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FIXED-POINT EXTENSIONS OF FIRST-ORDER LOGIC

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We prove that the three extensions of first-order logic by means of positive inductions, monotone inductions, and so-called non-monotone (in our terminology, inflationary) inductions respectively, all have the same expressive power in the case of finite structures.

0. Introduction

In 1979 Aho and Ullman [3] noted that the relational calculus is unable to express the transitive closure of a given relation, and suggested extending the relational calculus by adding the least fixed-point operator. The relational calculus [25] is a standard relational query language; from the point of view of expressive power, the relational calculus is exactly first-order logic. Aho and Ullman's paper triggered an extensive study of the expressive power of fixed-point extensions of first-order logic [5, 15, 26, 17, 9, 4, etc.] with emphasis on finite structures.

There are two fields where fixed-point extension of first-order logic were extensively studied earlier. One is the theory of inductive definitions [1, 10, 13, 19, 20, 22, 24, etc]. The other is semantics of programming languages where a fixed-point extension of first-order logic is known as first-order μ-calculus [7, 14, 21, 23, etc]. But neither of the two fields put finite structures into the center of attention.

Proviso. All structures are finite unless the contrary is said explicitly.

Let us explain how fixed-point operators arise in the frame of first-order logic. A first-order formula $\varphi(P, \boldsymbol{x})$ with a distinguished predicate variable P and a

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distinguished sequence $\boldsymbol{x}$ of free individual variables yields an operator $F(P) = \{\boldsymbol{x} : \varphi(P, \boldsymbol{x})\}$. The formula φ may have additional free individual variables; they are viewed as parameters. If the arity of P equals the length of $\boldsymbol{x}$, then the operator F can be applied repetitively. If this operator is monotone, then it has a least (with respect to the inclusion relation) fixed point LFP(F) which is the union of the predicates $\emptyset$, $F(\emptyset)$, $F(F(\emptyset))$, etc., see Section 1.

For example, if Edge is a binary predicate constant, P is a binary predicate variable, and $\varphi(P, x, y)$ is the formula

$$\text{Edge}(x, y) \text{ or } \exists z\ [P(x, z) \text{ and } P(z, y)],$$

then LPF(F) is the transitive closure of Edge. If f is a binary function symbol, P is a unary predicate variable and $\varphi(P, x, u, v)$ is the formula

$$x = u \text{ or } x = v \text{ or } \exists y\ \exists z\ [P(y) \text{ and } P(z) \text{ and } x = f(y, z)],$$

then LFP(F) is the closure of the set $\{u, v\}$ under the operation f.

This suggests extending first-order logic by the following formation rule: if $\varphi(P, \boldsymbol{x})$ is a well-formed formula, arity(P) = length($\boldsymbol{x}$) and the operator $F(P) = \{\boldsymbol{x} : \varphi(P, \boldsymbol{x})\}$ is monotone (on all structures where it is defined), then $\text{LFP}_{P;\boldsymbol{x}}\varphi(P, \boldsymbol{x})$ is a well-formed predicate. This extension (let us call it FO + LFP′) does not form a nice logic because recognizing well-formed formulas is undecidable (whether infinite structures are allowed or not) [9]. Fortunately, there is a simply recognizable syntactic property which is a sufficient condition for monotonicity: if a first-order formula $\varphi(P, \boldsymbol{x})$ is positive in P, i.e., every occurrence of P in $\varphi(P, \boldsymbol{x})$ is positive, then the operator $F(P) = \{\boldsymbol{x} : \varphi(P, \boldsymbol{x})\}$ is monotone on every structure where it is defined. Using positively instead of monotonicity gives the most popular fixed-point extension FO + LFP of first-order logic, see details in Section 2. Neil Immerman announced [15] that every FO + LFP formula is equivalent to an FO + LFP formula with only one application of LFP.

The monotonicity of an operator $F(P) = \{\boldsymbol{x} : \varphi(P, \boldsymbol{x})\}$ ensures that the sequence $F^{\alpha}(\emptyset)$ increases and the union is a least fixed point of F. Call F *inductive* if the sequence $F^{\alpha}(\emptyset)$ increases. If F is inductive, then $\bigcup_{\alpha} F^{\alpha}(\emptyset)$ is a fixed point of F that will be called the *inductive fixed point* IFP(F) of F; the inductive fixed point may be not a least fixed point of F (an inductive F may have no least fixed point) but it is very natural from the computational point of view. Call an operator F *inflationary* if $\forall P\ [P \subseteq F(P)]$. The inflation property guarantees that F is inductive. Note that the operator $F'(P) = \{\boldsymbol{x}\text{: } P(\boldsymbol{x}) \text{ or } \varphi(P, \boldsymbol{x})\}$ is always inflationary, and if F is monotone, then IFP(F') = LFP(F). This suggests the following formation rule: if $\varphi(P, \boldsymbol{x})$ is a well-formed formula and arity(P) = length($\boldsymbol{x}$), then $\text{IFP}_{P;\boldsymbol{x}}[P(\boldsymbol{x}) \text{ or } \varphi(P, \boldsymbol{x})]$ is a well-formed predicate. The resulting extension of first-order logic will be called FO + IFP, see details in Section 2.

Obviously, FO < FO + LFP ≤ FO + LFP′ ≤ FO + IFP by expressive power. The expressive power of FO + LFP vastly exceeds the expressive power of first-order logic. On the other hand, every FO + IFP query is computable within time polynomial in the size of a given structure. In the presence of linear order, every polynomial time computable relational query is expressible in FO + LFP [15, 26]; hence in the case of finite structures with linear order, FO + LFP and FO + IFP have the same expressive power. In general, however, not every polynomial time computable query is expressible in FO + LFP [5] or even in FO + IFP [4]. (This general case is important computationally: a query may depend on the isomorphism types of structures rather than the presentations.)

Main Theorem (see Section 3). *For every* FO + LFP *formula* $\varphi(P, \boldsymbol{x})$ *with* arity(P) = length$(\boldsymbol{x})$ *there is an* FO + LFP *formula* $\varphi^*(\boldsymbol{x})$ *that expresses the inductive fixed point of the inflationary operator* $P \mapsto \{\boldsymbol{x}: P(\boldsymbol{x}) \text{ or } \varphi(P, \boldsymbol{x})\}$.

Corollary. FO + LFP, FO + LFP′ *and* FO + IFP *have the same expressive power.*

Dana Scott has asked whether the proof gives $\varphi^*(\boldsymbol{x})$ as a formula with a parameter φ. The answer is yes except the parameter is not $\varphi(P, \boldsymbol{x})$ itself but the formula $\Phi(P, P', \boldsymbol{x})$ obtained from $\varphi(P, \boldsymbol{x})$ by replacing the negative occurrences of P by the negation of a new predicate variable P' of the same arity. To make this answer apparent we have changed the exposition. A stronger theorem is proved in Section 3 which implies Main Theorem. A related result is proved in Appendix.

Even though the expressive power of FO + LFP equals that of FO + IFP, sometimes things are naturally expressible in FO + IFP but not in FO + LFP. For example, Tim Fernando, a student of Kechris, proved that every polynomial time recognizable class of finite groups with a fixed number of generators is definable in FO + IFP.

In connection to the Corollary let us mention Lyndon's Theorem: If $\varphi(P, \boldsymbol{x})$ is first-order and the operator $F(P) = \{\boldsymbol{x}: \varphi(P, \boldsymbol{x})\}$ is monotone on all — finite or infinite — structures where it is defined, then $\varphi(P, \boldsymbol{x})$ is logically equivalent to a first-order formula $\varphi'(P, \boldsymbol{x})$ that is positive in P. (Lyndon's Theorem does not require arity(P) = length$(\boldsymbol{x})$.) However, there is no total recursive function that constructs the desired φ' from the given φ [8, 9, 16] (though Lyndon's proof provides a partial recursive function for the purpose). In the case of finite structures Lyndon's Theorem fails [2].

The proof of Main theorem uses finiteness of structures. We did not investigate the infinite case but on some point we had an impression that the proof of a weaker version of Main Theorem does not use finiteness; Alekos Kechris and Phokion Kolaitis caught the error. After seeing a version of this paper Kechris sent us unpublished manuscripts [11, 12, 13] with related results in the infinite case. Alekos Kechris and Yiannis Moschovakis informed us that the following seems to be deducible from those manuscripts: the expressive power of FO + LFP equals to that of FO + LFP′ on all (necessarily infinite) structures, called acceptable in [19]; and the expressive power of FO + LFP′ equals to that of FO + IFP on all (finite or infinite) structures.

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Added during proof-reading. Meantime the authors made some additional progress, see Proceedings of the 26th Annual Symposium on Foundation of Computer Science, IEEE Computer Society Press, 1985, 346–353.

1. The least fixed point and the inductive fixed point

We start with recalling some well known definitions and facts. (The proviso of Section 0 is not in force in this section.) A partially ordered set A is *complete* if every subset of A has a supremum and an infimum in A. It suffices to request the existence of suprema: the supremum of the set of lower bounds for a set X is the infimum of X. In particular, a complete partially ordered set A has the least element $\inf(A) = \sup(\emptyset)$ and the greatest element $\sup(A) = \inf(\emptyset)$. Thus the real line is not complete but the usual extension of the real line by means of $+\infty$ and $-\infty$ is complete.

A function f from a partially ordered set A to a partially ordered set B is *monotone* if for all x, y in A, $x \leq y$ implies $fx \leq fy$. Let f be a function from a partially ordered set A to the same partially ordered set A; an element x of A is a *fixed point* of f if $fx = x$, and a fixed point x of f is a *least fixed point* of f if for every fixed point y of f, $x \leq y$. To indicate that x is the least fixed point of f, we write $x = \mathrm{LFP}(f)$. The following theorem is well known.

Theorem 1. *Let A be a complete partial ordered set with a least element Λ, and let $f: A \to A$. If f is monotone, then it has a least fixed point.*

Proof. By a transfinite induction define $f^{\alpha}(\Lambda) = \sup\{f^{\beta}(\Lambda): \beta < \alpha\}$. There is an ordinal μ such that $\alpha < \beta \leq \mu \to f^{\alpha}(\Lambda) < f^{\beta}(\Lambda)$ and $\mu \leq \alpha \to f^{\alpha}(\Lambda) = f^{\mu}(\Lambda)$. In particular, $f^{\mu}(\Lambda)$ is a fixed point of f. Given any fixed point y of f, prove by induction on α that $f^{\alpha}(\Lambda) \leq y$. Thus $f^{\mu}(\Lambda) = \mathrm{LFP}(f)$. □

Recall that the *direct product* $A \times B$ of partially ordered sets A, B is the direct product of their universes ordered componentwise: $(x, y) \leq (x', y') \leftrightarrow [x \leq x'$ and $y \leq y']$. The direct product of complete partially ordered sets A, B is complete: for every $Z \subseteq A \times B$,

$$\sup(Z) = (\sup\{x : \exists y\ ((x, y) \in Z)\}, (\sup\{y : \exists x\ ((x, y) \in Z\})).$$

In our applications of Theorem 1, a typical complete partially ordered set is the collection $\mathrm{Pred}_r(U)$ of all predicates of a given arity r on a given nonempty set U ordered by inclusion. Theorem 1 allows to define new predicates by induction. The next theorem reduces an induction in $\mathrm{Pred}_l(U) \times \mathrm{Pred}_r(U)$, satisfying a certain restriction, to an induction in $\mathrm{Pred}_{l+r}(U)$. To simplify notation, we identify pairs $((x_1, \ldots, x_l), (y_1, \ldots, y_r))$ with tuples $(x_1, \ldots, x_l, y_1, \ldots, y_r)$; this makes $\mathrm{Pred}_l(U) \times \mathrm{Pred}_r(U)$ a subset of $\mathrm{Pred}_{l+r}(U)$.

Theorem 2. *Suppose*
U is a nonempty set, and l, r are positive integers,
$L: \mathrm{Pred}_l(U) \times \mathrm{Pred}_r(U) \to \mathrm{Pred}_l(U)$ is monotone,
$R: \mathrm{Pred}_l(U) \times \mathrm{Pred}_r(U) \to \mathrm{Pred}_r(U)$ is monotone,
$F(X, Y) = (L(X, Y), R(X, Y))$ for all $X \in \mathrm{Pred}_l(U)$ and $Y \in \mathrm{Pred}_r(U)$,
$G: \mathrm{Pred}_{l+r}(U) \to \mathrm{Pred}_{l+r}(U)$, and for every $Z \in \mathrm{Pred}_{l+r}(U)$,
$G(Z) = L(X, Y) \times R(X, Y)$ where $X = \{\boldsymbol{x} : \exists \boldsymbol{y}\, ((\boldsymbol{x}, \boldsymbol{y}) \in Z)\}$, $Y = \{\boldsymbol{y} : \exists \boldsymbol{x}\, ((\boldsymbol{x}, \boldsymbol{y}) \in Z)\}$.
Then F and G are monotone and have least fixed points, and if $L(\emptyset, \emptyset) \neq \emptyset$, $R(\emptyset, \emptyset) \neq \emptyset$, and (X^*, Y^*) is the least fixed point of F, then $X^* \times Y^*$ is the least fixed point of G.

Proof. Clearly, F and G are monotone. By Theorem 1 they have least fixed points. Suppose that $L(\emptyset, \emptyset) \neq \emptyset$, $R(\emptyset, \emptyset) \neq \emptyset$, and (X^*, Y^*) is the least fixed point of F. Since (X^*, Y^*) is a fixed point of F, we have $L(X^*, Y^*) = X^*$ and $R(X^*, Y^*) = Y^*$. Hence $G(X^* \times Y^*) = L(X^*, Y^*) \times R(X^*, Y^*) = X^* \times Y^*$, i.e., $X^* \times Y^*$ is a fixed point of F.

It remains to prove that every fixed point Z of G includes $X^* \times Y^*$. Let $X = \{\boldsymbol{x} : \exists \boldsymbol{y}\, ((\boldsymbol{x}, \boldsymbol{y}) \in Z)\}$, $Y = \{\boldsymbol{y} : \exists \boldsymbol{x}\, ((\boldsymbol{x}, \boldsymbol{y}) \in Z)\}$. Then $Z = G(Z) = L(X, Y) \times R(X, Y)$. Note that $L(X, Y) \times R(X, Y)$ is not empty because it includes $L(\emptyset, \emptyset) \times R(\emptyset, \emptyset)$ which is not empty. Hence $L(X, Y) = \{\boldsymbol{x} : \exists \boldsymbol{y}\, ((\boldsymbol{x}, \boldsymbol{y}) \in Z)\} = X$ and $R(X, Y) = \{\boldsymbol{y} : \exists \boldsymbol{x}\, ((\boldsymbol{x}, \boldsymbol{y}) \in Z)\} = Y$. Thus (X, Y) is a fixed point of F. Then $(X^*, Y^*) \leqslant (X, Y)$ and $X^* \times Y^* \subseteq X \times Y = Z$. □

Coming back to the proof of Theorem 1, let us note that the elements $f^\alpha(\Lambda)$ are defined in the general case when f is not necessarily monotone. If they form an increasing sequence then their supremum is a fixed point of f.

Definition. Let A be a partially ordered set with a least element Λ, and let $f: A \to A$. By induction on ordinal α define $f^\alpha(\Lambda) = \sup\{f^\beta(\Lambda) : \beta < \alpha\}$. If the sequence $f^\alpha(\Lambda)$ is (non-strictly) increasing, i.e., if $\alpha < \beta \to f^\alpha(\Lambda) \leqslant f^\beta(\Lambda)$, then the function f is *inductive*. If f is inductive, then there is $\mu = \inf\{\alpha : f^{\alpha+1}(\Lambda) = f^\alpha(\Lambda)\}$ and $f^\mu(\Lambda)$ is a fixed point of f; $f^\mu(\Lambda)$ is the *inductive fixed point* IFP(f) of f.

Definition. Let A be a partially ordered set. A function $f: A \to A$ is *inflationary* if $fx \geqslant x$ for every $x \in A$.

Theorem 3. *Let A be an arbitrary complete partially ordered set.*

(a) *Every inflationary function from A to A is inductive.*

(b) *If f is an arbitrary function from A to A, then the function $f' = \sup\{x, fx\}$ is inflationary.*

(c) *If $f: A \to A$ is monotone and $f' = \sup\{x, fx\}$, then* $\mathrm{IFP}(f') = \mathrm{LFP}(f)$.

Proof is clear. □

Examples. Let $U = \{0, 1, 2\}$ and X range over $A = \mathrm{Pred}_1(U)$.

(i) Define $F(X) = X \cup \{\text{the cardinality of } X\}$ if $X \neq U$, and $F(U) = U$. Clearly, F is inflationary. $F^i(\emptyset) = \{j : j < i\}$ for $i \leqslant 3$, and $\mathrm{IFP}(F) = U$. However, F is not monotone: $\{0, 2\}$ includes $\{0\}$ but $F\{0, 2\} = \{0, 2\}$ does not include $F\{0\} = \{0, 1\}$. Moreover, F does not have a least fixed point: $\{1\}$ and $\{0, 2\}$ are fixed points of F but $\emptyset$ is not a fixed point of F.

(ii) Define $G(X) = F(X)$ if X is an initial segment of U, and $G(X) = \emptyset$ otherwise. Then G is inductive but neither inflationary nor monotone.

(iii) Any constant function $H: A \to A$ with $H(X) \neq U$ is monotone but not inflationary (this example was suggested by several people).

Remark. Our treatment of inductive fixed points follows [9] but the phenomena of Theorem 3 were well known much earlier by the name of non-monotone induction [20].

2. Two fixed-points logics

We describe in this section the extension FO + LFP of first-order logic by the least fixed point operator and the extension FO + IFP of first-order logic by the inductive fixed point operator. The proviso of Section 0 is not in force in this section. Our treatment follows [9]. For definiteness we deal with the version of first-order logic that allows free and bound occurrences of the same individual variable in the same formula, and uses substitution as a formation rule.

The syntax of logic FO + LFP is the result of augmenting the syntax of first-order logic by the following formation rule.

The LFP formation rule. Let r be a positive integer, $\boldsymbol{x}$ be an r-tuple $x_1, \ldots, x_r$ of individual variables, P be an r-ary predicate variable, and $\varphi(P, \boldsymbol{x})$ be a well-formed formula where all free occurrences of P are positive. Then $\mathrm{LFP}_{P;\boldsymbol{x}}\varphi(P, \boldsymbol{x})$ is a well-formed predicate, and $[\mathrm{LFP}_{P;\boldsymbol{x}}\varphi(P, \boldsymbol{x})](\boldsymbol{x})$ is a well-formed formula.

All occurrences of P and $x_1, \ldots, x_r$ in the new predicate are bounded; the occurrences of individual variables in the tail $(\boldsymbol{x})$ of the new formula are free. If Q is a predicate variable different from P, then every free (respectively, bound) occurrence of Q in $\varphi(P, \boldsymbol{x})$ remains free (respectively, bound) in the new predicate and the new formula, and every positive (respectively, negative) occurrence of Q in $\varphi(P, \boldsymbol{x})$ remains positive (respectively, negative) in the new predicate and the new formula. If y is an individual variable different from $x_1, \ldots, x_r$, then every free (respectively, bound) occurrence of y in $\varphi(P, \boldsymbol{x})$ remains free (respectively, bound) in the new predicate and the new formula.

Remark. We do not give a complete definition of well-forned predicates: one can easily avoid speaking about well-formed predicates altogether and speak only about well-formed formulas (as it is customary in first-order logic). However, the LFP formation rule creates a new predicate more naturally than a new formula. Note that a simplified notation $\text{LFP}_P\varphi(P, \boldsymbol{x})$ for the formula $[\text{LFP}_{P;\boldsymbol{x}}\varphi(P, \boldsymbol{x})](\boldsymbol{x})$ is deficient: just try to express the formula $[\text{LFP}_{P;\boldsymbol{x}}\varphi(P, x)](t)$ in the simplified notation.

To be on the safe side, let us emphasize that logic FO + LFP allows interleaving of the LFP formation rule with propositional connectives (including negation) and quantifiers; in particular, one can negate an LFP formula then use the LFP formation rule again, etc.

Definition. Let φ be an FO + LFP formula or predicate. An individual (respectively predicate) variable with free occurrences in φ is a *free individual* (respectively *predicate*) *variable* of φ. The *vocabulary* of φ consists of:
the individual constants and the free individual variables of φ, the predicate constants and the free predicate variables of φ, and the function symbols of φ.

The meaning of the predicate $\text{LFP}_{P;\boldsymbol{x}}\varphi(P, \boldsymbol{x})$ is the least fixed point of the operator $F(P) = \{\boldsymbol{x} : \varphi(P, \boldsymbol{x})\}$. This operator is defined in every structure M whose vocabulary (also called signature and similarity type) includes the vocabulary of $\varphi(P, \boldsymbol{x})$ without the predicate symbol P and the individual variables $\boldsymbol{x}$. Since the formula $\varphi(P, \boldsymbol{x})$ is positive in P, the operator F is monotone in M and therefore has a least fixed point in M.

Logic FO + LFP is closed under simultaneous induction, see Simultaneous Induction Lemma in [19]. A minor drawback of that Lemma is the use of individual constants. The following theorem will suffice for our purposes here.

Theorem 1. *Suppose that*
$\varphi(P, Q, \boldsymbol{x})$ *and* $\psi(P, Q, \boldsymbol{y})$ *are* FO + LFP *formulas,*
$\text{arity}(P) = \text{length}(\boldsymbol{x}) = l$ *and* $\text{arity}(Q) = \text{length}(\boldsymbol{y}) = r$,
P, Q have only positive occurrences in the two formulas,
F is the operator $(P, Q) \mapsto (\{\boldsymbol{x} : \varphi(P, Q, \boldsymbol{x})\}, \{\boldsymbol{y} : \psi(P, Q, \boldsymbol{y})\})$,
R is a predicate variable of arity $l + r$ that occurs neither in φ nor in ψ,
$\chi(R, \boldsymbol{x}, \boldsymbol{y}) = \varphi(\exists \boldsymbol{y}\, R(\_, \boldsymbol{y}), \exists \boldsymbol{x}\, R(\boldsymbol{x}, \_), \boldsymbol{x})\ \&\ \psi(\exists \boldsymbol{y}\, R(\_, \boldsymbol{y}), \exists \boldsymbol{x}\, R(\boldsymbol{x}, \_), \boldsymbol{y})$.

Then F is monotone and has a least fixed point (X^*, Y^*), *and the conjunction* $[\exists x\,(\varphi(\emptyset, \emptyset, x))$ *and* $\exists y\,(\psi(\emptyset, \emptyset, y))]$ *implies the equivalence* $x \in X^* \leftrightarrow \exists y\,[\mathrm{LFP}_{R;x,y}\chi(R, x, y)](x, y)$.

Proof. Use Theorem 2 in Section 1. □

Extending the usual terminology, an FO + LFP formula φ will be called *positive* in a predicate symbol P if every free occurrence of P in φ is positive. Since the formula $\varphi(P, x)$ in the LFP formation rule is required to be positive in P, the operator $F(P) = \{x : \varphi(P, x)\}$ is monotone and therefore has a least fixed point. As we have mentioned in the introduction, direct replacing positivity by monotonicity does not lead to a nice logic. Note, however, that the operator $F'(P) = \{x : P(x) \text{ or } \varphi(P, x)\}$ is always inflationary and therefore has an inductive fixed point. According to Theorem 3 in Section 1, if F is monotone, then $\mathrm{IFP}(F') = \mathrm{LFP}(F)$. This leads to a more liberal extension FO + IFP of first-order logic. The syntax of logic FO + IFP is the result of augmenting the syntax of first-order logic by the following formation rule.

The IFP formation rule. Let r be a positive integer, x be an r-tuple of individual variables, P be an r-ary predicate variable, and $\varphi(P, x)$ be an arbitrary well-formed formula. Then $\mathrm{IFP}_{P;x}(P(x))$ is a well-formed predicate, and $[\mathrm{IFP}_{P;x}(P(x) \text{ or } \varphi(P, x))](x)$ is a well-formed formula.

With respect to free versus bound occurrences as well as positive versus negative occurrences the IFP formation rule behaves exactly as the LFP formation rule. The definition of vocabulary remains valid for FO + IFP formulas.

The meaning of the predicate $\mathrm{IFP}_{P;x}(P(x) \text{ or } \varphi(P, x))$ is the inductive fixed point of the operator $F'(P) = \{x : P(x) \text{ or } \varphi(P, x)\}$.

3. Expressing the inductive fixed point

Extend first-order logic by means of a symbol Γ of an operator that, given two unary relations and an element, produces a boolean value; formulas of the extended logic will be called *pseudo first-order*. Γ is supposed to be monotone in predicate arguments. View Γ as a positive (in predicate arguments) operator. The notion of positivity is generalized to pseudo first-order formulas in the obvious way. Let P and P' be unary predicate variables. The sign ~ will denote both the negation and the complementation. Let $\varphi(P, x) = [P(x) \text{ or } \Gamma(P, {\sim}P, x)]$. The operator $F(P) = \{x : \varphi(P, x)\}$ is inflationary. We express the inductive fixed point of F as (essentially) a projection of the least fixed point of a monotone operator definable by a positive pseudo first-order formula. Then we present this result in a vector from that implies Main Theorem.

The proviso of Section 0 is in force: all structures are finite. For expositary purposes we choose a nonempty finite set U as our universe of discourse. For every natural number n, let $P_n = F^n(\emptyset)$; thus $P_0 = \emptyset$ and $P_{n+1} = F(P_n)$. The sequence P_n is (non-strictly) increasing. Let $m = \min\{n : P_n = P_{n+1}\}$; P_m is the inductive fixed point of F. In addition, let $P_\infty = U$. For every $x \in U$, let $\text{stage}(x) = \min\{n : x \in P_n\}$. Note that $\text{stage}(x) > 0$. Let $x \leqslant y$ abbreviate the conjunction $[x \in P_m$ and $\text{stage}(x) \leqslant \text{stage}(y)]$, and let $x < y$ abbreviate $\text{stage}(x) < \text{stage}(y)$. Note that $x \leqslant x \leftrightarrow x \in P_m$. We start with constructing an inductive operator G whose inductive fixed point is the relation $\leqslant$.

Lemma 1. (Stage Comparison Theorem, [19])

$$x \leqslant y \leftrightarrow \varphi(\{x' : x' < y\}, x), \qquad x < y \leftrightarrow \sim\varphi(\{y' : \sim x \leqslant y'\}, y), \quad \textit{and}$$
$$x \leqslant y \leftrightarrow \varphi(\{x' : \sim\varphi(\{y' : \sim x' \leqslant y'\}, y)\}, x).$$

Proof sketch. To check the first equivalence, consider separately the cases $\text{stage}(y) < \infty$ and $\text{stage}(y) = \infty$. To check the second equivalence, consider separately the cases $\text{stage}(x) < \infty$ and $\text{stage}(x) = \infty$. The third equivalence follows from the first two. We skip details because, formally speaking, the lemma will be not used. But in essence the lemma gives the desired G. □

Let Q and Q' be binary predicate variables,

$$\Delta(Q, Q', x', y) = Q'(x', y) \text{ or } \Gamma(\{y' : Q'(x', y')\}, \{y' : Q(x', y')\}, y), \text{ and}$$
$$\Delta'(Q, Q', x', y) = \sim\Delta(\sim Q', \sim Q, x', y).$$

Obviously, Δ and Δ' are positive in Q and Q',

$$\Delta(Q, \sim Q, x', y) \leftrightarrow \varphi(\{y' : \sim Q(x', y')\}, y), \quad \text{and}$$
$$\Delta'(Q, \sim Q, x', y) \leftrightarrow \sim\Delta(Q, \sim Q, x', y) \leftrightarrow \sim\varphi(\{y' : \sim Q(x', y')\}, y).$$

Let $\Psi(Q, Q', x, y) = \Delta'(Q, Q', x', y)$ or $\Gamma(\{x' : \Delta'(Q, Q', x', y)\}, \{x' : \Delta(Q, Q', x', y)\}, x)$, $\psi(Q, x, y) = \Psi(Q, \sim Q, x, y)$. Obviously, Ψ is positive in Q and Q'.

Lemma 2. $\psi(Q, x, y) \leftrightarrow \varphi(\{x' : \sim\varphi(\{y' : \sim Q(x', y')\}, y), x)$.

Proof.

$$\begin{aligned}
&\Psi(Q, \sim Q, x, y) = \Delta'(Q, \sim Q, x', y) \text{ or} \\
&\quad \Gamma(\{x' : \Delta'(Q, \sim Q, x', y)\}, \{x' : \Delta(Q, \sim Q, x', y)\}, x) \\
&\leftrightarrow \sim\Delta(Q, \sim Q, x', y) \text{ or} \\
&\quad \Gamma(\{x' : \sim\Delta(Q, \sim Q, x', y)\}, \{x' : \Delta(Q, \sim Q, x', y)\}, x) \\
&\leftrightarrow \varphi(\{x' : \sim\Delta(Q, \sim Q, x', y)\}, x) \\
&\leftrightarrow \varphi(\{x' : \sim\varphi(\{y' : \sim Q(x', y')\}, y)\}, x). \quad \square
\end{aligned}$$

Let $G(Q) = \{(x, y) : \psi(Q, x, y)\}$ and $Q_k = G^k(\emptyset)$.

Lemma 3. *For every natural number k,* $Q_k = \bigcup\{(P_i \times P_\beta): k \geq i \leq \beta\}$ *where* β *may be equal to* ∞.

Proof. By induction on k. The case $k = 0$ is clear. We suppose

$$Q_k = \bigcup\{(P_i \times P_\beta): k \geq i \leq \beta\}$$

and prove $Q_{k+1} = \bigcup\{(P_i \times P_\beta): k+1 \geq i \leq \beta)$.

First, we analyze the formula $\sim\varphi(\{y' : \sim Q_k(x', y')\}, y)$. If $i' = \text{stage}(x') \leq k$, then $\sim Q_k(x', y') \leftrightarrow \text{stage}(y') < i'$, $\{y' : \sim Q_k(x', y')\} = P_{i'-1}$, and $\sim\varphi(P_{i'-1}, y) \leftrightarrow \text{stage}(y) > i'$. If $i' = \text{stage}(x') > k$, then $\sim Q_k(x', y') \leftrightarrow \text{TRUE}$, $\{y' : \sim Q_k(x', y')\} = U$, and $\sim\varphi(U, y) \leftrightarrow \text{FALSE}$. Thus, $\sim\varphi(\{y' : \sim Q_k(x', y')\}, y) \leftrightarrow \text{stage}(y) > \text{stage}(x') \leq k$.

Second, let $\beta = \text{stage}(y)$. We have $\{x' : \sim\varphi(\{y' : \sim Q_k(x', y')\}, y)\} = \{x' : \beta > \text{stage}(x') \leq k\} = P_j$ where $j + 1 = \min\{\beta, k+1\}$.

Third, let $i = \text{stage}(x)$. Then $(x, y) \in Q_{k+1} \leftrightarrow \varphi(P_j, x) \leftrightarrow i \leq j+1 \leftrightarrow (i \leq \beta$ and $i \leq k+1) \leftrightarrow (x, y) \in \bigcup\{(P_i \times P_\beta): \beta \geq i \leq k+1\}$. □

Corollary 4. *The operator G is inductive,* Q_m *is the inductive fixed point of G, and the relation* $\leq$ *coincides with* Q_m.

Let R and S be ternary predicate variables. Let $\rho(R, S, x, u, v)$ be the pseudo first-order formula

$x \in P_1$ and $(u, v) \in Q_1$, or $R(x, u, v)$, or there is y such that $R(y, y, y)$,
$\Psi(R(y, \_, \_), S(y, \_, \_), u, v)$, $S(y, x, x)$, and
$\Psi(R(y, \_, \_), S(y, \_, \_), x, x)$.

Let $\sigma(R, S, x, u, v)$ be the pseudo first-order formula

$x \in P_1$ and $(u, v) \notin Q_1$, or $S(x, u, v)$, or there is y such that $R(y, y, y)$,
$\sim\Psi(\sim S(y, \_, \_), \sim R(y, \_, \_), u, v)$, $S(y, x, x)$, and
$\Psi(R(y, \_, \_), S(y, \_, \_), x, x)$.

Here the expressions $x \in P_1$ and $(u, v) \in Q_1$ abbreviate pseudo first-order formulas $\varphi(\emptyset, x)$ and $\psi(\emptyset, u, v)$ respectively. Obviously, ρ and σ are positive in R and S. Therefore the operator

$$H(R, S) = (\{(x, u, v) : \rho(R, S, x, u, v)\}, \{(x, u, v) : \sigma(R, S, x, u, v)\}).$$

is monotone and has a least fixed point.

Lemma 5. *The least fixed point of H is*

$$\left(\bigcup_{k<m} [(P_{k+1} - P_k) \times Q_{k+1}], \bigcup_{k<m} [(P_{k+1} - P_k) \times \sim Q_{k+1}]\right).$$

Proof. For each natural number k, let $(R_k, S_k) = H^k(\emptyset, \emptyset)$. It suffices to prove that

$$R_k = \bigcup_{i<k} [(P_{i+1} - P_i) \times Q_{i+1}] \quad \text{and} \quad S_k = \bigcup_{i<k} [(P_{i+1} - P_i) \times {\sim}Q_{i+1}].$$

The case $k = 0$ is clear. The case $k = 1$ is clear too: the formulas $\rho(R_0, S_0, x, u, v)$, $\sigma(R_0, S_0, x, u, v)$ are equivalent to their first disjuncts, and those disjuncts describe R_1, S_1 explicitly. Assuming that the claim is proved for $k \geqslant 1$, we prove that

$$\rho(R_k, S_k, x, u, v) \leftrightarrow (x, u, v) \in R' \quad \text{where} \quad R' = R_k \cup [(P_{k+1} - P_k) \times Q_{k+1}],$$

and

$$\sigma(R_k, S_k, x, u, v) \leftrightarrow (x, u, v) \in S' \quad \text{where} \quad S' = S_k \cup [(P_{k+1} - P_k) \times {\sim}Q_{k+1}].$$

First, suppose $(x, u, v) \in R'$ and check $\rho(R_k, S_k, x, u, v)$. The case $(x, u, v) \in R_k$ is clear. If $(x, u, v) \in [(P_{k+1} - P_k) \times Q_{k+1}]$ choose any $y \in P_k - P_{k-1}$. Note that $R_k(y, \_, \_) = Q_k$, $S_k(y, \_, \_) = {\sim}Q_k$, and $\Psi(Q_k, {\sim}Q_k, u, v) \leftrightarrow \psi(Q_k, u, v) \leftrightarrow (u, v) \in Q_{k+1}$. It is easy to see that all statements $R_k(y, y, y)$, $\Psi(R_k(y, \_, \_), S_k(y, \_, \_), u, v)$, $S_k(y, x, x)$, and $\Psi(R_k(y, \_, \_), S_k(y, \_, \_), x, x)$ are true.

Second, the implication $(x, u, v) \in S' \rightarrow \sigma(R_k, S_k, x, u, v)$ is proved similarly. Note that ${\sim}\Psi({\sim}S(y, \_, \_), {\sim}R(y, \_, \_), u, v) \leftrightarrow {\sim}\Psi(Q_k, {\sim}Q_k, u, v) \leftrightarrow {\sim}\psi(Q_k, u, v) \leftrightarrow (u, v) \notin Q_{k+1}$.

Third, suppose $\rho(R_k, S_k, x, u, v)$ and check that $(x, u, v) \in R'$. The first disjunct of $\rho(R_k, S_k, X, u, v)$ obviously implies $(x, u, v) \in R'$, and the second disjunct of $\rho(R_k, S_k, x, u, v)$ obviously implies $(x, u, v) \in R'$. Let y be a witness for the third disjunct of $\rho(R_k, S_k, x, u, v)$. Note that $R_k(y, y, y)$ implies that $y \in P_i - P_{i-1}$ for some positive $i \leqslant k$, $R_k(y, \_, \_) = Q_i$, and $S_k(y, \_, \_)$ equals the complement of Q_i. Hence $\Psi(Q_i, {\sim}Q_i, u, v) \leftrightarrow \psi(Q_i, u, v) \leftrightarrow (u, v) \in Q_{i+1}$, $S_k(y, x, x) \rightarrow x \notin P_i$, and $\Psi(Q_i, {\sim}Q_i, x, x) \leftrightarrow (x, x) \in Q_{i+1} \leftrightarrow x \in P_{i+1}$; thus $(x, u, v) \in R'$.

Fourth, the implication $\sigma(R_k, S_k, x, u, v) \rightarrow (x, u, v) \in S'$. Note that if $R_k(y, \_, \_) = Q_i$ and $S_k(y, \_, \_)$ equals the complement of Q_i, then

$${\sim}\Psi({\sim}S_k(y, \_, \_), {\sim}R_k(y, \_, \_), u, v) \leftrightarrow {\sim}\psi(Q_i, u, v) \leftrightarrow (u, v) \notin Q_{i+1}. \quad \square$$

Let T be a 6-ary predicate, and $\tau(T, x, y, z, u, v, w)$ be the conjunction of variable positive pseudo first-order formulas

$$\rho(\exists u\, \exists v\, \exists w\, T(\_, \_, \_, u, v, w), \exists x\, \exists y\, \exists z\, T(x, y, z, \_, \_, \_), x, y, z)$$

and

$$\sigma(\exists u\, \exists v\, \exists w\, T(\_, \_, \_, u, v, w), \exists x\, \exists y\, \exists z\, T(x, y, z, \_, \_, \_), u, v, w).$$

Let $\pi(\Gamma, x)$ be disjunction

$$\forall x\, \varphi(\emptyset, x) \text{ or } \exists u\, \exists v\, \exists w\, ([\mathrm{LFP}_{T;x,y,z,u,v,w} \tau(T, x, y, z, u, v, w)](x, x, x, u, v, w)).$$

Theorem 1. $x \in \mathrm{IFP}(F) \leftrightarrow \pi(\Gamma, x)\}$.

Proof. If $P_1 = U$, then the equivalence is obvious. We assume $P_1 \neq U$ and prove

$$x \in \mathrm{LFP}(F) \leftrightarrow \exists u\, \exists v\, \exists w\, ([\mathrm{LFP}_{T;x,y,z,u,v,w} \tau(T, x, y, z, u, v, w)](x, x, x, u, v, w)).$$

If $P_1 = \emptyset$, then $\mathrm{LFP}(F) = \emptyset$, $R_1 = \emptyset$ hence $\rho(\emptyset, \emptyset, x, y, z) \leftrightarrow \text{FALSE}$, $\tau(\emptyset, x, y, z, u, v, w) \leftrightarrow \text{FALSE}$, $\mathrm{LFP}_{T;x,y,z,u,v,w} \tau(T, x, y, z, u, v, w) = \emptyset$, and the equivalence is clear.

Suppose that $P_1 \neq \emptyset$. Then $R_1 \neq \emptyset$ and $S_1 \neq \emptyset$. Let H be as in Lemma 2, and let $(X^*, Y^*) = \mathrm{LFP}(H)$. Clearly, $(x, x, x) \in X^*$ if and only if $x \in \mathrm{LFP}(F)$. But by Theorem 1 in Section 2 (with R, S, ρ, σ, H, T, τ playing the roles of P, Q, φ, ψ, F, R, χ respectively) we have

$$(x, x, x) \in X^* \leftrightarrow \exists u\, \exists v\, \exists w\, ([\mathrm{LFP}_{T;x,y,z,u,v,w} \tau(T, x, y, z, u, v, w)](x, x, x, u, v, w)).$$

□

Theorem 2. *Let $\Phi(P, P', x)$ be an arbitrary* FO + LFP *formula (such that substituting $\Phi(P, P', x)$ for Γ in $\pi(\Gamma, x)$ does not cause a collision of variables) which is positive in P, P'. Then the* FO + LFP *formula $\pi(\Phi, x)$ expresses the inductive fixed point of the operator $P \mapsto \{x : P(x)$* or *$\Phi(P, \sim P, x)\}$.*

Proof. This is an immediate consequence of Theorem 1. □

Let r be a positive integer, suppose that U is the cartesian product of r copies of a set V, and consider V as the main universe. Then Γ is an operator that, given two r-ary relations and an r-tuple of elements, produces a boolean value. The predicate variables P, Q, R, S and T are respectively r-ary, $2r$-ary, $3r$-ary, $3r$-ary and $6r$-ary. Individual variables in the formula π are abbreviations for r-tuples of individual variables. This turns $\pi(\Gamma, x)$ into a statement about V and Γ. Theorem 1 remains true and implies

Theorem 3. *Let $\Phi(P, P', x)$ be an* FO + LFP *formula where P and P' are r-ary predicate variables and x is an r-tuple of individual variables. Suppose that $\Phi(P, P', x)$ is positive in P and P', and substituting $\Phi(P, P', x)$ for Γ in $\pi(\Gamma, x)$ does not cause a collision of variables. Then the* FO + LFP *formula $\pi(\Phi, x)$ expresses the inductive fixed point of the operator $P \mapsto \{x : P(x)$* or *$\Phi(P, \sim P, x)\}$.*

Theorem 3 implies Main Theorem.

Appendix: From inflationary to monotone

We redefine Γ and present the formula of main interest in a more direct way. In this section *pseudo first-order* formulas are formulas of the extension of first-order logic by a symbol Γ of an operator that, given one unary relation and one element, produces a boolean value. Let P be a unary predicate variable, $\varphi(P, x) = [P(x) \text{ or } \Gamma(P, x)]$, and $F(P) = \{x : \varphi(P, x)\}$. We express the inductive fixed point of F as the diagonal of the least fixed point of a monotone operator definable by a pseudo first-order formula.

Again the proviso of Section 0 is in force, and again we choose a nonempty finite set U as our universe of discourse. Let $P_i = F^i(\emptyset)$ for every natural number i, and let $m = \min\{i : P_i = P_{i+1}\}$; P_m is the inductive fixed point of F. Let $\text{stage}(x) = \min\{i : x \in P_i\}$ for $x \in P_m$, and let $x \leqslant y$ mean that $x \in P_m$, $y \in P_m$, and $\text{stage}(x) \leqslant \text{stage}(y)$. (Note that the relation $\leqslant$ is defined here somewhat differently than in Section 3.)

Definition. A unary relation P is *downward closed* with respect to a binary relation Q if for all elements x and y, $(x, y) \in Q$ and $y \in P$ imply $x \in P$.

Recall that a binary relation Q on a nonempty set S is called a linear (reflexive) quasi-order if it is reflexive, transitive, and for all elements x, y of S, either xQy or yQx. If Q is a linear quasi-order on S, then
the relation $E = \{(x, y) : xQy \text{ and } yQx\}$ is an equivalence relation on S, and
the relation $\{(A, B) : \forall x \in A\ \forall y \in B\ (xQy)\}$ on the equivalence classes of E is a linear order.

Let Q be a binary predicate variable, and $\text{Nice}(Q, x)$ be a pseudo first-order formula saying that

the restriction of Q to the set $\{u : uQx\}$ is a linear quasi-order, and
for every $v \in \{u : uQx\}$, $F(\{u : uQv \text{ and } \sim(vQu)\}) = \{u : uQv\}$.

If $\text{Nice}(Q, x)$ holds we say that x is *nice* with respect to Q.

Lemma 1. *If P_i is downward closed with respect to a binary relation Q and if $uQv \leftrightarrow u \leqslant v$ for all u, v in P_i, then every $x \in P_i$ is Q-nice.*

Proof. Is clear. □

Lemma 2. *If an element x is nice with respect to a binary relation Q, then there is a positive integer $k \leqslant m$ such that $x \in P_k$, and P_k is downward closed with respect to Q, and $uQv \leftrightarrow u \leqslant v$ for all $u, v \in P_k$.*

Proof. Let $S = \{u : uQx\}$, E *be the equivalence relation* $\{(u, v) : uQv \text{ and } vQu\}$ on S, and $A_1, A_2, \ldots, A_k$ be the equivalence classes of E ordered with respect to Q (so that if $i < j$, $u \in A_i$, $v \in A_j$ then uQv but not vQu).

It suffices to prove that for every i and every $v \in A_i$, $\bigcup_{j \leqslant i} A_i = P_i$. If $i = 1$, then $A_1 = \{u : uQv\} = F(\{u : uQv \text{ and } \sim(vQu)\}) = F(\emptyset) = P_1$. Let $i > 1$, $v \in A_i$ and

$w \in A_{i-1}$. Then $\bigcup_{j \leqslant i} A_i = \{u : uQv\} = F(\{u : uQv$ and $\sim(vQu)\}) = F(\{u : uQw\}) = F(\bigcup_{j<i} A_j)$. By the induction hypothesis, $F(\bigcup_{j<i} A_j) = F(P_{i-1}) = P_i$. □

Corollary 3. *For every binary relation Q there is a natural number $i \leqslant m$ such that P_i is exactly the set of Q-nice elements, P_i is downward closed with respect to Q, and the restriction of Q to P_i coincides with the restriction of the relation $\leqslant$ to P_i.*

Let $\psi(Q, x, y)$ be a pseudo first-order formula saying the following where $S = \{x : \text{Nice}(Q, x)\}$ and $G(Q) = \{(x, y) : \psi(Q, x, y)\}$:
If $F(S) = S$, then $G(Q) = Q$,
Else, if $F(S)$ is not downward closed with respect to Q, then $G(Q) = U \times U$,
Else, $G(Q) = Q \cup [F(S) \times (F(S) - S)]$.

.emma 4. *The operator G is inflationary and monotone.*

'roof. The first statement is clear. Let $Q \subseteq Q'$. By Corollary 3, there are natural ıumbers $i, j \leqslant m$ such that $\{x : x \text{ is } Q\text{-nice}\} = P_i$ and $\{x : x \text{ is } Q'\text{-nice}\} = P_j$.
First suppose $i < j$. Then $F(P_i) = P_{i+1} \neq P_i$. Since $P_{i+1} \subseteq P_j$, P_{i+1} is downward losed with respect to Q', hence it is downward closed with respect to Q. Thus, $\tilde{}(Q) = Q \cup [P_{i+1} \times (P_{i+1} - P_i)]$ and therefore $G(Q)$ is included into the restric- ion of $\leqslant$ to P_j which is the restriction of Q' to P_j. Hence $G(Q) \subseteq Q' \subseteq G(Q')$.
Second suppose $j < i$. The restriction of Q' to P_{j+1} coincides with the restriction f Q to P_{j+1}; for, otherwise $xQ'y$ for some $x \in P_{j+1} - P_j$ and $y \in P_j$, which ontradicts the fact that P_j is downward closed with respect to Q'. If P_{j+1} is ownward closed with respect to Q', then every x in P_{j+1} is Q'-nice which is not ıe case. But $P_{j+1} = F(P_j) \neq P_j$. Hence $G(Q') = U \times U$ and $G(Q) \subseteq G(Q')$.
Third suppose $i = j$. If $F(P_i) = P_i$, then $G(Q) = Q \subseteq Q' \subseteq G(Q')$. Suppose that '$(P_i)$ properly includes P_i. If $F(P_i)$ is not downward closed with respect to Q', ıen $G(Q) \subseteq U \times U = G(Q')$. Suppose that $F(P_i)$ is downward closed with :spect to Q'. Then it is downward closed with respect to Q, and

$$G(Q) = Q \cup [F(P_i) \times (F(P_i) - P_i)]$$
$$\subseteq Q' \cup [F(P_i) \times (F(P_i) - P_i)] = G(Q'). \qquad \square$$

emma 5. *The relation $\leqslant$ is a least fixed point of G.*

roof. Obviously, the set of elements nice with respect to $\leqslant$, equals P_m. Since $(P_m) = P_m$, $G(\leqslant)$ coincides with $\leqslant$, i.e. $\leqslant$ is a fixed point of G.
For every natural number i, let $Q_i = G^i(\emptyset)$. By induction on i we prove that Q_i the restriction of $\leqslant$ to P_i. As a result, the relation $\leqslant$ coincide with Q_m and erefore is a least fixed point of G.
The case $i = 0$ is trivial. Suppose that $i < m$ and Q_i is the restriction of $\leqslant$ to P_i.

Obviously, the set of Q_i-nice elements equals P_i. Since $F(P_i) = P_{i+1} \notin P_i$ and P_{i+1} is downward closed with respect to Q_i, $G(Q_i) = Q_i \cup [P_{i+1} \times (P_{i+1} - P_i)]$ which is the restriction of $\leqslant$ to P_{i+1}. □

Let $\pi(\Gamma, x) = [\mathrm{LFP}_{Q;x,y}\psi(Q, x, y)](x, y)$.

Theorem 1. $x \in \mathrm{IFP}(F) \leftrightarrow \pi(\Gamma, x)\}$.

Proof. This is a consequence of Lemma 5 and the equivalence $x \in \mathrm{IFP}(F) \leftrightarrow x \leqslant x$. □

Theorem 2. *Let $\Phi(P, x)$ be an arbitrary* FO + LFP *formula (such that substituting $\Phi(P, x)$ for Γ in $\pi(\Gamma, x)$ does not cause a collision of variables). Then the* FO + LFP *formula $\pi(\Phi, x)$ expresses the inductive fixed point of the operator $P \mapsto \{x : P(x)$ or $\Phi(P, x)\}$.*

Theorem 2 is an obvious consequence on Theorem 1. It can be generalized in the same way that Theorem 2 of Section 3 was generalized in Section 3.

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