{\alpha}A,D_X^{\alpha\alpha\alpha}A,D_X^{\alpha}D_Y^{\beta\beta}A,D_X^{\alpha}D_Z^{\gamma\gamma}A\right)dX\,dY\,dZ\,dT. \quad (53)$$

Therefore, we can obtain the adjoint equation of (30) as the Euler-Lagrange equation

$$F^{*}=\frac{\delta\mathscr{L}}{\delta A}=0, \tag{54}$$

where $\delta/\delta m$ is the Euler-Lagrange operator defined as

$$\begin{aligned}\frac{\delta}{\delta A}&=\frac{\partial}{\partial A}+\left(D_T^{\omega}\right)^{*}\frac{\partial}{\partial D_T^{\omega}}+\left(D_X^{\alpha}\right)^{*}\frac{\partial}{\partial D_X^{\alpha}A}\\&-\left(D_X^{\alpha\alpha\alpha}\right)^{*}\frac{\partial}{\partial D_X^{\alpha\alpha\alpha}A}-\left(D_X^{\alpha}D_Y^{\beta\beta}\right)^{*}\frac{\partial}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\\&-\left(D_X^{\alpha}D_Z^{\gamma\gamma}\right)^{*}\frac{\partial}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A},\end{aligned} \tag{55}$$

where $(D_T^{\omega})^{*}$, $(D_X^{\alpha})^{*}$, $(D_X^{\alpha\alpha\alpha})^{*}$, $(D_X^{\alpha}D_Y^{\beta\beta})^{*}$, and $(D_X^{\alpha}D_Z^{\gamma\gamma})^{*}$ are the adjoint operators of the Riemann-Liouville fractional differential operators D_T^{ω}, D_X^{α}, $D_X^{\alpha\alpha\alpha}$, $D_X^{\alpha}D_Y^{\beta\beta}$, and $D_X^{\alpha}D_Z^{\gamma\gamma}$, respectively. These are given by

$$\begin{aligned}\left(D_T^{\omega}\right)^{*}&=(-1)^{n}I_p^{n-\omega}\left(D_T^{n}\right)={}_T^{C}D_p^{\omega},\\\left(D_X^{\alpha}\right)^{*}&=(-1)^{m}I_q^{m-\alpha}\left(D_X^{m}\right)={}_X^{C}D_q^{\alpha},\end{aligned} \tag{56}$$

where $I_p^{n-\omega}$ and $I_q^{m-\alpha}$ are the right-sided fractional integral operators of orders $n-\omega$ and $m-\beta$, respectively. ${}_T^{C}D_p^{\omega}$ and ${}_X^{C}D_q^{\alpha}$ are the right-sided Caputo fractional differential operators of orders ω and α, respectively. Therefore, the adjoint equation (54) can be rewritten as

$$\begin{aligned}F^{*}&=\left(D_T^{\gamma}\right)^{*}s+a_1\sqrt{A}\left(D_X^{\alpha}\right)^{*}s-a_2\left(D_X^{\alpha\alpha\alpha}\right)^{*}s\\&-a_3\left(D_X^{\alpha}D_Y^{\beta\beta}\right)^{*}s-a_4\left(D_X^{\alpha}D_Z^{\gamma\gamma}\right)^{*}s=0.\end{aligned} \tag{57}$$

Based on Section 4.1, we obtain infinitesimal symmetry of (30). We assume that the Lie characteristic function W is given by

$$W=\eta-\tau A_T-\xi A_X-\zeta A_Y-\psi A_Z. \tag{58}$$

Applying this on the M_5 of the symmetry (50), we obtain

$$\begin{aligned}W_1&=A_X,\\W_2&=A_Y,\\W_3&=A_Z,\\W_4&=A_T,\\W_5&=-\alpha(3-\omega)A-\alpha TA_T-\omega XA_X-\gamma YA_Y\\&-\beta ZA_Z.\end{aligned} \tag{59}$$

Using the Riemann-Liouville fractional derivative, the components of the conserved vectors of (30) are defined as

$$\begin{aligned}C^{T}&=\tau I+\sum_{k=0}^{n-1}(-1)_0^{k}D_T^{\gamma-1-k}(W)D_T^{k}\frac{\partial\mathscr{L}}{\partial\left({}_0D_T^{\gamma}A\right)}\\&-(-1)^{n}J\left(W,D_T^{n}\frac{\partial\mathscr{L}}{\partial\left({}_0D_T^{\gamma}A\right)}\right),\\C^{X}&=\xi I+W\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}A}+D_X^{\alpha\alpha}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha\alpha\alpha}A}\right.\\&\left.+D_Y^{\beta\beta}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}+D_Z^{\gamma\gamma}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right)-D_X^{\alpha}(W)\\&\cdot\left[D_X^{\alpha}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha\alpha\alpha}A}\right]-D_Y^{\beta}(W)\left[D_Y^{\beta}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right]\\&-D_Z^{\gamma}(W)\left[D_Z^{\gamma}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right]+D_X^{\alpha\alpha}(W)\\&\cdot\left[\frac{\partial\mathscr{L}}{\partial D_X^{\alpha\alpha\alpha}A}\right]+D_Y^{\beta\beta}(W)\left[\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right]\\&+D_Z^{\gamma\gamma}(W)\left[\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right],\\C^{Y}&=\zeta I+W\left[D_X^{\alpha}\left(D_Y^{\beta}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right)\right]-D_Y^{\beta}(W)\\&\cdot\left[D_X^{\alpha}\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right)\right]-D_X^{\alpha}(W)\\&\cdot\left[D_Y^{\beta}\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right)\right]+D_X^{\alpha}D_Y^{\beta}(W)\\&\cdot\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right),\\C^{Z}&=\zeta I+W\left[D_X^{\alpha}\left(D_Z^{\gamma}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right)\right]-D_Z^{\gamma}(W)\\&\cdot\left[D_X^{\alpha}\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}m}\right)\right]-D_X^{\alpha}(W)\\&\cdot\left[D_Z^{\gamma}\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right)\right]+D_X^{\alpha}D_Z^{\gamma}(W)\\&\cdot\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right),\end{aligned} \tag{60}$$

where $n=[\omega]+1$, and J is the integral given by

$$\begin{aligned}&J(a,b)\\&=\frac{1}{\Gamma(n-\omega)}\int_0^{T}\int_T^{T^0}\frac{f(\tau,X,Y)\,g(\mu,X,Y)}{(\mu-\tau)^{\omega+1-n}}d\mu\,d\tau,\end{aligned} \tag{61}$$

with the property that

$$D_T J(f,g) = f_T I_{T'}^{n-\gamma} g - g_0 I_T^{n-\gamma} f. \quad (62)$$

The conservation laws of the 3D TSF-Schamel-KdV equation are explained in detail below (see the appendix).

5. Multi-Soliton Solutions for the 3D TSF-Schamel-KdV Equation

The solution of the model is a relatively broad research area in science [50, 51]. In this section, using the simplified Hirota bilinear method [24, 52], we seek multiple soliton solutions of the 3D TSF-Schamel-KdV equation.

First, we introduce the following fractional transforms:

$$\begin{aligned} T' &= \frac{p_1 T^{\omega}}{\Gamma(1+\omega)}, \\ X' &= \frac{p_2 X^{\alpha}}{\Gamma(1+\alpha)}, \\ Y' &= \frac{p_3 Y^{\beta}}{\Gamma(1+\beta)}, \\ Z' &= \frac{p_4 Z^{\gamma}}{\Gamma(1+\gamma)}, \end{aligned} \quad (63)$$

where p_1, p_2, p_3 and p_4 are constants. Using the above transformations and omitting the apostrophe, we can convert the derivatives into classical derivatives,

$$\begin{aligned} \frac{\partial^{\omega} A}{\partial T^{\omega}} &= p_1 \frac{\partial A}{\partial T}, \\ \frac{\partial^{\alpha} A}{\partial X^{\alpha}} &= p_2 \frac{\partial A}{\partial X}, \\ \frac{\partial^{\beta} A}{\partial Y^{\beta}} &= p_3 \frac{\partial A}{\partial Y}, \\ \frac{\partial^{\gamma} A}{\partial Z^{\gamma}} &= p_4 \frac{\partial A}{\partial Z}. \end{aligned} \quad (64)$$

Then, (30) can be described as

$$A_T + a_1 \sqrt{A} A_X + a_2 A_{XXX} + a_3 A_{XYY} + a_4 A_{XZZ} = 0. \quad (65)$$

We assume that the solution of (65) has the form

$$A(X,Y,Z,T) = e^{\theta_i(X,Y,Z,T)}, \quad (66)$$

where

$$\theta_i(X,Y,Z,T) = k_i X + r_i Y + q_i Z - u_i T. \quad (67)$$

Substituting (66) and (67) into the linear term of (65), we can obtain the following dispersion relation:

$$u_i = a_2 k_i^3 + a_3 k_i r_i^2 + a_4 k_i q_i^2. \quad (68)$$

Hence, θ_i can be written as

$$\begin{aligned} \theta_i(X,Y,Z,T) &= k_i X + r_i Y + q_i Z \\ &\quad - \left(a_2 k_i^3 + a_3 k_i r_i^2 + a_4 k_i q_i^2\right) T. \end{aligned} \quad (69)$$

5.1. Single-Soliton Solution. We assume that the single-soliton solution of (65) has the following form:

$$A(X,Y,Z,T) = R(\ln f)_{XX}, \quad (70)$$

where $f(X,Y,Z,T)$ is the auxiliary function defined as

$$\begin{aligned} f(X,Y,Z,T) &= 1 + e^{\theta_1(X,Y,Z,T)} \\ &= 1 + e^{k_1 X + r_1 Y + q_1 Z - (a_2 k_1^3 + a_3 k_1 r_1^2 + a_4 k_1 q_1^2) T}. \end{aligned} \quad (71)$$

Substituting (70) into (65), we obtain

$$R = \left(\frac{18 a_2}{a_1}\right)^2. \quad (72)$$

Substituting (71) and (72) into (70), we obtain the following single-soliton solution:

$$\begin{aligned} A(X,Y,Z,T) &= \left(\frac{18 a_2}{a_1}\right)^2 k_1^2 \frac{e^{\theta_1(X,Y,Z,T)}}{\left(1 + e^{\theta_1(X,Y,Z,T)}\right)^2} \\ &= \left(\frac{18 a_2}{a_1}\right)^2 k_1^2 \\ &\quad \cdot \frac{e^{k_1 X + r_1 Y + q_1 Z - (a_2 k_1^3 + a_3 k_1 r_1^2 + a_4 k_1 q_1^2) T}}{\left(1 + e^{k_1 X + r_1 Y + q_1 Z - (a_2 k_1^3 + a_3 k_1 r_1^2 + a_4 k_1 q_1^2) T}\right)^2}. \end{aligned} \quad (73)$$

The above equation can be rewritten as

$$A(X,Y,Z,T) = \left(\frac{18 a_2}{a_1}\right)^2 k_1^2 \operatorname{sech}^2\left(\frac{\theta_1(X,Y,Z,T)}{2}\right), \quad (74)$$

where

$$\begin{aligned} \theta_1 &= k_1 \frac{X^{\alpha}}{\Gamma(1+\alpha)} + r_1 \frac{Y^{\beta}}{\Gamma(1+\beta)} + q_1 \frac{Z^{\gamma}}{\Gamma(1+\gamma)} \\ &\quad - \left(a_2 k_1^3 + a_3 k_1 r_1^2 + a_4 k_1 q_1^2\right) \frac{T^{\omega}}{\Gamma(1+\omega)}. \end{aligned} \quad (75)$$

5.2. Two-Soliton Solution. We assume that the two-soliton solution has the following form:

$$\begin{aligned} f(X,Y,Z,T) &= 1 + e^{\theta_1(X,Y,Z,T)} + e^{\theta_2(X,Y,Z,T)} \\ &\quad + a_{12} e^{\theta_1(X,Y,Z,T) + \theta_2(X,Y,Z,T)}, \end{aligned} \quad (76)$$

where θ_1 and θ_2 are defined in (69). We know that

$$A(X,Y,Z,T) = \left(\frac{18 a_2}{a_1}\right)^2 (\ln f)_{XX}, \quad (77)$$

and substituting this expression into (65), the coefficient a_{12} can be obtained as

$$a_{12} = \frac{k_1^2 + k_2^2 - k_1 k_2}{k_1^2 + k_2^2 + 2 k_1 k_2}. \quad (78)$$

Therefore, the two-soliton solution for (65) has the following form:

$$A(X,Y,Z,T) = \left(\frac{18a_2}{a_1}\right)^2 \frac{k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + \left[a_{12}\left(k_2^2 e^{\theta_1} + k_1^2 e^{\theta_2}\right) + a_{12}\left(k_1+k_2\right)^2 + \left(k_1-k_2\right)^2\right] e^{\theta_1+\theta_2}}{\left(1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2}\right)^2}, \quad (79)$$

where

$$\theta_1 = \frac{k_1 X^\alpha}{\Gamma(1+\alpha)} + \frac{r_1 Y^\beta}{\Gamma(1+\beta)} + \frac{q_1 Z^\gamma}{\Gamma(1+\gamma)} - \frac{\left(a_2 k_1^3 + a_3 k_1 r_1^2 + a_4 k_1 q_1^2\right) T^\omega}{\Gamma(1+\omega)},$$
$$\theta_2 = \frac{k_2 X^\alpha}{\Gamma(1+\alpha)} + \frac{r_2 Y^\beta}{\Gamma(1+\beta)} + \frac{q_2 Z^\gamma}{\Gamma(1+\gamma)} - \frac{\left(a_2 k_2^3 + a_3 k_2 r_2^2 + a_4 k_2 q_2^2\right) T^\omega}{\Gamma(1+\omega)}. \quad (80)$$

5.3. Three-Soliton Solution. To investigate the three-soliton solution of (65), we assume that the auxiliary function $f(X,Y,Z,T)$ has the following form:

$$f(X,Y,Z,T) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} a_{13} e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{123}e^{\theta_1+\theta_2+\theta_3}, \quad (81)$$

where

$$a_{ij} = \frac{k_i^2 + k_j^2 - k_i k_j}{k_i^2 + k_j^2 + 2k_i k_j}. \quad (82)$$

Substituting (77) and (81) into (65), we find the following pattern:

$$a_{123} = a_{12}a_{13}a_{123}. \quad (83)$$

According to the pattern obtained in Section 5.3, the N-soliton solutions for the $3D$ TSF-Schamel-KdV equation can be obtained, where $N \geq 1$. Based on the single-soliton solution and the two-soliton solution, we can study the characteristics of the motion of the ion-acoustic solitary waves.

In this section, we describe the interaction of two small ion-acoustic solitary waves with finite amplitude in a weakly relativistic 3D magnetic plasma. Then, we can study the characteristics of motion of the solitary waves by changing the coefficients. Based on the single-soliton solution of ion-acoustic solitary waves, we obtain the evolution plots of the ion-acoustic solitary waves (see Figure 1). Figure 1 shows that the solitonic amplitude increases with an increase in the a_2/a_1 ratio, and the initial superimposed solitons travel different distances over a period of time for the different choices of a_1 and a_2. Therefore, we conclude that the soliton moves along the positive x-axis with constant amplitude and velocity.

Examination of Figure 2(a) shows that the propagation trajectory of the soliton exhibits a periodic oscillation. Figure 2(a) shows the curve propagation trajectory with constant amplitude and constantly changing velocity, where the velocity changes with time. Furthermore, Figure 2(b) shows the two-soliton interaction with constantly changing velocity. When $T \to 0$, the trajectory is sinusoidal with periodic oscillation. Otherwise, when T is far from the origin, the trajectory is parabolic-like. It can be seen from Figure 2(c) that the soliton generates a peak at the time of the interaction. Based on this, we conclude that, in addition to the periodic oscillation of the solitons in the local region, the large-scale propagation trajectories for such a structure show parabolic-type curves. Thus, if the variable coefficients are taken to have other forms, the corresponding characteristic curves will present different behaviors.

Remark 4. The present study describes the propagation and interaction of two small but finite amplitude ion-acoustic solitary waves in a weakly relativistic 3D magnetized plasma. Our conclusions can be considered as a generalization of the model suggested by Nejoh [53] and Pakira and Chowdhury [54] by including the effect of different coefficients. It has been found that the initial superposed solitons travel different distances over a period of time for the different choices of a_1 and a_2 and that the solitonic amplitude increases with an increase in the a_2/a_1 ratio. Moreover, the time fractional order ω and space fractional orders α, β, and γ play an important role in higher-order perturbation theory in the variation of the soliton amplitude. We believe that our research may be of basic interest for particle trapping experiments. Compared to other solitary waves, the unique features of the ion-acoustic solitary waves are the existence of an ultra-low frequency regime for wave propagation and the high charging of the grains, which can fluctuate because of the collection of plasma currents onto dust surfaces. It is difficult to carry out a reasonable comparison between previous studies and the present work. Nevertheless, due to the flexibility provided by the nonextensive method, we suggest that the quantitative discrepancies between the theory and experiment can be reduced. Thus, the application of our model may be particularly interesting in some plasma environments, such as space-plasmas, laser-plasma interactions, and the plasma sheet boundary layer of the earth's magnetosphere.

6. Conclusions

In this paper, based on the basic system of equations of ion-acoustic solitary waves, we have obtained a new 3D Schamel-KdV equation by applying multi-scale analysis and the perturbation method. Then, based on the newly developed model and using the semi-inverse method and the fractional variational principle, a new 3D TSF-Schamel-KdV equation is obtained. We study the conservation laws and

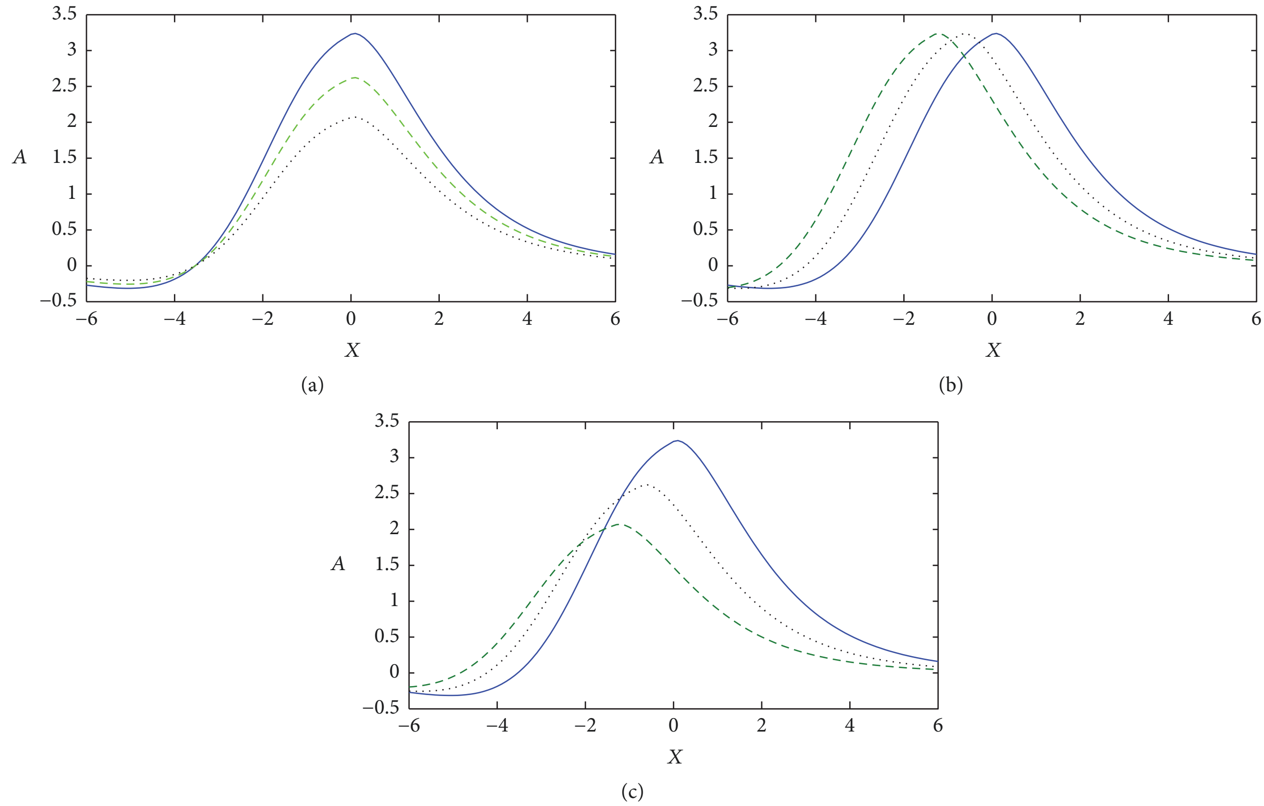

Figure 1: Plots for the evolution of the one-soliton solution of ion-acoustic solitary waves with (a): $t = 1$, $a_1 = 1$, $a_3 = 0$, $a_4 = 1$, $k_1 = 1$, $r_1 = 0.4$, $q_1 = 0.4$, $\alpha = 0.8$, $\beta = 1$, $\gamma = 1$, $\omega = 2$; (b): $t = 3$, $a_1 = 1$, $a_3 = 0$, $a_4 = 1$, $k_1 = 1$, $r_1 = 0.4$, $q_1 = 0.4$, $\alpha = 0.8$, $\beta = 1$, $\gamma = 1$, $\omega = 1.5$; (c): $t = 5$, $a_1 = 1$, $a_3 = 0$, $a_4 = 1$, $k_1 = 1$, $r_1 = 0.4$, $q_1 = 0.4$, $\alpha = 0.8$, $\beta = 1$, $\gamma = 1$, $\omega = 1.5$, and $a_2 = 0.1$ (solid line), $a_2 = 0.09$ (dotted line), $a_2 = 0.08$ (dashed line).

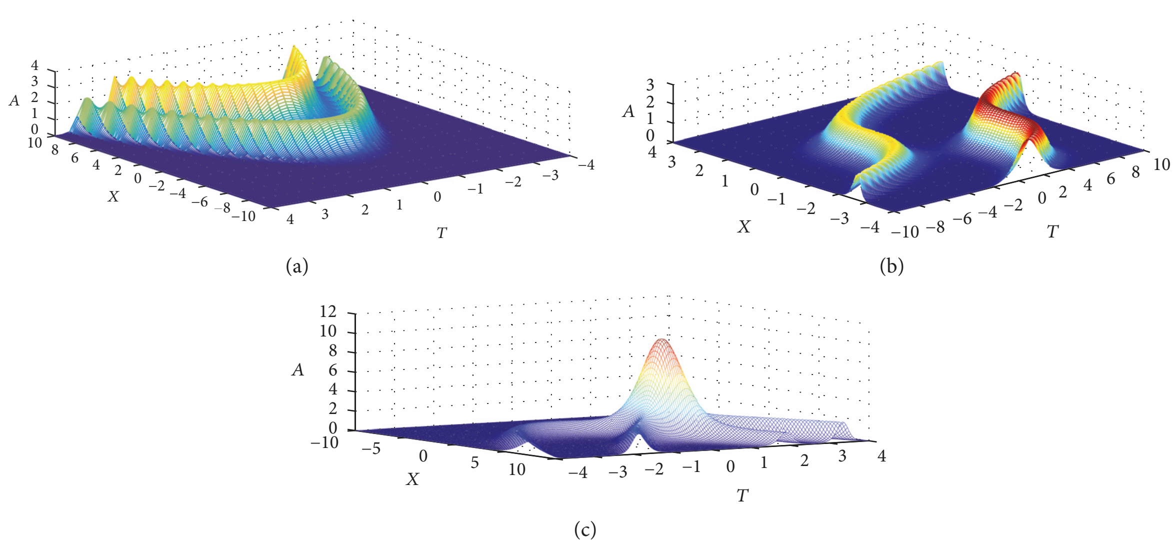

Figure 2: Interaction of the two solitons of ion-acoustic solitary waves with (a): $a_1 = 1$, $a_2 = 0.1$, $a_3 = 1$, $a_4 = 1$, $k_1 = 1.6$, $r_1 = 1$, $q_1 = 1$, $k_2 = -1.9$, $r_2 = 1$, $q_2 = 1$, $\alpha = 0.9$, $\beta = 0.8$, $\gamma = 0.9$, $\omega = 1.9$; (b): $a_1 = 1$, $a_2 = 0.1$, $a_3 = 1$, $a_4 = 1$, $k_1 = 1.6$, $r_1 = 1$, $q_1 = 1$, $k_2 = -1.9$, $r_2 = 1$, $q_2 = 1$, $\alpha = 0.7$, $\beta = 1$, $\gamma = 1$, $\omega = 2.8$; (c): $a_1 = 1$, $a_2 = 0.1$, $a_3 = 1$, $a_4 = 1$, $k_1 = 1.6$, $r_1 = 1$, $q_1 = 1$, $k_2 = -1.9$, $r_2 = 1$, $q_2 = 1$, $\alpha = 0.8$, $\beta = 1$, $\gamma = 1$, $\omega = 2.8$.

soliton solutions of the 3D TSF-Schamel-KdV equation. By theory and image analysis, the following conclusions can be obtained:

(1) Based on the basic system of equations and using multi-scale analysis and the perturbation method, we have obtained a new 3D Schamel-KdV equation. This equation is more suitable than other models for the study of ion-acoustic solitary waves. Furthermore, based on the new integer-order model and using the semi-inverse method and the fractional variational principle, we obtain the 3D TSF-Schamel-KdV equation. The fractional model opens the door to the study of ion-acoustic solitary waves.

(2) Using the Riemann-Liouville fractional derivative, we study the conservation laws of the 3D TSF-Schamel-KdV equation. Then, we discuss the soliton solutions of the new fractional model. Using the multi-soliton solutions, we study the characteristics of motion of ion-acoustic solitary waves.

Appendix

When $\omega \in (0,1)$ and $n = 1$, we can obtain the following components of the conserved vectors:

$$\begin{aligned}
C_i^T &= \tau I + (-1)\,{}_0^0D_T^{\omega-1}(W_i)\,D_T^0\frac{\partial \mathscr{L}}{\partial\left({}_0D_T^{\gamma}m\right)} - (-1)^1 \\
&\quad \cdot J\left(W_i, D_T^1\frac{\partial \mathscr{L}}{\partial\left({}_0D_T^{\omega}m\right)}\right) = {}_0D_T^{\omega-1}(W_i)\,s \\
&\quad + J\left(W_i, D_T^1 s\right), \\
C_i^X &= \xi I + W_i\left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}A} + D_X^{\alpha\alpha}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha\alpha\alpha}A}\right. \\
&\quad \left. + D_Y^{\beta\beta}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A} + D_Z^{\gamma\gamma}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right) - D_X^{\alpha}(W_i) \\
&\quad \cdot \left[D_X^{\alpha}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha\alpha\alpha}A}\right] - D_Y^{\beta}(W_i)\left[D_Y^{\beta}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right] \\
&\quad - D_Z^{\gamma}(W_i)\left[D_Z^{\gamma}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right] + D_X^{\alpha\alpha}(W_i) \\
&\quad \cdot \left[\frac{\partial \mathscr{L}}{\partial D_X^{\alpha\alpha\alpha}A}\right] + D_Y^{\beta\beta}(W_i)\left[\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right] \\
&\quad + D_Z^{\gamma\gamma}(W_i)\left[\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right], \\
C_i^Y &= \zeta I + W_i\left[D_X^{\alpha}\left(D_Y^{\beta}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right)\right] - D_Y^{\beta}(W_i) \\
&\quad \cdot \left[D_X^{\alpha}\left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right)\right] - D_X^{\alpha}(W_i) \\
&\quad \cdot \left[D_Y^{\beta}\left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right)\right] + D_X^{\alpha}D_Y^{\beta}(W_i) \\
&\quad \cdot \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A}\right), \\
C_i^Z &= \zeta I + W_i\left[D_X^{\alpha}\left(D_Z^{\gamma}\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right)\right] - D_Z^{\gamma}(W_i) \\
&\quad \cdot \left[D_X^{\alpha}\left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}m}\right)\right] - D_X^{\alpha}(W_i) \\
&\quad \cdot \left[D_Z^{\gamma}\left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right)\right] + D_X^{\alpha}D_Z^{\gamma}(W_i) \\
&\quad \cdot \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right),
\end{aligned} \tag{A.1}$$

When $i = 1$ and $W_1 = A_X$, we can obtain the following components of the conserved vectors:

$$\begin{aligned}
C_1^T &= {}_0D_T^{\omega-1}(A_X)\,s + J\left(A_X, D_T^1 s\right), \\
C_1^X &= A_X\left(a\sqrt{A}s + a_2D_X^{\alpha\alpha}s + a_3D_Y^{\beta\beta}s + a_4D_Z^{\gamma\gamma}s\right) \\
&\quad - a_2D_X^{\alpha}(A_X)\,D_X^{\alpha}s - a_3D_Y^{\beta}(A_X)\,D_Y^{\beta}s \\
&\quad - a_4D_Z^{\gamma}(A_X)\,D_Z^{\gamma}s + \left[D_X^{\alpha\alpha}(A_X) + D_Y^{\beta\beta}(A_X)\right. \\
&\quad \left. + D_Z^{\gamma\gamma}(A_X)\right]s, \\
C_1^Y &= a_3\left[A_XD_X^{\alpha}D_Y^{\beta}s - D_Y^{\beta}(A_X)\,D_X^{\alpha}s\right. \\
&\quad \left. - D_X^{\alpha}(A_X)\,D_Y^{\beta}s + D_X^{\alpha}D_Y^{\beta}(A_X)\,s\right], \\
C_1^Y &= a_4\left[A_XD_X^{\alpha}D_Z^{\gamma}s - D_Z^{\beta}(A_X)\,D_X^{\alpha}s\right. \\
&\quad \left. - D_X^{\alpha}(A_X)\,D_Z^{\gamma}s + D_X^{\alpha}D_Z^{\gamma}(A_X)\,s\right].
\end{aligned} \tag{A.2}$$

When $i = 2$ and $W_2 = A_Y$, we can obtain the following components of conserved vectors:

$$\begin{aligned}
C_2^Y &= {}_0D_T^{\omega-1}(A_Y)\,s + J\left(A_Y, D_T^1 s\right), \\
C_2^X &= A_Y\left(a\sqrt{A}s + a_2D_X^{\alpha\alpha}s + a_3D_Y^{\beta\beta}s + a_4D_Z^{\gamma\gamma}s\right) \\
&\quad - a_2D_X^{\alpha}(A_Y)\,D_X^{\alpha}s - a_3D_Y^{\beta}(A_Y)\,D_Y^{\beta}s \\
&\quad - a_4D_Z^{\gamma}(A_Y)\,D_Z^{\gamma}s + \left[D_X^{\alpha\alpha}(A_Y) + D_Y^{\beta\beta}(A_Y)\right. \\
&\quad \left. + D_Z^{\gamma\gamma}(A_Y)\right]s, \\
C_2^Y &= a_3\left[A_YD_X^{\alpha}D_Y^{\beta}s - D_Y^{\beta}(A_Y)\,D_X^{\alpha}s\right. \\
&\quad \left. - D_X^{\alpha}(A_Y)\,D_Y^{\beta}s + D_X^{\alpha}D_Y^{\beta}(A_Y)\,s\right], \\
C_2^Y &= a_4\left[A_YD_X^{\alpha}D_Z^{\gamma}s - D_Z^{\beta}(A_Y)\,D_X^{\alpha}s\right. \\
&\quad \left. - D_X^{\alpha}(A_Y)\,D_Z^{\gamma}s + D_X^{\alpha}D_Z^{\gamma}(A_Y)\,s\right].
\end{aligned} \tag{A.3}$$

When $i = 3$ and $W_2 = A_Z$, we can obtain the following components of the conserved vectors:

$$\begin{aligned}
C_3^T &= {}_0D_T^{\omega-1}(A_Z)s + J(A_Z, D_T^1 s),\\
C_3^X &= A_Z(a\sqrt{A}s + a_2D_X^{\alpha\alpha}s + a_3D_Y^{\beta\beta}s + a_4D_Z^{\gamma\gamma}s)\\
&\quad - a_2D_X^{\alpha}(A_Z)D_X^{\alpha}s - a_3D_Y^{\beta}(A_Z)D_Y^{\beta}s\\
&\quad - a_4D_Z^{\gamma}(A_Z)D_Z^{\gamma}s + [D_X^{\alpha\alpha}(A_Z) + D_Y^{\beta\beta}(A_Z)\\
&\quad + D_Z^{\gamma\gamma}(A_Z)]s, \qquad (A.4)\\
C_3^Y &= a_3[A_ZD_X^{\alpha}D_Y^{\beta}s - D_Y^{\beta}(A_Z)D_X^{\alpha}s\\
&\quad - D_X^{\alpha}(A_Z)D_Y^{\beta}s + D_X^{\alpha}D_Y^{\beta}(A_Z)s],\\
C_3^Z &= a_4[A_ZD_X^{\alpha}D_Z^{\gamma}s - D_Z^{\beta}(A_Z)D_X^{\alpha}s\\
&\quad - D_X^{\alpha}(A_Z)D_Z^{\gamma}s + D_X^{\alpha}D_Z^{\gamma}(A_Z)s].
\end{aligned}$$

When $i = 4$ and $W_4 = A_T$, we can obtain the following components of the conserved vectors:

$$\begin{aligned}
C_4^T &= {}_0D_T^{\omega-1}(A_T)s + J(A_T, D_T^1 s),\\
C_4^X &= A_T(a\sqrt{A}s + a_2D_X^{\alpha\alpha}s + a_3D_Y^{\beta\beta}s + a_4D_Z^{\gamma\gamma}s)\\
&\quad - a_2D_X^{\alpha}(A_T)D_X^{\alpha}s - a_3D_Y^{\beta}(A_T)D_Y^{\beta}s\\
&\quad - a_4D_Z^{\gamma}(A_T)D_Z^{\gamma}s + [D_X^{\alpha\alpha}(A_T) + D_Y^{\beta\beta}(A_T)\\
&\quad + D_Z^{\gamma\gamma}(A_T)]s, \qquad (A.5)\\
C_4^Y &= a_3[A_TD_X^{\alpha}D_Y^{\beta}s - D_T^{\beta}(A_X)D_X^{\alpha}s\\
&\quad - D_X^{\alpha}(A_T)D_Y^{\beta}s + D_X^{\alpha}D_T^{\beta}(A_X)s],\\
C_4^Z &= a_4[A_TD_X^{\alpha}D_Z^{\gamma}s - D_Z^{\beta}(A_T)D_X^{\alpha}s\\
&\quad - D_X^{\alpha}(A_T)D_Z^{\gamma}s + D_X^{\alpha}D_Z^{\gamma}(A_T)s].
\end{aligned}$$

When $i = 5$ and $W_5 = -\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y - \beta ZA_Z$, we can obtain the following components of the conserved vectors:

$$\begin{aligned}
C_5^T &= {}_0D_T^{\omega-1}(-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X\\
&\quad - \gamma YA_Y - \beta ZA_Z)s + J(-\alpha(3-\omega)A - \alpha TA_T\\
&\quad - \omega XA_X - \gamma YA_Y - \beta ZA_Z, D_T^1 s),\\
C_5^X &= (-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y\\
&\quad - \beta ZA_Z)(a\sqrt{A}s + a_2D_X^{\alpha\alpha}s + a_3D_Y^{\beta\beta}s + a_4D_Z^{\gamma\gamma}s)\\
&\quad - a_2D_X^{\alpha}(-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y\\
&\quad - \beta ZA_Z)D_X^{\alpha}s - a_3D_Y^{\beta}(-\alpha(3-\omega)A - \alpha TA_T\\
&\quad - \omega XA_X - \gamma YA_Y - \beta ZA_Z)\cdot D_Y^{\beta}s - a_4D_Z^{\gamma}(-\alpha(3\\
&\quad - \omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y - \beta ZA_Z)D_Z^{\gamma}s\\
&\quad + [D_X^{\alpha\alpha}\cdot(-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X\\
&\quad - \gamma YA_Y - \beta ZA_Z) + D_Y^{\beta\beta}(A_X) + D_Z^{\gamma\gamma}\\
&\quad \cdot(-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y\\
&\quad - \beta ZA_Z)]s,\\
C_5^Y &= a_3[(-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y\\
&\quad - \beta ZA_Z)D_X^{\alpha}D_Y^{\beta}s - D_Y^{\beta}\cdot(-\alpha(3-\omega)A - \alpha TA_T\\
&\quad - \omega XA_X - \gamma YA_Y - \beta ZA_Z)D_X^{\alpha}s\\
&\quad - D_X^{\alpha}[-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y\\
&\quad - \beta ZA_Z]D_Y^{\beta}s + D_X^{\alpha}D_Y^{\beta}(-\alpha(3-\omega)A - \alpha TA_T\\
&\quad - \omega XA_X - \gamma YA_Y - \beta ZA_Z)s],\\
C_5^Y &= a_4[(-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y\\
&\quad - \beta ZA_Z)D_X^{\alpha}D_Z^{\gamma}s - D_Z^{\beta}\cdot(-\alpha(3-\omega)A - \alpha TA_T\\
&\quad - \omega XA_X - \gamma YA_Y - \beta ZA_Z)D_X^{\alpha}s\\
&\quad - D_X^{\alpha}(-\alpha(3-\omega)A - \alpha TA_T - \omega XA_X - \gamma YA_Y\\
&\quad - \beta ZA_Z)D_Z^{\gamma}s + D_X^{\alpha}D_Z^{\gamma}(-\alpha(3-\omega)A - \alpha TA_T\\
&\quad - \omega XA_X - \gamma YA_Y - \beta ZA_Z)s].
\end{aligned} \qquad (A.6)$$

When $\omega \in (1,2)$ and $n = 2$, we can obtain the following components of the conserved vectors:

$$\begin{aligned}
C_i^T &= \tau I + (-1)\,{}_0^0D_T^{\omega-1}(W_i)D_T^0\frac{\partial\mathscr{L}}{\partial({}_0D_T^{\gamma}m)} - (-1)^1\\
&\quad \cdot J\left(W_i, D_T^1\frac{\partial\mathscr{L}}{\partial({}_0D_T^{\omega}m)}\right) + (-1)\,{}_0^1D_T^{\omega-2}(W_i)\\
&\quad \cdot D_T^1\frac{\partial L}{\partial_0 D_T^{\omega}} - (-1)^2 J\left(W_i, D_T^2\frac{\partial\mathscr{L}}{\partial({}_0D_T^{\omega}m)}\right)\\
&= {}_0D_T^{\omega-1}(W_i)s + J(W_i, D_T^1 s) - {}_0D_T^{\omega-1}(W_i)s_T\\
&\quad - J(W_i, D_T^2 s),\\
C_i^X &= \xi I + W_i\left(\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}A} + D_X^{\alpha\alpha}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha\alpha\alpha}A}\right.\\
&\quad \left. + D_Y^{\beta\beta}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Y^{\beta\beta}A} + D_Z^{\gamma\gamma}\frac{\partial\mathscr{L}}{\partial D_X^{\alpha}D_Z^{\gamma\gamma}A}\right) - D_X^{\alpha}(W_i)
\end{aligned}$$

$$\cdot \left[D_X^{\alpha} \frac{\partial \mathscr{L}}{\partial D_X^{\alpha\alpha\alpha} A} \right] - D_Y^{\beta}(W_i) \left[D_Y^{\beta} \frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Y^{\beta\beta} A} \right] - D_Z^{\gamma}(W_i) \left[D_Z^{\gamma} \frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Z^{\gamma\gamma} A} \right] + D_X^{\alpha\alpha}(W_i) \cdot \left[\frac{\partial \mathscr{L}}{\partial D_X^{\alpha\alpha\alpha} A} \right] + D_Y^{\beta\beta}(W_i) \left[\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Y^{\beta\beta} A} \right] + D_Z^{\gamma\gamma}(W_i) \left[\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Z^{\gamma\gamma} A} \right],$$

$$C_i^Y = \zeta I + W_i \left[D_X^{\alpha} \left(D_Y^{\beta} \frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Y^{\beta\beta} A} \right) \right] - D_Y^{\beta}(W_i) \cdot \left[D_X^{\alpha} \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Y^{\beta\beta} A} \right) \right] - D_X^{\alpha}(W_i) \cdot \left[D_Y^{\beta} \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Y^{\beta\beta} A} \right) \right] + D_X^{\alpha} D_Y^{\beta}(W_i) \cdot \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Y^{\beta\beta} A} \right),$$

$$C_i^Z = \zeta I + W_i \left[D_X^{\alpha} \left(D_Z^{\gamma} \frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Z^{\gamma\gamma} A} \right) \right] - D_Z^{\gamma}(W_i) \cdot \left[D_X^{\alpha} \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Z^{\gamma\gamma} m} \right) \right] - D_X^{\alpha}(W_i) \cdot \left[D_Z^{\gamma} \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Z^{\gamma\gamma} A} \right) \right] + D_X^{\alpha} D_Z^{\gamma}(W_i) \cdot \left(\frac{\partial \mathscr{L}}{\partial D_X^{\alpha} D_Z^{\gamma\gamma} A} \right). \tag{A.7}$$

When $i = 1$ and $W_1 = A_X$, we can obtain the following components of the conserved vectors:

$$C_1^T = {}_0D_T^{\omega-1}(A_X)\,s + J\left(A_X, D_T^1 s\right) - {}_0D_T^{\omega-1}(A_X) \cdot s_T - J\left(A_X, D_T^2 s\right),$$

$$C_1^X = A_X\left(a\sqrt{A}s + a_2 D_X^{\alpha\alpha} s + a_3 D_Y^{\beta\beta} s + a_4 D_Z^{\gamma\gamma} s\right) - a_2 D_X^{\alpha}(A_X)\, D_X^{\alpha} s - a_3 D_Y^{\beta}(A_X)\, D_Y^{\beta} s - a_4 D_Z^{\gamma}(A_X)\, D_Z^{\gamma} s + \left[D_X^{\alpha\alpha}(A_X) + D_Y^{\beta\beta}(A_X) + D_Z^{\gamma\gamma}(A_X)\right] s,$$

$$C_1^Y = a_3\left[A_X D_X^{\alpha} D_Y^{\beta} s - D_Y^{\beta}(A_X)\, D_X^{\alpha} s - D_X^{\alpha}(A_X)\, D_Y^{\beta} s + D_X^{\alpha} D_Y^{\beta}(A_X)\, s\right],$$

$$C_1^Y = a_4\left[A_X D_X^{\alpha} D_Z^{\gamma} s - D_Z^{\beta}(A_X)\, D_X^{\alpha} s - D_X^{\alpha}(A_X)\, D_Z^{\gamma} s + D_X^{\alpha} D_Z^{\gamma}(A_X)\, s\right]. \tag{A.8}$$

When $i = 2$ and $W_2 = A_Y$, we can obtain the following components of the conserved vectors:

$$C_2^Y = {}_0D_T^{\omega-1}(A_Y)\,s + J\left(A_Y, D_T^1 s\right) - {}_0D_T^{\omega-1}(A_Y)\,s_T - J\left(A_Y, D_T^2 s\right),$$

$$C_2^X = A_Y\left(a\sqrt{A}s + a_2 D_X^{\alpha\alpha} s + a_3 D_Y^{\beta\beta} s + a_4 D_Z^{\gamma\gamma} s\right) - a_2 D_X^{\alpha}(A_Y)\, D_X^{\alpha} s - a_3 D_Y^{\beta}(A_Y)\, D_Y^{\beta} s - a_4 D_Z^{\gamma}(A_Y)\, D_Z^{\gamma} s + \left[D_X^{\alpha\alpha}(A_Y) + D_Y^{\beta\beta}(A_Y) + D_Z^{\gamma\gamma}(A_Y)\right] s,$$

$$C_2^Y = a_3\left[A_Y D_X^{\alpha} D_Y^{\beta} s - D_Y^{\beta}(A_Y)\, D_X^{\alpha} s - D_X^{\alpha}(A_Y)\, D_Y^{\beta} s + D_X^{\alpha} D_Y^{\beta}(A_Y)\, s\right],$$

$$C_2^Y = a_4\left[A_Y D_X^{\alpha} D_Z^{\gamma} s - D_Z^{\beta}(A_Y)\, D_X^{\alpha} s - D_X^{\alpha}(A_Y)\, D_Z^{\gamma} s + D_X^{\alpha} D_Z^{\gamma}(A_Y)\, s\right]. \tag{A.9}$$

When $i = 3$ and $W_2 = A_Z$, we can obtain the following components of the conserved vectors:

$$C_3^T = {}_0D_T^{\omega-1}(A_Z)\,s + J\left(A_Z, D_T^1 s\right) - {}_0D_T^{\omega-1}(A_Z) \cdot s_T - J\left(A_Z, D_T^2 s\right),$$

$$C_3^X = A_Z\left(a\sqrt{A}s + a_2 D_X^{\alpha\alpha} s + a_3 D_Y^{\beta\beta} s + a_4 D_Z^{\gamma\gamma} s\right) - a_2 D_X^{\alpha}(A_Z)\, D_X^{\alpha} s - a_3 D_Y^{\beta}(A_Z)\, D_Y^{\beta} s - a_4 D_Z^{\gamma}(A_Z)\, D_Z^{\gamma} s + \left[D_X^{\alpha\alpha}(A_Z) + D_Y^{\beta\beta}(A_Z) + D_Z^{\gamma\gamma}(A_Z)\right] s,$$

$$C_3^Y = a_3\left[A_Z D_X^{\alpha} D_Y^{\beta} s - D_Y^{\beta}(A_Z)\, D_X^{\alpha} s - D_X^{\alpha}(A_Z)\, D_Y^{\beta} s + D_X^{\alpha} D_Y^{\beta}(A_Z)\, s\right],$$

$$C_3^Z = a_4\left[A_Z D_X^{\alpha} D_Z^{\gamma} s - D_Z^{\beta}(A_Z)\, D_X^{\alpha} s - D_X^{\alpha}(A_Z)\, D_Z^{\gamma} s + D_X^{\alpha} D_Z^{\gamma}(A_Z)\, s\right]. \tag{A.10}$$

When $i = 4$ and $W_4 = A_T$, we can obtain the following components of the conserved vectors:

$$C_4^T = {}_0D_T^{\omega-1}(A_T)\,s + J\left(A_T, D_T^1 s\right) - {}_0D_T^{\omega-1}(A_T)\,s_T - J\left(A_T, D_T^2 s\right),$$

$$
\begin{aligned}
C_4^X &= A_T\left(a\sqrt{A}s + a_2 D_X^{\alpha\alpha}s + a_3 D_Y^{\beta\beta}s + a_4 D_Z^{\gamma\gamma}s\right) \\
&\quad - a_2 D_X^{\alpha}\left(A_T\right) D_X^{\alpha}s - a_3 D_Y^{\beta}\left(A_T\right) D_Y^{\beta}s \\
&\quad - a_4 D_Z^{\gamma}\left(A_T\right) D_Z^{\gamma}s + \left[D_X^{\alpha\alpha}\left(A_T\right) + D_Y^{\beta\beta}\left(A_T\right)\right. \\
&\quad \left. + D_Z^{\gamma\gamma}\left(A_T\right)\right] s, \\
C_4^Y &= a_3\left[A_T D_X^{\alpha} D_Y^{\beta}s - D_T^{\beta}\left(A_X\right) D_X^{\alpha}s\right. \\
&\quad \left. - D_X^{\alpha}\left(A_T\right) D_Y^{\beta}s + D_X^{\alpha} D_T^{\beta}\left(A_X\right)s\right], \\
C_4^Y &= a_4\left[A_T D_X^{\alpha} D_Z^{\gamma}s - D_Z^{\beta}\left(A_T\right) D_X^{\alpha}s\right. \\
&\quad \left. - D_X^{\alpha}\left(A_T\right) D_Z^{\gamma}s + D_X^{\alpha} D_Z^{\gamma}\left(A_T\right)s\right].
\end{aligned}
\tag{A.11}
$$

When $i = 5$ and $W_5 = -\alpha(3-\omega)A - \alpha T A_T - \omega X A_X - \gamma Y A_Y - \beta Z A_Z$, we can obtain the following components of the conserved vectors:

$$
\begin{aligned}
C_5^T &= {}_0D_T^{\omega-1}\left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X\right. \\
&\quad \left. - \gamma Y A_Y - \beta Z A_Z\right)s + J\left(-\alpha(3-\omega)A - \alpha T A_T\right. \\
&\quad \left. - \omega X A_X - \gamma Y A_Y - \beta Z A_Z, D_T^1 s\right) \\
&\quad - {}_0D_T^{\omega-1}\left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X\right. \\
&\quad \left. - \gamma Y A_Y - \beta Z A_Z\right)s_T - J\left(-\alpha(3-\omega)A - \alpha T A_T\right. \\
&\quad \left. - \omega X A_X - \gamma Y A_Y - \beta Z A_Z, D_T^2 s\right), \\
C_5^X &= \left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X - \gamma Y A_Y\right. \\
&\quad \left. - \beta Z A_Z\right)\left(a\sqrt{A}s + a_2 D_X^{\alpha\alpha}s + a_3 D_Y^{\beta\beta}s + a_4 D_Z^{\gamma\gamma}s\right) \\
&\quad - a_2 D_X^{\alpha}\left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X\right. \\
&\quad \left. - \gamma Y A_Y - \beta Z A_Z\right) D_X^{\alpha}s - a_3 D_Y^{\beta}\left(-\alpha(3-\omega)A\right. \\
&\quad \left. - \alpha T A_T - \omega X A_X - \gamma Y A_Y - \beta Z A_Z\right)\cdot D_Y^{\beta}s \\
&\quad - a_4 D_Z^{\gamma}\left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X\right. \\
&\quad \left. - \gamma Y A_Y - \beta Z A_Z\right) D_Z^{\gamma}s + \left[D_X^{\alpha\alpha}\cdot\left(-\alpha(3-\omega)A\right.\right. \\
&\quad \left. - \alpha T A_T - \omega X A_X - \gamma Y A_Y - \beta Z A_Z\right) \\
&\quad + D_Y^{\beta\beta}\left(A_X\right) + D_Z^{\gamma\gamma}\cdot\left(-\alpha(3-\omega)A - \alpha T A_T\right. \\
&\quad \left.\left. - \omega X A_X - \gamma Y A_Y - \beta Z A_Z\right)\right] s, \\
C_5^Y &= a_3\left[\left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X - \gamma Y A_Y\right.\right. \\
&\quad \left. - \beta Z A_Z\right) D_X^{\alpha} D_Y^{\beta}s - D_Y^{\beta}\cdot\left(-\alpha(3-\omega)A - \alpha T A_T\right. \\
&\quad \left. - \omega X A_X - \gamma Y A_Y - \beta Z A_Z\right) D_X^{\alpha}s \\
&\quad - D_X^{\alpha}\left[-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X - \gamma Y A_Y\right. \\
&\quad \left. - \beta Z A_Z\right] D_Y^{\beta}s + D_X^{\alpha} D_Y^{\beta}\left(-\alpha(3-\omega)A - \alpha T A_T\right. \\
&\quad \left.\left. - \omega X A_X - \gamma Y A_Y - \beta Z A_Z\right)s\right], \\
C_5^Y &= a_4\left[\left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X - \gamma Y A_Y\right.\right. \\
&\quad \left. - \beta Z A_Z\right) D_X^{\alpha} D_Z^{\gamma}s - D_Z^{\beta}\cdot\left(-\alpha(3-\omega)A - \alpha T A_T\right. \\
&\quad \left. - \omega X A_X - \gamma Y A_Y - \beta Z A_Z\right) D_X^{\alpha}s \\
&\quad - D_X^{\alpha}\left(-\alpha(3-\omega)A - \alpha T A_T - \omega X A_X - \gamma Y A_Y\right. \\
&\quad \left. - \beta Z A_Z\right) D_Z^{\gamma}s + D_X^{\alpha} D_Z^{\gamma}\left(-\alpha(3-\omega)A - \alpha T A_T\right. \\
&\quad \left.\left. - \omega X A_X - \gamma Y A_Y - \beta Z A_Z\right)s\right].
\end{aligned}
\tag{A.12}
$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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