The modal logic of Bayesian belief revision

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Abstract

In Bayesian belief revision a Bayesian agent revises his prior belief by conditionalizing the prior on some evidence using Bayes' rule. We define a hierarchy of modal logics that capture the logical features of Bayesian belief revision. Elements in the hierarchy are distinguished by the cardinality of the set of elementary propositions on which the agent's prior is defined. The containment relations among the modal logics in the hierarchy are determined. By linking the modal logics in the hierarchy to Medvedev's logic of finite problems and to Skvortsov's logic of infinite problems it is shown that the modal logic of Belief revision determined by probabilities on a finite set of elementary propositions is not finitely axiomatizable.

Keywords: Modal logic, Bayesian inference, Bayes learning, Bayes logic, Medvedev frames.

1 Introduction and overview

Let $(X, \mathcal{B}, p)$ be a classical probability measure space with $\mathcal{B}$ a Boolean algebra of subsets of set X and p a probability measure on $\mathcal{B}$. In Bayesian belief revision elements in $\mathcal{B}$ stand for the propositions that an agent regards as possible statements about the world, and the probability measure p represents an agent's prior degree of belief in the truth of these propositions. Learning proposition A in $\mathcal{B}$ to be true, the agent revises his prior p on the basis of this evidence and replaces p with $q(\cdot) = p(\cdot \mid A)$, where $p(\cdot \mid A)$ is the conditional probability given by Bayes' rule:

$$p(B \mid A) \doteq \frac{p(B \cap A)}{p(A)} \qquad \forall B \in \mathcal{B} \tag{1}$$

This new probability measure q can be regarded as the probability measure that the agent *infers* from p on the basis of the information (evidence) that A is true. The aim of this paper is to study the logical aspects of this type of inference from the perspective of modal logic.

Why *modal* logic? We will see in section 2 that it is very natural to regard the move from p to q in terms of modal logic: The core idea is to view A in the Bayes' rule (1) as a variable and say that a probability measure q *can* be inferred from p if there exits an A in $\mathcal{B}$ such that

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$q(\cdot) = p(\cdot \mid A)$. Equivalently, we will say in this situation that "q *can* be (Bayes) learned from p". That "it is *possible* to obtain/learn q from p" is clearly a modal talk and calls for a logical modeling in terms of concepts of modal logic.

Bayesian belief revision is just a particular type of belief revision: Various rules replacing the Bayes's rule have been considered in the context of belief change (e.g. Jeffrey conditionalization, maxent principle; see [20] and [6]), and there is a huge literature on other types of belief revision as well. Without completeness we mention: the AGM postulates in the seminal work of Alchurrón–Gärdenfors–Makinson [1]; the dynamic epistemic logic [19]; van Benthem's dynamic logic for belief revision [18]; probabilistic logics, e.g. Nilsson [14]; and probabilistic belief logics [2]. For an extensive overview we refer to Gärdenfors [7]. Typically, in this literature beliefs are modeled by sets of formulas defined by the syntax of a given logic and axioms about modalities are intended to prescribe how a belief represented by a formula should be modified when new information and evidence are provided.

Viewed from the perspective of such theories of belief revision our intention in this paper is very different: Rather than trying to give a plausible set of axioms intended to capture desired features of statistical inference we take the standard Bayes model and we aim at an in-depth study of this model from a purely logical perspective. Our investigation is motivated by two observations. First, the logical properties of this type of belief change do not seem to have been studied in terms of the modal logic that we see emerging naturally in connection with Bayesian belief revision. Second, Bayesian probabilistic inference is relevant not only for belief change: Bayesian conditionalization is the typical and widely applied inference rule also in situations where probability is interpreted not as subjective degree of belief but as representing objective matters of fact. Finding out the logical properties of this type of probabilistic inference has thus a wide interest going way beyond the confines of belief revision.

The structure of the paper is the following. After some motivation, in section 2 the modal logic of Bayesian probabilistic inference (we call it "Bayes logic") is defined in terms of possible world semantics. The set of possible worlds will be the set of all probability measures on a measurable space $(X, \mathcal{B})$. The accessibility relation among probability measures will be the "Bayes accessibility" relation, which expresses that the probability measure q is accessible from p if $q(\cdot) = p(\cdot \mid A)$ for some A (Definition 2.1). We will see that probability measures on $(X, \mathcal{B})$ with X having different cardinalities determine different Bayes logics. The containment relation of these Bayes logics is clarified by Theorem 4.1 in section 4: the different Bayes logics are all comparable, and the larger the cardinality of X, the smaller the logic. The standard modal logical features of the Bayes logics are determined in section 3 (see Proposition 3.1). In section 5 we establish a connection between Bayes logics and Medvedev's logic of finite problems [13]. We will prove (Theorem 5.2) that the Bayes logic determined by the set of probability measures over $(X, \mathcal{B})$ with a finite or countable X coincides with Medvedev's logic. This entails that the Bayes logic determined by a probability space on a *finite* X (hence with finite Boolean algebra $\mathcal{B}$) is *not* finitely axiomatizable (Proposition 5.9). This result is clearly significant because it indicates that axiomatic approaches to belief revision might be severely limited. It remains an open question whether general Bayes logics are finitely axiomatizable (Problem 5.10). Section 6 indicates future directions of research.

2 Motivation and basic definitions

Let $\langle X, \mathcal{B}\rangle$ be a measurable space and consider statements such as

$$\phi \doteq \text{“the probability of } A \text{ is at least } 1/4 \text{ and at most } 1/2\text{”} \tag{2}$$

$$\psi \doteq \text{“the probability of } B \text{ is } 1/7\text{”} \tag{3}$$

where A and B are in $\mathcal{B}$. Truth-values of propositions ϕ and ψ can be meaningfully defined with respect to a probability measure p on $\mathcal{B}$:

$$p \Vdash \phi \quad \text{if and only if} \quad 1/4 \leq p(A) \leq 1/2 \tag{4}$$

$$p \Vdash \psi \quad \text{if and only if} \quad p(B) = 1/7 \tag{5}$$

Consider now a statement χ such as

$$\chi \doteq \text{“it \textit{can be learned} that the probability of } A \text{ is at least } 1/4 \text{ and at most } 1/2\text{”} \tag{6}$$

$$= \text{“it \textit{can be learned} that } \phi\text{”} \tag{7}$$

In view of the interpretation of Bayes' rule formulated in the Introduction, it is very natural to define χ to be true at probability p if there is a B in $\mathcal{B}$ such that the conditional probability $q(\cdot) \doteq p(\cdot \mid B)$ makes true the proposition

$$\phi = \text{“the probability of } A \text{ is at least } 1/4 \text{ and at most } 1/2\text{”} \tag{8}$$

where true is understood in the sense of (4); i.e. if for some $B \in \mathcal{B}$ we have

$$1/4 \leq q(A) = p(A \mid B) \leq 1/2 \tag{9}$$

Propositions such as χ in (6)-(7) are obviously of modal character and it is thus very natural to express this modality formally using the modal operator $\Diamond$ by writing the sentence χ as $\Diamond\phi$. In view of (7) the reading of $\Diamond\phi$ is "ϕ can be learned in a Bayesian manner".

Thus we model Bayesian learning by specifying a standard unimodal language given by the grammar

$$a \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \tag{10}$$

defining formulas φ, where a belongs to a nonempty countable set Φ of propositional letters. As usual $\Box$ abbreviates $\neg\Diamond\neg$. (We refer to the books [3, 5] concerning basic notions in modal logic).

Models of such a language are tuples $\mathfrak{M} = \langle W, R, [\,\cdot\,]\rangle$ based on frames $\mathcal{F} = \langle W, R\rangle$, where W is a non-empty set, R a binary relation on W and $[\,\cdot\,] : \Phi \to \wp(W)$ is an evaluation of propositional letters. Truth of a formula φ at world w is defined in the usual way

- $\mathfrak{M}, w \Vdash a \iff w \in [\,a\,]$ for propositional letters $a \in \Phi$.
- $\mathfrak{M}, w \Vdash \varphi \wedge \psi \iff \mathfrak{M}, w \Vdash \varphi$ AND $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \neg\varphi \iff \mathfrak{M}, w \nVdash \varphi$.
- $\mathfrak{M}, w \Vdash \Diamond\varphi \iff$ there is v such that wRv and $\mathfrak{M}, v \Vdash \varphi$.

By definition a formula is valid over a frame if and only if it is true at every point in every model based on the frame. For a class C of frames the modal logic of C is the set of all modal formulas that are valid on every frame in C:

$$\Lambda(\mathsf{C}) = \big\{\phi : \ (\forall \mathcal{F} \in \mathsf{C})\ \mathcal{F} \Vdash \phi\big\} \tag{11}$$

We denote by $M(X,\mathcal{B})$ the set of all probability measures over $\langle X,\mathcal{B}\rangle$. Note that $M(X,\mathcal{B})$ is non-empty as the Dirac measures δ_x for $x \in X$ always belong to $M(X,\mathcal{B})$.

For a fixed $\langle X,\mathcal{B}\rangle$ the set of possible worlds W is defined to be the set of probability measures $M(X,\mathcal{B})$. Consider again the sentences

$$\phi \ \doteq\ \text{“the probability of } A \text{ is at least } 1/4 \text{ and at most } 1/2\text{”} \tag{12}$$

$$\psi \ \doteq\ \text{“the probability of } B \text{ is } 1/7\text{”} \tag{13}$$

The core idea of the semantic of the introduced modal language describing Bayesian statistical inference is the following:

- The intended interpretation of ϕ and ψ are the sets

$$[\,\phi\,] \ = \ \big\{p \in M(X,\mathcal{B}) : \ 1/4 \le p(A) \le 1/2\big\} \tag{14}$$

$$[\,\psi\,] \ = \ \big\{p \in M(X,\mathcal{B}) : \ p(B) = 1/7\big\} \tag{15}$$

- The intended interpretation of $\Diamond\phi$ is that “ϕ can be learned in a Bayesian manner”:

$$[\,\Diamond\phi\,] \ = \ \big\{p \in M(X,\mathcal{B}) : \text{ there is } A \in \mathcal{B} \text{ such that } p(\cdot \mid A) \Vdash \phi\ \big\} \tag{16}$$

This intended interpretation suggests the following definition of the accessibility relation R on $W = M(X,\mathcal{B})$:

Definition 2.1. For $v, w \in M(X,\mathcal{B})$ we say that w is *Bayes accessible* from v if there is an $A \in \mathcal{B}$ such that $w(\cdot) = v(\ \cdot \mid A)$. We denote the Bayes accessibility relation on $W = M(X,\mathcal{B})$ by $R(X,\mathcal{B})$.

We are now in the position to give the definition of one of the central concepts of this paper.

Definition 2.2 (Bayes frames)**.** A Bayes frame is a frame $\langle W, R\rangle$ that is isomorphic, as a directed graph, to $\mathcal{F}(X,\mathcal{B}) = \langle M(X,\mathcal{B}), R(X,\mathcal{B})\rangle$ for a measurable space $\langle X,\mathcal{B}\rangle$. $\Box$

A Bayes model is a model $\mathfrak{M} = \langle M(X,\mathcal{B}),\ R(X,\mathcal{B}),\ [\,\cdot\,]\rangle$ based on a Bayes frame $\mathcal{F}(X,\mathcal{B})$. The modal logic $\Lambda(\mathcal{F}(X,\mathcal{B}))$ corresponds then to the set of laws of Bayesian learning based on the frame $\mathcal{F}(X,\mathcal{B})$. The *general laws of Bayesian learning* independent of the particular representation $\langle X,\mathcal{B}\rangle$ of the events is then the modal logic

$$\mathbf{BL} = \{\phi : \ (\forall \text{ Bayes frames } \mathcal{F})\ \mathcal{F} \Vdash \phi\} \tag{17}$$

From the point of view of applications the most important classes of Bayes frames $\mathcal{F}(X,\mathcal{B})$ are Bayes frames determined by measurable spaces $\langle X,\mathcal{B}\rangle$ having a finite or a countable X. We will see that finiteness of X serves as a dividing line when defining the logic of Bayes frames.

We make use of the following notation

$$\mathcal{F}_n = \langle M(X, \wp(X)),\ R(X, \wp(X))\rangle, \quad \text{where } X = \{1, \ldots, n\} \tag{18}$$

$$\mathcal{F}_\omega = \langle M(X, \wp(X)),\ R(X, \wp(X))\rangle, \quad \text{where } X = \mathbb{N} \tag{19}$$

Note that if the measurable space $\langle X, \mathcal{B}\rangle$ is finite or countable, then $\mathcal{B}$ is the powerset algebra $\wp(X)$ (because we rely on the convention that elementary events $\{x\}$ for $x \in X$ always belong to the algebra $\mathcal{B}$).

We denote by BF the class of all Bayes frames. BF_n denotes the class of Bayes frames isomorphic to $\mathcal{F}_n$, while $\mathsf{BF}_{<\omega}$ contains those Bayes frames that are isomorphic to $\mathcal{F}_k$ for some natural number $k > 0$. Elements of BF_ω are isomorphic to $\mathcal{F}_\omega$.

Definition 2.3 (Bayes logics)**.** We define a family of normal modal logics based on finite or countable or countably infinite or all Bayes frames as follows.

$$\mathbf{BL}_n = \{\phi :\ \mathcal{F}_n \Vdash \phi\} = \Lambda(\mathsf{BF}_n) \tag{20}$$

$$\mathbf{BL}_{<\omega} = \{\phi :\ (\forall n \in \mathbb{N})\mathcal{F}_n \Vdash \phi\} = \Lambda(\mathsf{BF}_{<\omega}) = \bigcap_n \Lambda(\mathsf{BF}_n) \tag{21}$$

$$\mathbf{BL}_\omega = \{\phi :\ \mathcal{F}_\omega \Vdash \phi\} = \Lambda(\mathsf{BF}_\omega) \tag{22}$$

$$\mathbf{BL}_{\leq\omega} = \Lambda(\mathsf{BF}_{<\omega} \cup \mathsf{BF}_\omega) = \mathbf{BL}_{<\omega} \cap \mathbf{BL}_\omega \tag{23}$$

$$\mathbf{BL} = \{\phi :\ (\forall \text{ Bayes frames } \mathcal{F})\ \mathcal{F} \Vdash \phi\} = \Lambda(\mathsf{BF}) \tag{24}$$

We call $\mathbf{BL}_{<\omega}$ (resp. $\mathbf{BL}_{\leq\omega}$) the logic of finite (resp. countable) Bayes frames; however, observe that the set of possible worlds $M(X, \mathcal{B})$ of a Bayes frame $\mathcal{F}(X, \mathcal{B})$ is finite if and only if X is a one-element set, otherwise it is at least of cardinality continuum. $\square$

One can easily check the containments

$$\mathbf{BL} \subseteq \mathbf{BL}_{\leq\omega} \subseteq \mathbf{BL}_\omega \quad \text{and} \quad \mathbf{BL} \subseteq \mathbf{BL}_{<\omega} \subseteq \mathbf{BL}_n \tag{25}$$

using the very definition of Bayes logics.

3 Modal principles of Bayes learning

In this section we discuss the connections of Bayes logic to a list of modal axioms that are usually considered in the literature. Let us recall some of the standard frame properties (cf. [3] and [5]).

Basic frame properties		
Name	Formula	Corresponding frame property
T	$\Box\phi \to \phi$	accessibility relation R is reflexive
4	$\Box\phi \to \Box\Box\phi$	accessibility relation R is transitive
M	$\Box\Diamond\phi \to \Diamond\Box\phi$	2nd order property not to be covered here
Grz	$\Box(\Box(\phi \to \Box\phi) \to \phi) \to \phi$	$\mathbf{T4} + \neg\exists P(\forall w \in P)(\exists v\ wRv)(v \neq w\ \wedge\ P(v))$
S4	$\mathbf{T} + \mathbf{4}$	preorder
S4.1	$\mathbf{T} + \mathbf{4} + \mathbf{M}$	preorder having endpoints

As Bayes logics were defined to be the modal logics of certain frames, these logics are normal modal logics. The next proposition establishes the connection between the Bayes logics and the usual frame properties.

Proposition 3.1. The following statements hold:

- $\mathbf{BL} \vdash \mathbf{S4}$ but $\mathbf{BL} \nvdash \mathbf{M}$
- $\mathbf{BL}_{\leq\omega} \vdash \mathbf{S4.1}$
- $\mathbf{BL}_{<\omega} \vdash \mathbf{Grz}$ while $\mathbf{BL}_{\omega} \nvdash \mathbf{Grz}$

Proof. Let $\mathcal{F} = \langle M(X, \mathcal{B}), R(X, \mathcal{B})\rangle$ be an arbitrary Bayes frame. To see that $\mathcal{F}$ validates **S4** we need to verify that $R = R(X, \mathcal{B})$ is a preorder (reflexive and transitive).

- Reflexivity: for all measures $w \in M(X, \mathcal{B})$ we have $w(\cdot) = w(\,\cdot \mid X)$.
- Transitivity: suppose $u, v, w \in M(X, \mathcal{B})$ with uRv and vRw, i.e. there are $A, B \in \mathcal{B}$ with $u(A) \neq 0$, $v(B) \neq 0$ and we have $v(\cdot) = u(\,\cdot \mid A)$ and $w(\cdot) = v(\,\cdot \mid B)$. As $v(B) = u(B \mid A) \neq 0$ we also get $u(B) \neq 0$ and thus $u(A \cap B) \neq 0$. Therefore $w(\cdot) = u(\,\cdot \mid A \cap B)$ which means uRw.

The accessibility relation is also antisymmetric:

- Antisymmetry: If $v(\cdot) = w(\,\cdot \mid A)$ and $w(\cdot) = v(\,\cdot \mid B)$, then $v(\cdot) = v(\,\cdot \mid A \cap B)$. This ensures $v(A \cap B) = 1$, whence $v(B) = 1$ and thus $v = w$.

In order to show $\mathbf{BL}_{\leq\omega} \vdash \mathbf{S4.1}$ it is enough to verify that for a countable measurable space $\langle X, \mathcal{B}\rangle$, the frame $\mathcal{F}(X, \mathcal{B})$ has end-points in the following sense.

- Endpoints: That R has endpoints means $\forall w \exists u (wRu \ \wedge \ \forall v(uRv \rightarrow u = v))$. Pick an arbitrary w and let $x \in X$ be such that $w(\{x\}) \neq 0$. Such an x must exist because X is countable. We claim that $u = w(\,\cdot \mid \{x\})$ will be suitable. For $H \in \wp(X)$ we have

  $$w(H \mid \{x\}) = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{otherwise} \end{cases}$$

  Thus $w(\,\cdot \mid \{x\})$ is the Dirac measure δ_x. If a measure is Bayes accessible from δ_x, then it must be absolutely continuous with respect to δ_x and it is clear that δ_x is the only such probability measure.

To see that $\mathbf{BL} \nvdash \mathbf{M}$ it is enough to give an example for a Bayes frame in which there are paths without endpoints. Consider the frame $\mathcal{F} = \langle M([0,1], \mathcal{B}), R([0,1], \mathcal{B})\rangle$ where $[0,1]$ is the unit interval and $\mathcal{B}$ is the Borel σ-algebra. Let w be the Lebesgue measure. We claim that

$$\mathcal{F} \not\models \exists u (wRu \ \wedge \ \forall v(uRv \rightarrow u = v)) \tag{26}$$

For, suppose for some probability u we have wRu. Then $u(\cdot) = w(\,\cdot \mid A)$ for some Borel set A with $w(A) \neq 0$. Each Borel set A with non-zero Lebesgue measure contains a Borel subset $B \subset A$

with a strictly smaller but non-zero Lebesgue measure: $0 < w(B) < w(A)$. From $u = w(\,\cdot \mid A)$ we can Bayes access $w(\,\cdot \mid B)$ and since $w(B) < w(A)$ we also have $w(\,\cdot \mid A) \neq w(\,\cdot \mid B)$.

Next, let us verify $\mathbf{BL}_{<\omega} \vdash \mathbf{Grz}$. To this end it is enough to show that no Bayes frame $\mathcal{F}(X, \mathcal{B})$ with a finite X can contain an infinite $R(X, \mathcal{B})$-path. But this follows from the fact that finiteness of X implies finiteness of $\mathcal{B} = \wp(X)$, whence there are only finitely many elements in $\mathcal{B}$ that can serve as possible evidence for conditionalizing a probability.

Finally, we prove $\mathcal{F}_\omega \not\Vdash \mathbf{Grz}$ (thus $\mathbf{BL}_\omega \not\vdash \mathbf{Grz}$). Let $w \in M(\mathbb{N}, \wp(\mathbb{N}))$ be a measure such that for all $x \in \mathbb{N}$ we have $w(\{x\}) \neq 0$. Fix a sequence $A_i = \mathbb{N} - \{0, \ldots, i\}$ for $i \in \mathbb{N}$. Then

$$w(\cdot)\; R\; w(\cdot \mid A_0)\; R\; w(\cdot \mid A_1)\; R\; w(\cdot \mid A_2) \cdots \tag{27}$$

shows the failure of the Grzegorczyk axiom $\mathbf{Grz}$ in $\mathcal{F}_\omega$. ∎

4 Containments between Bayes logics

Recall the containments that follow directly from the definition of Bayes logics:

$$\mathbf{BL} \subseteq \mathbf{BL}_{\leq\omega} \subseteq \mathbf{BL}_\omega \quad \text{and} \quad \mathbf{BL} \subseteq \mathbf{BL}_{<\omega} \subseteq \mathbf{BL}_n \tag{28}$$

In this section we prove the following theorem:

Theorem 4.1. $\mathbf{BL} \subsetneq \mathbf{BL}_\omega = \mathbf{BL}_{\leq\omega} \subsetneq \mathbf{BL}_{<\omega} \subsetneq \mathbf{BL}_{n+k} \subsetneq \mathbf{BL}_n$

Some of the containments in the above theorem follow from Proposition 3.1. For instance $\mathbf{BL} \subsetneq \mathbf{BL}_{\leq\omega}$ is witnessed by $\mathbf{S4.1} \in \mathbf{BL}_{\leq\omega} \smallsetminus \mathbf{BL}$. To prove the other containments, we establish several lemmas first.

For two frames $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{G} = \langle W', R'\rangle$ we write $\mathcal{F} \trianglelefteq \mathcal{G}$ if $\mathcal{F}$ is (isomorphic as a frame to) a generated subframe of $\mathcal{G}$. We recall that if $\mathcal{F} \trianglelefteq \mathcal{G}$, then $\mathcal{G} \Vdash \phi$ implies $\mathcal{F} \Vdash \phi$, whence $\Lambda(\mathcal{G}) \subseteq \Lambda(\mathcal{F})$ (see Theorem 3.14 in [3]).

Lemma 4.2. $\mathcal{F}_n \trianglelefteq \mathcal{F}_{n+k} \trianglelefteq \mathcal{F}_\omega$, consequently $\mathbf{BL}_\omega \subseteq \mathbf{BL}_{n+k} \subseteq \mathbf{BL}_n$.

Proof. Let $X_n = \{1, \ldots, n\}$ and $\mathcal{F}_n = \langle M(X_n), R(X_n)\rangle$. To each $w \in M(X_n, \wp(X_n))$ assign $\alpha(w) \in M(X_{n+k}, \wp(X_{n+k}))$ defined by

$$\alpha(w)(x) = \begin{cases} w(x) & \text{if } x = 1, \ldots, n \\ 0 & \text{if } x = n+1, \ldots, n+k \end{cases}$$

It can be checked that α establishes $\mathcal{F}_n \trianglelefteq \mathcal{F}_{n+k}$. The case $\mathcal{F}_n \trianglelefteq \mathcal{F}_\omega$ is similar. ∎

To see why the non-equality $\mathbf{BL}_{n+k} \subsetneq \mathbf{BL}_n$ holds we need some preparation. In a frame $\mathcal{F} = \langle W, R\rangle$ let us call a sequence x_0, x_1, …, x_k a *path* if $x_i R x_{i+1}$ for $i < k$ and $x_i \neq x_j$ for $i \neq j$. The length of a path is the number of the x_i's in the sequence. Define by recursion the following formulas

$$\pi_1 = p_1 \tag{29}$$

$$\pi_2 = (p_2 \wedge \neg p_1) \wedge \Diamond\pi_1 \tag{30}$$

$$\pi_{n+1} = (p_{n+1} \wedge \neg p_n \wedge \cdots \wedge \neg p_1) \wedge \Diamond\pi_n \tag{31}$$

The proof of the next lemma is easy and is left to the Reader.

Lemma 4.3. Let $\mathcal{F} = \langle W, R\rangle$ be a frame, $\mathfrak{M} = \langle \mathcal{F}, [\,\cdot\,]\rangle$ be a model, and $x \in W$.

- $\mathfrak{M}, x \Vdash \pi_n$ if and only if there is in $\mathcal{F}$ a path of length n starting from x.
- If there is in $\mathcal{F}$ a path of length n starting from x, then there is an evaluation $[\,\cdot\,]$, such that in the corresponding model $\mathfrak{M}$ we have $\mathfrak{M}, x \Vdash \pi_n$.
- If $\mathcal{F} \Vdash \neg\pi_n$, then there is no path of length n in $\mathcal{F}$. ∎

We have seen in the proof of Proposition 3.1 that if $\mathcal{F}(X, \wp(X))$ is a Bayes frame with a finite X, then there are only finitely many elements in $\mathcal{B}$ that can serve as a possible evidence for conditionalizing a probability. From this it follows, that in these finite cases the maximal length of a path in $\mathcal{F}(X, \wp(X))$ is smaller then the cardinality of the power set $\wp(X)$. Therefore, for every $n < m$ there exists k such that

$$\mathbf{BL}_n \vdash \neg\pi_k \quad \text{while} \quad \mathbf{BL}_m \nvdash \neg\pi_k \tag{32}$$

This proves $\mathbf{BL}_m \neq \mathbf{BL}_n$.

Lemma 4.4. $\mathbf{BL}_{\leq\omega} = \mathbf{BL}_\omega \subsetneq \mathbf{BL}_{<\omega}$

Proof. By Lemma 4.2 for each n we have $\mathbf{BL}_\omega \subseteq \mathbf{BL}_n$; so we also obtain $\mathbf{BL}_\omega \subseteq \bigcap_n \mathbf{BL}_n = \mathbf{BL}_{<\omega}$. By Proposition 3.1 we have $\mathbf{Grz} \in \mathbf{BL}_{<\omega}$ while $\notin \mathbf{BL}_\omega$; thus $\mathbf{BL}_\omega \subsetneq \mathbf{BL}_{<\omega}$. Since $\mathbf{BL}_{\leq\omega} = \mathbf{BL}_\omega \cap \mathbf{BL}_{<\omega}$ and $\mathbf{BL}_\omega \subseteq \mathbf{BL}_{<\omega}$, we obtain $\mathbf{BL}_{\leq\omega} = \mathbf{BL}_\omega$. ∎

5 Connection to Medvedev's logic of finite problems

Let us recall briefly Medvedev's logic of finite problems and its origins in intuitionistic logic. We rely on and refer to the book [5] and to Shehtman [16]. (Medvedev's logic of finite problems and it's extension to infinite problems by Skvortsov are covered in the papers [13, 17, 15, 16, 12, 9]). Roughly speaking, the main principle of intuitionism is that the truth of a mathematical statement can be established only by producing a constructive proof of the statement. If the notion of proof and construction are taken to be primary, then, one can establish a proof of $\phi \to \psi$ if and only if there is a construction which, given a proof of ϕ, yields a proof of ψ. Medvedev's formalization of the proof as interpreted in intuitionistic logic relies on the idea to treat intuitionistic formulas as finite problems. A finite problem, according to Medvedev's definition, is a tuple $\langle X, Y\rangle$ of finite sets such that $Y \subseteq X$ and $X \neq \emptyset$. Elements of X are regarded as "possible solutions" while elements of Y are "solutions". Medvedev then defines logical operations on finite problems as follows:

$$
\begin{aligned}
\langle X_1, Y_1\rangle \wedge \langle X_2, Y_2\rangle &= \langle X_1 \times X_2, Y_1 \times Y_2\rangle && (33)\\
\langle X_1, Y_1\rangle \vee \langle X_2, Y_2\rangle &= \langle X_1 \sqcup X_2, Y_1 \sqcup Y_2\rangle && (34)\\
\langle X_1, Y_1\rangle \to \langle X_2, Y_2\rangle &= \left\langle X_2^{X_1},\ \{f \in X_2^{X_1} :\ f(Y_1) \subseteq Y_2\}\right\rangle && (35)\\
\perp &= \langle X, \emptyset\rangle && (36)
\end{aligned}
$$

Here $\sqcup$ is the disjoint union of sets. Given a propositional formula ϕ, after replacing its variables with finite problems and performing the operations, one can check whether the result, a finite problem, has a non-empty set of solutions. In this case ϕ is called "finitely valid". Finitely valid formulas are closed under modus ponens and substitutions.

Using the Gödel–McKinsey–Tarski translation of intuitionistic language into the unimodal language, Medvedev's logic of finite problems is recursively embedded into the unimodal logic of Medvedev frames, definition of which we recall below.

Definition 5.1 (Medvedev frame)**.** A *Medvedev frame* is a frame that is isomorphic (as a directed graph) to $\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle$ for a non-empty *finite* set X. □

Medvedev's logic $\mathbf{ML}_{<\omega}$ is the modal logic that corresponds to the Medvedev frames:

$$
\begin{aligned}
\mathbf{ML}_n &= \bigcap \left\{\Lambda\big(\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle\big) :\ |X| = n\right\} && (37)\\
\mathbf{ML}_{<\omega} &= \bigcap \left\{\Lambda\big(\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle\big) :\ |X| \text{ non-empty, finite}\right\} && (38)
\end{aligned}
$$

A *Skvortsov frame* is defined in the same way except with X is a non-empty set of *any cardinality.* We denote the corresponding Skvortsov logics by $\mathbf{ML}_\alpha$ for sets X of cardinality α. It has been proved (see Theorem 2.2 in [17]) that

$$
\mathbf{ML} \overset{def}{=} \bigcap_{\alpha} \mathbf{ML}_\alpha = \mathbf{ML}_\omega \tag{39}
$$

The main result of this section is the following theorem:

Theorem 5.2. *Countable Bayes and Medvedev's logics coincide.*

$$
\begin{array}{ccccccccc}
\mathbf{ML} & = & \mathbf{ML}_\omega & = & \mathbf{ML}_{\leq\omega} & \subsetneq & \mathbf{ML}_{<\omega} & \subsetneq & \mathbf{ML}_n \\
\cup & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
\mathbf{BL} & \subsetneq & \mathbf{BL}_\omega & = & \mathbf{BL}_{\leq\omega} & \subsetneq & \mathbf{BL}_{<\omega} & \subsetneq & \mathbf{BL}_n
\end{array} \tag{40}
$$

We prove Theorem 5.2 through a series of lemmas. To simplify writing we introduce the following notations: For a finite or countably infinite set X denote by $\mathcal{P}^0(X)$ the Medvedev (Skvortsov) frame $\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle$, and we use $\mathcal{F}(X)$ to denote the Bayes frame $\langle M(X, \wp(X)), R(X, \wp(X)\rangle$. Recall that if X is finite or countable then the support of $v \in M(X, \wp(X))$ is defined to be the set $\mathrm{supp}(v) = \{x \in X :\ v(\{x\}) \neq 0\}$.

Lemma 5.3. $\mathcal{P}^0(X) \trianglelefteq \mathcal{F}(X)$ *for all finite or countably infinite set* X.

Proof. Take any $w \in M(X, \wp(X))$ with full support $\operatorname{supp}(w) = X$, and consider the subframe $\mathcal{F}_w = \langle W, R\rangle$ of $\mathcal{F}(X)$ generated by w. As $\mathcal{F}(X)$ is transitive, elements of W are of the form $w(\,\cdot \mid H)$ for some non-empty $H \subseteq \operatorname{supp}(w)$. Now, if $H, H' \subseteq \operatorname{supp}(w)$ are different subsets, then $1 = w(H \mid H) \neq w(H' \mid H) < 1$. Therefore each element $v \in W$ can be identified with a non-empty subset $H \subseteq \operatorname{supp}(w)$. It follows that $\mathcal{F}_w$ is isomorphic, as a directed graph, to $\mathcal{P}^0(X)$, which completes the proof. ∎

Lemma 5.3 implies $\mathbf{BL}_\omega \subseteq \mathbf{ML}_\omega$ (with X countably infinite) and $\mathbf{BL}_n \subseteq \mathbf{ML}_n$ for all $n > 0$ (with $|X| = n$) and therefore $\mathbf{BL}_{<\omega} \subseteq \mathbf{ML}_{<\omega}$. Next, we want to establish the converse containments.

Let $\mathcal{F} \twoheadrightarrow \mathcal{G}$ denote a surjective, bounded morphism between frames $\mathcal{F}$ and $\mathcal{G}$. Recall that if $\mathcal{F} \twoheadrightarrow \mathcal{G}$, then $\mathcal{F} \Vdash \phi$ implies $\mathcal{G} \Vdash \phi$, whence $\Lambda(\mathcal{F}) \subseteq \Lambda(\mathcal{G})$ (see Theorem 3.14 in [3]). We also recall that $(\forall i)\ \mathcal{F}_i \Vdash \phi$ implies $\biguplus \mathcal{F}_i \Vdash \phi$ (for the definition of the disjoint union $\biguplus$ of frames see Definition 3.13 in [3]). In the special case when $\mathcal{F}_i = \mathcal{F}$ it follows that $\Lambda(\mathcal{F}) \subseteq \Lambda(\biguplus \mathcal{F})$ (Theorem 3.14 in [3]).

Note that neither $\mathcal{F}(X) \trianglelefteq \mathcal{P}^0(X)$ nor $\mathcal{P}^0(X) \twoheadrightarrow \mathcal{F}(X)$ can hold if X is finite because the underlying set $M(X, \mathcal{B})$ of $\mathcal{F}(X)$ has the cardinality of continuum (for $n > 1$) while $\wp(X)$ is finite.

Lemma 5.4. $\mathbf{ML}_\omega \subseteq \mathbf{BL}_\omega$ and $\mathbf{ML}_n \subseteq \mathbf{BL}_n$ for all $n > 0$.

Proof. Let X be a finite or countably infinite set. We prove

$$\biguplus_{v \in F} \mathcal{P}^0(X) \twoheadrightarrow \mathcal{F}(X) \tag{41}$$

for a suitable set F. This is enough since $\biguplus_v \mathcal{P}^0(X) \twoheadrightarrow \mathcal{F}(X)$ implies

$$\Lambda\big(\mathcal{P}^0(X)\big) \subseteq \Lambda\big(\biguplus \mathcal{P}^0(X)\big) \subseteq \Lambda\big(\mathcal{F}(X)\big) \tag{42}$$

For $|X| = n$ this means $\mathbf{ML}_n \subseteq \mathbf{BL}_n$ and for X countably infinite it is $\mathbf{ML}_\omega \subseteq \mathbf{BL}_\omega$.

Consider the Bayes frame $\mathcal{F}(X) = \langle M(X, \wp(X)), R(X, \wp(X)\rangle$. A measure $v \in M(X, \wp(X))$ is faithful if $v(H) = 0$ if and only if $H = \emptyset$, equivalently $v(H) = 1$ if and only if $H = X$. Each faithful measure v has full support $\operatorname{supp}(v) = X$.

Claim A. Faithful measures cannot be Bayes accessed. More precisely, if v is faithful, then $\forall u(uRv \to u = v)$. Indeed, suppose $v(\cdot) = u(\,\cdot \mid A)$ for some $A \subseteq X$, $u(A) \neq 0$. Then $v(A) = u(A \mid A) = 1$ thus faithfulness of v ensures $A = X$. But then $v = u$.

Claim B. If u is not faithful, then there is a faithful v such that vRu. Suppose u is not faithful. Take any faithful measure over $X \smallsetminus \operatorname{supp}(u)$ and pick a real number $0 < c < 1$. Write

$$v(H) = c \cdot u(H \cap A) + (1 - c) \cdot r(H \cap (X \smallsetminus \operatorname{supp}(u))) \tag{43}$$

Then v is a faithful measure over X and $v(H \mid \operatorname{supp}(u)) = u(H)$.

Let $F \subseteq M(X, \wp(X))$ be the set of faithful measures. We claim that $\mathcal{F}(X)$ is a surjective, bounded morphic image of $\biguplus_{v \in F} \mathcal{P}^0(X)$. Let us denote the copy of $\mathcal{P}^0(X)$ corresponding to $v \in F$ in the disjoint union by $\mathcal{P}^0_v = \langle P_v, \supseteq\rangle$, where $P_v = \wp(X) \smallsetminus \{\emptyset\}$. Define the mapping f as follows

$$f : \biguplus_{v \in F} \mathcal{P}^0_v \to \mathcal{F}(X), \qquad P_v \supseteq A \mapsto v(\,\cdot \mid A) \tag{44}$$

Let us verify that f is a surjective, bounded morphism $\biguplus_{v\in F}\mathcal{P}^0(X) \twoheadrightarrow \mathcal{F}(X)$.

Surjectivity. Pick a probability $u \in M(X, \wp(X))$. By Claim B there is a faithful v from which u is accessible by an $A \subseteq X$. Then $A \subseteq P_v$ and $f(A) = v(\,\cdot \mid A) = u(\cdot)$.

Homomorphism. We have to show that $P_v \supseteq A \supseteq B$ implies $f(B)$ is Bayes accessible from $f(A)$. Indeed, $f(A) = v(\,\cdot \mid A)$ and $f(B) = v(\,\cdot \mid B)$ and $v(\,\cdot \mid A\cap B) = v(\,\cdot \mid B)$.

Zig–zag property. We have to verify that if $f(A)Rw$, then there is a C such that $w = f(C)$ and $A \supseteq C$. Denote $f(A)$ by u. Let v be the faithful measure such that $A \subseteq P_v$. Then $u = v(\,\cdot \mid A)$, and by the assumption uRw there is $B \subseteq X$ such that $w(\cdot) = u(\,\cdot \mid B)$. Then $w(\cdot) = v(\,\cdot \mid A\cap B) = f(A\cap B)$, therefore setting $C = A\cap B$ completes the proof. ∎

So far we have proved $\mathbf{BL}_\omega = \mathbf{ML}_\omega$, $\mathbf{BL}_{<\omega} = \mathbf{ML}_{<\omega}$ and $\mathbf{BL}_n = \mathbf{ML}_n$ for all $n > 0$. To complete the proof of Theorem 5.2 it remains to show $\mathbf{BL} \subsetneq \mathbf{ML}$.

Lemma 5.5. $\mathbf{BL} \subsetneq \mathbf{ML}$

Proof. That $\mathbf{BL} \neq \mathbf{ML}$ follows from

$$\mathbf{S4.1} \subset \mathbf{ML} \quad \text{while} \quad \mathbf{S4.1} \not\subset \mathbf{BL} \tag{45}$$

By Proposition 3.1 we have $\mathbf{S4.1} \not\subset \mathbf{BL}$. On the other hand, every Medvedev frame $\mathcal{P}^0$ has endpoints in the sense

$$\mathcal{P}^0 \models \forall w \exists u (wRu \ \wedge\ \forall v(uRv \to u = v)) \tag{46}$$

For, if $\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle$ is a Medvedev frame, then $\{x\}$ is an endpoint, for all $x \in X$.

So it remains to show $\mathbf{BL} \subseteq \mathbf{ML}$. But this follows from our previously proven containments: $\mathbf{ML} = \mathbf{ML}_\omega = \mathbf{BL}_\omega \supseteq \mathbf{BL}$. ∎

Putting together all the previous lemmas we arrive at Theorem 5.2:

$$\begin{array}{ccccccccc}
\mathbf{ML} & = & \mathbf{ML}_\omega & = & \mathbf{ML}_{\le\omega} & \subsetneq & \mathbf{ML}_{<\omega} & \subsetneq & \mathbf{ML}_n \\
\cup & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
\mathbf{BL} & \subsetneq & \mathbf{BL}_\omega & = & \mathbf{BL}_{\le\omega} & \subsetneq & \mathbf{BL}_{<\omega} & \subsetneq & \mathbf{BL}_n
\end{array} \tag{47}$$

Though we established $\mathbf{BL} \neq \mathbf{ML}$, the two logics are "close" to each other in the sense of the following proposition.

Proposition 5.6. The logic of each Bayes frame can be dominated by the logic of a Medvedev frame, and vice versa.

Proof.

#1: Proving that for all $\mathcal{F}(X, \mathcal{B})$ there exists $\mathcal{P}^0(Y)$ such that $\Lambda(\mathcal{F}) \subseteq \Lambda(\mathcal{P}^0)$:

Take any $\mathcal{F}(X, \mathcal{B})$ and let $Y \subseteq X$ be a finite, non-empty subset. Let $v \in M(X, \mathcal{B})$ be a probability measure such that $\mathrm{supp}(v) = Y$. Then the subframe $\mathcal{F}_v$ generated by v is isomorphic (as a directed graph) to $\mathcal{P}^0(Y)$ (cf. the proof of Lemma 5.3), whence $\mathcal{P}^0(Y) \trianglelefteq \mathcal{F}(X, \mathcal{B})$. This implies $\Lambda(\mathcal{F}(X, \mathcal{B})) \subseteq \Lambda(\mathcal{P}^0(Y))$, as desired.

#2: Proving that for all $\mathcal{P}^0(Y)$ there exists $\mathcal{F}(X,\mathcal{B})$ such that $\Lambda(\mathcal{P}^0) \subseteq \Lambda(\mathcal{F})$:

The proof is similar to that of Lemma 5.4. Take any $\mathcal{P}^0(Y)$ and let $X \subseteq Y$ be a finite, non-empty subset. We need the following Lemma:

Lemma 5.7. If $X \supseteq Y$, then $\mathcal{P}^0(X) \twoheadrightarrow \mathcal{P}^0(Y)$.

Proof.[of Lemma 5.7] Any surjection $f : X \to Y$ can be lifted up to a surjection $f^+ : \wp(X) \to \wp(Y)$ via $f^+(H) = \{f(h) : h \in H\}$. It can be checked that f^+ is a bounded morphism $\mathcal{P}^0(X) \twoheadrightarrow \mathcal{P}^0(Y)$. ∎

Lemma 5.7 applies and we get $\mathcal{P}^0(Y) \twoheadrightarrow \mathcal{P}^0(X)$. With $\mathcal{F}(X) = \langle M(X,\wp(X)), R(X,\wp(X))\rangle$, X being finite, following the proof of Lemma 5.4 one obtains $\biguplus \mathcal{P}^0(X) \twoheadrightarrow \mathcal{F}(X)$. Consequently

$$\biguplus \mathcal{P}^0(Y) \;\twoheadrightarrow\; \biguplus \mathcal{P}^0(X) \;\twoheadrightarrow\; \mathcal{F}(X) \tag{48}$$

which implies

$$\Lambda(\mathcal{P}^0(Y)) \;\subseteq\; \Lambda(\biguplus \mathcal{P}^0(Y)) \;\subseteq\; \Lambda(\mathcal{F}(X)) \tag{49}$$

∎

Recall that if $\langle X, \mathcal{B}\rangle$ is a finite probability space (with $|X| > 1$), then the set of probability measures $M(X,\mathcal{B})$ has cardinality continuum. Therefore Bayes frames $\mathcal{F}(X,\mathcal{B})$ over finite probability spaces are uncountable. Thus it is surprising that despite uncountability of Bayes frames the corresponding logic has the finite frame property:

Proposition 5.8. The modal logic $\mathbf{BL}_{<\omega}$ of Bayes frames over a finite probability space has the finite frame property.

Proof. $\mathbf{ML}_{<\omega}$ is complete with respect to the set of (finite) Medvedev frames by definition, and $\mathbf{BL}_{<\omega} = \mathbf{ML}_{<\omega}$. ∎

An immediate consequence is that $\mathbf{BL}_{<\omega}$ is complete with respect to a recursive set of finite frames. Therefore, non-validities can be witnessed by finite counterexamples.

The most remarkable consequence of the identification of Bayes logic with Medvedev's logic concerns the (non-)axiomatizability properties of Bayes logics:

Proposition 5.9. The modal logic $\mathbf{BL}_{<\omega}$ of Bayes frames over finite probability spaces is not finitely axiomatizable.

Proof. As $\mathbf{BL}_{<\omega} = \mathbf{ML}_{<\omega}$, this is essentially Shehtman's result [16] for the logic of Medvedev's frames. ∎

The previous proposition is philosophically significant: it tells us that there is no finite set of formulas from which all general laws of Bayesian belief revision and Bayesian learning based on probability spaces with a finite set of propositions can be deduced. Bayesian learning and belief revision based on such simple probability spaces are among the most important instances of probabilistic updatings because they are widely used in applications. Proposition 5.9 says that the logic of such very basic belief revisions cannot be captured by a finite set of axioms. If the axiomatic approach to belief revision is not capable to characterize the logic of the simplest, paradigm form of belief revision, then this casts doubt on the general enterprize that aims at axiomatizations of belief revision systems.

It is a longstanding open question whether $\mathbf{ML}_{<\omega}$ (and thus $\mathbf{BL}_{<\omega}$) is recursively axiomatizable (see [5], Chapter 2). Since the class of Medvedev frames is a recursive class of finite frames, $\mathbf{BL}_{<\omega}$ is co-recursively enumerable. It follows that if $\mathbf{ML}_{<\omega}$ is recursively axiomatizable, then $\mathbf{BL}_{<\omega}$ is decidable.

Countable Bayes logics can be characterized not only by Medvedev's logic but also by the logic of Kubiński frames: Łazarz [12] proved that Medvedev's and Kubiński's logic coincide (even in the infinite case). Taking into account Theorem 5.2, Łazarz's result gives a lattice characterization of countable Bayes logics. For the necessary definitions we refer to [12].

It is also known that $\mathbf{ML}_\omega = \mathbf{ML}$ $(= \mathbf{BL}_\omega)$ is not finitely axiomatizable [16, 15]. As $\mathbf{BL} \subsetneq \mathbf{ML}$ the question arises whether $\mathbf{BL}$ is finitely axiomatizable. This we leave an open problem:

Problem 5.10. Is $\mathbf{BL}$ finitely axiomatizable?

A central problem when considering logics based on frames is whether the class of frames in question can be defined by formulas. Holliday [9] proved that the class of Medvedev frames is definable by a formula in an extended language, the language of tense logic extended with a converse modality and with any of nominals, difference modality or complement modality.

Recall that for classes C and D of frames, formula ϕ defines C relative to D if each frame in D belongs to C if and only if the frame validates ϕ.

Proposition 5.11. $\mathsf{BF}_{<\omega}$ and BF_ω are definable over BF; and $\mathsf{BF}_{<\omega}$ is definable over $\mathsf{BF}_{\leq\omega}$.

Proof. Straightforward from Proposition 3.1. ∎

We have no results concerning absolute definability of Bayes frames.

Problem 5.12. Are any of the classes of frames BF, $\mathsf{BF}_{\leq\omega}$, BF_ω or $\mathsf{BF}_{<\omega}$ definable in a "reasonable" extended modal language?

6 Closing words and open problems

Apart from the standard Bayes conditionalization there are other Bayesian methods, extensions of the standard one, of updating a probability measure: The Jeffrey's conditionalization and conditionalization based on the concept of conditional expectations (cf. [10, 6, 8]).

Let us first recall Jeffrey's conditionalization. Suppose $p \in M(X, \mathcal{B})$ is a prior probability, $\{E_i\}_{i<n}$ is a finite partition of X with $p(E_i) \neq 0$ and we are given a probability measure $r : \mathcal{A} \to [0, 1]$, called the *uncertain* evidence, on the subalgebra $\mathcal{A}$ of $\mathcal{B}$ generated by this partition. The Bayesian Agent updates his prior probability p using the evidence r to get the posterior probability defined by the "Jeffrey rule":

$$q(H) = \sum_{i<n} p(H \mid E_i) r(E_i) \tag{50}$$

Given two measures $p, q \in M(X, \mathcal{B})$ one can define Jeffrey accessibility in a manner similar to Bayes accessibility: q is Jeffrey accessible from p if there is a partition $\{E_i\}_{i<n}$ and uncertain evidence r such that eq. (50) holds.

Jeffrey's conditionalization is just a special case of the general conditionalization based on the concept of conditional expectation introduced by Kolmogorov [11] already (see [8] as well): Let $\mathcal{S}$ be the Borel σ-algebra of $\mathbb{R}$. Recall that for $p \in M(X, \mathcal{B})$ and $\mathcal{A} \leq \mathcal{B}$ the conditional expectation $\mathrm{E}_p(\cdot \mid \mathcal{A}) : X \to \mathbb{R}$ is any $(\mathcal{A}, \mathcal{S})$-measurable function that satisfies eq. (51) below for all $(\mathcal{B}, \mathcal{S})$-measurable $f : X \to \mathbb{R}$

$$\int_Z \mathrm{E}_p(f \mid \mathcal{A}) \, dp = \int_Z f \, dp \qquad \text{for each } Z \in \mathcal{A} \tag{51}$$

Such a function exists and is unique p-almost everywhere. let $\frac{dq}{dp} : X \to \mathbb{R}$ denote the Radon–Nikodym derivative of q wit respect to p. We say that q can be inferred from p using general conditionalization if q is absolutely continuous with respect to p and there is a σ-subalgebra $\mathcal{A}$ of $\mathcal{B}$ such that

$$q(H) = \int_H \mathrm{E}_p \left(\frac{dq}{dp} \middle| \mathcal{A} \right) \, dp \tag{52}$$

for all $H \in \mathcal{B}$. If q can be inferred from p in this way, we say that q is generally Bayes accessible from p.

One can now define the modal logics based on Bayes frames $\mathcal{F}(X, \mathcal{B})$, where the accessibility relation is replaced with either the Jeffrey accessibility or with the more general accessibility using conditional expectations. Basic logical properties of the Jeffrey accessibility has been studied in the manuscript [4] and frame properties of accessibility using conditional expectations has been investigated in [8]. It has been proven in [4] that Jeffrey accessibility is also not finitely axiomatizable in the finite case; however, we do not yet have results about the infinite case or about decidability questions.

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