Against Eliminating Sorts

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As is well-known, any theory formulated with sort symbols is equivalent to a theory formulated without sort symbols.[1] In fact, this fact is so well-known that most philosophers, even the technically inclined, ignore sort symbols and the role they play in our theoretical representations.

What is less well-known is that the recipe for converting a many-sorted theory to an unsorted theory is not uniquely determined. In other words, a many sorted theory can be converted in different ways into an unsorted theory, and most metaphysicians would take the resulting unsorted theories to be strongly inequivalent to each other.

It will be easiest to understand the issue by looking at a specific example. Let T be the theory that has two sort symbols: σ_0 for physical objects, and σ_1 for mathematical objects. Suppose, moreover, that T says that there are baboons: $\exists x B(x)$. We will take T to specify that both sorts σ_0 and σ_1 are non-empty, but otherwise we will leave it open for T to be developed in greater detail. (No amount of further detail will change the upshot of this discussion.)

Those who would eliminate sorts now tell us that there is an obvious way to convert T into an unsorted theory: add predicates U_0 for physical objects, U_1 for mathematical objects, and then add the axiom "no mathematical objects are baboons". Quine says as much explicitly:

> ...the many-sorted [logic] is translatable into one-sorted. Generally such translation has the side effect of admitting as meaningful some erstwhile meaningless predications. E.g. if the predicate

[1] For an explicit proof see (Barrett and Halvorson, 2017). But this fact has been known by logicians since at least the 1950s.

> "divisible by 3" is henceforth to be trained on general variables instead of number variables, we must make sense of calling things other than numbers divisible by 3. But this is easy; we may count such attributions false instead of meaningless. ... Carnap's reservations over *Allwörter* now cease to apply, and so his special strictures against philosophical questions of existence lapse as well. (Quine, 1969, p 96)

A similar sentiment is expressed by Peter Geach in reference to a case where the predicate "hungry" applies to the sort "animal".[2]

> Let us now introduce an artificial predicate "hungery"; "hungery" is to be true of just what "hungry" is true of, "not hungery" is to be true of whatever "not hungry" is true of and also of whatever "animal" is not true of. Surely the new word will allow us to say whatever we wanted to say with the old word; and it cuts out some troublesome restrictions on the use of the old word (trouble-some to a formal logician, that is). (Geach and Bednarowski, 1956, p 67)

These sort-eliminators are correct that the recommended procedure yields a theory T^- that is, in one precise sense, equivalent to the original theory T. What they glossed over, however, is the question of whether the unsorted theory that they offer is the *unique* unsorted representation of the content of T. In one important sense, it is not.

Indeed, there is another unsorted theory T^+ that is also equivalent to T. In particular, instead of adding the axiom "no mathematical objects are baboons", it would do just as well to add the axiom "all mathematical objects are baboons". From a logical point of view, this second procedure (positive extension) is just as impeccable, and no less well-motivated, than the first procedure (negative extension). Nothing in the logical situation tells us to prefer the negative extension T^- over the positive extension T^+. In fact, to see the symmetry clearly, imagine that the original theory T had also specified that not every physical thing is a baboon, i.e. some things are not-baboons. If one attempted to follow the Geach-Quine recommendation of always extending negatively, then we should also extend the predicate "is a non-baboons" negatively, which would yield the axiom: "no mathematical

[2]Thanks to Dana Goswick for bringing Geach's statement to my attention.

objects are non-baboons", or in other words, "all mathematical objects are baboons". Thus, if we apply negative extension to "is a non-baboon" then we get a positive extension of "is a baboon".

Nor will it help Geach and Quine to say that we should apply negative extension to elementary predicates, but not to negated predicates. Indeed, our original theory T could very well have had another elementary predicate $A(x)$ and an axiom of the form $\forall x(B(x) \leftrightarrow \neg A(x))$. In this case, attempting to apply negative extension to both B and A would lead to inconsistency.

Thus, there are two theories T^- and T^+ with equal rights to be called the unsorted version of the theory T. How should we choose between them? Here I can offer some consolation: the theories T^- and T^+ are themselves equivalent — at least if one allows oneself the more liberal notion of Morita equivalence (see Barrett and Halvorson, 2016). This equivalence was, of course, to be expected: since both T^- and T^+ are Morita equivalent to T, and Morita equivalence is symmetric and transitive, T^- and T^+ are Morita equivalent to each other. If one adopts Morita equivalence, then there is no puzzle left. If one adopts a more strict notion of equivalence, and if one wishes to eliminate sorts, then one has some work to do in determining the "one true way" of replacing a sorted theory with an unsorted theory.

References

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