The ubiquitous defeaters: no admissibility troubles for Bayesian accounts of direct inference

Zalán Gyenis, Leszek Wroński
Jagiellonian University

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Abstract

In this paper we dispel the supposed "admissibility troubles" for Bayesian accounts of direct inference proposed by Wallmann & Hawthorne (2018), which concern the existence of surprising, unintuitive defeaters even for mundane cases of direct inference. We show that if one follows the majority of authors in the field in using classical probability spaces unimbued with any additional structure, one should expect similar phenomena to arise and should consider them unproblematic in themselves: defeaters abound! We then show that the framework of Higher Probability Spaces (Gaifman, 1988) allows the natural modelling of the discussed cases which produces no troubles of this kind.

1. Introduction: defeaters for direct inference

There is a number of ways in which the term "defeater" has been used in philosophy. We can divide them into two families: nonprobabilistic and probabilistic. The former typically involves speaking about some piece of evidence defeating the justification a subject has for believing some proposition (Kelly, 2016). The latter involves conditional probability, interpreted as degree of belief; however, it is still internally divided. Sometimes, e.g. in some corners of the intersection of philosophy of religion and philosophy of science[1] the term is used so that a subject acquires a defeater for his or hers belief A if he or she acquires a belief B such that $P(A \mid B)$ is low or inscrutable (see Plantinga (2003) and Merricks (2002)). The topic of this paper is, however, the usage of the notion familiar for readers of modern formal epistemology: it also involves conditional probability, albeit in a little bit more complicated way.

The context in which the notion of defeater we are concerned with appears is usually that of direct inference (Levi, 1977): roughly, if a subject knows that the chance of A is x, and if he or she knows no other relevant information, then he or she should assign to A the degree of belief x. That is, with the "relevant information" proviso in mind, the rational degree of belief in A conditional on that the chance of A is x should be x. A defeater is then a proposition such that if it is additionally conditionalized upon, that is, if it is added to the proposition about chance to form the body of propositions given which the conditional subjective probability of A is considered, then that probability becomes something else than x. In other words, we call a piece of evidence a

[1] But not only there: see Pollock (1983).

defeater if a certain conditional probability is not resilient under it.[2] This is how Hawthorne et al. (2017) formulate the idea (where P is a subjective probability function):

> "[Suppose] X says that the chance (...) of A is x (...) We shall take the claim that E is not a defeater to hold just when $P(A \mid XE) = x = P(A \mid X)$." (Hawthorne et al., 2017, p. 123-124)

It is typical to discuss the notion in the context of the Principal Principle ("PP", Lewis (1986)), which is where the notoriously vague notion of admissibility comes into play: using the notation as before, $P(A \mid XE) = x$ provided E is *admissible*. Without going into the jungle of details about admissibility, we hope it is evident at this point that whatever the notion really comes down to, we can agree that (keeping the notation introduced so far and being somewhat charitable with implicit quantification) if something is a defeater, then it is inadmissible, which is exactly the route taken in the recent paper by Wallmann & Hawthorne (2018).

The prevailing intuition among the authors writing about admissibility seems to be that inadmissibility should be somewhat rare, since it involves impacting the credence in A by some other way than impacting the credence about the chance of A (Lewis, 1986, p. 92). For examples, barring some very special cases, historical information and hypothetical information about chance should be admissible. The clear examples of inadmissible evidence occurring in the literature invariably involve some sort of soothsaying device. This is from where the results in Wallmann & Hawthorne (2018) receive at least a part of their bite: seemingly, defeaters abound! A key example concerns Maria, a craps player, who by way of direct inference assigns credence $1/6$ to the proposition that the outcome of the next toss of two fair dice will be seven, based on her knowledge of the chances involved. It turns out, Wallmann & Hawthorne claim, that something so seemingly innocent as the proposition uttered by John, who is standing nearby and says "I'll buy you dinner this evening if and only if the next toss comes up seven", is a defeater for that credence: given Maria's knowledge about the relevant chances, and that the next toss comes up seven if and only if John buys her dinner this evening, her credence that the next toss comes up seven is not equal to $1/6$. And so in this situation this biconditional is a defeater—an inadmissible proposition.

If Wallmann & Hawthorne—who proceed to generalize the above example to a theorem about inadmissible biconditionals—are right, then inadmissibility is a lot more frequent than everyone assumed. This may pose some troubles for Bayesian accounts of direct inference (though, it has to be mentioned, the two authors disagree on the import of their results).

In this paper we will argue that this view is mistaken: the fact that defeaters for direct inference are so easy to come by should come as no surprise, and is actually a consequence of using the traditional, Kolmogorovian notion of probability whereas a different approach should better be employed. The key problem is that the event algebra[3] of a classical probability space does not contain events which would in any formal sense be related to the idea that the probability of some event has some value. That is, for any event A, the probability space assigns it some probability $P(A)$, but if we want to use that space to model a degree of belief function of an agent who not only has a credence in A but also in that the probability of A is, say, .3, we seemingly have no structural features to turn to which would help us in identifying which of the elements of the event algebra is the proposition "that the probability of A is .3". Some additional structure involving higher order probabilities seems needed at this point.

[2] This is essentially how Lewis (1986) uses the notion of resiliency, see e.g. p. 85-86.

[3] We will be using the notions "event" and "proposition" interchangeably in this paper, always refering to an element of the second element of some probability space (the field of sets).

Still, many authors in modern formal epistemology proceed to, in a completely informal move, claim that some element of the event algebra *is* "the proposition that the chance of A is x" and then prove theorems which at their core are simply results about various conditional dependences in classical probability spaces and not really about propositions about probabilities (see e.g. the discussion of the Hawthorne et al. (2017) paper in Gyenis & Wroński (2017)).

In the following section we will show that if we stick to the classical probability space approach then in quite frequent cases we should expect defeaters to abound, and so it is doubtful whether the results by Wallmann & Hawthorne (2018) should really surprise us. This suggests that a serious discussion of probabilistic defeaters for direct inference should take place in some non-classical setting. In Section 3 we recall one such proposal by Gaifman, that of Higher Order Probability spaces (HOPs). Then, in Section 4, we return to Wallmann & Hawthorne's result about biconditionals and show how to model the phenomena under discussion using HOPs: no "unwanted" defeaters appear, and so, we believe, no admissibility troubles for Bayesian accounts are encountered in this way.

2. Defeaters in classical probability spaces

We will first show that if one wishes to use classical probability spaces to non-trivially speak about defeaters, one should better consider certain cardinality features (virtually never discussed in the literature) of the events in question. We are assuming that the traditional notion of a classical probability space is to be used, that is, a triple $(W, \mathcal{F}, P)$ consisting of a nonempty set, a field of its subsets, and a probability measure. Wallmann & Hawthorne could perhaps respond that the phenomena we will be describing disappear if we consider probability functions defined not on fields of sets, but on languages, which they seem to suggest (see e.g. p. 2 of their paper). However, carrying out this move would require its proponents to precisely define an algebra of statements on which the probability function is defined on, and this in turn would require a formulation of some condition linking the proposition A with the proposition that the probability of A is (say) .3. None of this is present in the discussed paper.

When it comes to notation, assume the following: AB means $A \cap B$; $|A|$ is the cardinality of A; A' is the complement of A.

We will now define a general notion of defeater, applicable also outside of the context of direct inference. (To reiterate the point, from the perspective of a classical probability space unimbued with any additional structure the phenomenon of direct inference is essentially invisible, because no events are really events about probabilities: so, there is — at least not until the structure is carefully interpreted or extended — no defeat for direct inference, there is just non-resilient conditional probability.)

Definition 1. Assume a probability space $(W, \mathcal{F}, P)$ is given. For any $A, B \in \mathcal{F}$ such that $P(B) > 0$ an event $D \in \mathcal{F}$ is called a:

- *non-trivial defeater for* $P(A \mid B)$ if $P(BD) > 0$ and $P(A \mid B) \neq P(A \mid BD)$;
- *trivial defeater* for $P(A \mid B)$ if $P(BD) = 0$.

D is a defeater for $P(A \mid B)$ if it is either a trivial or a non-trivial defeater for $P(A \mid B)$. If D is neither a trivial nor a non-trivial defeater for $P(A \mid B)$ we say that it is a *non-defeater for* $P(A \mid B)$.

If there is no risk of confusion about the measure, we will use the terms "defeater for $P(A \mid B)$" and "defeater for A and B" interchangeably. Just to be sure concerning how negation works here: if something is not a non-defeater for A and B then it is a defeater for those events; also, any defeater for A and B is not a non-defeater for those events.

Are defeaters rare? Should we expect conditional probabilities to be resilient? We will now argue that in finite uniform probability spaces, for (literally) most pairs of events, (literally) most events are defeaters. Therefore in general we should be surprised if something is *not* a defeater for some given two events, and not when it is.

Suppose a finite probability space $(W, \mathcal{F}, P)$ is given with P uniform, $|W| = N$. Take two events $A, B \in \mathcal{F}$, such that $P(B) > 0$. By uniformity of P this latter condition amounts to $B \neq \emptyset$. First, we determine the number of non-defeaters for A and B. We have two main cases depending on whether or not A and B are disjoint. Let us write $|AB| = k$, $|A'B| = l$.

Case #1: Suppose $A \cap B = \emptyset$. D is a non-defeater (for A and B) if $|AB|/|B| = |ABD|/|BD|$. As the numerators are zero, in order to have this equality we need to make sure $|BD| > 0$ (otherwise the left-hand side is zero, the right-hand side is undefined). Therefore each D which overlaps B is a non-defeater. The number of such events is $(2^{|B|} - 1)2^{|B'|}$, which we can write as $2^{N-l}(2^l - 1)$.

Case #2: Suppose $A \cap B \neq \emptyset$. Let D be an event and put $|ABD| = n$, $|BD| = n + c$. That D is a non-defeater means $|AB|/|B| = |ABD|/|BD|$, i.e. $k/k+l = n/n+c$. Suppose D is a non-defeater. Then $k > 0$ and the previous equality implies $n > 0$. We have two subcases.

Subcase #1: suppose $l = 0$ (that is, $B \subseteq A$). Then D is a non-defeater if and only if $ABD \neq \emptyset$. Thus, D should contain arbitrarily many but at least one element from AB, and any number of elements from $(AB)'$. The number of such D's is $(2^{|AB|} - 1)2^{|(AB)'|} = 2^{N-k}(2^k - 1)$.

Subcase #2: suppose $l > 0$. That D is a non-defeater, in particular the condition $k/k+l = n/n+c$, is equivalent to $0 < n \leqslant k$ and $c = nl/k \in \mathbb{N}$. Let us write

$$G(k, l) = \{0 < n \leqslant k : \ nl/k \in \mathbb{N}\}.$$

Now, D is a non-defeater for A and B if and only if for some $n \in G(k, l)$, D contains

- n elements from $A \cap B$;
- nl/k elements from $A' \cap B$;
- arbitrary many elements from B'.

Therefore the number of non-defeaters for A and B in this case is

$$2^{N-(k+l)} \sum_{n \in G(k,l)} \binom{k}{n} \binom{l}{\frac{nl}{k}}.$$

Summing up, let $\mathrm{NonDef}(N, k, l)$ denote the number of non-defeaters for A and B, where $k = |AB|$ and $l = |A'B|$ and N is the size of the sample space. Given the uniform probability over the sample space, combining the cases above we obtain

$$\mathrm{NonDef}(N, k, l) = \begin{cases} 2^{N-l}(2^l - 1) & \text{if } k = 0, l > 0; \\ 2^{N-k}(2^k - 1) & \text{if } k > 0, l = 0; \\ 2^{N-(k+l)} \sum\limits_{n \in G(k,l)} \binom{k}{n} \binom{l}{\frac{nl}{k}} & \text{if } k > 0, l > 0. \end{cases} \tag{1}$$

(Note that $\mathrm{NonDef}(N,0,0)$ is undefined as we require $P(B) > 0$).

Let us return to the original question: for given N,k,l, which of the two, $\mathrm{NonDef}(N,k,l)$ or $\mathrm{Def}(N,k,l)$, is higher? ($\mathrm{Def}(N,k,l)$ is the respective number of defeaters).

In the case of disjoint A and B (i.e. $k=0$) or the case of $B \subseteq A$ (i.e. $l=0$), almost all events are non-defeaters. If one of the sets $|AB|$ and $|A'B|$ is large enough (but the other is empty), then the proportion of the cardinality of the set of nondefeaters to 2^N is very close to 1. (It is approximately 0.97 already if that cardinality is as low as 5.) However, we can dismiss these cases as unimportant to our argument, for two reasons.

First, in the context of direct inference, we are interested in nontrivial conditionalization, where some proposition A and some proposition specifying the chance of A are such that, first, they are in general not mutually exclusive, and second, the latter does not logically imply the former. We are pursuing here a general mathematical question about the resiliency of conditional probability, but the two cases outlined above will not be important for the philosophical goal.

Second, such cases are in a precise sense rare. Suppose we chose two subsets (at random with uniform probability) A and B of W. What is the probability that they are disjoint (respectively, $B \subseteq A$)? We have 3^N possibilities to choose a disjoint pair, as for each element in W we have three choices: in A; in B; in neither A nor B (respectively, 3^N possibilities to choose a pair with $B \subseteq A$, as for each element in W we have three choices again: in A but not in B; in both; in neither). There are 2^{2N} possibilities in total to choose a pair of events, therefore the probability that A and B are disjoint (respectively, $B \subseteq A$) is $3^N/2^{2N} = (3/4)^N$. As $3/4 < 1$, for large enough N this probability gets very close to 0. (It equals approximately 0.06 for $N = 10$ already.) This means it is very unlikely to pick sets at random that are disjoint (resp. one contains the other), provided the sample space is large enough.

To reiterate the point, in Section 3 we move away from classical probability spaces towards structures tailored for meta-level probabilistic phenomena. There A and B are not chosen independently: B will be the event $pr(A,\alpha)$ expressing "the probability of A is α" (cf. Definition 2). It can be shown that A and $pr(A,\alpha)$ cannot be disjoint, provided Miller's principle holds, which will be the axiom (VI_w) on p. 9 below. In the remainder of this section we concentrate, then, on the "non-trivial" case only, when $k,l > 0$.

Let us then proceed, then, under assumption that A and B are not disjoint. We know how many nondefeaters there are, namely, $\mathrm{NonDef}(N,k,l)$. Is this number lower or higher than the number of defeaters, $\mathrm{Def}(N,k,l)$? Take a non-defeater D; there is an n such that $|ABD| = n$. Consider the family

$$\mathrm{Def}_D(N,k,l) = \{E : \ ABE = ABD \text{ and } |A'BE| \neq nl/k\}$$

of defeaters "generated by D". Note that:

- for distinct D_1 and D_2, $\mathrm{Def}_{D_1}(N,k,l) \cap \mathrm{Def}_{D_2}(N,k,l) = \emptyset$;
- $|\,\mathrm{Def}_D(N,k,l)| = 2^l - \binom{l}{nl/k} > 1$.

Therefore

$$\mathrm{NonDef}(N,k,l) < \mathrm{Def}(N,k,l). \tag{2}$$

Let us now assume that for given A and B (with $|AB|, |A'B| > 0$) we randomly choose an event D with uniform probability Prob over all events. Which is more likely: that D is a defeater or a non-defeater? The probability that D is a non-defeater is the ratio of $\mathrm{NonDef}(N,k,l)$ and the number of all events 2^N. Since there are more defeaters than non-defeaters (inequality (2)), we immediately get

$$\mathrm{Prob}(D \text{ is a non-defeater for } A, B) < \mathrm{Prob}(D \text{ is a defeater for } A, B). \tag{3}$$

As every event D is either a defeater or a non-defeater for A and B, it follows that

$$\mathrm{Prob}(D \text{ is a non-defeater for } A, B) < {}^1\!/_2. \tag{4}$$

Our counting arguments above allow us to express the precise value of the probability in (4), namely,

$$\mathrm{Prob}(D \text{ is a non-defeater for } A, B) = \frac{\sum\limits_{n \in G(k,l)} \binom{k}{n}\binom{l}{\frac{nl}{k}}}{2^{k+l}}. \tag{5}$$

Note that the probability of D being a (non)-defeater (for A and B) depends only on the size of AB and $A'B$.

The gist of inequality (3) is this: assume a finite uniform probability space is given. Suppose you are choosing two events at random. If you are not lucky, you will end up with two events A and B such that a vast majority of events in the space will be defeaters for them.

By how much the probability of being a non-defeater is smaller than ${}^1\!/_2$? The general answer to this question seems hard[4] so we discuss below a single easy-to-handle case which turns out to be more likely than it's not.

Assume k and l are coprime. Then $G(k,l) = \{k\}$, and thus

$$\sum_{n \in G(k,l)} \binom{k}{n}\binom{l}{\frac{nl}{k}} = \binom{k}{k}\binom{l}{l} = 1.$$

The only (and all) D's which are *not* defeaters for A and B are such that $B \subseteq D$. The proportion of non-defeaters for A and B is then

$$\mathrm{Prob}(D \text{ is a non-defeater for coprime } k, l) = \frac{\sum\limits_{n \in G(k,l)} \binom{k}{n}\binom{l}{\frac{nl}{k}}}{2^{k+l}} = \frac{1}{2^{k+l}} = \frac{1}{2^{|B|}}.$$

If B is a singleton, then $\frac{1}{2^{|B|}}$ equals .5. But in this case every D avoiding B is a trivial defeater and every D containing B is a non-defeater, thus, there are no non-trivial defeaters for singleton B's. In all other cases ($|B| > 1$) the value of the fraction is at most $\frac{1}{3}$. The value "quickly" diminishes if the cardinality (of $|B|$) increases.

Among randomly selected integers two co-primes are *not* so hard to come by: If we consider sets of integers $\{1, \ldots, N\}$, then with N approaching infinity the probability of randomly selecting

[4] Let us note that $G(k,l)$ can be rewritten as $G(k,l) = \{x \cdot {}^k\!/_{\gcd(k,l)} :\ 0 < x \leqslant \gcd(k,l), x \in \mathbb{N}\}$, thus $|G(k,l)| = \gcd(k,l)$. Assuming $k = l$ the sum $\sum_{n \in G(k,l)} \binom{k}{n}\binom{l}{nl/k}$ simplifies to $\sum_{n \leqslant k} \binom{k}{n}^2$ which is equal to $\binom{2k}{k}$. Calculus shows $\lim_{k\to\infty} \binom{2k}{k}/2^{k+k} = 0$, consequently for large enough $k = l$ the probability $\mathrm{Prob}(D$ is non-defeater for $A, B)$ is very close to 0, way much smaller than ${}^1\!/_2$. We conjecture that $\lim_{k+l\to\infty} \sum_{n \in G(k,l)} \binom{k}{n}\binom{l}{nl/k} = 0$ and thus the argument that $\mathrm{Prob}(D$ is non-defeater for $A, B)$ is negligible can be extended to all the cases, irrespective of the value of k and l. We do not pursue this issue any further here.

two co-primes from such a set is $6/\pi^2$ (Hardy & Wright, 2008, Theorem 332, p. 354), that is, about .61. Elaborated number theoretical results[5] show that this probability is always strictly larger than .5. The situation, then, is more likely than not!

To recap: if you are choosing two events at random from a finite probability space (say, you go over all elements of the sample space and individually decide to include them in the given event or not depending on a fair coin toss) with the uniform measure, you should expect ending up with two events such that whatever third event you draw, it is more often than not a defeater for them.

There is at least one obvious generalization which the Reader might feel to be needed at this point, namely, one could loosen the uniformity assumption regarding the measure. This we have not attempted. We do not wish to consider these issues here, because our results above already persuasively suggest the main philosophical point we would like to make: before investigating any defeater-related phenomena in classical probability spaces, and certainly before we allow ourselves to be surprised by a general theorem regarding the apparent abundance or lack of defeaters, we should make sure that the relationships between the events involved are not such that they would trivialize the matters. Who knows? Maybe for some deep metaphysical reason the cardinalities of the proposition A and any proposition about the chance of A are such that almost any event will be a defeater for them (assuming uniformity of the measure; we already know that most of the events are defeaters for them). Can we exclude that? Should we? We don't know, but in our opinion it seems a lot more fruitful to move away from classical probability spaces and towards structures tailored for meta-level probabilistic phenomena. An example of such an approach is that of Higher Order Probability spaces from Gaifman (1988), to which we now turn.[6]

3. Defeaters in Higher-Order Probability Spaces

We will abbreviate "Higher-order probability space" with "HOP". To quote (Gaifman, 1988, p. 197), a HOP is a 4-tuple $(W, \mathcal{F}, P, pr)$—where $\mathcal{F}$ is a field of subsets of W, to be called "events", P is a probability over $\mathcal{F}$ and pr is a mapping associating with every $A \in \mathcal{F}$ and every real closed interval Δ an event $pr(A, \Delta)$—which satisfies axioms (I)-(V) below. The initially intended interpretation is that P is the agent's subjective probability and $pr(A, \Delta)$ is the event that "the expert probability of A lies in Δ".

We will adopt the convention that $\mathbf{1} = W$ and $\mathbf{0} = \emptyset$ and will omit the curly brackets when dealing with singletons without further commentary (e.g., $pr(w_1, [.5, .6])$ means $pr(\{w_1\}, [.5, .6])$). Also, closed intervals may be single points, and when talking about such cases we will use α instead of Δ: $pr(A, \alpha)$ makes sense for $\alpha = .3$ and means then the same thing as $pr(A, [.3, .3])$.

Each HOP satisfies the following five axioms, which are actually axiom schemes, with implicit universal quantification over A:

(I) $pr(A, [0, 1]) = pr(W, [1.1]) = \mathbf{1}$;

[5] Namely, results by Mertens 1874 and Erdős and Shapiro 1951. Mertens proves that the number of coprimes in $\{1, \dots n\}$ is $S(n) = \frac{6}{\pi^2}n^2 + O(n \log n)$. The probability of two coprimes is then $P(n) = \frac{S(n)}{n^2}$. According to Erdős and Shapiro, the error function $S(n) - \frac{6}{\pi^2}n^2$ is in between $\pm cn \log \log \log \log n$ for some constant c. Then, the error of $P(n)$ is smaller than $\frac{\log \log \log \log n}{n}$, which is smaller than .005 for all $n > 1$. Thus $P(n)$ is always larger than $\frac{6}{\pi^2} - .005 > .5$.

[6] Note that the paper, despite the identical title, is not a simple reprint of Gaifman (1986). We recommend reading the later version, even if it is the earlier one which has been carefully LaTeXed and recently reprinted in Arló-Costa et al. (2016).

(II) $pr(A, \emptyset) = \mathbf{0}$;

(III) If $\Delta_1 \cup \Delta_2$ is an interval, then

$$pr(A, \Delta_1 \cup \Delta_2) = pr(A, \Delta_1) \cup pr(A, \Delta_2);$$

(IV) $\cap_n pr(A, \Delta_n) = pr(A, \cap_n \Delta_n)$;

(V) If, for all $n \neq m$, $A_n \cap A_m = \emptyset$, then:

$$\cap_n pr(A_n, [\alpha_n, \beta_n]) \subseteq pr(\cup_n A_n, [\textstyle\sum_n \alpha_n, \sum_n \beta_n]).$$

Let us briefly mention two points which might not be evident on first reading Gaifman's paper:

- even though the author writes about extending the function *pr* so that it would deal with arbitrary (Borel) subsets of $\mathbb{R}$ (p. 199), the axioms only work for convex sets, and in fact the only generalization explicitly considered is one which allows the Δ to be a half-open or open interval (p. 201);
- Gaifman calls anything which satisfies axioms (I)-(V) a HOP, even if it does not satisfy (VI) or (VI_w), that is, even if no condition connecting P and *pr* is stipulated to hold in the structure under consideration.

HOPs become workable when we deal with their *kernels*. A kernel of a HOP (having finite W) is a $|W| \times |W|$ matrix of real numbers interpreted as a mapping p which associates with each $x \in W$ a probability p_x over $\mathcal{F}$ such that

$$pr(A, \Delta) = \{x : p_x(A) \in \Delta\}. \tag{6}$$

In a kernel, a row corresponding to some $x \in W$ contains the values of p_x for all singletons of elements of W (in effect defining p_x as a probability on $\mathcal{F}$; a kernel is, then, a *stochastic* matrix, that is, each entry is non-negative and the sum of each row is 1). That axioms (I) - (V) suffice for the existence of kernels so understood, with p_x's connected with *pr* as stated in equation (6), is the subject of Gaifman's Theorem 1. For later purposes we introduce the notion of a *reduced kernel*. Let W' contain $x \in W$ if and only if $P(x) = 0$. The reduced kernel is obtained from the kernel matrix by deleting columns and rows corresponding to $x \in W'$. The reduced kernel is thus the kernel "modulo P-zero".

For a formally trivial, but perhaps conceptually beneficial observation, note that for each choice of A and Δ, $pr(A, \Delta)$ is a single event (possibly the empty set). The answer to, say, "Is the fact that the expert probability of B lies in $[.2, .4]$ represented by one or more distinct events in $\mathcal{F}$?" is "By one and only one".

We will now introduce formally the notion of a defeater in the context of HOPs—a "HOP-defeater".

Definition 2. Assume a HOP $(W, \mathcal{F}, P, pr)$ is given. For any $A \in \mathcal{F}$ and closed real interval Δ such that $P(pr(A, \Delta)) > 0$ an event $D \in \mathcal{F}$ is called a

- *non-trivial HOP-defeater for A and Δ* if $P(pr(A, \Delta) \cap D) > 0$ and

$$P(A \mid pr(A, \Delta) \cap D) \notin \Delta; \tag{7}$$

- *trivial HOP-defeater for A and Δ* if $P(pr(A,\Delta) \cap D) = 0$.

D is a HOP-defeater for A and Δ if it is either a trivial or a non-trivial HOP-defeater for A and Δ.
Note that the conventions we adopted dictate that a D is a HOP-defeater for A and α if $P(A \mid pr(A,\alpha) \cap D) \neq \alpha$.

What questions can be meaningfully asked about defeaters in a given HOP depends on whether it satisfies any of the following two axioms (p. 200), of which (VI) logically implies (VI_w):

(VI) If C is a finite intersection of events of the form $pr(B_i, \Delta_i)$, and if $P(pr(A,\Delta) \cap C) \neq 0$, then:

$$P(A \mid pr(A,\Delta) \cap C) \in \Delta.$$

(VI_w) If $P(pr(A,\Delta)) \neq 0$, then:

$$P(A \mid pr(A,\Delta)) \in \Delta.$$

If we consider single point intervals, (VI_w) becomes Miller's Principle: $P(A \mid pr(A,\alpha)) = \alpha$.
Note that if a HOP satisfies (VI), then its reduced kernel is an idempotent matrix (i.e. it is equal to its own square)[7]. The reduced kernel of a HOP satisfying (VI) is, then, an idempotent stochastic matrix.[8]

Now, if any event C is a finite intersection of events of the form $pr(B_i, \Delta_i)$, then if the HOP satisfies (VI), no non-trivial questions about defeaters can be asked. It will already be interesting to consider which events are not of the form $pr(A,\Delta)$ for any possible choice of A and Δ:

Definition 3. Assume a HOP $H = (W, \mathcal{F}, P, pr)$ is given. We will say that an event $E \in \mathcal{F}$ is *metarepresentable in H* if and only if there exist $A \in \mathcal{F}$ and a closed real interval Δ such that $E = pr(A,\Delta)$.

Example 1. Consider a HOP $(W, \mathcal{F}, P, pr)$ with $W = \{w_1, w_2, w_3\}$, $\mathcal{F} = \mathcal{P}(W)$, P uniform, and pr given by the following kernel:

$$\begin{matrix} .5 & .5 & 0 \\ 0 & .5 & .5 \\ .5 & 0 & .5 \end{matrix}$$

(This is Gaifman's Example 1, p. 208). A direct check will ensure the Reader that in this HOP all events are metarepresentable. Axiom (VI) fails, while axiom (VI_w), and so Miller's principle, holds.

It turns out that satisfaction of axiom (VI) is enough for some events not to be metarepresentable, and what's more, for some events not to be intersections of families of metarepresentable

[7] We stress that the result predicates idempotence of the reduced kernel only. Example 2 (Gaifman, 1988, p. 206) features a HOP which satisfies (VI) and has a kernel which is not idempotent but its reduced version is.

[8] If the HOP satisfies axiom (VI_w), then P (considered as a row-vector) is a left-eigenvector of the kernel matrix corresponding to eigenvalue 1. There is a serious typo in (3_d) of Lemma 1 in Gaifman (1988). The correct equation is $P(y) = \sum_x p(x,y) \cdot P(x)$, thus P is a left-eigenvector of the matrix p, instead of being a (right-)eigenvector. A HOP satisfies axiom (VI) if and only if its *reduced* kernel is idempotent and P is a left-eigenvector of the kernel (Gaifman, 1988, Theorem 2).

events. In what follows we recall (Högnäs & Mukherjea, 2011, Theorem 1.16) about the shape of idempotent stochastic matrices. Let M be a $d \times d$ idempotent stochastic matrix of rank $k > 0$. Then there is a unique partition of $\{1, \ldots, d\}$ into classes $\{T, C_1, \ldots, C_k\}$[9] such that the following hold (see Figure 1):

1. $T = \{i : \text{ the } i^{\text{th}} \text{ column of } M \text{ is a zero column}\}$
2. $M\restriction_{C_s \times C_s}$ has identical positive rows of sum 1 and $M\restriction_{C_s \times C_t} = 0$ for $s \neq t$
3. If $i \in T$, then

$$\frac{M_{ij}}{M_{jj}} = \frac{M_{ih}}{M_{jh}}, \quad j, h \in C_s \tag{8}$$

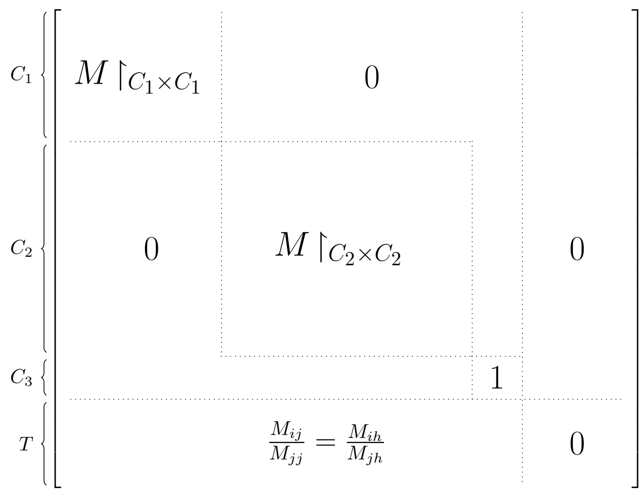

Figure 1: The form of an idempotent stochastic matrix of rank 3

(Notice that T or the C_i's might contain non-consecutive numbers (rows). In this sense Figure 1 might be a little misleading, as the blocks $M\restriction_{C_1 \times C_1}$ might not be "connected". On the other hand, there always exists a permutation of $\{1, \ldots, d\}$ such that after it is applied to the rows and columns of the matrix M, the matrix achieves the shape as displayed in Figure 1.)

The "partition" $\{T, C_1, \ldots, C_k\}$ can be of course thought of as a set of sets of rows of M, which will be the reading we ask the Reader to adopt in the proof of the following lemma.

Lemma 1. *Assume a HOP $(W, \mathcal{F}, P, pr)$ is given. If this HOP satisfies (VI) and its reduced kernel is not the identity matrix, then some event E in $\mathcal{F}$ with $P(E) > 0$ is not metarepresentable.*

Proof. Since the HOP satisfies (VI), then, as already mentioned, its reduced kernel M is an idempotent stochastic matrix. Take the partition $\{T, C_1, \ldots, C_k\}$ of the reduced kernel according to (Högnäs & Mukherjea, 2011, Theorem 1.16). We note first that T of this partition must be empty. As axiom (VI) is satisfied, P is the mixture of the rows of the kernel. Suppose the i^{th} column of the kernel is filled with 0's. Then $P(w_i)$ must be zero as it is a mixture of 0's. It follows that when creating the reduced kernel we delete the i^{th} row and column of the kernel.

[9] *Caveat*: as will be clearly visible soon, T may be empty, so this is not a partition in the strict set-theoretic sense.

As the reduced kernel is not the identity matrix, it is impossible that $|C_s| = 1$ for all s. Thus, there is at least one C_s having at least two elements, say $i, j \in C_s$. As rows of C_s are identical, it is immediate to note from equation (6) that for any A and Δ, if w_i belongs to $pr(A, \Delta)$, then $w_j \in pr(A, \Delta)$, too. Thus $E = \{w_i\}$ is not metarepresentable. $\square$

(Note that, in contrast with (VI), (VI$_w$) does *not* imply non-metarepresentability: Example 1 describes a HOP which satisfies (VI$_w$), whose kernel contains no linearly dependent rows, and in which all events are metarepresentable.)

We now know that it follows from (VI) that some events are not metarepresentable. But to assess whether the notion of a HOP-defeater is not trivialised, we need to check whether some events are not finite intersection of metarepresentable events (which is a weaker condition, since *a priori* some events which are not themselves metarepresentable might be finite intersections of metarepresentable ones). Fortunately this is also the case in general. To see this, define a function $pr : W \to \mathcal{F}$ as follows:

$$pr(w) := \{pr(A, \Delta) : w \in pr(A, \Delta)\}. \tag{9}$$

If in the given HOP's kernel rows corresponding to distinct worlds w and v are identical, then $p_w = p_v$ and so for any $A \in \mathcal{F}$ and closed real interval Δ it holds that $w \in pr(A, \Delta)$ iff $v \in pr(A, \Delta)$. That is, when w and v correspond to identical rows, $pr(w) = pr(v)$. Then the singleton $\{w\}$ is not only not metarepresentable but also is not an intersection of metarepresentable events: if it was, then for some family of events of the form $pr(B_i, \Delta_i)$ the singleton $\{w\}$ would be the set of all elements belonging to each event from the aforementioned family, but then that singleton would also contain v: contradiction. We know, then, that if a HOP satisfies (VI), then if its kernel contains identical rows, it follows that some events are not intersections of metarepresentable events.

For the remaining case, assume a HOP satisfies (VI) but its kernel does not contain identical rows. Consider its reduced kernel and its partition $\{T, C_1, \dots, C_k\}$ from (Högnäs & Mukherjea, 2011, Theorem 1.16) which we have just used. Recall that in this case, since we are talking about the reduced kernel, T is empty. Since no rows are identical, the reduced kernel is the identity matrix. We have, then, established the following Fact:

Fact 1. *Assume a HOP $(W, \mathcal{F}, P, pr)$ is given. If this HOP satisfies (VI) and its reduced kernel is not the identity matrix, then some event E in $\mathcal{F}$ with $P(E) > 0$ is not an intersection of metarepresentable events.*

On the one hand, Fact 1 brings a positive message: assuming (VI)—which, if P is interpreted as degree of belief, is a requirement of rationality, for which a coherence argument in the form of a Dutch Book construction is given in Gaifman's paper on p. 201-204, and which is similar in spirit to Lewis' Principal Principle (see p. 201 of Gaifman's paper and its 4^{th} footnote)—guarantees that problems regarding defeaters will not be trivialised just by structural infelicities. To be sure, some cardinality-related troubles similar to the ones we talked about in the previous section might probably be reformulated in the context of HOPs. Such problems, if they are found, will not, however, originate from the label "the proposition that the chance of A belongs to Δ" being slapped on some more or less arbitrarily chosen event. On the contrary, in the context of HOPs we can say something substantial about the relationship of A and $pr(A, \Delta)$. A big part of it is captured by (VI$_w$): P has to satisfy a certain conditional probability requirement. Axiom (VI) adds to this the requirement that certain events cannot be HOP-defeaters: namely, events of the form $pr(B_i, \Delta_i)$ for some B_i and Δ_i—that is, metarepresentable events—and finite conjuctions of them. (In the

Lewisian parlance: chance information is always admissible.) All remaining phenomena regarding HOP-defeating will depend on the particular HOP; at this moment it is not clear whether there are any worrying general results on the horizon (that e.g. all events of some seemingly innocent form would be HOP-defeaters, similarly to what Wallmann & Hawthorne suggest for the classical probabilistic variant of the notion), while the interested parties can continue philosophizing assured that they have a formal grasp of how a degree of belief function of a subject who satisfies the Principal Principle looks like, without resorting to hand-waving about admissibility or being informal when talking about how some events are actually "about" chances of other events.

On the other hand, if, as suggested by Gaifman himself (p. 193), we take rows to signify objective chance functions at various possible worlds, we seem to have stumbled on a weird consequence: an *a priori* constraint on the possible relationships between objective chance functions as considered by a rational subject. We are planning to revisit these issues in a paper devoted exclusively to the details of HOPs, and in the current article we will now return to the "admissibility troubles" which according to Wallmann & Hawthorne supposedly plague Bayesian accounts of direct inference.

4. Admissibility troubles revisited

Recall the problem posed in Wallmann & Hawthorne (2018): Bayesian accounts of direct inference seem to be in trouble, since seemingly innocent propositions turn out to be defeaters. In Section 2.1 of their paper the authors tell a story about supposedly inadmissible biconditionals. If a fair pair of dice is tossed on a flat surface in a fair way, the chance that the outcome of the toss is seven is $1/6$. Maria, the subject, forms a direct inference, and sets her credence in that the outcome of the next toss is seven conditional on that setup to $1/6$. Nothing out of ordinary so far.

But now John says to Maria "I'll buy you dinner this evening if and only if the next toss comes up seven". A surprising claim of Wallmann & Hawthorne is that if Maria conditionalises additionally on *that*, her credence in that the next toss comes up seven moves away from $1/6$. The result is that the seemingly innocent biconditional ends up being a defeater for the probability that the next toss comes up seven given that the chance setup is like described above—which is highly unintuitive. Such biconditionals should not be defeaters for such probabilities; in this simple case, because we can assume that Maria believes that having dinner with John in the future is not probabilistically relevant for the outcome of the toss. We will now see that this biconditional is not a defeater if we think about the situation using HOPs.

Example 2. We will capture the original example from Section 2.1 of Wallmann & Hawthorne (2018) in a 4-world HOP. The worlds are as follows:

	next toss is 7?	John buys dinner?
w_1	yes	yes
w_2	no	yes
w_3	yes	no
w_4	no	no

and the kernel is as follows:

$$\begin{matrix} 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{matrix}$$

with P being any mixture of the rows, ending up of course as the vector $(^1/_{12}, ^5/_{12}, ^1/_{12}, ^5/_{12})$. Call the proposition "next toss comes up seven" N and "John buys Maria dinner" D. From the setup we see immediately that $N = \{w_1, w_3\}$, $D = \{w_1, w_2\}$, $N \leftrightarrow D = \{w_1, w_4\}$, $pr(N, ^1/_6) = W$, and so $P(N \mid pr(N, ^1/_6)) = ^1/_6$, in accordance with (VI_w). What's more, $N \leftrightarrow D$ is not a HOP-defeater for N and $^1/_6$, since $P(N \mid pr(N, ^1/_6) \cap N \leftrightarrow D) = ^1/_6$, just like it should.

The approach is easily generalised in a number of natural aspects. For example, Maria can entertain various hypotheses about the chance function, can have an arbitrary prior for D, and may but also may not assume initially that D is probabilistically independent from the toss result or from which chance function the dice are governed by. Since describing an appropriate HOP at that level of generality would require the production of tediously detailed calculations which in themselves would be quite mundane (which is as it should be, since the situation modelled falls short of being spectacular), we will show now only a modest generalisation: it concerns Maria who entertains two hypotheses about the possible chance function governing the dice, but keeps e.g. the assumption that according to her prior John buying dinner is probabilistically independent from the toss result, and that her prior of him buying dinner is .5. Both of these requirements can be relaxed by the interested Reader.

Example 3. Assume the setup is similar, but now Maria considers two chance hypotheses: that the dice are fair (which she gives credence $^2/_3$) or that they are skewed so that the chance of the next toss coming up seven is $^1/_4$ (let us call it the "skewed chance function"). We will use 8 worlds, which are as follows:

	next toss is 7?	John buys dinner?	chance?
w_1	yes	yes	fair
w_2	no	yes	fair
w_3	yes	no	fair
w_4	no	no	fair
w_5	yes	yes	skewed
w_6	no	yes	skewed
w_7	yes	no	skewed
w_8	no	no	skewed

and the kernel is as follows:

$$\begin{matrix} ^1/_{12} & ^5/_{12} & ^1/_{12} & ^5/_{12} & 0 & 0 & 0 & 0 \\ ^1/_{12} & ^5/_{12} & ^1/_{12} & ^5/_{12} & 0 & 0 & 0 & 0 \\ ^1/_{12} & ^5/_{12} & ^1/_{12} & ^5/_{12} & 0 & 0 & 0 & 0 \\ ^1/_{12} & ^5/_{12} & ^1/_{12} & ^5/_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ^1/_8 & ^3/_8 & ^1/_8 & ^3/_8 \\ 0 & 0 & 0 & 0 & ^1/_8 & ^3/_8 & ^1/_8 & ^3/_8 \\ 0 & 0 & 0 & 0 & ^1/_8 & ^3/_8 & ^1/_8 & ^3/_8 \\ 0 & 0 & 0 & 0 & ^1/_8 & ^3/_8 & ^1/_8 & ^3/_8 \end{matrix}$$

with P being a mixture of the rows of the kernels with weights $^1/_6$ (for the first four rows) and $^1/_{12}$ (for the next four rows), ending up with the vector $(^1/_{18}, ^5/_{18}, ^1/_{18}, ^5/_{18}, ^1/_{24}, ^3/_{24}, ^1/_{24}, ^3/_{24})$. As before, call the proposition "next toss comes up seven" N and "John will buy Maria dinner" D. From the setup we see immediately that $N = \{w_1, w_3, w_5, w_7\}$, $D = \{w_1, w_2, w_5, w_6\}$, $N \leftrightarrow D = \{w_1, w_4, w_5, w_8\}$, $pr(N, ^1/_6) = \{w_1, w_2, w_3, w_4\}$, $pr(N, ^1/_4) = \{w_5, w_6, w_7, w_8\}$ and so $P(N \mid pr(N, ^1/_6)) = ^1/_6$ and

$P(N \mid pr(N, 1/4)) = 1/4$, in accordance with ($\text{VI}_w$). What's more, $N \leftrightarrow D$ is not a HOP-defeater for N and $1/6$, since $P(N \mid pr(N, 1/6) \cap N \leftrightarrow D) = 1/6$; it is also not a HOP-defeater for N and $1/4$, since $P(N \mid pr(N, 1/4) \cap N \leftrightarrow D) = 1/4$, just like it should.

In our opinion all the intuitive independencies are modelled by this HOP and no unintuitive dependencies are introduced by conditionalising on the $N \leftrightarrow D$ proposition. Consider, still, that D is initially probabilistically independent of whether the chance function is fair or skewed, but ends up being dependent after conditionalization on $N \leftrightarrow D$. This is as it should be: Maria may initially think that whether John buys her dinner has no bearing on the toss result, and *vice versa*, but once John makes his promise, the details of the chances start being relevant to the dinner plans!

5. Conclusion

In this paper we believe we have shown that the supposed "admissibility troubles" for Bayesian accounts of direct inference proposed by Wallmann & Hawthorne are an illusion. One possible source of it might be the usage of classical probability spaces unimbued with any additional structure; the practice which, as we argue in Section 2, should be abandoned in any discourse dealing with probabilities of propositions about the probabilities of other propositions. We have pointed out that there is at least one meta-probabilistic framework on the philosophical market, the HOP approach by Gaifman, which can model the supposedly problematic situations without generating the troublesome consequences.

Bayesianism may, of course, be deeply mistaken, but not for the reasons suggested by Wallmann & Hawthorne (2018): it can model direct inference, given that we provide it enough structure for that task.

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