Maximal Beable Subalgebras of Quantum-Mechanical Observables

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May 13, 1999

ABSTRACT:

Given a state on an algebra of bounded quantum-mechanical observables, we investigate those subalgebras that are maximal with respect to the property that the given state's restriction to the subalgebra is a mixture of dispersion-free states—what we call maximal *beable* subalgebras (borrowing terminology due to J. S. Bell). We also extend our results to the theory of algebras of unbounded observables (as developed by Kadison), and show how our results articulate a solid mathematical foundation for certain tenets of the orthodox 'Copenhagen' interpretation of quantum theory.

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1. Introduction

A number of results in the theory of operator algebras establish the impossibility of assigning simultaneously determinate values to all observables of a quantum system. Von Neumann first observed that the algebra of bounded operators $\mathfrak{L}(\mathcal{H})$ on a separable Hilbert space with $\dim \mathcal{H} > 1$ does not admit any dispersion-free normal state [36, p. 320], a result which was only somewhat later extended by Misra to arbitrary dispersion-free states [29, Cor. 2]. From general algebraic postulates for observables, and without recourse to a Hilbert space representation, Segal deduced that an algebra of quantum-mechanical observables possesses a *full set* of dispersion-free states if and only if it is commutative [34, Thm. 3]. Kochen and Specker [25] relaxed von Neumann's requirement that values of observables be given by a linear functional on $\mathfrak{L}(\mathcal{H})$, and regarded the latter as, instead, a partial algebra, with the product and sum of two elements defined only if they commute. For $2 < \dim \mathcal{H} < \infty$, they constructed finitely generated partial subalgebras of self-adjoint operators in $\mathfrak{L}(\mathcal{H})$ that possess no *partial* dispersion-free states. This result was established independently by J. S. Bell [3] (cf. [12]), who also noticed that the nonexistence of a partial dispersion-free state on *all* self-adjoint elements of $\mathfrak{L}(\mathcal{H})$ is an immediate corollary of Gleason's theorem [18]. More recently, these results have been extended to the case of infinite-dimensional $\mathcal{H}$, by reduction to the finite case [21, 2]; and, in the latter case, many more examples have been uncovered of partial subalgebras of observables without partial dispersion-free states (see [30, Ch. 7], [10, Ch. 3] for reviews).

Evidently none of these negative results settle the positive question of which subalgebras of quantum-mechanical observables (apart from commutative ones) *can* be taken to have simultaneously determinate values. Bell [3, Sec. 3], a well-known critic of the foundational importance of von Neumann's (and, indeed, his own) result (cf. [15, Sec. 3]), was the first to raise the importance of this positive question. With the aim of avoiding primitive reference to the term 'measurement' in the axiomatic foundations of quantum theory, Bell forcefully argued (see also [5, Chs. 7,19]) that it ought to suffice to assign simultaneous values to some appropriate proper subset of all quantum-mechanical observables—which he distinguished from the latter by calling them '*be*ables': "Could one not just promote *some* of the observables of the present quantum theory to the status of beables? The beables would then be represented by linear operators in the state space. The values which they are allowed to *be* would be the eigenvalues of those operators. For the general state the probability of a beable *being* a particular value would be calculated just as was formerly calculated the probability of *observing* that value" [4, p. 688]. Bell's remarks here suggest the following problem (that has received scant attention in the mathematical literature; but see [37, 15]): Given a state on an algebra of observables, characterize those subalgebras, of 'beables,'

that are maximal with respect to the property that the given state's restriction to the subalgebra is a mixture of dispersion-free states. Such *maximal beable subalgebras* could then represent maximal sets of observables with simultaneously determinate values distributed in accordance with the state's expectation values. The aim of the present paper is to investigate maximal beable subalgebras and establish their importance for the foundations of quantum theory. Later on, we shall extend our analysis of maximal beable subalgebras to include the case where sets of *un*bounded observables are assigned simultaneously determinate values consistent with a state's expectation values. Though some open problems remain, our results suffice to articulate certain aspects of the orthodox 'Copenhagen' interpretation of quantum theory (such as the joint indeterminacy of canonically conjugate variables, and Bohr's defense of the 'completeness' of quantum theory against the argument of Einstein-Podolsky-Rosen) in a mathematically rigorous way.

1.1. **From JLB- to C^*-algebras.** In the first instance, our investigation concerns *algebras* of bounded quantum-mechanical observables, which immediately raises the question of what sort of algebraic structure should be assumed. Following [26, 27, 15], we choose to regard the observables of a quantum system as a JLB-algebra. In brief, a JLB-algebra is any real Banach space $(\mathfrak{X}, \|\cdot\|, \circ, \bullet)$ such that the Jordan product $\circ$ is symmetric, the Lie product $\bullet$ is antisymmetric and satisfies the Jacobi identity, $\bullet$ is a derivation with respect to $\circ$, and $\circ$ and $\bullet$ together respect the associator identity:

$$(A \circ B) \circ C - A \circ (B \circ C) = r((A \bullet C) \bullet B), \tag{1.1}$$

for some $r \in [0, \infty)$. Moreover, defining $A^2 \equiv A \circ A$ the norm on $\mathfrak{X}$ must satisfy

$$\|A \circ B\| \leq \|A\| \, \|B\|, \quad \|A^2\| = \|A\|^2, \quad \|A^2\| \leq \|A^2 + B^2\|, \tag{1.2}$$

for all $A, B \in \mathfrak{X}$.

A JLB-algebra $\mathfrak{X}$ has a positive cone $\mathfrak{X}^+$ consisting of elements of the form $\{A^2 : A \in \mathfrak{X}\}$. A linear functional ρ of $\mathfrak{X}$ is said to be *positive* just in case $\rho(A) \geq 0$, for all $A \in \mathfrak{X}^+$. If $\mathfrak{X}$ has a unit I, a positive linear functional ρ of $\mathfrak{X}$ is said to be a state just in case $\rho(I) = 1$.

We have not provided any sort of axiomatic or operational derivation to justify our choice of JLB-algebras over the various other sorts of algebraic structures we might have used. For example, we might well have chosen to set our investigation in the context of Segal algebras [34], which admit neither a Lie product nor a *distributive* Jordan product. However, the choice of JLB-algebras has an extremely strong pragmatic justification, owing to the fact that the theory of JLB-algebras (unlike the case of Segal algebras—see [28]) may essentially be reduced to the theory of C^*-algebras.

First, if $\mathfrak{X}$ is a JLB-algebra, its complex span $\mathfrak{X}_{\mathbb{C}}$ is canonically isomorphic to a C^*-algebra. In particular, for $A, A' \in \mathfrak{X}$, we define a C^* product by

$$AA' \equiv (A \circ A') - i\sqrt{r}(A \bullet A'), \tag{1.3}$$

and, for $A + iB, A' + iB' \in \mathfrak{X}_{\mathbb{C}}$,

$$(A + iB)(A' + iB') \equiv (AA' - BB') + i(AB' + BA'). \tag{1.4}$$

(The associativity of the C^* product follows from the Jacobi and associator identities together with the fact that $\bullet$ is a derivation with respect to $\circ$.) We define an involution $*$ on $\mathfrak{X}_{\mathbb{C}}$ by

$$(A + iB)^* \equiv A - iB. \tag{1.5}$$

It can then be shown that the norm on $\mathfrak{X}$ extends uniquely so that $(\mathfrak{X}_{\mathbb{C}}, \|\cdot\|)$ is a C^*-algebra (see [26, Sec. 3.8]).

Conversely, if $(\mathfrak{A}, \|\cdot\|)$ is a C^*-algebra, then the set of self-adjoint elements of $\mathfrak{A}$, denoted by $\mathfrak{A}_{\mathrm{sa}}$, forms a real Banach space with norm $\|\cdot\|$. We can then equip $\mathfrak{A}_{\mathrm{sa}}$ with a Jordan product $\circ$ defined by

$$A \circ B \equiv \frac{1}{2}[A, B]_+ = \frac{1}{2}(AB + BA), \tag{1.6}$$

and with a Lie product $\bullet$ defined by

$$A \bullet B \equiv \frac{i}{2}[A, B] = \frac{i}{2}(AB - BA). \tag{1.7}$$

The resulting object $(\mathfrak{A}_{\mathrm{sa}}, \|\cdot\|, \circ, \bullet)$ can then be shown to satisfy the axioms which define a JLB-algebra [17, 26, 27].

Recall that a state of a C^*-algebra is a positive linear functional of norm-1. It then follows that there is a natural bijective correspondence between states of a C^*-algebra $\mathfrak{A}$ and the states of the JLB-algebra $\mathfrak{A}_{\mathrm{sa}}$. Indeed, if ω is a state of $\mathfrak{A}$, then $\omega|_{\mathfrak{A}_{\mathrm{sa}}}$ is a state of $\mathfrak{A}_{\mathrm{sa}}$. Conversely, if ρ is a state of $\mathfrak{A}_{\mathrm{sa}}$, then the unique linear extension of ρ to $\mathfrak{A}$ is a state of $\mathfrak{A}$. (That the extension is indeed a state follows from the fact that any positive element in a C^*-algebra is the square of a self-adjoint element.)

We note two further parallels between JLB- and C^*-algebras:

(i) A JLB-algebra $\mathfrak{X}$ is called *abelian* just in case $A \bullet B = 0$ for all $A, B \in \mathfrak{X}$. Clearly, $\mathfrak{X}$ is abelian if and only if $\mathfrak{X}_{\mathbb{C}}$ is an abelian C^*-algebra.

(ii) Let $\mathfrak{A}$ be a *concrete* C^*-algebra, acting on some Hilbert space $\mathcal{H}$. Let $\mathfrak{A}^-$ denote the weak-operator topology (WOT) closure of $\mathfrak{A}$ in $\mathfrak{L}(\mathcal{H})$. It then follows easily (from the WOT-continuity of $*$ and von Neumann's double commutant theorem) that $(\mathfrak{A}_{\mathrm{sa}})^- = (\mathfrak{A}^-)_{\mathrm{sa}}$. Consequently, $\mathfrak{A}$ is a von Neumann algebra if and only if $\mathfrak{A}_{\mathrm{sa}}$ is WOT-closed.

In the remainder of this paper, then, we will carry out our inquiry in the setting of the theory of C^*- and von Neumann algebras. If the reader is disturbed by the use of complex *-algebras in a discussion of assigning determinate values to quantum-mechanical *observables* (which, of course, have to be self-adjoint), the above results can be used to translate what follows into the language of JLB-algebras.

1.2. **Dispersion-free states.** Let $\mathfrak{A}$ be a unital C^*-algebra and let ω be a state of $\mathfrak{A}$. Following [1, p. 304], we define the *definite algebra* of ω by:

$$\mathfrak{D}_\omega \equiv \{A \in \mathfrak{A} : \omega(AX) = \omega(A)\omega(X) \text{ for all } X \in \mathfrak{A}\}. \tag{1.8}$$

It is not difficult to show that $\mathfrak{D}_\omega$ is a unital subalgebra of $\mathfrak{A}$. (Indeed (cf. Exercise 4.6.16 in [23]) $\mathfrak{D}_\omega$ is none other than the complex span of the Kadison-Singer definite set [24, p. 398]:

$$\{A \in \mathfrak{A}_{\mathrm{sa}} : \omega(A^2) = (\omega(A))^2\} \tag{1.9}$$

which is the JLB-algebra canonically determined, via (1.6) and (1.7), by $\mathfrak{D}_\omega$.) For $A \in \mathfrak{A}$, we say that ω is *dispersion-free* on A just in case $A \in \mathfrak{D}_\omega$. If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, we say that ω is *dispersion-free* on $\mathfrak{B}$ just in case $\mathfrak{B} \subseteq \mathfrak{D}_\omega$. With this notation, we have the following:

Proposition 1.1.

   (i) *If $\mathfrak{B} \subseteq \mathfrak{D}_\omega$, then $\omega|_{\mathfrak{B}}$ is a pure state. The converse holds if $\mathfrak{B}$ is abelian.*
   (ii) *If $A \in \mathfrak{D}_\omega$, then $\omega(A) \in \mathrm{sp}(A)$.*

Proof. (i) follows immediately from Proposition 4.4.1 in [23], and the comments following the proof of that proposition. (ii) follows immediately from Remark 3.2.11 of [23]. ☐

Remark 1.2. The fact that $\mathfrak{L}(\mathcal{H})$ (with $\mathcal{H}$ nontrivial and separable) possesses no dispersion-free states can now be easily seen to follow from (i) and the fact (cf. [1, p. 305]) that commutators—i.e., operators expressible as $[X, Y]$ for some $X, Y \in \mathfrak{L}(\mathcal{H})$—are norm dense in $\mathfrak{L}(\mathcal{H})$. Note that if (ii) did not hold, it would not make physical sense to use the value of a dispersion-free state to represent the intrinsic, possessed value of an observable (assuming, that is, that when an observable with a determinate value is measured, its value is faithfully revealed by the result of the measurement).

In their partial algebraic approach, Kochen and Specker explicitly require, not just that (partial) dispersion-free states preserve the continuous functional relations between observables, but all *Borel* functional relations [25, Eqn. 4]. If one restricts to the case of observables on finite-dimensional spaces (as they eventually do [25, Sec. 3]), then this extra assumption is redundant, since every Borel function of such an observable is a polynomial function. However, we certainly want to allow as 'beables' observables with continuous

spectra, and also assign them (precise point) values via dispersion-free states. We end this section by showing how this allowance forces one to give up requiring that the values of beables preserve all Borel functional relations.

Definition. If $\mathfrak{V}$ is a von Neumann algebra, and ω is a dispersion-free state of $\mathfrak{V}$, we say that ω satisfies Borel-FUNC on $\mathfrak{V}$ just in case $\omega(f(A)) = f(\omega(A))$, for each $A \in \mathfrak{V}_{\mathrm{sa}}$ and each bounded Borel function f on $\mathrm{sp}(A)$.

Recall, a $*$-homomorphism Φ from von Neumann algebra $\mathfrak{V}_1$ to von Neumann algebra $\mathfrak{V}_2$ is called σ-*normal* when Φ maps the least upper bound of each increasing sequence of self-adjoint operators bounded above in $\mathfrak{V}_1$ onto the least upper bound of the image sequence in $\mathfrak{V}_2$. Recall also that a state ω on a von Neumann algebra $\mathfrak{V}$ is called *normal* just in case $\omega(H_a) \to \omega(H)$ for each monotone increasing net of self-adjoint operators $\{H_a\}$ in $\mathfrak{V}$ with least upper bound H. If ω is a dispersion-free state of $\mathfrak{V}$, then ω (being hermitian) is a $*$-homomorphism of $\mathfrak{V}$ onto the (von Neumann algebra of) complex numbers. Thus, a dispersion-free normal state ω of $\mathfrak{V}$ is a σ-normal homomorphism of $\mathfrak{V}$ onto $\mathbb{C}$.

Notation. If x is a unit vector in $\mathcal{H}$, ω_x denotes the *vector state* of $\mathfrak{L}(\mathcal{H})$ defined by $\omega_x(A) = \langle Ax, x \rangle$, for each $A \in \mathfrak{L}(\mathcal{H})$.

Theorem 1.3. *Let $\mathfrak{V}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, and let ω be a dispersion-free state of $\mathfrak{V}$. Then ω satisfies Borel-FUNC on $\mathfrak{V}$ if and only if there is a unit vector $x \in \mathcal{H}$ such that $\omega = \omega_x|_{\mathfrak{V}}$.*

Proof. "$\Leftarrow$" If $\omega = \omega_x|_{\mathfrak{V}}$, then ω is a normal state of $\mathfrak{V}$ (cf. [23, Thm. 7.1.12]). Since ω is (by hypothesis) dispersion-free on $\mathfrak{V}$, ω is a σ-normal homomorphism of $\mathfrak{V}$ onto $\mathbb{C}$, and the conclusion that ω satisfies Borel-FUNC on $\mathfrak{V}$ follows immediately from [23, Prop. 5.2.14].

"$\Rightarrow$" Suppose that ω satisfies Borel-FUNC on $\mathfrak{V}$. Let $\{P_a : a \in \mathbb{A}\}$ be any family of mutually orthogonal projections in $\mathfrak{V}$. Since $\mathcal{H}$ is separable, $\mathbb{A}$ must be countable, and we may assume that $\mathbb{A} = \mathbb{N}$. Let $A = \sum_{n=1}^{\infty} 3^{-n} P_n$. Since $\mathfrak{V}$ is SOT-closed, $A \in \mathfrak{V}_{\mathrm{sa}}$. Further, $f_n(A) = P_n$, where f_n is the characteristic function of the (singleton) set $\{3^{-n}\}$. Let f be the characteristic function of the (entire) set $\{3^{-n} : n \in \mathbb{N}\}$. Now, $\sum_{n=1}^{\infty} f_n = f$ in the sense of pointwise convergence of partial sums. Since the map $g \to g(A)$ from Borel functions on $\mathrm{sp}(A)$ into $\mathfrak{V}$ is a σ-normal homomorphism [23, p. 320], $\sum_{n=1}^{\infty} f_n(A) = f(A)$. Using the previous two facts, we may compute:

$$\omega(\sum P_n) = \omega(\sum f_n(A)) = \omega(f(A)) = f(\omega(A)) \tag{1.10}$$

$$= \sum f_n(\omega(A)) = \sum \omega(f_n(A)) = \sum \omega(P_n), \tag{1.11}$$

where we used Borel-FUNC in the third and fifth equalities. Since $\{P_n\}$ is an arbitrary family of orthogonal projections in $\mathfrak{V}$, ω is totally additive on $\mathfrak{V}$. By [23, Thm. 7.1.9], there

is a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$, and sequence $\{\lambda_n\}$ of non-negative real numbers with sum 1, such that $\omega = \sum_{n=1}^{\infty} \lambda_n \omega_{x_n}|_{\mathfrak{V}}$. We may assume that $0 < \lambda_1 \leq 1$, and the last equation can then be written as $\omega = \lambda_1 \omega_{x_1}|_{\mathfrak{V}} + (1 - \lambda_1)\rho$, with ρ a state of $\mathfrak{V}$. Since ω is dispersion-free on $\mathfrak{V}$, it is pure on $\mathfrak{V}$ (Proposition 1.1.(i)). Thus, $\omega = \omega_{x_1}|_{\mathfrak{V}}$, and ω is a vector state. $\qquad\square$

Evidently a dispersion-free state on a concrete C^*-algebra is a vector state only if that vector is a common eigenvector for every observable in the algebra. Thus Theorem 1.3 shows (*pace* [25, 35]) that it would be too strong to require dispersion-free states on beable subalgebras to satisfy Borel-FUNC, for that would have the effect of excluding continuous spectrum observables from beable subalgebras by fiat.

2. Beable and Maximal Beable Subalgebras of Observables

We start by formalizing the idea of a beable subalgebra, and then fully characterize both beable and maximal beable subalgebras relative to a normal (i.e. ultraweakly continuous) state on a concrete C^*-algebra (generalizing [15, Thm. 10]).

Definition. Let $\mathfrak{A}$ be a unital C^*-algebra, let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ such that $I \in \mathfrak{B}$, and let ρ be a state of $\mathfrak{A}$. Following [15], we say that $\mathfrak{B}$ is *beable for ρ* if $\rho|_{\mathfrak{B}}$ is a mixture of dispersion-free states; i.e. if and only if there is a probability measure μ on the space S of dispersion-free states of $\mathfrak{B}$ such that

$$\rho(A) = \int_{\mathsf{S}} \omega_s(A) d\mu(s), \qquad\qquad (A \in \mathfrak{B}). \qquad\qquad (2.1)$$

Physically, $\mathfrak{B}$ is beable for ρ just in case the observables in $\mathfrak{B}$ can be taken to have determinate values statistically distributed in accordance with ρ's expectation values. We say that $\mathfrak{B}$ is *maximal beable* for ρ if $\mathfrak{B}$ is beable for ρ and $\mathfrak{B}$ is not properly contained in any other subalgebra of $\mathfrak{A}$ that is beable for ρ. (An easy application of Zorn's lemma, using the characterization in Prop. 2.2 (ii) below, establishes that maximal beable subalgebras always exist for any state.)

Example (Definite Algebra). Let $\mathfrak{A}$ be a C^*-algebra and let ρ be a state of $\mathfrak{A}$. Clearly $\mathfrak{D}_\rho$ is beable for ρ, since ρ itself is dispersion-free on $\mathfrak{D}_\rho$. Although it requires a non-trivial result [24, Thm. 4], it can also be shown that for any pure state ρ, $\mathfrak{D}_\rho$ is *maximal* beable for ρ [15, Thm. 11]. In the case when $\rho = \omega_x$, a vector state on a concrete C^*-algebra, $\mathfrak{D}_{\omega_x}$ consists of exactly those observables with x as an eigenvector. For example, Dirac [16, Sec. 12] takes for granted that the observables determinate for a quantum system in a pure state ω_x coincide with $\mathfrak{D}_{\omega_x}$—an assumption sometimes called the 'eigenstate-eigenvalue link'.

Notation. If $\mathcal{M}$ is a subset of some Hilbert space $\mathcal{H}$, we let $[\mathcal{M}]$ denote its closed, linear span. If $A \in \mathfrak{L}(\mathcal{H})$, we let $\mathcal{R}(A)$ denote the closure of the range of A, and we let $\mathcal{N}(A)$ denote the null-space of A. If $\mathcal{T}$ is a closed subspace of $\mathcal{H}$, we let $P_\mathcal{T}$ denote the projection onto $\mathcal{T}$. For $x \in \mathcal{H}$, we abbreviate $P_{[x]}$ by P_x.

We will make frequent use of the following simple Lemma.

Lemma 2.1. *Let $\mathcal{H}$ be a Hilbert space, and let $x \in \mathcal{H}$. Suppose that $\mathfrak{A}$ is a C^*-algebra acting on $\mathcal{H}$. Then, for any $T \in \mathfrak{L}(\mathcal{H})$, if $Tx \in [\mathfrak{A}x]$ and $TAx = ATx$ for all $A \in \mathfrak{A}$, then (i) T leaves $[\mathfrak{A}x]$ invariant, and (ii) $TAy = ATy$ for all $A \in \mathfrak{A}$ and for all $y \in [\mathfrak{A}x]$.*

Proof. (i) Suppose that $Tx \in [\mathfrak{A}x]$ and that $TAx = ATx$ for all $A \in \mathfrak{A}$. Fix $A \in \mathfrak{A}$. Clearly A itself leaves $[\mathfrak{A}x]$ invariant (since $\mathfrak{A}$ is a C^*-algebra and A is continuous). Thus, $TAx = A(Tx) \in [\mathfrak{A}x]$. Since $A \in \mathfrak{A}$ was arbitrary, it follows by the linearity and continuity of T that T leaves $[\mathfrak{A}x]$ invariant.

(ii) Let $A \in \mathfrak{A}$, and let $y \in [\mathfrak{A}x]$. Since $[T, A]$ is linear and continuous, it is sufficient to show that $[T, A]Bx = 0$ for any $B \in \mathfrak{A}$. But, this is immediate from the fact that $[T, AB]x = 0$ and $[T, B]x = 0$. $\square$

Let $\mathfrak{A}$ be a C^*-algebra, and let ρ be a state of $\mathfrak{A}$. Recall that the *left-kernel* $\mathfrak{I}_\rho$ of ρ is the set of elements $A \in \mathfrak{A}$ such that $\rho(A^*A) = 0$. We may then formulate the following equivalent conditions for a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ to be beable for ρ:

Proposition 2.2. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$. Let ρ be a state of $\mathfrak{A}$, let $(\pi_\rho, \mathcal{H}_\rho, x_\rho)$ be the GNS representation of $\mathfrak{A}$ induced by the state ρ, and let $\mathcal{T} \equiv [\pi_\rho(\mathfrak{B})x_\rho] \subseteq \mathcal{H}_\rho$. Let $(\phi_\rho, \mathcal{G}_\rho, v_\rho)$ be the GNS representation of $\mathfrak{B}$ induced by $\rho|_\mathfrak{B}$. Then, the following are equivalent:*

(i) $\mathfrak{B}$ *is beable for ρ.*
(ii) $[A, B] \in \mathfrak{I}_\rho$ *for all $A, B \in \mathfrak{B}$.*
(iii) $\phi_\rho(\mathfrak{B})$ *is abelian.*
(iv) $\pi_\rho(\mathfrak{B})P_\mathcal{T}$ *is abelian.*

Remark 2.3. [15, Thm. 7] contains an alternate proof of (i) $\Leftrightarrow$ (ii).

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and then (iii) $\Leftrightarrow$ (iv).

"(i)$\Rightarrow$(ii)" Suppose that $\mathfrak{B}$ is beable for ρ. Then, there is a measure μ on the set S of dispersion-free states of $\mathfrak{B}$ such that (2.1) holds. Fix arbitrary $A, B \in \mathfrak{B}$. Since each ω in S is a $*$-homomorphism of $\mathfrak{B}$ into $\mathbb{C}$, and states are hermitian, $\omega([A, B]^*[A, B]) = |\omega([A, B])|^2 = |\omega(A)\omega(B) - \omega(B)\omega(A)|^2 = 0$ for each $\omega \in \mathsf{S}$, and thus $\rho([A, B]^*[A, B]) = 0$ by (2.1).

"(ii)⇒(iii)" Suppose that $[A, B] \in \mathfrak{I}_\rho$ for all $A, B \in \mathfrak{B}$. In order to show that $\phi_\rho(\mathfrak{B})$ is abelian, let $\phi_\rho(A) \in \phi_\rho(\mathfrak{B})$. Thus, for any $\phi_\rho(B) \in \phi_\rho(\mathfrak{B})$,

$$\left\langle \left[\phi_\rho(A), \phi_\rho(B)\right]v_\rho, \left[\phi_\rho(A), \phi_\rho(B)\right]v_\rho \right\rangle = \left\langle \phi_\rho\Big([A, B]^*[A, B]\Big)v_\rho, v_\rho \right\rangle \qquad (2.2)$$

$$= \rho\Big([A, B]^*[A, B]\Big) = 0. \qquad (2.3)$$

Thus, $[\phi_\rho(A), \phi_\rho(B)]v_\rho = 0$. Now, since $[\phi_\rho(\mathfrak{B})v_\rho] = \mathcal{G}_\rho$ we may apply Lemma 2.1, with C^*-algebra $\phi_\rho(\mathfrak{B})$ and vector v_ρ, to conclude that $\phi_\rho(A) \in \phi_\rho(\mathfrak{B})'$. Therefore, $\phi_\rho(\mathfrak{B}) \subseteq \phi_\rho(\mathfrak{B})'$, and $\phi_\rho(\mathfrak{B})$ is abelian.

"(iii)⇒(i)" If $\phi_\rho(\mathfrak{B})$ is abelian, we may identify it with the set of continuous, complex-valued functions, $\mathcal{C}(\mathsf{S})$, on some compact Hausdorff space S [23, Thm. 4.4.3]. Consider the vector state ω_{v_ρ} on $\phi_\rho(\mathfrak{B}) \simeq \mathcal{C}(\mathsf{S})$ induced by v_ρ. By the Riesz Representation Theorem [32, Thm. 2.14], there is a probability measure μ on S such that

$$\omega_{v_\rho}(\phi(A)) = \int_S [\phi_\rho(A)](s)d\mu(s), \qquad (\phi_\rho(A) \in \phi_\rho(\mathfrak{B}) \simeq \mathcal{C}(\mathsf{S})). \qquad (2.4)$$

For each $s \in \mathsf{S}$, define $\omega_s : \mathfrak{B} \to \mathbb{C}$ by $\omega_s(A) = [\phi_\rho(A)](s)$, $(A \in \mathfrak{B})$. The reader may verify without difficulty that each ω_s defines a dispersion-free state on $\mathfrak{B}$ (using the fact that ϕ_ρ is a $*$-homomorphism, and the definition of multiplication on $\mathcal{C}(\mathsf{S})$). Finally, for each $A \in \mathfrak{B}$,

$$\rho(A) \equiv (\omega_{v_\rho} \circ \phi_\rho)(A) = \omega_{v_\rho}(\phi_\rho(A))$$
$$= \int_{\mathsf{S}} [\phi_\rho(A)](s)d\mu(s) = \int_{\mathsf{S}} \omega_s(A)d\mu(s). \qquad (2.5)$$

Therefore, $\mathfrak{B}$ is beable for ρ.

"(iii)⇔(iv)" $\pi_\rho(\mathfrak{B})$ is a C^*-subalgebra of $\pi_\rho(\mathfrak{A})$, which is in turn a C^*-subalgebra of $\mathfrak{L}(\mathcal{H}_\rho)$. Thus, the mapping $\pi_\rho(A) \overset{\xi}{\to} \pi_\rho(A)|_\mathcal{T}$ is a representation of $\pi_\rho(\mathfrak{B})$ on $\mathcal{T}$ with cyclic vector x_ρ (cf. [23, p. 276]), and the composition map $\widehat{\pi}_\rho \equiv \xi \circ \pi_\rho|_\mathfrak{B}$ is a cyclic representation of $\mathfrak{B}$ on $\mathcal{T}$.

We now show that $(\widehat{\pi}_\rho, \mathcal{T})$ is unitarily equivalent to $(\phi_\rho, \mathcal{G}_\rho)$. Recall that $\mathcal{G}_\rho$ is the completion of the pre-hilbert space $\{A + \mathfrak{I}_\rho^\mathfrak{B} : A \in \mathfrak{B}\}$, where $\mathfrak{I}_\rho^\mathfrak{B} \equiv \mathfrak{I}_\rho \cap \mathfrak{B}$. For elements of this latter set, there is a natural isometric mapping $\overline{U}$ into $\mathcal{T}$; namely the mapping that takes $A + \mathfrak{I}_\rho^\mathfrak{B}$ to $A + \mathfrak{I}_\rho$. It is not difficult to verify that $\overline{U}$ extends uniquely to a unitary operator U from $\mathcal{G}_\rho$ onto $\mathcal{T}$, and that $\widehat{\pi}_\rho(A)U = U\phi_\rho(A)$ for all A in $\mathfrak{B}$. Thus, $(\widehat{\pi}_\rho, \mathcal{T})$ is unitarily equivalent to $(\phi_\rho, \mathcal{G}_\rho)$.

The equivalence of (iii) and (iv) now follows from the fact that $\pi_\rho(\mathfrak{B})P_\mathcal{T}$ is $*$-isomorphic to $\widehat{\pi}_\rho(\mathfrak{B})$. □

Recall that a state ρ on a C^*-algebra $\mathfrak{B}$ is called *faithful* just in case whenever $A \in \mathfrak{B}^+$ and $\rho(A) = 0$, then $A = 0$.

Corollary 2.4. *Suppose that $\mathfrak{B}$ is beable for ρ and that ρ is a faithful state of $\mathfrak{B}$. Then $\mathfrak{B}$ is abelian.*

Proof. Since ρ is faithful, ϕ_ρ is an isomorphism of $\mathfrak{B}$ onto $\phi_\rho(\mathfrak{B})$ [23, Exercise 4.6.15]. However, $\phi_\rho(\mathfrak{B})$ is abelian (Prop. 2.2 (iii)). $\square$

Example (Vacuum State). Let $\{\mathfrak{A}(O)\}_{O \subseteq M}$ be a net of local von Neumann algebras over Minkowski spacetime M, and let ρ be the vacuum state (cf. [19, p. 23]). If O has nonempty spacelike complement in M, it follows by the Reeh-Schlieder Theorem [19, Thm. 1.3.1] that ρ is a faithful state of $\mathfrak{A}(O)$ (since ρ is induced by the vacuum vector Ω which is separating for $\mathfrak{A}(O)$). Suppose that $\mathfrak{B} \subseteq \mathfrak{A}(O)$ and that $\mathfrak{B}$ is beable for ρ. Then $\rho\,|\,_{\mathfrak{B}}$ is faithful, and it follows from Corollary 2.4 that $\mathfrak{B}$ is abelian.

2.1. **Beable Algebras for Normal States.** We have defined the beable status of a C^*-algebra $\mathfrak{B}$ with respect to an arbitrary state ρ of $\mathfrak{B}$. In what follows, we specialize to the concrete case where $\mathfrak{B}$ is acting on some (fixed) Hilbert space $\mathcal{H}$ (not necessarily separable). If ρ is a normal state of $\mathfrak{L}(\mathcal{H})$, it follows that there is a positive trace-1 operator $K \in \mathfrak{L}(\mathcal{H})$ such that $\rho(A) = \mathrm{Tr}(KA)$ for each $A \in \mathfrak{L}(\mathcal{H})$ [23, Remark 7.1.10, Thm. 7.1.12]. With this in mind, we will freely interchange "$\mathfrak{B}$ is beable for ρ," with "$\mathfrak{B}$ is beable for K."

Notation. In what follows, we will abbreviate $\mathcal{R}(K)$ by $\mathcal{K}$.

Remark 2.5. In the special case where $K = P_v$ for some unit vector $v \in \mathcal{H}$, $\mathfrak{B}$ is beable for P_v just in case $ABv = BAv$ for each $A, B \in \mathfrak{B}$. This follows by Proposition 2.2 (ii) since $\mathrm{Tr}(P_v[A, B]^*[A, B]) = \langle [A, B]v, [A, B]v \rangle$.

Lemma 2.6. *Suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{L}(\mathcal{H})$, K is a positive, trace-1 operator on $\mathcal{H}$, $\mathcal{M}$ is a subset of $\mathcal{H}$, and $0 \neq v \in \mathcal{H}$.*

(i) *$\mathfrak{B}$ is beable for P_x, for all $x \in \mathcal{M}$, if and only if $\mathfrak{B}$ is beable for P_y, for all $y \in [\mathcal{M}]$.*
(ii) *$\mathfrak{B}$ is beable for P_v if and only if $\mathfrak{B}$ is beable for P_x, for all $x \in [\mathfrak{B}v]$.*
(iii) *$\mathfrak{B}$ is beable for K if and only if $\mathfrak{B}$ is beable for P_x, for all $x \in \mathcal{K}$.*

Proof. (i) The "if" implication is trivial. Suppose then that $\mathfrak{B}$ is beable for P_x, for all $x \in \mathcal{M}$. Consider the closed subspace of $\mathcal{H}$ given by

$$\mathcal{Y} \equiv \bigwedge \{\mathcal{N}([A, B]) \; : \; A, B \in \mathfrak{B}\}. \tag{2.6}$$

Clearly, $\mathcal{Y}$ is precisely the set of all $x \in \mathcal{H}$ such that $\mathfrak{B}$ is beable for P_x (see Remark 2.5). By supposition, $\mathcal{M} \subseteq \mathcal{Y}$; thus, $\mathcal{Y}$ will also contain $\mathcal{M}$'s closed, linear span $[\mathcal{M}]$.

(ii) The "if" implication is trivial, since $\mathfrak{B}$ contains the identity. Conversely, suppose that $\mathfrak{B}$ is beable for P_v. Fix $A \in \mathfrak{B}$. Then, for any $B \in \mathfrak{B}$, $[A, B]v = 0$, and moreover $Av \in [\mathfrak{B}v]$. Thus, we may apply Lemma 2.1 to conclude that $ABx = BAx$ for any $B \in \mathfrak{B}$ and for any $x \in [\mathfrak{B}v]$. Since A was an arbitrary element of $\mathfrak{B}$, it follows (Remark 2.5) that $\mathfrak{B}$ is beable for P_x whenever $x \in [\mathfrak{B}v]$.

(iii) Recall, first, that as a positive, trace-1 operator, K has a pure-point spectrum [36, pp. 188-191]. By the spectral theorem (and the fact that K leaves $\mathcal{K}$ invariant), $\mathcal{K}$ is the closed span of the eigenvectors of K in its range. Thus, there is a countable set $\{x_n\} \subseteq \mathcal{K}$, such that $\|x_n\| = 1$ for all n, $K = \sum_n \lambda_n P_{x_n}$, where $\lambda_n \in (0, 1]$, and $\sum_n \lambda_n = 1$.

"$\Rightarrow$" Suppose that $\mathfrak{B}$ is beable for K. Recall from Proposition 2.2 (ii) that $\mathfrak{B}$ is beable for K if and only if $\mathrm{Tr}(K[A, B]^*[A, B]) = 0$ for all $A, B \in \mathfrak{B}$. Given any eigenvector y in $\mathcal{K}$, we may write $K = \lambda P_y + (1 - \lambda)K'$ for some positive, trace-1 operator K', and $\lambda \in (0, 1]$. Thus, by the linearity of the trace,

$$\lambda \mathrm{Tr}(P_y[A, B]^*[A, B]) = \mathrm{Tr}(K[A, B]^*[A, B]) - (1 - \lambda)\mathrm{Tr}(K'[A, B]^*[A, B]) \qquad (2.7)$$

$$\leq \mathrm{Tr}(K[A, B]^*[A, B]) = 0, \qquad (2.8)$$

where the inequality in (2.8) follows since $\lambda \in (0, 1]$ and $\mathrm{Tr}(K'[A, B]^*[A, B]) \geq 0$. Thus, $\mathrm{Tr}(P_y[A, B]^*[A, B]) = 0$ for any eigenvector y of K in its range. Since the closed linear span of these eigenvectors is just $\mathcal{K}$, the conclusion follows by (i).

"$\Leftarrow$" Let $A, B \in \mathfrak{B}$. Then, by hypothesis, $\mathrm{Tr}(P_x[A, B]^*[A, B]) = 0$ whenever $x \in \mathcal{K}$. In particular, $\mathrm{Tr}(P_{x_n}[A, B]^*[A, B])$, for each n, where $K = \sum_n \lambda_n P_{x_n}$. Therefore, $\mathrm{Tr}(K[A, B]^*[A, B]) = \sum_n \lambda_n \mathrm{Tr}(P_{x_n}[A, B]^*[A, B]) = 0$. Since $A, B \in \mathfrak{B}$ were arbitrary, $\mathfrak{B}$ is beable for K. $\qquad \square$

Lemma 2.7. $\mathfrak{B}$ *is beable for* K *if and only if* $\mathfrak{B}$ *is beable for* P_x, *for all* $x \in [\mathfrak{B}\mathcal{K}]$.

Proof. The "if" implication follows trivially from Lemma 2.6.(iii). Conversely, suppose $\mathfrak{B}$ is beable for K. By (iii), $\mathfrak{B}$ is beable for P_y, for all $y \in \mathcal{K}$. Fix y. By (ii), $\mathfrak{B}$ is beable for P_z, for all $z \in [\mathfrak{B}y]$. Finally, $[\mathfrak{B}\mathcal{K}] = \bigvee_{y \in \mathcal{K}}[\mathfrak{B}y]$, so by (i), $\mathfrak{B}$ is beable for P_x, for all $x \in [\mathfrak{B}\mathcal{K}]$. $\qquad \square$

We turn now to providing intrinsic operator algebraic characterizations of beable, and maximal beable, status with respect to a normal state.

Theorem 2.8. *Let* $\mathfrak{B}$ *be a* C^*-*algebra acting on* $\mathcal{H}$, *and let* $\mathcal{T} \equiv [\mathfrak{B}\mathcal{K}]$. *Then,*

(i) $\mathfrak{B}$ *is beable for* K *if and only if* $\mathfrak{B} \subseteq \mathfrak{L}(\mathcal{T}^\perp) \oplus \mathfrak{N}$, *where* $\mathfrak{N}$ *is an abelian subalgebra of* $\mathfrak{L}(\mathcal{T})$.

(ii) $\mathfrak{B}$ *is maximal beable for* K *if and only if* $\mathfrak{B} = \mathfrak{L}(\mathcal{T}^\perp) \oplus \mathfrak{N}$, *where* $\mathfrak{N}$ *is a maximal abelian subalgebra of* $\mathfrak{L}(\mathcal{T})$.

Proof. (i) "⇒" Suppose that $\mathfrak{B}$ is beable for K. Clearly, we have defined $\mathcal{T}$ in such a way that $\mathcal{T}$ reduces $\mathfrak{B}$. Thus, each element of $\mathfrak{B}$ will decompose uniquely into the direct sum of an operator on $\mathcal{T}^{\perp}$ and an operator on $\mathcal{T}$. We must show that whenever $A_1 \oplus A_2$ and $B_1 \oplus B_2$ are in $\mathfrak{B}$, then $A_2 B_2 = B_2 A_2$. In other words, we must show that elements of the set $P_{\mathcal{T}} \mathfrak{B} P_{\mathcal{T}}$ commute with each other.

Let $A, B \in P_{\mathcal{T}} \mathfrak{B} P_{\mathcal{T}}$. Thus, $A = P_{\mathcal{T}} A' P_{\mathcal{T}}$ for some $A' \in \mathfrak{B}$ and $B = P_{\mathcal{T}} B' P_{\mathcal{T}}$ for some $B' \in \mathfrak{B}$. Let $x \in \mathcal{H}$ be arbitrary. Then, $x = y + z$ for (unique) $y \in \mathcal{T}^{\perp}$ and $z \in \mathcal{T}$. Since $z \in \mathcal{T}$, $\mathfrak{B}$ is beable for P_z (Lemma 2.7). Thus,

$$ABx = (P_{\mathcal{T}} A' P_{\mathcal{T}} P_{\mathcal{T}} B') P_{\mathcal{T}} x \; = \; (P_{\mathcal{T}} A' P_{\mathcal{T}} P_{\mathcal{T}} B') z \tag{2.9}$$

$$= (P_{\mathcal{T}} A' P_{\mathcal{T}}) B' z \; = \; (P_{\mathcal{T}} A') B' z \; = \; A' B' z, \tag{2.10}$$

where the last two equalities hold because both A' and B' leave $\mathcal{T}$ invariant. By symmetry, $BAx = B' A' z$. But, $A' B' z = B' A' z$ since $A', B' \in \mathfrak{B}$, $z \in \mathcal{T}$, and $\mathfrak{B}$ is beable for P_z. Thus, $ABx = BAx$, and since x was arbitrary, $AB = BA$. Since $A, B \in P_{\mathcal{T}} \mathfrak{B} P_{\mathcal{T}}$ were arbitrary, any two elements of $P_{\mathcal{T}} \mathfrak{B} P_{\mathcal{T}}$ commute.

"⇐" Suppose that $P_{\mathcal{T}} \mathfrak{B} P_{\mathcal{T}}$ consists of mutually commuting operators. Let $x \in \mathcal{K}$. Then, since $\mathfrak{B}$ contains the identity, $x \in \mathcal{T}$. Let $A, B \in \mathfrak{B}$. Then, we may write $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. Hence, $ABx = (A_1 \oplus A_2)(B_1 \oplus B_2)x = (A_1 \oplus A_2)(0 + B_2 x) = A_2 B_2 x$. By symmetry, $BAx = B_2 A_2 x$. But, since elements of $P_{\mathcal{T}} \mathfrak{B} P_{\mathcal{T}}$ commute, $A_2 B_2 x = B_2 A_2 x$. Thus, $ABx = BAx$ for any $A, B \in \mathfrak{B}$; that is, $\mathfrak{B}$ is beable for P_x. Furthermore, since x was an arbitrary element of $\mathcal{K}$, we see that $\mathfrak{B}$ is beable for every state defined by a (unit) vector in $\mathcal{K}$. By Lemma 2.6.(iii), $\mathfrak{B}$ is beable for K.

(ii) We have proved in (i) that any algebra $\mathfrak{B}$ which is beable for K will be commutative in its action on $[\mathfrak{B}\mathcal{K}]$, and that any algebra $\mathfrak{B}$ which is commutative in its action on $[\mathfrak{B}\mathcal{K}]$ will be beable for K. To complete the proof, then, it will suffice to show that if $\mathfrak{B} = \mathfrak{L}(\mathcal{T}^{\perp}) \oplus \mathfrak{N}$, where $\mathfrak{N}$ is maximal abelian, then $\mathfrak{B}$ is not properly contained in any beable algebra for K.

Suppose then that $\mathfrak{B} \subseteq \mathfrak{C}$, and that $\mathfrak{C}$ is beable for K. (We show that $\mathfrak{C} = \mathfrak{B}$.) Since $\mathfrak{C}$ is beable for K, $\mathfrak{C}$ is beable for P_y whenever $y \in [\mathfrak{C}\mathcal{K}]$ (Lemma 2.7). Furthermore, $[\mathfrak{B}\mathcal{K}] \subseteq [\mathfrak{C}\mathcal{K}]$. Thus, $\mathfrak{C}$ is beable for P_z whenever $z \in [\mathfrak{B}\mathcal{K}] \equiv \mathcal{T}$. Now, $P_{\mathcal{T}} \in \mathfrak{N}$ since the latter is maximal abelian and since the former is the identity on $\mathcal{T}$. Thus, $P_{\mathcal{T}} \in \mathfrak{B} \subseteq \mathfrak{C}$. Let D be a self-adjoint element of $\mathfrak{C}$. Then, for all $z \in \mathcal{T}$, $Dz = DP_{\mathcal{T}} z = P_{\mathcal{T}} Dz$, since $\mathfrak{C}$ is beable for P_z. That is, D leaves $\mathcal{T}$ invariant. However, since D is self-adjoint, it also leaves $\mathcal{T}^{\perp}$ invariant, and therefore $D = (I - P_{\mathcal{T}}) D (I - P_{\mathcal{T}}) \oplus P_{\mathcal{T}} D P_{\mathcal{T}} \in \mathfrak{L}(\mathcal{T}^{\perp}) \oplus \mathfrak{L}(\mathcal{T})$.

Finally, let $A \in \mathfrak{N}$. For any $z \in \mathcal{T}$,

$$P_{\mathcal{T}} D P_{\mathcal{T}} A z = P_{\mathcal{T}} D A z = D A z = A D z \qquad (2.11)$$

$$= A D P_{\mathcal{T}} z = A P_{\mathcal{T}} D P_{\mathcal{T}} z. \qquad (2.12)$$

The first, second, and fifth equalities hold since A and D leave $\mathcal{T}$ invariant. The third equality holds since $A, D \in \mathfrak{C}$, and $\mathfrak{C}$ is beable for P_z. Hence, $P_{\mathcal{T}} D P_{\mathcal{T}} \in \mathfrak{N}' \subseteq \mathfrak{N}$ and thus $D \in \mathfrak{L}(\mathcal{T}^{\perp}) \oplus \mathfrak{N}$. We have shown that $\mathfrak{C}_{\mathrm{sa}} \subseteq \mathfrak{B}$, from which it follows that $\mathfrak{C} \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is maximal beable for K. $\qquad \square$

Example (Multiplication Algebra). Let $\mathfrak{M}$ be the von Neumann algebra of multiplications by essentially bounded (measurable) functions on $L_2(\mathbb{R})$, generated by the unbounded 'multiplication by x' (position) operator. Let ψ be any (wave) function in L_2 that is non-zero almost everywhere. It follows then that $\mathfrak{M}$ is maximal beable for ψ. Indeed, an elementary measure-theoretic argument proves that ψ is a separating vector for $\mathfrak{M}$. Moreover, since $\mathfrak{M}$ is maximal abelian, $\mathfrak{M} = \mathfrak{M}'$ and ψ is a generating vector for $\mathfrak{M}$ [23, Cor. 5.5.12]. Thus, $\mathcal{T} \equiv [\mathfrak{M}\psi] = L_2$ and the maximal beable status of $\mathfrak{M}$ for ψ follows from Theorem 2.8 (ii). Bohm's 'causal' interpretation of quantum theory [6]—which only grants beable status to a particle's position—can be understood as privileging $\mathfrak{M}$ (see [15, Sec. 5]).

Corollary 2.9. *Let ρ be a normal state on $\mathfrak{L}(\mathcal{H})$.*

   (i) *If $\mathfrak{B}$ is maximal beable for ρ, then $\mathfrak{B} = \mathfrak{B}^{-}$.*
   (ii) *If $\mathfrak{B}$ is beable for ρ, then $\mathfrak{B}^{-}$ is beable for ρ as well.*

Proof. (i) Let K be a positive trace-1 operator that induces the state ρ on $\mathfrak{L}(\mathcal{H})$. If $\mathfrak{B}$ is maximal beable for K, then $\mathfrak{B} = \mathfrak{L}(\mathcal{T}^{\perp}) \oplus \mathfrak{N}$, where $\mathfrak{N}$ is a maximal abelian subalgebra of $\mathfrak{L}(\mathcal{T})$. Since $\mathfrak{N}$ is a maximal abelian subalgebra of $\mathfrak{L}(\mathcal{T})$, it follows that $\mathfrak{N}$ is a von Neumann algebra. Therefore, $\mathfrak{B}$ is a von Neumann algebra.

   (ii) Now suppose that $\mathfrak{B}$ is beable for ρ. Then, $\mathfrak{B}$ is contained in some maximal beable algebra $\mathfrak{C}$ for ρ. By part (i) of this Corollary, $\mathfrak{C} = \mathfrak{C}^{-}$. Thus, $\mathfrak{B}^{-} \subseteq \mathfrak{C}^{-} = \mathfrak{C}$, and since beable status is hereditary, the conclusion follows. $\qquad \square$

Recall that a pure state ρ on a concrete C^*-algebra $\mathfrak{A}$ is called *singular* just in case it is *not* ultraweakly continuous. Thus, a singular state is a pure, non-normal state.

Remark 2.10. Both parts of the above Corollary, in particular (i), fail if ρ is not assumed to be a normal state of $\mathfrak{L}(\mathcal{H})$. For example, if ρ is a singular state of $\mathfrak{L}(\mathcal{H})$, then $\mathfrak{D}_\rho$ is maximal beable for ρ (see Example 2). However, $\mathfrak{D}_\rho$ is not WOT-closed. For recall that $\rho|_{\mathfrak{K}} = 0$, where $\mathfrak{K}$ is the ideal of compact operators in $\mathfrak{L}(\mathcal{H})$ [23, Cor. 10.4.4]. Thus, $\mathfrak{K} \subseteq \mathfrak{D}_\rho$, since $\rho(AX) = 0 = \rho(A)\rho(X)$ for any $A \in \mathfrak{K}$ and for any $X \in \mathfrak{L}(\mathcal{H})$. Moreover, $\mathfrak{K}^{-} = \mathfrak{L}(\mathcal{H})$, and

it follows that $(\mathfrak{D}_\rho)^- = \mathfrak{L}(\mathcal{H})$. But, clearly, $\mathfrak{D}_\rho \neq \mathfrak{L}(\mathcal{H})$ ($\mathcal{H}$ separable), since there are no states dispersion-free on all of $\mathfrak{L}(\mathcal{H})$ (Remark 1.2).

3. Beable Status for Unbounded Observables

To this point, we have restricted discussion of "beable status" to bounded operators. Of course, many of the observables of interest in quantum theory, such as position and momentum, are represented by unbounded operators. Thus, in this section we make use of the theory of algebras of unbounded functions and operators (as expounded in [23, Sec. 5.6] and Kadison [22]) in order to articulate the sense in which an unbounded operator can have beable status with respect to a state. The section ends with results that capture the essential content of the Heisenberg-Bohr indeterminacy principle for canonically conjugate observables.

Let $\mathfrak{V}$ be a von Neumann algebra acting on $\mathcal{H}$ and let R be a (possibly unbounded) normal operator on $\mathcal{H}$. R is said to be *affiliated* with $\mathfrak{V}$ just in case $U^* R U = R$ whenever U is a unitary operator in $\mathfrak{V}'$. Frequently this relation is denoted by $R \,\eta\, \mathfrak{V}$. Now, if $\mathfrak{V}$ is an *abelian* von Neumann algebra, the set S of *pure* states of $\mathfrak{V}$, with the weak-$*$ (i.e. pointwise convergence) topology, is an extremely disconnected compact Hausdorff space, and $\mathfrak{V}$ is $*$-isomorphic to $\mathcal{C}(\mathsf{S})$ [23, Thm. 4.4.3, Thm. 5.2.1]. Under this isomorphism, $A \in \mathfrak{V}$ goes to $\phi(A) \in \mathcal{C}(\mathsf{S})$ defined by $\phi(A)(\omega) = \omega(A)$, for all $\omega \in \mathsf{S}$. A *normal function* on an extremely disconnected compact Hausdorff space S is defined as a continuous complex-valued function f defined on an open dense subset $\mathsf{S} \backslash \mathsf{Z}$ of S such that $\lim_{\omega \to \tau} |f(\omega)| = \infty$ for each τ in Z (where $\omega \in \mathsf{S} \backslash \mathsf{Z}$), and a *self-adjoint function* on S is a real-valued normal function on S [23, Def. 5.6.5]. Let $\mathcal{N}(\mathfrak{V})$ be the set of (normal) operators affiliated with $\mathfrak{V}$. Then, $\mathcal{N}(\mathfrak{V})$ may be equipped with two operations $\widehat{+}$ (closed addition) and $\widehat{\cdot}$ (closed multiplication) under which it is a commutative $*$-algebra [23, Thm. 5.6.15]. Similarly, if $\mathcal{N}(\mathsf{S})$ is the set of normal functions on S, then there are operations $\widehat{+}, \widehat{\cdot}$ and $*$ that extend the standard pointwise operations. Moreover, the $*$-isomorphism ϕ from $\mathfrak{V}$ onto $C(\mathsf{S})$ extends to a $*$-isomorphism (which we denote again by ϕ) from $\mathcal{N}(\mathfrak{V})$ onto $\mathcal{N}(\mathsf{S})$, providing us with what we might call the "extended function representation" of the abelian von Neumann algebra $\mathfrak{V}$ [23, Thm. 5.6.19].

For each family $\mathfrak{F}$ of normal operators, there will be a unique smallest (not necessarily abelian) von Neumann algebra $W^*(\mathfrak{F})$ such that R is affiliated with $W^*(\mathfrak{F})$ for all $R \in \mathfrak{F}$. We may call $W^*(\mathfrak{F})$ the von Neumann algebra generated by $\mathfrak{F}$. If $\mathfrak{F}$ consists of a single normal operator R, then it follows that $W^*(R)$ is an *abelian* von Neumann algebra [23, Thm. 5.6.18]. Thus, R is represented by a normal function $\phi(R)$ on S, where S is now the set of pure states of $W^*(R)$. If, as usual, $\mathrm{sp}(R)$ is defined to be the set of real numbers λ

such that $R - \lambda I$ is not a one-to-one mapping of the domain of R onto $\mathcal{H}$, it follows that the range of the function $\phi(R)$ is identical to $\mathrm{sp}(R)$ [23, Proposition 5.6.20]. It is not difficult to see that the range of a normal function is a closed (compact only if R is bounded) subset of $\mathbb{C}$ [23, p. 356]. Thus, $\mathrm{sp}(R)$ is closed in $\mathbb{C}$.

Borel functions of R may be defined, via the isomorphism of $\mathcal{N}(W^*(R))$ and $\mathcal{N}(\mathsf{S})$, as follows [23, Remark 5.6.25]. Let Z be the closed nowhere dense subset of S such that $\phi(R)$ is defined and continuous on $\mathsf{S}\backslash\mathsf{Z}$. Let g be an arbitrary element of $\mathcal{B}_u(\mathrm{sp}(R))$, the algebra of complex-valued Borel functions (finite almost everywhere) on $\mathrm{sp}(R)$. Define $\widetilde{g}$ by:

$$\widetilde{g}(\omega) \equiv \begin{cases} (g \circ \phi(R))(\omega) & \omega \in \mathsf{S}\backslash\mathsf{Z}, \\ 0 & \omega \in \mathsf{Z}. \end{cases} \tag{3.1}$$

Then $\widetilde{g}$ is in $\mathcal{B}_u(\mathsf{S})$, and there is a *unique* function $h \in \mathcal{N}(\mathsf{S})$ such that $\widetilde{g}$ and h agree on the complement of a meager (i.e. first category) set M [23, Lemma 5.6.22]. Note that since S is compact Hausdorff, the Baire Category Theorem ensures us that $\mathsf{S}\backslash\mathsf{M}$ is dense in S. Thus, $\widetilde{g}$ and h may not disagree on any non-empty open set—a fact we shall make frequent use of in what follows. Finally, $g(R)$ is defined as $\phi^{-1}(h)$, as represented in the diagram below:

$$
\begin{array}{ccc}
\mathcal{B}_u(\mathrm{sp}(R)) & \xrightarrow{\;g \to \widetilde{g}\;} & \mathcal{B}_u(\mathsf{S}) \\
{\scriptstyle g \to g(R)}\downarrow & & \downarrow{\scriptstyle \widetilde{g} \to h} \\
\mathcal{N}(W^*(R)) & \xrightarrow[\;\phi\;]{} & \mathcal{N}(\mathsf{S})
\end{array}
$$

The Borel functional calculus also provides a method of defining a projection-valued measure E on $\mathrm{sp}(R)$ and, by extension, a projection-valued measure on $\mathbb{C}$ [23, Thm. 5.6.26]. If C is a Borel subset of $\mathrm{sp}(R)$, then $E(\mathsf{C})$ is defined to be $\chi_\mathsf{C}(R)$, where χ_C is the characteristic function of C. If C is any Borel subset of $\mathbb{C}$, then $E(\mathsf{C})$ is defined to be $E(\mathsf{C} \cap \mathrm{sp}(R))$. Note that for any $\mathsf{C} \subseteq \mathbb{C}$, $\phi(E(\mathsf{C}))$ is a characteristic function (since $E(\mathsf{C})$ is a projection) and is actually *continuous* on S (since $\phi(E(\mathsf{C})) \in \mathcal{C}(S)$).

In what follows, we specialize to the case where R is *self-adjoint*, so that $\mathrm{sp}(R) \subseteq \mathbb{R}$, and $\phi(R)$ is a self-adjoint function on S. We consider $\mathrm{sp}(R)$ with the order relation inherited from $\mathbb{R}$ and with the relative topology inherited from $\mathbb{R}$. Recall that a *convex* subset of $\mathrm{sp}(R)$ is any subset C with the following property: If $a, b \in \mathsf{C}$ and there is a $c \in \mathrm{sp}(R)$ such that $a < c < b$, then $c \in \mathsf{C}$. Note that the relative basis of $\mathrm{sp}(R)$ consists of convex sets with compact closure. If $\mathsf{C} \subseteq \mathrm{sp}(R)$, we let $\mathrm{clo}\,\mathsf{C}$ denote the closure of C with respect to the relative topology.

Lemma 3.1. *Let ω be a pure state of $W^*(R)$, and let C be a convex subset of $\mathrm{sp}(R)$ with compact closure.*

(i) *If $\omega(E(\mathsf{C})) = 1$, then $\phi(R)$ is defined at ω and $\phi(R)(\omega) \in \mathrm{clo}\,\mathsf{C}$.*

(ii) *If C is open and $\phi(R)(\omega) \in \mathsf{C}$, then $\omega(E(\mathsf{C})) = 1$.*

Proof. (i) Suppose that $\omega(E(\mathsf{C})) = 1$, and consider $\chi_{\mathsf{C}} \in \mathcal{B}_u(\mathrm{sp}(R))$, the characteristic function of C. Define $\widetilde{\chi_{\mathsf{C}}} \in \mathcal{B}_u(\mathsf{S})$ as in (3.1), so $\widetilde{\chi_{\mathsf{C}}}(\omega) = 1$ if $\phi(R)(\omega) \in \mathsf{C}$, $= 0$ otherwise. Let h be the unique function in $\mathcal{C}(\mathsf{S})$ that agrees with $\widetilde{\chi_{\mathsf{C}}}$ on the complement of a meager set. Thus, $E(\mathsf{C}) \equiv \phi^{-1}(h)$ and $h(\omega) = \omega(E(\mathsf{C})) = 1$.

Suppose, for reductio ad absurdum, that $\phi(R)$ is not defined at ω, so that $\widetilde{\chi_{\mathsf{C}}}(\omega) = 0$. Since $\phi(R)$ is self-adjoint, $\lim_{\tau \to \omega} |\phi(R)(\tau)| \to \infty$. Since C is bounded, there is an open neighborhood U of ω such that $\phi(R)(\tau) \notin \mathsf{C}$, for all $\tau \in \mathsf{U}$. Thus, $\widetilde{\chi_{\mathsf{C}}}(\mathsf{U}) = \{0\}$. However, since h is a continuous map from S into $\{0, 1\}$ and $h(\omega) = 1$, there is an open neighborhood V of ω such that $h(\mathsf{V}) = \{1\}$. But then $\widetilde{\chi_{\mathsf{C}}}$ and h disagree on the non-empty open set $\mathsf{U} \cap \mathsf{V}$, which is impossible. Therefore, $\phi(R)$ is defined at ω.

Again, suppose for reductio that $\phi(R)$ is defined at ω but that $\phi(R)(\omega) \notin \mathsf{C}^-$. Since $\phi(R)$ is defined at ω, it is continuous at ω. Hence $\omega \notin [\phi(R)^{-1}(\mathsf{C})]^-$ and there is an open neighborhood U of ω such that $\mathsf{U} \cap [\phi(R)^{-1}(\mathsf{C})] = \emptyset$. Thus, $\phi(R)(\mathsf{U}) \cap \mathsf{C} = \emptyset$ and $\widetilde{\chi_{\mathsf{C}}}(\mathsf{U}) = \{0\}$. Since h is continuous, there is an open neighborhood V of ω such that $h(\mathsf{V}) = \{1\}$. But then $\widetilde{\chi_{\mathsf{C}}}$ and h disagree on the non-empty open set $\mathsf{U} \cap \mathsf{V}$, which again is impossible. Therefore $\phi(R)(\omega) \in \mathrm{clo}\,\mathsf{C}$.

(ii) Suppose that C is open and $\phi(R)(\omega) \in \mathsf{C}$. Let $h \equiv \phi(E(\mathsf{C}))$. We must show that $h(\omega) = 1$. Recall from (i) that h agrees on the complement of a meager set with $\widetilde{\chi_{\mathsf{C}}} \in \mathcal{B}_u(\mathsf{S})$. By assumption, then, $\widetilde{\chi_{\mathsf{C}}}(\omega) = 1$. Suppose, for reductio, that $h(\omega) = 0$. Since h is continuous, there is an open neighborhood U of ω such that $h(\mathsf{U}) = \{0\}$. Since $\phi(R)$ is continuous on $\mathsf{S}\backslash\mathsf{Z}$, $\mathsf{V} \equiv \phi(R)^{-1}(\mathsf{C})$ is open in $\mathsf{S}\backslash\mathsf{Z}$ (and thus open in S, since $\mathsf{S}\backslash\mathsf{Z}$ is open in S). Then, $\widetilde{\chi_{\mathsf{C}}}(\mathsf{V}) = \{1\}$ and $\mathsf{U} \cap \mathsf{V}$ (which contains ω) is a non-empty open set on which h and $\widetilde{\chi_{\mathsf{C}}}$ disagree—a contradiction. Therefore, $h(\omega) = \omega(E(\mathsf{C})) = 1$. $\qquad\square$

The next proposition confirms what might otherwise expect: that R may be assigned a dispersion-free value $\ell \in \mathrm{sp}(R)$ exactly when all propositions of the form 'the value of R lies in K', for all compact convex $\mathsf{K} \subset \mathrm{sp}(R)$ that contain ℓ, are true. (Clearly if this were not so—in particular, if *no* proposition of that form were true—then it would make no physical sense to assign R any value whatsoever.)

Proposition 3.2. *Let ω be a pure (dispersion-free) state of $W^*(R)$. Then $\phi(R)$ is defined at ω if and only if there is a compact convex set $\mathsf{K} \subset \mathrm{sp}(R)$ such that $\omega(E(\mathsf{K})) = 1$. If these conditions hold, then*

$$\{\phi(R)(\omega)\} = \bigcap \Big\{ \mathsf{K} : \mathsf{K} \text{ is a compact convex set in } \mathrm{sp}(R) \text{ and } \omega(E(\mathsf{K})) = 1 \Big\}.$$

Proof. "$\Leftarrow$" Immediate from Lemma 3.1.(i).

"$\Rightarrow$" Suppose that $\phi(R)$ is defined at $\omega \in \mathsf{S}$. Then, since there is an open convex neighborhood C of $\phi(R)(\omega)$ such that clo C is compact (and convex), $\omega(E(\mathsf{C})) = 1$ (by Lemma 3.1.(ii)). Moreover, since a projection-valued measure is monotone and states are order-preserving,

$$1 \;=\; \|E(\text{clo}\,\mathsf{C})\| \;\geq\; \omega(E(\text{clo}\,\mathsf{C})) \;\geq\; \omega(E(\mathsf{C})) \;=\; 1, \tag{3.2}$$

which entails $\omega(E(\text{clo}\,\mathsf{C})) = 1$.

Suppose now that the the above equivalent conditions hold for $\phi(R)$ and ω, and let Y denote the intersection in the statement of this proposition. By assumption, there is at least one compact convex set K such that $\omega(E(\mathsf{K})) = 1$, so that Y is nonempty. Let L be any other such set where $\omega(E(\mathsf{L})) = 1$. Then, by Lemma 3.1.(i), $\phi(R)(\omega) \in \text{clo}\,\mathsf{L} = \mathsf{L}$. Therefore, $\phi(R)(\omega) \in \mathsf{Y}$.

Finally, to see that $\phi(R)(\omega)$ is the unique point in Y, suppose that $\lambda \in \mathsf{Y}$ yet $\lambda \neq \phi(R)(\omega)$. Since $\text{sp}(R)$ is Hausdorff, there is an open convex neighborhood C in $\text{sp}(R)$ such that $\phi(R)(\omega) \in \mathsf{C}$ but $\lambda \notin \text{clo}\,\mathsf{C}$. (We choose C such that its closure is compact.) Since $\phi(R)(\omega) \in \mathsf{C}$ and C is open, $\omega(E(\text{clo}\,\mathsf{C})) = 1$ (by Lemma 3.1.(ii)). Therefore, $\lambda \notin \mathsf{Y}$, contradicting our assumption. It follows that $\phi(R)(\omega)$ is the unique element of Y. $\qquad\square$

Given a von Neumann algebra $\mathfrak{B}$, beable for a state ρ, it is natural to ask when an unbounded self-adjoint operator R (or a family of such observables) affiliated with $\mathfrak{B}$ can be taken to have beable status for ρ *together with the observables in* $\mathfrak{B}$. This would require that ρ be a mixture of dispersion-free states on $\mathfrak{B}$ each of which restricts to a pure state on $W^*(R) \subseteq \mathfrak{B}$ that permits a value for R to be defined in accordance with the above proposition. As we show in Theorem 3.6 below, a sufficient condition for this is that ρ determine a finite expectation value for R.

While pure states of $W^*(R)$ correspond to points of S, the general state ρ (pure or mixed) of $W^*(R)$ corresponds uniquely, via the Riesz Representation Theorem, to a probability measure μ_ρ on S. (A pure state corresponds to a measure concentrated at a single point.) That is,

$$\rho(A) = \int_{\mathsf{S}} \phi(A)(s)d\mu_\rho(s) \qquad\qquad (A \in W^*(R)). \tag{3.3}$$

Since $\phi(R)$ is an unbounded function on S, its integral with respect to μ_ρ may or may not converge to a finite value. In order to capture the idea that some states may be used consistently to assign finite (not necessarily dispersion-free) expectation values to unbounded operators, we introduce the following notion of a well-defined state:

Definition. Suppose that ρ is a state of $W^*(R)$ and that μ_ρ is the measure on S corresponding to ρ. If $\int_\mathsf{S} \phi(R)d\mu_\rho < \infty$, we say that ρ is a *well-defined* state for R.

As should be the case, this definition entails that if ρ is a pure state, then ρ is well-defined for R if and only if $\phi(R)$ is defined at ρ. Moreover, by [23, Thm. 5.6.26], a vector state ω_x is well-defined for R if and only if x is in the domain of R. Of course, this definition may easily be extended to any von Neumann algebra $\mathfrak{V}$, such that R is affiliated with $\mathfrak{V}$. If ρ is a state of $\mathfrak{V}$, we say that ρ is well-defined for R just in case $\rho|_{W^*(R)}$ is well-defined for R.

Remark 3.3. Of course it is possible for ρ to be well-defined for R, but not for polynomials in R. For example, let $R = Q$ be the the multiplication by x operator on $L_2(\mathbb{R})$. Then, one can easily construct unit vectors in $\mathcal{D}(Q) - \mathcal{D}(Q^2)$ whose corresponding states will be well-defined for Q but not for Q^2.

Pure well-defined states, however, are extremely well-behaved:

Proposition 3.4 (Cont-FUNC). *Let R be a (possibly unbounded) self-adjoint operator on $\mathcal{H}$ and suppose that ω is a pure state of $\mathfrak{L}(\mathcal{H})$ that is well-defined for R. Then, $\omega(f(R)) = f(\omega(R))$, for any $f \in \mathcal{C}(\mathrm{sp}(R))$.*

Proof. Note that $\widetilde{f}|_{\mathsf{S}\backslash\mathsf{Z}}$ is continuous, being the composition of two continuous functions, f and $\phi(R)|_{\mathsf{S}\backslash\mathsf{Z}}$. Moreover, since the normal function h agrees with $\widetilde{f}$ on the complement of a meager set, h must agree with $\widetilde{f}$ throughout $\mathsf{S}\backslash\mathsf{Z}$. Thus, $\omega(f(R)) = h(\omega) = \widetilde{f}(\omega) = f(\omega(R))$. $\qquad\square$

Lemma 3.5. *Let ρ be any state (pure or mixed) of $W^*(R)$, and let $F_n \equiv E((-n, n])$, where E is the projection-valued measure associated with R. If ρ is well-defined for R, then $\lim_{n\to\infty} \rho(F_n) = 1$.*

Proof. Let $\widetilde{g}_n$ be defined as $\widetilde{g}_n(\omega) = 1$ if $\phi(R)(\omega) \in (-n, n]$ and $\widetilde{g}_n(\omega) = 0$ otherwise. Then, $F_n \equiv \phi^{-1}(h_n)$ where h_n is the unique function in $\mathcal{C}(\mathsf{S})$ which agrees with $\widetilde{g}_n$ on the complement of a meager set. Clearly then, $\{h_n\}$ converges pointwise to $\chi_{(\mathsf{S}\backslash\mathsf{Z})}$, the characteristic function of $\mathsf{S}\backslash\mathsf{Z}$, as $n \to \infty$. Thus, $\rho(F_n) = \int_\mathsf{S} h_n d\mu_\rho$, and

$$\lim_{n\to\infty} \rho(F_n) = \lim_{n\to\infty} \int_\mathsf{S} h_n d\mu_\rho = \int_\mathsf{S} \chi_{(\mathsf{S}\backslash\mathsf{Z})} d\mu_\rho = \mu_\rho(\mathsf{S}\backslash\mathsf{Z}), \tag{3.4}$$

where the second equality follows from the Monotone Convergence Theorem [32, Thm. 1.26]. Hence, $\mu_\rho(\mathsf{Z}) = 1 - \lim_{n\to\infty} \rho(F_n)$.

Since $\phi(R)$ is a self-adjoint function on X, we may decompose Z as $\mathsf{Z}_+ \cup \mathsf{Z}_-$ where Z_+ is the set of points ω of S such that $\lim_{\tau\to\omega} \phi(R)(\tau) = +\infty$ and Z_- is the set of points ω of S such that $\lim_{\tau\to\omega} \phi(R)(\tau) = -\infty$ [23, p. 344]. Now, let $f_+ \equiv \max\{\phi(R), 0\}$ be the positive

part of $\phi(R)$ and let $f_- \equiv -\min\{\phi(R), 0\}$ the negative part. Then,

$$\int_S f_+ d\mu_\rho = \int_{(S\setminus Z_+)} f_+ d\mu_\rho + \int_{Z_+} f_+ d\mu_\rho \tag{3.5}$$

$$= \int_{(S\setminus Z_+)} f_+ d\mu_\rho + (\mu_\rho(Z_+) \cdot +\infty), \tag{3.6}$$

and similarly,

$$\int_S f_- d\mu_\rho = \int_{(S\setminus Z_-)} f_- d\mu_\rho + (\mu_\rho(Z_-) \cdot -\infty). \tag{3.7}$$

By definition, $\int_S \phi(R) d\mu_\rho$ is defined only if either (3.6) or (3.7) is finite, and then,

$$\int_S \phi(R) d\mu_\rho \equiv \int_S f_+ d\mu_\rho - \int_S f_- d\mu_\rho. \tag{3.8}$$

Thus, $\mu_\rho(Z) = \mu_\rho(Z_+) + \mu_\rho(Z_-)$, and if $\mu_\rho(Z) > 0$ then either (3.6) or (3.7) is infinite and either $\int_S \phi(R) d\mu_\rho$ is undefined or $= \pm\infty$. Therefore, $\int_S \phi(R) d\mu_\rho$ has a finite value only if $\lim_{n\to\infty} \rho(F_n) = 1$. $\qquad\square$

Theorem 3.6. *Suppose $\mathfrak{B}$ is a von Neumann algebra and $\mathfrak{B}$ is beable for ρ. Suppose that $\{R_j\}$ is a countable family of self-adjoint operators affiliated with $\mathfrak{B}$ such that, for all $j \in \mathbb{N}$, ρ is a well-defined state for R_j. Then, there is a probability measure μ on the set of dispersion-free states S of $\mathfrak{B}$ such that*

$$\rho(A) = \int_S \omega_s(A) d\mu(s) \qquad (A \in \mathfrak{B}), \tag{3.9}$$

and for every $\omega_s \in S$ and $j \in \mathbb{N}$, ω_s is well-defined for R_j.

Proof. Since $\mathfrak{B}$ is beable for ρ, we have a probability measure μ on the set of dispersion-free states T of $\mathfrak{B}$ such that

$$\rho(A) = \int_T \omega_t(A) d\mu(t) \qquad (A \in \mathfrak{B}). \tag{3.10}$$

We will show that $\mu(S) = 1$, where S is the subset of T consisting of those states that are well-defined for each R_j.

Fix $j \in \mathbb{N}$. Let $W^*(R_j)$ be the von Neumann algebra generated by R_j, let E^j be the projection-valued measure on $\mathbb{R}$ induced by R_j, and let $F_n^j \equiv E^j((-n, n])$. Let

$$Z_j \equiv \{t \in T : \omega_t(F_n^j) = 0, \text{ for all } n \in \mathbb{N}\}. \tag{3.11}$$

It is not difficult to verify that Z_j is a measurable subset of T. Suppose, for reductio, that $\mu(\mathsf{Z}_j) = \delta > 0$ so that $\mu(\mathsf{T}\backslash\mathsf{Z}_j) = 1 - \delta$. Choose any $m \in \mathbb{N}$. Then,

$$\rho(F_m^j) = \int_{\mathsf{Z}_j} \omega_t(F_m^j)d\mu(t) + \int_{(\mathsf{T}\backslash\mathsf{Z}_j)} \omega_t(F_m^j)d\mu(t) \tag{3.12}$$

$$= 0 + \int_{(\mathsf{T}\backslash\mathsf{Z}_j)} \omega_t(F_m^j)d\mu(t) \tag{3.13}$$

$$\leq \mu(\mathsf{T}\backslash\mathsf{Z}_j) = 1 - \delta. \tag{3.14}$$

(3.13) follows by the definition of Z_j, and the inequality in (3.14) follows since $\omega_t(F_m^j) \leq \|F_m^j\| = 1$ for all $t \in \mathsf{T}$. Since m was arbitrary $\lim_{n\to\infty} \rho(F_n^j) \leq 1 - \delta$. By Proposition 3.5, $\rho|_{\mathfrak{A}_j}$ does not correspond to a convergent measure, contradicting our assumption that ρ is well-defined for R_j. Thus, $\mu(\mathsf{Z}_j) = 0$. Since μ is countably additive, $\mu(\cup_{j=1}^\infty \mathsf{Z}_j) = 0$.

Let $\mathsf{S} = \mathsf{T}\backslash(\cup_{j=1}^\infty \mathsf{Z}_j)$. Then, for $A \in \mathfrak{B}$,

$$\rho(A) = \int_{\mathsf{T}} \omega_t(A)d\mu(t) \ = \ \int_{\mathsf{S}} \omega_t(A)d\mu(t) + \int_{\cup \mathsf{Z}_j} \omega_t(A)d\mu(t) \tag{3.15}$$

$$= \int_{\mathsf{S}} \omega_t(A)d\mu(t) \ = \ \int_{\mathsf{S}} \omega_s(A)d\mu(s), \tag{3.16}$$

where the penultimate equality follows since $\mu(\cup_{j=1}^\infty \mathsf{Z}_j) = 0$. Finally, suppose that $\omega \in \mathsf{S}$; that is, for each j there is an m such that $\omega(F_m^j) \equiv \omega(E^j((-m, m])) > 0$. But $\omega(F_m^j) \in \{0, 1\}$ since ω is dispersion-free on $\mathfrak{A}_j$ and F_m^j is a projection. Thus, for each j there is an m such that $\omega(F_m^j) = 1$, and by Lemma 3.1.(i), ω is well-defined for each R_j. Therefore, ρ is a mixture of dispersion-free states of $\mathfrak{B}$, all of which are well-defined for each R_j. $\qquad\square$

Definition. Let $\{R_\lambda : \lambda \in \Lambda\}$ be a family of (possibly unbounded) self-adjoint operators acting on a Hilbert space $\mathcal{H}$, and ρ a state of $\mathfrak{L}(\mathcal{H})$. We say that the observables $\{R_\lambda : \lambda \in \Lambda\}$ have *joint beable status* for ρ if there is a subalgebra $\mathfrak{B} \subseteq \mathfrak{L}(\mathcal{H})$, to which each R_λ is affiliated, such that ρ is a mixture of dispersion-free states on $\mathfrak{B}$ μ_ρ-measure-one of which are well-defined for each R_λ. Thus, a family of observables has joint beable status in a state just in case it is possible to think of the observables as possessing simultaneously determinate values without contradicting the state's expectation values. In particular, when ρ is well-defined on all the observables $\{R_\lambda : \lambda \in \Lambda\}$, their joint beable status for ρ (sufficient conditions for which are identified in Theorem 3.6 above) guarantees that the expectation values ρ assigns to each R_λ can be interpreted as arising due to ignorance about the precise values jointly possessed by the observables in $\{R_\lambda : \lambda \in \Lambda\}$.

As should be the case, the bounded observables in any subalgebra $\mathfrak{B} \subseteq \mathfrak{L}(\mathcal{H})$ beable for ρ have joint beable status for ρ. And, of course, any single bounded observable R—being

affiliated with the abelian von Neumann algebra it generates—has beable status in any state. However, when R is unbounded, this need not be true, as the next results show.

Proposition 3.7. *Let A, B be canonically conjugate self-adjoint unbounded operators on some Hilbert space $\mathcal{H}$, that is, they satisfy $[A, B] = \pm iI$ with $\mathrm{sp}(A) = \mathrm{sp}(B) = \mathbb{R}$. Let ρ be a state of $\mathfrak{L}(\mathcal{H})$ such that $\rho|_{W^*(A)}$ is pure, and ρ is well-defined for A. Then $\rho(E(\mathsf{C})) = 0$ for any compact interval C in $\mathbb{R}$, where E is the projection-valued measure for B.*

Proof. We show first that $\rho(\cos tB) = 0$ for all $t \in \mathbb{R}\backslash\{0\}$. For this, let $U_s \equiv e^{isA}$ and let $W_t \equiv e^{itB}$. Then, invoking the Weyl form of $[A, B] = \pm iI$ (taking either sign), we have $U_s W_t = e^{\pm ist} W_t U_s$ for all $s, t \in \mathbb{R}$. Thus, $\rho(U_s W_t) = e^{\pm ist} \rho(W_t U_s)$. Moreover, since $U_s \in W^*(A)$, ρ is dispersion-free on U_s and $\rho(U_s)\rho(W_t) = e^{\pm ist} \rho(W_t)\rho(U_s)$. Again, since ρ is dispersion-free on $W^*(A)$, $\rho(U_s) \neq 0$ for all $s \in \mathbb{R}$, and $\rho(W_t) = e^{\pm ist}\rho(W_t)$ for all $s, t \in \mathbb{R}$. Let $t = t_0 \neq 0$. Then we may choose s such that $e^{\pm ist_0} \neq 1$, and hence $\rho(W_{t_0}) = 0$. But t_0 was an arbitrary non-zero number; thus, $\rho(W_t) = 0$ for all $t \neq 0$. Moreover, $\rho(W_t) = \rho(\cos tB) + i\rho(\sin tB)$, from which it follows that $\rho(\cos tB) = 0$ for all $t \neq 0$.

Recall that

$$\cos^{2n}\theta = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{m=0}^{n-1}\binom{2n}{m}\cos 2(n-m)\theta. \qquad (3.17)$$

Let $F_t^n \equiv \cos^{2n} tB$. From (3.17) we may deduce the operator identity:

$$\cos^{2n} tB = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{m=0}^{n-1}\binom{2n}{m}\cos 2(n-m)tB. \qquad (3.18)$$

Thus, from the linearity of ρ, in combination with the result of the previous paragraph, we may conclude that $\rho(F_t^n) = 2^{-2n}\binom{2n}{n} \equiv k(n)$ whenever $t \neq 0$. And, using Stirling's approximation for the factorial, $k(n) \approx (\pi n)^{-1/2}$ for large n, whence $\lim_{n\to\infty} k(n) = 0$.

Now let C be a compact interval in $\mathbb{R}$. Then, $\rho(F_t^n E(\mathsf{C})) \leq \rho(F_t^n) = k(n)$, for all $n \in \mathbb{N}$ and all $t \neq 0$. Consider the extended function representation $\mathcal{N}(\mathsf{S})$ of the abelian von Neumann algebra $W^*(B)$. Let $\mathsf{T} \equiv \{\omega \in \mathsf{S} : \omega(E(\mathsf{C})) = 1\}$. ($\mathsf{T}$ is clopen since it is the support of the continuous idempotent function $\phi(E(\mathsf{C}))$.) Fix $n \in \mathbb{N}$ and let $f_t \equiv \phi(F_t^n E(\mathsf{C}))$ for each $t \in \mathbb{R}\backslash\{0\}$. We claim that f_t converges pointwise to χ_T as $t \to 0$. Note first that $\phi(B)$ is defined at all points of T and $\phi(B)(\mathsf{T}) \subseteq \mathsf{C}$ by Lemma 3.1.(i). Now, for any $\epsilon > 0$, we may choose t small enough that $1 - \cos^{2n} tx < \epsilon$ for all $x \in \mathsf{C}$ (since C is compact and n is fixed). Thus, for any $\omega \in \mathsf{T}$ (and using Cont-FUNC in the fourth step),

$$\chi_\mathsf{T}(\omega) - f_t(\omega) = 1 - \phi(F_t^n E(\mathsf{C}))(\omega) = 1 - \omega(F_t^n E(\mathsf{C})) \qquad (3.19)$$

$$= 1 - \omega(F_t^n) = 1 - \cos^{2n}(t\omega(B)) = 1 - \cos^{2n}(t\phi(B)(\omega)) < \epsilon, \qquad (3.20)$$

which is what we needed to show.

Since f_t converges pointwise to χ_{T} we may apply the Dominated Convergence Theorem [32, Thm. 1.34] to conclude that

$$\lim_{t \to 0} \int_{\mathsf{S}} f_t d\mu_\rho \; = \; \int_{\mathsf{S}} \chi_{\mathsf{T}} d\mu_\rho \; = \; \mu_\rho(\mathsf{T}) \; = \; \rho(E(\mathsf{C})). \tag{3.21}$$

However, since $k(n) \geq \rho(F_t^n E(\mathsf{C})) = \int f_t d\mu_\rho$, for all $t \neq 0$, it follows that $k(n) \geq \rho(E(\mathsf{C}))$. Since this is true for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} k(n) = 0$, it follows that $\rho(E(\mathsf{C})) = 0$. $\qquad\square$

Corollary 3.8. *Let A, B be as above. Then $\mu_\rho(\mathsf{Z}) = 1$, where Z is the set of states at which B is not defined. In particular, when ρ is a state of $\mathfrak{L}(\mathcal{H})$ such that $\rho\,|_{W^*(A)}$ is pure and ρ is well-defined for A, then B does* not *have beable status for ρ.*

Proof. Let $E_n \equiv E([-n, n])$. Let $\mathsf{S}_n \equiv \{\omega \in \mathsf{S} : \omega(E_n) = 1\}$. Then, from the preceding Proposition, $\rho(E_n) = 0$, and thus $\mu_\rho(\mathsf{S}_n) = 0$, for all $n \in \mathbb{N}$. However, $\cup_{n=1}^{\infty} \mathsf{S}_n = \mathsf{S} \backslash \mathsf{Z}$, and it follows from the countable additivity of μ_ρ that $\mu_\rho(\mathsf{S} \backslash \mathsf{Z}) = 0$. $\qquad\square$

Example (Heisenberg-Bohr Indeterminacy Principle)**.** Let D and Q be the momentum and position operators for a particle in one-dimension with state space $L_2(\mathbb{R})$. It is a well-known consequence of $[Q, D] = i\hbar I$ that the product of the dispersions of Q and D, for all wavefunctions $\psi \in \mathcal{D}(QD) \cap \mathcal{D}(DQ)$, is bounded below by $\hbar$. The standard Copenhagen interpretation of this uncertainty principle is not simply that a precision momentum measurement necessarily and uncontrollably disturbs the value of position, and vice-versa, but that D and Q can never in reality be thought of as simultaneously determinate. The warrant for this stronger 'indeterminacy principle' is not obvious, since there appears to be nothing preventing the view that the dispersion required in (say) a particle's momentum when its position is measured simply reflects our loss of knowledge about that momentum—not any breakdown in the applicability of the momentum concept itself. However, the foregoing results allow us to exhibit the indeterminacy principle as a direct mathematical consequence of $[Q, D] = i\hbar I$ (and without taking any a priori stand on precisely which (if any) of the many subalgebras with beable status for a given state should be taken to represent observables that *actually* possess determinate values). As we have seen, a necessary (and sufficient) condition for thinking of Q and D as having simultaneously determinate values in a state ρ is that they have joint beable status for ρ. This, in turn, requires that ρ be a mixture of states (on some subalgebra of $\mathfrak{L}(L_2)$) each of which is pure on both $W^*(Q)$ and $W^*(D)$ and well-defined on both Q and D. Yet, as Proposition 3.7 and its Corollary make clear, satisfaction of these requirements for Q precludes their satisfaction for D, and vice-versa. It follows that there is *no* state ρ for which Q and D have joint beable status, and the indeterminacy principle is proved.

4. Beable Subalgebras Determined by a Family of Privileged Observables

It is evident from Theorem 2.8 that any subspace $\mathcal{T} \subseteq \mathcal{H}$ containing $\mathcal{K}$, together with any maximal abelian subalgebra of $\mathfrak{L}(\mathcal{T})$, determines a maximal beable subalgebra $\mathfrak{B} \subseteq \mathfrak{L}(\mathcal{H})$ for K. In the present section we take steps to eliminate this arbitrariness. Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{R}$ be a mutually commuting family of "privileged" observables drawn from $\mathfrak{A}$. We may then inquire into the structure of all beable algebras for a given state that contain the commuting family $\mathfrak{R}$.

The reasons why one might want to demand a priori that certain preferred observables $\mathfrak{R}$ be included in the subalgebra with beable status will become apparent when we apply our results to the orthodox Copenhagen interpretation of quantum theory below. We shall also be requiring that a beable subalgebra $\mathfrak{B}$ for ρ containing some set of observables $\mathfrak{R}$ be (at least implicitly) definable in terms of $\mathfrak{R}$, ρ, and the algebraic operations available within $\mathfrak{A}$. This idea is captured by requiring that $\mathfrak{B}$ be invariant under spatial automorphisms of $\mathfrak{A}$ that fix both $\mathfrak{R}$ and the state ρ. (We say that the spatial automorphism Φ induced by unitary U *fixes* ρ just in case $\rho_U = \rho$, where ρ_U is defined by $\rho_U(A) = \rho(U^*AU)$ for all A in $\mathfrak{A}$.)

Definition. Let $\mathfrak{A}$ be a C^*-algebra, let $\mathfrak{R}$ be any mutually commuting family of observables in $\mathfrak{A}$, and let ρ be a state of $\mathfrak{A}$. Then, for any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, we say that $\mathfrak{B}$ is $\mathfrak{R}$-*beable* for ρ just in case:

> **(Beable):** $\mathfrak{B}$ is beable for ρ.
> **($\mathfrak{R}$-Priv):** $\mathfrak{R} \subseteq \mathfrak{B}$.
> **(Def):** For any unitary $U \in \mathfrak{A}$, if $U \in \mathfrak{R}'$ and $\rho_U = \rho$, then $U\mathfrak{B}U^* = \mathfrak{B}$.

We say that $\mathfrak{B}$ is *maximal $\mathfrak{R}$-beable* for ρ if and only if $\mathfrak{B}$ is maximal with respect to the properties (Beable), ($\mathfrak{R}$-Priv), and (Def) (noting that, by Zorn's lemma, maximal $\mathfrak{R}$-beable algebras exist for any state).

4.1. **$\mathfrak{R}$-beable algebras for normal states.** We now specialize to the case where $\mathfrak{A} = \mathfrak{L}(\mathcal{H})$, and where ρ is a normal state of $\mathfrak{L}(\mathcal{H})$. In this case, we may replace $\rho_U = \rho$ in (Def) by $UKU^* = K$, where K is the trace-1 operator that defines the state ρ. We shall soon see that the above requirements, for certain $\mathfrak{R}$ and K, suffice to determine a *unique* maximal $\mathfrak{R}$-beable algebra for K (cf. Corollary 4.6 below).

Lemma 4.1. *Suppose that $\mathfrak{B}$ is a C^*-algebra acting on some Hilbert space $\mathcal{H}$, and that ρ is a normal state of $\mathfrak{B}$. If $\mathfrak{B}$ is $\mathfrak{R}$-beable for ρ, then $\mathfrak{B}^-$ is also $\mathfrak{R}$-beable for ρ.*

Proof. ($\mathfrak{R}$-Priv) $\mathfrak{R} \subseteq \mathfrak{B} \subseteq \mathfrak{B}^-$. (Beable) See Corollary 2.9.(ii). (Def) Suppose that U is a unitary element of $\mathfrak{L}(\mathcal{H})$ such that $U \in \mathfrak{R}'$ and $UKU^* = K$. Then, since $\mathfrak{B}$ satisfies

(Def), $U\mathfrak{B}U^* = \mathfrak{B}$. Since the spatial automorphism Φ of $\mathfrak{L}(\mathcal{H})$ induced by U is a WOT-homeomorphism from $\mathfrak{L}(\mathcal{H})$ to $\mathfrak{L}(\mathcal{H})$, it follows that $\Phi(\mathfrak{B}^-) = \Phi(\mathfrak{B})^- = \mathfrak{B}^-$. $\qquad\square$

In order to prove the main result of this section, we will need to make use of the following lemma:

Lemma 4.2. *Let $Q \in \mathfrak{L}(\mathcal{H})$ be a projection, and let $\mathfrak{V}$ be a von Neumann algebra acting on $\mathcal{H}$. Suppose that for every unitary operator $U \in \mathfrak{V}'$, $[UQU^*, Q] = 0$. Then, $Q \in \mathfrak{V}$.*

Remark 4.3. Recall that every element of a C^*-algebra (such as $\mathfrak{V}'$) is expressible as a linear combination of (four) unitary elements in that algebra [23, Theorem 4.1.7]. Thus, we may reformulate Lemma 4.2 equivalently as: If $[UQU^*, Q] = 0$ for each $U \in \mathfrak{V}'$, then $[U, Q] = 0$ for each $U \in \mathfrak{V}'$.

Proof. To show that $Q \in \mathfrak{V}(= \mathfrak{V}'')$, it will suffice to show that $[Q, H] = 0$ for any self-adjoint $H \in \mathfrak{V}'$ (since $\mathfrak{V}'$ is a $*$-algebra). If $H = H^* \in \mathfrak{V}'$, then $U_t \equiv e^{itH} \in \mathfrak{V}'$ is unitary for all $t \in \mathbb{R}$. By hypothesis, then, $[U_t Q U_{-t}, Q] = 0$, for all $t \in \mathbb{R}$.

Since H is bounded, $\mathrm{sp}(H)$ is a compact subset of $\mathbb{R}$. Consider the one-parameter family $\{e^{itx}\}_{t \in \mathbb{R}}$ of (complex valued) continuous functions on $\mathrm{sp}(H)$. Clearly, this family converges uniformly to the constant 1 function as $t \to 0$. Employing the continuous function calculus [23, p. 239], it follows that e^{itH} converges uniformly to I as $t \to 0$. Thus, $\lim_{t \to 0}(U_t Q U_{-t}) = Q$. Since $U_t Q U_{-t}$ and Q commute, we may write $Q = A_t + B_t$, $U_t Q U_{-t} = A_t + C_t$, where A_t, B_t and C_t are pairwise orthogonal projections. Then, $0 = \lim_{t \to 0}\|U_t Q U_{-t} - Q\| = \lim_{t \to 0}\|B_t - C_t\|$. Choose $s > 0$ such that $\|B_t - C_t\| < \frac{1}{2}$ for all $t < s$. Suppose that $B_t \neq 0$ for some $t < s$. Then $\mathcal{R}(B_t) \neq \{0\}$ and we may choose a unit vector $x \in \mathcal{R}(B_t)$. But then $\|(B_t - C_t)x\| = \|x\| = 1$, which contradicts the fact that $\|B_t - C_t\| < \frac{1}{2}$. Thus, $B_t = 0$ for all $t < s$, and by symmetry $C_t = 0$ for all $t < s$. Hence, for all $t < s$, $U_t Q U_{-t} = A_t = Q$, i.e. $[U_t, Q] = 0$.

Employing the functional calculus for $\mathrm{sp}(H)$ again, we see that $t^{-1}(e^{itx} - 1)$ converges uniformly to ix as $t \to 0$; thus, $t^{-1}(e^{itH} - I) \to iH$ uniformly as $t \to 0$. We may then compute,

$$(-i)(iH)Q = -i\left[\lim_{t \to 0}(t^{-1}(U_t - I))\right]Q = -i\left[\lim_{t \to 0}(t^{-1}(U_t Q - Q))\right] \tag{4.1}$$

$$= -i\left[\lim_{t \to 0}(t^{-1}(QU_t - Q))\right] = -iQ\left[\lim_{t \to 0}(t^{-1}(U_t - I))\right] \tag{4.2}$$

$$= -iQ(iH). \tag{4.3}$$

The second (and fourth) equalities follow since right (and left) multiplication by Q is norm continuous. The third equality follows since there is an $s > 0$ such that $[U_t, Q] = 0$ for all $t < s$. Therefore, $[H, Q] = 0$. $\qquad\square$

Remark 4.4. As before, let $\mathcal{K} \equiv \mathcal{R}(K)$. Consider the family of subspaces $\mathcal{Y}$ of $\mathcal{H}$ such that $\mathcal{Y}$ contains $\mathcal{K}$ and $\mathcal{Y}$ is invariant under each element of $\mathfrak{R}$. Since this family is closed under intersection, it will contain a unique smallest element which we denote by $\mathcal{S}$. It is not difficult to see then that $\mathcal{S} = [\mathfrak{R}''\mathcal{K}]$. (Note that since $\mathfrak{R}$ is (trivially) closed under taking adjoints, it follows that $W^*(\mathfrak{R}) = \mathfrak{R}''$.) Indeed, $[\mathfrak{R}''\mathcal{K}]$ contains $\mathcal{K}$ and is invariant under each element of $\mathfrak{R}$. Thus, $\mathcal{S} \subseteq [\mathfrak{R}''\mathcal{K}]$. Conversely, $[\mathfrak{R}''\mathcal{K}]$ is the smallest subspace of $\mathcal{H}$ that contains $\mathcal{K}$ and that is invariant under each element in $\mathfrak{R}''$. However, $\mathcal{S}$ contains $\mathcal{K}$ and $\mathcal{S}$ is invariant under each element in $\mathfrak{R}''$ (since $P_{\mathcal{S}} \in \mathfrak{R}' = \mathfrak{R}'''$). Therefore, $\mathcal{S} = [\mathfrak{R}''\mathcal{K}]$.

Theorem 4.5. *Let $\mathcal{S}$ be the smallest subspace of $\mathcal{H}$ such that $\mathcal{S}$ contains $\mathcal{R}(K)$ and $\mathcal{S}$ is invariant under $\mathfrak{R}$ (so $\mathcal{S} = [\mathfrak{R}''\mathcal{K}]$). Then, every maximal $\mathfrak{R}$-beable algebra for K has the form $\mathfrak{L}(\mathcal{S}^\perp) \oplus \mathfrak{M}$, where $W^*(\mathfrak{R})P_{\mathcal{S}} \subseteq \mathfrak{M} \subseteq W^*(\mathfrak{R}, K)P_{\mathcal{S}}$, and $\mathfrak{M}$ is maximal abelian in $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$.*

For convenience, we call algebras of the above form *MRB-algebras* for K. (With this proof in hand, we can justifiably call these algebras maximal $\mathfrak{R}$-beable algebras.)

Proof. The proof proceeds in two stages. First, we show that every MRB-algebra for K is, in fact, a maximal $\mathfrak{R}$-beable algebra for K ((i) below). Second, we show that every $\mathfrak{R}$-beable algebra for K is contained in some MRB-algebra for K ((ii) below).

(i) Suppose that $\mathfrak{B}$ is an MRB-algebra for K. That is $\mathfrak{B} = \mathfrak{L}(\mathcal{S}^\perp) \oplus \mathfrak{M}$, where $\mathfrak{M}$ is a maximal abelian subalgebra of $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$, and $W^*(\mathfrak{R})P_{\mathcal{S}} \subseteq \mathfrak{M}$.

($\mathfrak{R}$-Priv) Since $\mathfrak{R}$ leaves $\mathcal{S}$ invariant and $\mathfrak{R}P_{\mathcal{S}} \subseteq W^*(\mathfrak{R})P_{\mathcal{S}} \subseteq \mathfrak{M}$, it follows that $\mathfrak{R} \subseteq \mathfrak{B}$.

(Beable) Let $\mathcal{T} \equiv [\mathfrak{B}\mathcal{K}]$. By construction of $\mathcal{S}$, $\mathcal{S}$ contains $\mathcal{K}$ and is invariant under each element of $\mathfrak{B}$. However, $\mathcal{T}$ is the smallest subspace of $\mathcal{H}$ that contains $\mathcal{K}$ and is invariant under each element in $\mathfrak{B}$. Hence, $\mathcal{T} \subseteq \mathcal{S}$. Conversely, $\mathcal{T}$ is invariant under each element of $\mathfrak{R}$ since $\mathfrak{R}$ is contained in $\mathfrak{B}$. Thus, $\mathcal{S} = \mathcal{T}$ and $\mathfrak{B} = \mathfrak{L}(\mathcal{T}^\perp) \oplus \mathfrak{M}$. But since $\mathfrak{M}$ is an abelian subalgebra of $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$, it is an abelian subalgebra of $\mathfrak{L}(\mathcal{T})$. By Theorem 2.8.(i), $\mathfrak{B}$ is beable for K.

(Def) We show first that $P_{\mathcal{S}}$ is in the center of $W^*(\mathfrak{R}, K)$. Since $\mathfrak{R} \cup \{K\}$ is a self-adjoint set, $W^*(\mathfrak{R}, K) = (\mathfrak{R} \cup \{K\})''$. Let $A \in (\mathfrak{R} \cup \{K\})' = \mathfrak{R}' \cap \{K\}'$, and let Bx be a generator of $\mathcal{S}$. (That is, $x \in \mathcal{K}$ and $B \in \mathfrak{R}''$.) Since A commutes with K, A leaves $\mathcal{K}$ invariant. Further, $[A, B] = 0$ since $A \in \mathfrak{R}' = \mathfrak{R}'''$. Thus, $A(Bx) = B(Ax) \in [\mathfrak{R}''\mathcal{K}] \equiv \mathcal{S}$, and we may conclude (by linearity and continuity of A) that $A(\mathcal{S}) \subseteq \mathcal{S}$. Since the same argument applies to A^* (which is also contained in $\mathfrak{R}' \cap \{K\}'$), $\mathcal{S}$ reduces A; and thus, $P_{\mathcal{S}} \in (\mathfrak{R} \cup \{K\})''$. On the other hand, $P_{\mathcal{S}}$ is clearly contained in $(\mathfrak{R} \cup \{K\})'$ since $\mathcal{S}$ is invariant under the action of K and under the action of the self-adjoint set $\mathfrak{R}$.

Let $U \in (\mathfrak{R} \cup \{K\})' = \mathfrak{R}' \cap \{K\}'$. Since $P_{\mathcal{S}} \in (\mathfrak{R} \cup \{K\})''$, it follows that $UP_{\mathcal{S}} = P_{\mathcal{S}}U$. Now let $B \in \mathfrak{B}$. Then,

$$B = (I - P_{\mathcal{S}})B(I - P_{\mathcal{S}}) \oplus P_{\mathcal{S}}BP_{\mathcal{S}}, \tag{4.4}$$

where $P_{\mathcal{S}}BP_{\mathcal{S}} \in W^*(\mathfrak{R}, K)P_{\mathcal{S}}$. Since $\mathcal{S}$ reduces U, the spatial isomorphism Φ induced by U factors into Φ_1, the spatial automorphism on $\mathfrak{L}(\mathcal{S}^\perp)$ induced by $(I - P_{\mathcal{S}})U(I - P_{\mathcal{S}})$, and Φ_2, the spatial automorphism on $\mathfrak{L}(\mathcal{S})$ induced by $P_{\mathcal{S}}UP_{\mathcal{S}}$. Hence,

$$\Phi(B) = \Phi_1((I - P_{\mathcal{S}})B(I - P_{\mathcal{S}})) \oplus \Phi_2(P_{\mathcal{S}}BP_{\mathcal{S}}). \tag{4.5}$$

Trivially, $\Phi_1((I - P_{\mathcal{S}})B(I - P_{\mathcal{S}})) \in \mathfrak{L}(\mathcal{S}^\perp)$. Furthermore, since $\mathfrak{M}$ is a subset of $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$, it follows that Φ_2 is the identity automorphism on $\mathfrak{M}$. To see this, note that,

$$P_{\mathcal{S}}UP_{\mathcal{S}} \in P_{\mathcal{S}}(\mathfrak{R} \cup \{K\})'P_{\mathcal{S}} = [W^*(\mathfrak{R}, K)P_{\mathcal{S}}]', \tag{4.6}$$

where the final equality follows from [23, Proposition 5.5.6] and the fact that $P_{\mathcal{S}} \in (\mathfrak{R} \cup \{K\})'$. Thus, $P_{\mathcal{S}}UP_{\mathcal{S}}$ commutes with every operator in $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$, and Φ_2 is the identity automorphism on $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$. It then follows that

$$\Phi(B) = \Phi_1((I - P_{\mathcal{S}})B(I - P_{\mathcal{S}})) \oplus \Phi_2(P_{\mathcal{S}}BP_{\mathcal{S}}) \tag{4.7}$$

$$= \Phi_1((I - P_{\mathcal{S}})B(I - P_{\mathcal{S}})) \oplus P_{\mathcal{S}}BP_{\mathcal{S}}, \tag{4.8}$$

which is obviously contained in $\mathfrak{B}$. Since B was an arbitrary element of $\mathfrak{B}$, it follows that $\Phi(\mathfrak{B}) \subseteq \mathfrak{B}$. Moreover, this map is onto, for given $A \in \mathfrak{B}$,

$$\Phi((I - P_{\mathcal{S}})\Phi^{-1}(A)(I - P_{\mathcal{S}}) \oplus P_{\mathcal{S}}AP_{\mathcal{S}}) = (I - P_{\mathcal{S}})A(I - P_{\mathcal{S}}) \oplus \Phi_2(P_{\mathcal{S}}AP_{\mathcal{S}}) \tag{4.9}$$

$$= (I - P_{\mathcal{S}})A(I - P_{\mathcal{S}}) \oplus P_{\mathcal{S}}AP_{\mathcal{S}} = A. \tag{4.10}$$

Therefore, $\Phi(\mathfrak{B}) = \mathfrak{B}$.

(Maximality) To see that $\mathfrak{B}$ is a maximal $\mathfrak{R}$-beable algebra for K, it suffices to show that (a) every $\mathfrak{R}$-beable algebra for K is contained in an MRB-algebra for K, and (b) if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are distinct MRB-algebras for K, then $\mathfrak{B}_1 \nsubseteq \mathfrak{B}_2$. We establish (a) in (ii) below. For (b) it suffices to note that if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are distinct MRB-algebras for K, then $\mathfrak{B}_1 = \mathfrak{L}(\mathcal{S}^\perp) \oplus \mathfrak{M}_1$ and $\mathfrak{B}_2 = \mathfrak{L}(\mathcal{S}^\perp) \oplus \mathfrak{M}_2$, where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are distinct maximal abelian subalgebras of $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$ (each containing $W^*(\mathfrak{R})P_{\mathcal{S}}$). Thus, $\mathfrak{M}_1 \nsubseteq \mathfrak{M}_2$ and $\mathfrak{B}_1 \nsubseteq \mathfrak{B}_2$.

(ii) Suppose that $\mathfrak{B}$ is $\mathfrak{R}$-beable for K. Since $\mathfrak{B} \subseteq \mathfrak{B}^-$, it will suffice to show that $\mathfrak{B}^-$ is contained in an MRB-algebra for K because, by Lemma 4.1, $\mathfrak{B}^-$ is $\mathfrak{R}$-beable for K. Thus, we may assume that $\mathfrak{B}$ is a von Neumann algebra.

Once again, let $\mathcal{T} \equiv [\mathfrak{B}K]$. Obviously, $\mathcal{T}$ reduces $\mathfrak{B}$, and $P_{\mathcal{T}} \in \mathfrak{B}'$. Since $\mathfrak{B}$ is a von Neumann algebra $\mathfrak{B}P_{\mathcal{T}}$ is a von Neumann algebra acting on $\mathcal{T}$ [23, Proposition 5.5.6]. Likewise, $\mathfrak{B}(I - P_{\mathcal{T}})$ is a von Neumann algebra acting on $\mathcal{T}^\perp$. Let $\mathfrak{M} \equiv \mathfrak{B}P_{\mathcal{T}}$. Then we have $\mathfrak{B} = \mathfrak{B}(I - P_{\mathcal{T}}) \oplus \mathfrak{M}$, where each summand is a von Neumann algebra. Since $\mathfrak{B}$ is

beable for K, $\mathfrak{M}$ is in fact an abelian subalgebra of $\mathfrak{L}(\mathcal{T})$ (Theorem 2.8.(i)). We show that $\mathcal{T} = \mathcal{S}$ and that $\mathfrak{M} \subseteq W^*(\mathfrak{R}, K) P_{\mathcal{S}}$. (Clearly, once $\mathcal{T} = \mathcal{S}$ has been established, we will automatically have $W^*(\mathfrak{R}) P_{\mathcal{S}} \subseteq \mathfrak{M}$, since $W^*(\mathfrak{R}) \subseteq \mathfrak{B}$ and $W^*(\mathfrak{R})$ leaves $\mathcal{S}$ invariant.)

$\mathcal{S}$ is clearly a subspace of $\mathcal{T}$ since $\mathfrak{R} \subseteq \mathfrak{B}$. In order to show that $\mathcal{T} \subseteq \mathcal{S}$, let $\overline{F}$ be a projection in $\mathfrak{B}$. Since $\mathcal{T}$ reduces $\overline{F}$, $\overline{F} = F_0 \oplus F \in \mathfrak{L}(\mathcal{T}^\perp) \oplus \mathfrak{M}$. Choose $\theta \in \mathbb{R}$ such that $e^{-i\theta} \neq \pm 1$. Let $\overline{U} = P_{\mathcal{T}^\perp} \oplus P_{\mathcal{S}} \oplus (e^{i\theta} P_{\mathcal{T} \wedge \mathcal{S}^\perp})$, and let $U = P_{\mathcal{S}} \oplus (e^{i\theta} P_{\mathcal{T} \wedge \mathcal{S}^\perp}) \in \mathfrak{L}(\mathcal{T})$. Since $\mathfrak{R} \subseteq \mathfrak{B}$, $\mathfrak{R}$ leaves $\mathcal{T}$ invariant, and (by construction) $\mathfrak{R}$ leaves $\mathcal{S}$ invariant. Thus, $\mathfrak{R}$ leaves $\mathcal{S}^\perp \wedge \mathcal{T}$ invariant, and $\overline{U} \in \mathfrak{R}'$. Furthermore, $\mathcal{K} \subseteq \mathcal{S}$, and $\overline{U}|_{\mathcal{S}} = I$. Thus, $\overline{U} \in \{K\}'$. Since $\overline{U} \in (\mathfrak{R}' \cap \{K\}')$, it follows by (Def) that $\overline{U} \mathfrak{B} \overline{U}^* = \mathfrak{B}$; and since $\mathcal{T}$ reduces $\overline{U}$, it also follows that $U \mathfrak{M} U^* = \mathfrak{M}$. In particular, both F and $U F U^*$ are in the abelian algebra $\mathfrak{M}$.

Since F and $U F U^*$ commute, there are mutually orthogonal projections A, B, C on $\mathcal{T}$ such that $F = A + B$, $U F U^* = A + C$. To see that $B = 0$, let $v \in \mathcal{R}(B) \subseteq \mathcal{T}$. Then,

$$Fv = v, \tag{4.11}$$

$$U F U^* v = 0. \tag{4.12}$$

Now, we may also write $v = w + w'$ (uniquely), where $w \in \mathcal{S}$ and $w' \in (\mathcal{S}^\perp \wedge \mathcal{T})$. Using (4.12) (and $U^{-1}(0) = 0$), we get:

$$F U^* (w + w') = 0. \tag{4.13}$$

But, by the definition of U (and using $v = w + w'$), this implies that

$$F(w + e^{-i\theta} w') = 0 \tag{4.14}$$

and

$$F(v - w' + e^{-i\theta} w') = 0. \tag{4.15}$$

Next, using (4.11) and $e^{-i\theta} \neq 1$

$$Fw' = (1 - e^{-i\theta})^{-1} v, \tag{4.16}$$

thus,

$$U F U^* F w' = (1 - e^{-i\theta})^{-1} U F U^* v, \tag{4.17}$$

and we can see by (4.12) that this last expression vanishes. However, UFU^* and F commute on $w'(\in \mathcal{T})$. It follows that $F(UFU^*)w' = 0$ as well. We can then compute, using the definition of U and $e^{-i\theta} \neq 0$,

$$FUFw' = 0. \tag{4.18}$$

By (4.16),

$$FU[(1 - e^{-i\theta})^{-1}v] = 0. \tag{4.19}$$

But since $(1 - e^{-i\theta})^{-1} \neq 0$,

$$FUv = 0, \tag{4.20}$$
$$FU(w + w') = 0, \tag{4.21}$$
$$F(w + e^{i\theta}w') = 0, \tag{4.22}$$

again using the definition of U in the move to (4.22). But now (4.14) and (4.22) together entail:

$$F(w + e^{-i\theta}w' - (w + e^{i\theta}w')) = 0, \tag{4.23}$$
$$F((e^{-i\theta} - e^{i\theta})w') = 0. \tag{4.24}$$

However, $e^{-i\theta} - e^{i\theta} \neq 0$ (since $e^{-i\theta} \neq \pm 1$). Thus, by (4.22) and (4.24),

$$Fw = 0, \tag{4.25}$$
$$Fw' = 0, \tag{4.26}$$
$$Fv = F(w + w') \tag{4.27}$$
$$= 0. \tag{4.28}$$

Thus, $Fv = 0$. But, by (4.11), $Fv = v$. Hence, $v = 0$ and $B = 0$.

Now we may repeat a similar argument with F replaced by UFU^*, and $U^*(UFU^*)U = F$. (The only change to the argument is that, throughout, θ must be interchanged with $-\theta$, since U is interchanged with U^*). It follows that $C = 0$ as well, and thus $UFU^* = F$.

We chose U, however, so that if $UF = FU$, then $P_{\mathcal{S}}F = FP_{\mathcal{S}}$. Indeed, a routine calculation shows that $P_{\mathcal{S}} = (e^{i\theta} - 1)^{-1}[e^{i\theta}P_{\mathcal{T}} - U]$. Furthermore,

$$\overline{F}P_{\mathcal{S}} = [(I - P_{\mathcal{T}})\overline{F}(I - P_{\mathcal{T}}) + P_{\mathcal{T}}\overline{F}P_{\mathcal{T}}]P_{\mathcal{S}} \tag{4.29}$$
$$= [(I - P_{\mathcal{T}})\overline{F}(I - P_{\mathcal{T}}) + F]P_{\mathcal{S}} \tag{4.30}$$
$$= FP_{\mathcal{S}} = P_{\mathcal{S}}F = P_{\mathcal{S}}\overline{F} \tag{4.31}$$

since $P_{\mathcal{S}}(I - P_{\mathcal{T}}) = 0$. Thus, $\overline{F}P_{\mathcal{S}} = P_{\mathcal{S}}\overline{F}$, for all projections $\overline{F} \in \mathfrak{B}$. Since $\mathfrak{B}$ is (by hypothesis) a von Neumann algebra, each $A \in \mathfrak{B}$ is a norm-limit of linear combinations of projections in $\mathfrak{B}$. Thus, $AP_{\mathcal{S}} = P_{\mathcal{S}}A$, for all $A \in \mathfrak{B}$ and $\mathcal{S}$ reduces $\mathfrak{B}$. Since $\mathcal{K} \subseteq \mathcal{S}$, and since $\mathcal{T}$ is the smallest subspace that contains $\mathcal{K}$ and reduces $\mathfrak{B}$, it follows that $\mathcal{T} \subseteq \mathcal{S}$.

We have now shown that $\mathcal{T} = \mathcal{S}$ and that, accordingly, $\mathfrak{M}$ is an abelian von Neumann subalgebra of $\mathfrak{L}(\mathcal{S})$. All that remains is to show that $\mathfrak{M} \subseteq W^*(\mathfrak{R}, K)P_{\mathcal{S}}$. Since $\mathfrak{M}$ is a von Neumann algebra, it will suffice to show that for any projection $Q \in \mathfrak{M}$, $Q \in W^*(\mathfrak{R}, K)P_{\mathcal{S}}$. Let $U \in (\mathfrak{R}' \cap \{K\}')$. Then, by (Def), $U\mathfrak{B}U^* = \mathfrak{B}$. Since U is reduced by $\mathcal{S} = [\mathfrak{R}''\mathcal{K}]$, $U\mathfrak{M}U^* = \mathfrak{M}$. In particular, $Q, UQU^* \in \mathfrak{M}$. Since $\mathfrak{M}$ is abelian, $UQU^*Q = QUQU^*$. But U was an arbitrary unitary operator in $\mathfrak{R}' \cap \{K\}'$. Applying Lemma 4.2, with $\mathfrak{V} = W^*(\mathfrak{R}, K)$, we may conclude that $Q \in W^*(\mathfrak{R}, K)$. Moreover, $QP_{\mathcal{S}} = Q$, since $Q \leq P_{\mathcal{S}}$. Thus, $Q \in W^*(\mathfrak{R}, K)P_{\mathcal{S}}$ and $\mathfrak{B}$ is contained in an MRB-algebra for K. $\qquad \square$

Theorem 4.5 shows that the requirement that $\mathfrak{B}$ be maximal $\mathfrak{R}$-beable places a significant restriction on the structure of $\mathfrak{B}$. However, it still does not follow that there is always a unique maximal $\mathfrak{R}$-beable algebra for K.

Example. Let $\mathcal{H}$ be three-dimensional, choose an orthonormal basis $\{r_1, r_2, r_3\}$, and let $\mathfrak{R} = \{R\}$, where R is a self-adjoint operator on $\mathcal{H}$ with only two eigenvalues and corresponding eigenspaces, $[r_1]$ and $[r_2, r_3]$. Choose another orthonormal basis $\{w_1, w_2, w_3\}$ so that the vectors w_1 and w_2 do not lie inside either of R's eigenspaces, and $P_{[r_2,r_3]}w_1$ and $P_{[r_2,r_3]}w_2$—the orthogonal projections of w_1 and w_2 onto the plane $[r_2, r_3]$—are neither parallel nor orthogonal. Let K be any positive, trace-1 operator with three distinct nonzero eigenvalues corresponding to the eigenspaces $[w_1]$, $[w_2]$, and $[w_3]$. By construction, $\{P_{[r_1]}w_1, P_{[r_2,r_3]}w_1, P_{[r_2,r_3]}w_2\}$ spans $\mathcal{H}$, and thus $\mathcal{S} = [\mathfrak{R}''\mathcal{K}] = \mathcal{H}$. It follows from Theorem 4.5 that any maximal abelian subalgebra of $W^*(R, K)$ containing R is maximal R-beable for K. In fact, $W^*(R, K)$ contains *two* such subalgebras.

Let A_1 be any nondegenerate, self-adjoint operator with one-dimensional (mutually orthogonal) eigenspaces $[P_{[r_2,r_3]}w_1]$, $[P_{[r_2,r_3]}w_1]^{\perp} \wedge [r_2, r_3]$, and $[r_1]$. Let A_2 be any nondegenerate, self-adjoint operator with one-dimensional eigenspaces $[P_{[r_2,r_3]}w_2]$, $[P_{[r_2,r_3]}w_2]^{\perp} \wedge [r_2, r_3]$ and $[r_1]$. Since $W^*(R, K)$ contains the spectral projections of both R and K, and the projections in $W^*(R, K)$ form an ortholattice, the projections onto all the eigenspaces of each A_i lie in $W^*(R, K)$ (for example, $[P_{[r_2,r_3]}w_1]$ may be expressed as $([w_1] \vee [r_1]) \wedge [r_2, r_3]$, and similarly for $[P_{[r_2,r_3]}w_2]$). It follows that each $A_i \in W^*(R, K)$. Let $W^*(A_i)$ be the von Neumann algebra generated by A_i. Since the projections onto $[P_{[r_2,r_3]}w_1]$ and $[P_{[r_2,r_3]}w_2]$ fail to commute (by construction of w_1 and w_2), A_1 and A_2 do not commute and $W^*(A_1)$ and $W^*(A_2)$ are distinct. And, since each A_i is nondegenerate, each $W^*(A_i)$ is maximal abelian in $\mathfrak{L}(\mathcal{H})$, and thus maximal abelian in $W^*(R, K)$. Moreover, $R \in W^*(A_i)$ since R

commutes with each A_i. Therefore, $W^*(A_1)$ and $W^*(A_2)$ are distinct maximal R-beable algebras for K.

Although the above example shows that we cannot always expect there to be a unique maximal $\mathfrak{R}$-beable algebra for K, there are at least two important cases where uniqueness does hold:

Corollary 4.6.

    (i) *If $K \in \mathfrak{R}'$, then the* unique *maximal $\mathfrak{R}$-beable algebra for K is $\mathfrak{L}(\mathcal{S}^\perp) \oplus W^*(\mathfrak{R}, K)P_{\mathcal{S}}$.*

    (ii) *If $K = P_v$, for some $v \in \mathcal{H}$, then the* unique *maximal $\mathfrak{R}$-beable algebra for K is $\mathfrak{L}(\mathcal{S}^\perp) \oplus W^*(\mathfrak{R})P_{\mathcal{S}}$.*

Proof. (i) Since elements of $\mathfrak{R}$ pairwise commute, and $K \in \mathfrak{R}'$, it follows that $W^*(\mathfrak{R}, K)$ is abelian, as is $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$. Therefore, $\mathfrak{L}(\mathcal{S}^\perp) \oplus W^*(\mathfrak{R}, K)P_{\mathcal{S}}$ is itself the unique maximal $\mathfrak{R}$-beable algebra for K.

(ii) Recall that $\mathcal{S} = [\mathfrak{R}''\mathcal{K}]$. Thus, in this case, $\mathcal{S} = [\mathfrak{R}''v]$. Since $W^*(\mathfrak{R})P_{\mathcal{S}}$ is abelian and v is a cyclic vector for $W^*(\mathfrak{R})P_{\mathcal{S}}$, it follows that $W^*(\mathfrak{R})P_{\mathcal{S}}$ is maximal abelian as a subalgebra of $\mathfrak{L}(\mathcal{S})$ [23, Corollary 7.2.16]. Accordingly, $W^*(\mathfrak{R})P_{\mathcal{S}}$ is the unique maximal abelian subalgebra $\mathfrak{M}$ of $W^*(\mathfrak{R}, K)P_{\mathcal{S}}$ with the property that $W^*(\mathfrak{R})P_{\mathcal{S}} \subseteq \mathfrak{M}$. $\qquad\square$

Remark 4.7. When $\mathfrak{R} = \{K\}$, case (i) applies, and the maximal $\mathfrak{R}$-beable subalgebra consists of exactly those observables that share with K the spectral projections that project onto K's range. This set of observables are those taken to be determinate in most 'modal' interpretations of quantum theory [13, 14, 37]. On the other hand, case (ii) strengthens and generalizes (to observables with continuous spectra) the theorem proved in [11], which is the basis for the alternative modal interpretation of quantum theory developed by Bub [10].

In what follows, we will denote the von Neumann algebra referred to in Corollary 4.6.(ii) by $\mathfrak{B}(\mathfrak{R}, v)$. That is, $\mathfrak{B}(\mathfrak{R}, v) = \mathfrak{L}(\mathcal{S}^\perp) \oplus W^*(\mathfrak{R})P_{\mathcal{S}}$, where $\mathcal{S} = [\mathfrak{R}''v]$. We end this section with two applications of Corollary 4.6.(ii) to the Copenhagen interpretation of quantum theory that are facilitated by the following more tractable characterization:

Proposition 4.8. *Let A be in $\mathfrak{L}(\mathcal{H})_{\mathrm{sa}}$.*

    (i) *If $A \in \mathfrak{R}'$ and $Av \in [\mathfrak{R}''v]$, then $A \in \mathfrak{B}(\mathfrak{R}, v)$.*

    (ii) *If A does not leave $[\mathfrak{R}''v]$ invariant, then $A \notin \mathfrak{B}(\mathfrak{R}, v)$.*

Proof. (i) Suppose that $A \in \mathfrak{R}' = \mathfrak{R}'''$ and that $Av \in [\mathfrak{R}''v]$. Using Lemma 2.1 for the C^*-algebra $\mathfrak{R}''$, it follows that A leaves $[\mathfrak{R}''v] = \mathcal{S}$ invariant. Since A is self-adjoint, A also leaves $\mathcal{S}^\perp$ invariant and $A \in \mathfrak{L}(\mathcal{S}^\perp) \oplus \mathfrak{L}(\mathcal{S})$.

Since $P_{\mathcal{S}} \in W^*(\mathfrak{R})'$, the commutant of $W^*(\mathfrak{R})P_{\mathcal{S}}$ relative to $\mathfrak{L}(\mathcal{S})$ is $P_{\mathcal{S}}W^*(\mathfrak{R})'P_{\mathcal{S}} = P_{\mathcal{S}}\mathfrak{R}'P_{\mathcal{S}}$ [23, Prop. 5.5.6]. Clearly, then, $P_{\mathcal{S}}AP_{\mathcal{S}}$ is in the commutant of $W^*(\mathfrak{R})P_{\mathcal{S}}$. However, since $W^*(\mathfrak{R})P_{\mathcal{S}}$ is maximal abelian, $P_{\mathcal{S}}AP_{\mathcal{S}} \in W^*(\mathfrak{R})P_{\mathcal{S}}$. Therefore, $A \in \mathfrak{L}(\mathcal{S}^{\perp}) \oplus W^*(\mathfrak{R})P_{\mathcal{S}} = \mathfrak{B}(\mathfrak{R}, v)$.

(ii) is trivial, since each element in $\mathfrak{B}(\mathfrak{R}, v)$ leaves $[\mathfrak{R}''v]$ invariant. $\qquad\square$

One of Bell's motivations for distinguishing beables from observables was that the distinction makes "...explicit some notions already implicit in, and basic to, ordinary quantum theory. For, in the words of Bohr, 'it is decisive to recognize that, however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms'. It is the ambition of the theory of local beables to bring these 'classical terms' into the equations, and not relegate them entirely to the surrounding talk" [5, p. 52]. One can fulfil this ambition by understanding Bohr's assertions, about the possibility of attributing certain observables determinate values in certain measurement contexts, as arising from selecting the maximal set of observables that can be determinate together with the determinacy of whatever measurement results are actually obtained in a given measurement context. Thus we propose to understand the Copenhagen interpretation, *not* as relying on a collapse of the state vector when a measurement occurs, but rather as selecting for beable status the maximal R-beable subalgebra determined by the "privileged" pointer observable R of the measuring system and the pure entangled state P_v of the composite measured/measuring system (see [9, 20] for related proposals that do not adopt an algebraic approach). It then becomes possible to make precise the (hitherto obscure) sense in which a measurement, for Bohr, can 'make determinate' the observable that was measured, as well as make determinate certain observables of spacelike-separated systems.

Example (Ideal Measurement). Let $\mathcal{H}$ and $\mathcal{G}$ be separable Hilbert spaces for an apparatus and object respectively, let $\overline{R}$ be the apparatus pointer observable on $\mathcal{H}$ with eigenvectors x_n, and let $\overline{M}$ be the measured observable on $\mathcal{G}$ with eigenvectors y_n and respective eigenvalues λ_n. Let R be the self-adjoint operator $\overline{R} \otimes I$ on $\mathcal{H} \otimes \mathcal{G}$ and let M be the self-adjoint operator $I \otimes \overline{M}$ on $\mathcal{H} \otimes \mathcal{G}$. Note that $W^*(R) = W^*(\overline{R}) \otimes I$.

Prior to an entangling measurement interaction that strictly correlates the values of $\overline{M}$ with $\overline{R}$, the total state will be $v_0 = x_0 \otimes \sum c_n y_n$, where $\sum |c_n|^2 = 1$ and x_0 is the 'ground state' of the pointer observable. When two or more of the coefficients $\{c_n\}$ are nonzero, and two or more of the $\{\lambda_n\}$ unequal, then the pre-measurement maximal R-beable algebra for P_{v_0}, $\mathfrak{B}(R, v_0)$, will fail to contain M. For every element of $\mathcal{S} = [W^*(R)v_0]$ has the form $x \otimes \sum c_n y_n$ for some $x \in \mathcal{H}$, yet $Mv_0 = x_0 \otimes \sum c_n \lambda_n y_n$ which is not of the required form. Thus M fails to leave $\mathcal{S}$ invariant, and $M \notin \mathfrak{B}(R, v_0)$ by Corollary 4.8.(ii).

On the other hand, after the unitary evolution that affects the measurement, the state is $v = \sum c_n(x_n \otimes y_n)$. If Q_n is the projection onto the one-dimensional subspace $[x_n]$ of $\mathcal{H}$, it follows that $Q_n \otimes I \in W^*(R)$. We then have, $Mv = \sum c_n\lambda_n(x_n \otimes y_n) = \left[\sum \lambda_n(Q_n \otimes I)\right]v$, and $\sum \lambda_n(Q_n \otimes I) \in W^*(R)$. Since M commutes with R, both conditions of Corollary 4.8.(i) are satisfied, and $M \in \mathfrak{B}(R, v)$! Thus the act of measuring M has, in a sense, *made M determinate*, but not via any physical disturbance (cf. Bohr's [8, p. 317] well-known and oft-repeated caution against speaking of 'creation of physical attributes of objects by measurements'). Rather, both before and after the measurement one constructs the maximal set of observables that, together with the pointer observable R, can have simultaneously determinate values, and these purely formal constructions, designed to secure a maximally complete account of each stage of the measurement process in classical terms, forces one to different verdicts concerning the determinacy of M.

Example (EPR Correlations—Spin Case). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two-dimensional Hilbert spaces and let $\sigma_{xi}, \sigma_{yi}, \sigma_{zi}$ be the Pauli spin operators on $\mathcal{H}_i$, for $i = 1, 2$. For convenience, we write vectors in Dirac's ket notation and suppress tensor products between vectors; for example, $|\sigma_{x1} = +1\rangle|\sigma_{x2} = -1\rangle$ denotes an eigenvector of $\sigma_{x1} \otimes \sigma_{x2}$ with eigenvalue -1. Let $|\mathsf{s}\rangle$ be the singlet state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ which, expanded in the basis of eigenvectors for σ_{x1}, is

$$|\mathsf{s}\rangle = \frac{1}{\sqrt{2}}\Big(|\sigma_{x1} = +1\rangle|\sigma_{x2} = -1\rangle - |\sigma_{x1} = -1\rangle|\sigma_{x2} = +1\rangle\Big). \qquad (4.32)$$

This state also assumes the same form relative to the y- and z-bases, thus it predicts that identical spin components of the two particles will always be found on measurement to be anti-correlated long after the particles have interacted and separated. Exploiting correlations of the exact same kind between the positions and momenta of two particles (whose analysis we defer until the next section), Einstein, Podolsky, and Rosen [31] argued for the joint determinacy of incompatible observables on the basis of the following 'reality' criterion: *"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity"* [31, p. 777]. In the case of incompatible spin components in the state (4.32), EPR's argument is straightforward. If σ_{x1} were measured, then, regardless of the value obtained, σ_{x2}'s value could be predicted with certainty without in any way disturbing particle 2, spacelike-separated from 1. It then would follow from the reality criterion that σ_{x2} has a value (is an 'element of reality') quite apart from whether σ_{x1} is actually measured on particle 1. But then, by exactly parallel reasoning from the possibility of measuring σ_{y1}, σ_{y2} must have a value as well—and yet it fails to commute (or, indeed, share any eigenvectors) with σ_{x2}. Bohr's response to EPR's argument pointed to an ambiguity in their phrase "without in any way disturbing a

system": "Of course there is in a case like that just considered no question of a mechanical disturbance of the system....[but] there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behaviour of the system*" [7, p. 148]. The phrase Bohr italicizes here has seemed opaque to many commentators (not least, Bell [5, p. 155]). Yet by employing the appropriate maximal R-beable subalgebras, it is possible to understand how measuring the x-spin (respectively, y-spin) of particle 1 can have an affect on the conditions that permit the ascription of a definite value to the x-spin (respectively, y-spin) of particle 2.

Let $\mathcal{H}_0$ be a three-dimensional Hilbert space, and let R_0 be a self-adjoint operator on $\mathcal{H}_0$ whose eigenvalues $-1, 0$ and 1 represent the different possible states of the pointer observable on an apparatus ready to measure σ_{x1}. Prior to the measurement, the total state of apparatus and particles is $|v_0\rangle = |R_0 = 0\rangle|\mathsf{s}\rangle$ with $|R_0 = 0\rangle$ the apparatus ground state. As before, take $R \equiv R_0 \otimes I \otimes I$. Clearly, $[W^*(R)|w\rangle]$ consists only of vectors of the form $|t\rangle|\mathsf{s}\rangle$ for some $|t\rangle \in \mathcal{H}_0$, since $W^*(R) = W^*(R_0) \otimes (I \otimes I) = W^*(R_0) \otimes I$. However, $(I \otimes \sigma_{x1} \otimes I)|v_0\rangle = |R_0 = 0\rangle|u\rangle$, where

$$|u\rangle = \frac{1}{\sqrt{2}}\Big(|\sigma_{x1} = +1\rangle|\sigma_{x2} = -1\rangle + |\sigma_{x1} = -1\rangle|\sigma_{x2} = +1\rangle\Big), \tag{4.33}$$

and $|u\rangle \perp |\mathsf{s}\rangle$. Thus, by Corollary 4.8.(ii), $I \otimes \sigma_{x1} \otimes I \notin \mathfrak{B}(R, v_0)$. A similar argument shows that none of $(I \otimes I \otimes \sigma_{x2}), (I \otimes \sigma_{y1} \otimes I)$, or $(I \otimes I \otimes \sigma_{y2})$ lie in $\mathfrak{B}(R, v_0)$.

However, after the measurement of σ_{x1} actually occurs, it results in the entangled state

$$|v\rangle \equiv \frac{1}{\sqrt{2}}\Big(|R_0 = +1\rangle|\sigma_{x1} = +1\rangle|\sigma_{x2} = -1\rangle - |R_0 = -1\rangle|\sigma_{x1} = -1\rangle|\sigma_{x2} = +1\rangle\Big). \tag{4.34}$$

Now, $I \otimes \sigma_{x1} \otimes \sigma_{x2}$ commutes with R, and $(I \otimes \sigma_{x1} \otimes \sigma_{x2})|v\rangle = -|v\rangle \in [W^*(R)|v\rangle]$. Thus, by Corollary 4.8.(i), $I \otimes \sigma_{x1} \otimes \sigma_{x2} \in \mathfrak{B}(R, v)$. Moreover, it is easy to see that $(I \otimes \sigma_{x1} \otimes I)|v\rangle = (R_0 \otimes I \otimes I)|v\rangle \in [W^*(R)|v\rangle]$. Thus, $I \otimes \sigma_{x1} \otimes I \in \mathfrak{B}(R, v)$. But since $\sigma_{x1}^2 = I$,

$$(I \otimes \sigma_{x1} \otimes \sigma_{x2})(I \otimes \sigma_{x1} \otimes I) = I \otimes I \otimes \sigma_{x2}, \tag{4.35}$$

and the latter lies in $\mathfrak{B}(R, v)$ as well. On the other hand, it is not difficult to show that $I \otimes \sigma_{y1} \otimes I \notin \mathfrak{B}(R, v)$. First,

$$(I \otimes \sigma_{y1} \otimes I)|v\rangle = -i\Big(|R_0 = +1\rangle|\sigma_{x1} = -1\rangle|\sigma_{x2} = -1\rangle$$
$$+ |R_0 = -1\rangle|\sigma_{x1} = +1\rangle|\sigma_{x2} = +1\rangle\Big). \tag{4.36}$$

However, since $W^*(R) = W^*(R_0) \otimes I \otimes I$, the generic element of $[W^*(R)|v\rangle]$ has the form,

$$|t\rangle|\sigma_{x1} = +1\rangle|\sigma_{x2} = -1\rangle - |u\rangle|\sigma_{x1} = -1\rangle|\sigma_{x2} = +1\rangle, \tag{4.37}$$

for some $|t\rangle, |u\rangle \in \mathcal{H}_0$. Thus, $(I \otimes \sigma_{y1} \otimes I)|v\rangle \notin [W^*(R)|v\rangle]$, and it follows from Corollary 4.8.(ii) that $I \otimes \sigma_{y1} \otimes I \notin \mathfrak{B}(R, v)$. A similar argument shows that $I \otimes I \otimes \sigma_{y2} \notin \mathfrak{B}(R, v)$. Thus we see how *once σ_{x1} is actually measured*, both it and σ_{x2} 'become determinate' in just the same nonmechanical sense as explained at the end of the previous example—and this occurs at the expense of the determinacy of the y-spins of the particles. Of course, a parallel analysis applies, by symmetry, if σ_{y1} were actually measured instead; in that case, it would be legitimate to ascribe determinacy to both y-spins of the particles at the expense of their x-spins. In *neither* case (i.e. in neither the x_1- or y_1-spin measurement context) does it follow that *both* σ_{x2} and σ_{y2} are determinate. Thus the EPR argument fails for exactly the reason suggested by the phrase Bohr sets in italics in the passage cited above.

Finally, it is worth addressing Schrödinger's [33, Sec. 12] clever modification of the EPR argument. In terms of spin, his proposal was that one consider measuring σ_{x1} at the same time t (in some frame) as σ_{y2} is measured. The latter measurement allows one to directly ascertain the value of σ_{y2} while the former's measurement result, obtained at a distance ('without in any way disturbing the system'), allows one to infer the value of σ_{x2} indirectly via (4.32)'s strict x-spin correlation and the EPR reality criterion. It would thus appear, not only that σ_{x2} and σ_{y2} must by simultaneously determinate at t, but can be simultaneously *known*! (According to Schrödinger, we have 'hypermaximal' knowledge of the state of particle 2.)

Of course, the value of σ_{x2} that becomes 'known' by such a procedure will have no predictive significance for a measurement of σ_{x2} that occurs later than t, thus the uncertainty principle is not contradicted. More importantly, an analysis along the lines set forth above shows that Schrödinger's experiment cannot be used to contradict the indeterminacy principle for σ_{x2} and σ_{y2} either. Assuming both σ_{x1} and σ_{y2} are actually measured at time t in state (4.32), and modelling the two measurements in terms of strict correlations to the values of two pointer observables R_1 and R_2, let the final post-measurement state be $|v_t\rangle$. To ascertain which observables can be regarded as determinate in this measurement context, we must now take our set of preferred observables to include *both* pointer observables. It is then easy to show that σ_{y2} (and σ_{x1}) lies in $\mathfrak{B}(\{R_1, R_2\}, v_t)$ but *not* σ_{x2} (or σ_{y1}). Thus performing a direct measurement of σ_{y2} renders invalid Schrödinger's use of the EPR reality criterion to secure a value for σ_{x2} in the given measurement context. It follows that the EPR reality criterion cannot be part of the Copenhagen interpretation (if our reconstruction of the interpretation is correct), but is valid only in the special case where there is no direct measurement being made of observables incompatible with ones whose values are predictable with certainty on the basis of the criterion.

4.2. $\mathfrak{R}$-beable algebras for arbitrary pure states. We now discuss the extension of Theorem 4.5 to the case of an arbitrary (not necessarily normal) pure state on a C^*-algebra $\mathfrak{A}$ (either abstract or concrete). Although our results are limited, they still permit a characterization of Bohr's response to the *original* EPR argument, which in fact employed a singular state of two particles with strictly correlated positions and momenta.

Let $\mathfrak{A}$ be a C^*-algebra and let $(\pi_\rho, \mathcal{H}_\rho, x_\rho)$ be the GNS triple for $\mathfrak{A}$ induced by the pure state ρ. Once more, let $\mathfrak{R}$ be a family of mutually commuting observables drawn from $\mathfrak{A}$. Now, $\pi_\rho(\mathfrak{R})$ is a mutually commuting family of observables in $\mathfrak{L}(\mathcal{H}_\rho)$. Thus, we may apply Corollary 4.6.(ii) to conclude that $\mathfrak{B}(\pi_\rho(\mathfrak{R}), x_\rho) \equiv \mathfrak{L}(\mathcal{S}^\perp) \oplus W^*(\pi_\rho(\mathfrak{R}))P_\mathcal{S}$ is the unique maximal (in $\mathfrak{L}(\mathcal{H}_\rho)$) $\pi_\rho(\mathfrak{R})$-beable algebra for ω_{x_ρ}. (In this case, $\mathcal{S}$ is the smallest subspace of $\mathcal{H}_\rho$ such that $x_\rho \in \mathcal{S}$ and $\pi_\rho(\mathfrak{R})$ leaves $\mathcal{S}$ invariant; i.e. $\mathcal{S} = [\pi_\rho(\mathfrak{R})'' x_\rho]$.) We now verify that the inverse image of $\mathfrak{B}(\pi_\rho(\mathfrak{R}), x_\rho)$ under π_ρ is $\mathfrak{R}$-beable for ρ.

Notation. We define $\mathfrak{B}(\mathfrak{R}, \rho) \equiv \pi_\rho^{-1}[\mathfrak{B}(\pi_\rho(\mathfrak{R}), x_\rho)]$. This is meant to extend our earlier (concrete) notation $\mathfrak{B}(\mathfrak{R}, v)$ since, when $\mathfrak{A} = \mathfrak{L}(\mathcal{H})$ and ρ is induced by a unit vector $v \in \mathcal{H}$, $\mathfrak{L}(\mathcal{H})$ and $\mathfrak{L}(\mathcal{H}_\rho)$ are unitarily equivalent, from which it follows that $\mathfrak{B}(\mathfrak{R}, \rho) = \mathfrak{B}(\mathfrak{R}, v)$.

Proposition 4.9. *Let ρ be a pure state of $\mathfrak{A}$ and let $\mathfrak{R}$ be a mutually commuting family of observables in $\mathfrak{A}$. Then, $\mathfrak{B}(\mathfrak{R}, \rho)$ is $\mathfrak{R}$-beable for ρ.*

Proof. Clearly, $\mathfrak{B}(\mathfrak{R}, \rho)$ is a C^*-algebra, since it is the inverse image under π_ρ of a C^*-algebra. Furthermore, ($\mathfrak{R}$-Priv) follows by the construction of $\mathfrak{B}(\mathfrak{R}, \rho)$.

(Beable) Let $\mathcal{T} \equiv [\pi_\rho(\mathfrak{B}(\mathfrak{R}, \rho)) x_\rho]$. By Prop. 2.2.(iv), it will be sufficient to show that $\pi_\rho(\mathfrak{B}(\mathfrak{R}, \rho)) P_\mathcal{T}$ is abelian. Clearly, $\mathcal{T}$ is the smallest subspace (of $\mathcal{H}_\rho$) that contains x_ρ and that is invariant under $\pi_\rho(\mathfrak{B}(\mathfrak{R}, \rho))$. However, $\mathcal{S}$ contains x_ρ by construction, and $\mathcal{S}$ is invariant under $\pi_\rho(\mathfrak{B}(\mathfrak{R}, \rho))$ (since π_ρ maps $\mathfrak{B}(\mathfrak{R}, \rho)$ into $\mathfrak{L}(\mathcal{S}^\perp) \oplus \mathfrak{L}(\mathcal{S})$). Thus $\mathcal{T} \subseteq \mathcal{S}$. Conversely, $\pi_\rho(\mathfrak{R})$ leaves $\mathcal{T}$ invariant, since $\pi_\rho(\mathfrak{R}) \subseteq \pi_\rho(\mathfrak{B}(\mathfrak{R}, \rho))$. Therefore $\mathcal{S} = \mathcal{T}$ and $\pi_\rho(\mathfrak{B}(\mathfrak{R}, \rho)) P_\mathcal{T} = \pi_\rho(\mathfrak{B}(\mathfrak{R}, \rho)) P_\mathcal{S} \subseteq W^*(\pi_\rho(\mathfrak{R})) P_\mathcal{S}$. The conclusion then follows by noting that $W^*(\pi_\rho(\mathfrak{R})) P_\mathcal{S}$ is abelian (since $\mathfrak{R}$ is a mutually commuting family of operators).

(Def) Let U be a unitary element of $\mathfrak{A}$ such that $U \in \mathfrak{R}'$ and $\rho_U = \rho$. In this case (i.e. where ρ is pure), we can actually prove the stronger result that $U \in \mathfrak{B}(\mathfrak{R}, \rho)$, from which it follows immediately that $U \mathfrak{B}(\mathfrak{R}, \rho) U^* = \mathfrak{B}(\mathfrak{R}, \rho)$.

We show first that x_ρ is an eigenvector of $\pi_\rho(U)$. For this, let $x \equiv x_\rho$ and let $y \equiv \pi_\rho(U) x_\rho$. Since ρ is pure, the representation $(\pi_\rho, \mathcal{H}_\rho)$ of $\mathfrak{A}$ is irreducible [23, Thm. 10.2.3]. Thus, $\pi_\rho(\mathfrak{A})^- = \mathfrak{L}(\mathcal{H}_\rho)$. In particular, there is a net $(\pi_\rho(H_a)) \subseteq \pi_\rho(\mathfrak{A})$ such that WOT-$\lim_a \pi_\rho(H_a) = P_x$. However, $\langle \pi_\rho(H_a) x, x \rangle = \rho(H_a) = \rho_U(H_a) = \langle \pi_\rho(H_a) y, y \rangle$, for all a. Since ω_x and ω_y are WOT-continuous, it follows that

$$1 \;=\; \langle P_x x, x \rangle \;=\; \langle P_x y, y \rangle \;=\; |\langle x, y \rangle|^2. \tag{4.38}$$

Hence, $|\langle x, y \rangle|^2 = \|x\| \cdot \|y\|$, and by the Cauchy-Schwarz inequality, $y = cx$ for some $c \in \mathbb{C}^1$, which is what we wanted to show.

Now, since x_ρ is an eigenvector of $\pi_\rho(U)$, it follows that $\pi_\rho(U)x_\rho \in \mathcal{S}$. Moreover, since $U \in \mathfrak{R}'$, it follows that $[\pi_\rho(U), \pi_\rho(\mathfrak{R})] = \{0\}$. Thus, by Prop. 4.8.(i), $\pi_\rho(U) \in \mathfrak{B}(\pi_\rho(\mathfrak{R}), x_\rho)$ and $U \in \mathfrak{B}(\mathfrak{R}, \rho)$. $\qquad\square$

Open Problem. Let $\mathcal{H}$ be a Hilbert space, and let ρ be a state of $\mathfrak{L}(\mathcal{H})$.

   (i) When ρ is singular: Do all its maximal $\mathfrak{R}$-beable algebras contain $\mathfrak{B}(\mathfrak{R}, \rho)$? Is $\mathfrak{B}(\mathfrak{R}, \rho)$ itself maximal? Is it unique?

   (ii) When ρ is not normal (pure or mixed): Classify the maximal $\mathfrak{R}$-beable algebras for ρ along the lines of Theorem 4.5.

   (iii) When ρ is not a vector state: Give necessary and sufficient conditions for there to be a unique maximal $\mathfrak{R}$-beable algebra for ρ (cf. Corollary 4.6).

In our final section, we reconstruct Bohr's reply to the original EPR argument, in terms of maximal $\mathfrak{R}$-beable algebras by employing $\mathfrak{B}(\mathfrak{R}, \rho)$, which is well-defined when ρ is taken to be the singular EPR state. Should the answer to the second and third questions in (i) above prove negative, the results of our reconstruction will still remain valid if the first question can be answered positively.

4.3. **EPR Correlations: Position/Momentum Case.** We begin by defining the EPR state ρ. Let $\mathcal{G} \equiv L_2(\mathbb{R})$, and let Q be the unbounded, self-adjoint position operator on $\mathcal{G}$ defined by $Q\psi = x\psi$, where $\mathcal{D}(Q)$ consists of those functions $\psi \in L_2(\mathbb{R})$ such that $x\psi \in L_2(\mathbb{R})$. Let T be the $L_2(\mathbb{R})$ Fourier transform, a unitary operator on $\mathcal{G}$ (cf. [23, Thm. 3.2.31]). Let D be the unbounded, self-adjoint operator on $\mathcal{G}$ defined to be $T^{-1}QT$ on domain $T^{-1}(\mathcal{D}(Q))$ (cf. [23, Exercise 5.7.49]). One may show, then, that $D\psi = i(d\psi/dx)$ when $\psi \in \mathcal{D}(D)$ is differentiable, i.e., D is the momentum operator.

Since Q is affiliated with the abelian von Neumann algebra $W^*(Q)$, Q is represented by a self-adjoint function $\phi(Q)$ on S_0 (the space of pure states of $W^*(Q)$). Since $\mathrm{sp}(Q) = \mathbb{R}$ and the range of $\phi(Q)$ is equal to $\mathrm{sp}(Q)$, we are guaranteed that for each $\ell \in \mathbb{R}$, there is a (necessarily singular) pure state $\alpha \in \mathsf{S}_0$ such that $\phi(Q)(\alpha) = \ell$. We may apply the same procedure for D in order to obtain a pure state β of $W^*(D)$. In keeping with the original EPR argument, we shall choose β (as we may) such that $\phi(D)(\beta) = 0$.

Now, we may think of $W^*(Q)$ and $W^*(D)$ as acting on two different copies $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{G}$. In this case, we may form the C^*-tensor product $W^*(Q) \otimes W^*(D)$, which acts on $\mathcal{G}_1 \otimes \mathcal{G}_2$ [23, Section 11.1]. It then follows that there is a *unique* pure state $\alpha \otimes \beta$ on the $W^*(Q) \otimes W^*(D)$ [23, Prop. 11.1.1]. Moreover, since $\alpha \otimes \beta$ is pure, we may extend it to a pure state ω of $\mathfrak{L}(\mathcal{G}_1 \otimes \mathcal{G}_2)$. (Note, however, that there is no guarantee of the uniqueness of

our choice of ω. In particular, arbitrariness entered into our choice of α and β as well as into our extension of $\alpha \otimes \beta$ to $\mathfrak{L}(\mathcal{G}_1 \otimes \mathcal{G}_2)$.)

Notation. Let $\mathcal{H} \equiv \mathcal{G}_1 \otimes \mathcal{G}_2 \simeq L_2(\mathbb{R}^2)$. Let Q_1 be the unbounded operator $Q \otimes I$ acting on $\mathcal{H}$. Let Q_2 be the operator $I \otimes Q$. Define D_1 and D_2 similarly.

For $\theta \in \mathbb{R}$, let $R_{-\theta}$ be the rotation of $\mathbb{R}^2$ through $-\theta$. Let $U(\theta)$ be the unitary rotation operator on $\mathcal{H}$ defined by $U(\theta)\psi \equiv \psi \circ R_{-\theta}$. If we let $X \equiv U(\theta)^{-1}Q_1 U(\theta)$ and $P \equiv U(\theta)^{-1}D_2 U(\theta)$, it follows that

$$X = Q_1 \cos\theta \,\widehat{-}\, Q_2 \sin\theta \qquad\qquad P = D_1 \sin\theta \,\widehat{+}\, D_2 \cos\theta. \qquad (4.39)$$

(That is, these pairs of unbounded operators have the same domain and agree on this domain. Cf. Bohr [7, p. 696(note)].) With this in mind, we define the singular state ρ of $\mathfrak{L}(\mathcal{H})$ by $\rho \equiv \omega_{U(\theta)}$. One can then show that for any $f \in \mathcal{B}(\mathbb{R})$,

$$\rho(f(X)) = \omega(f(Q_1)) \qquad\qquad \rho(f(P)) = \omega(f(D_2)). \qquad (4.40)$$

This makes precise the sense in which the behavior of ρ relative to X and P is identical to the behavior of ω relative to Q_1 and D_2. In particular, it follows that $\rho|_{W^*(X)}$ and $\rho|_{W^*(P)}$ are dispersion-free and are, respectively, well-defined for X and P. Moreover, one can show that $\phi(X)(\rho_1) = \ell$ and $\phi(P)(\rho_2) = 0$, where $\rho_1 = \rho|_{W^*(X)}$ and $\rho_2 = \rho|_{W^*(P)}$. In what follows we will fix $\theta = \pi/4$. However, instead of letting $X \equiv 2^{-1/2}(Q_1 \widehat{-} Q_2)$, we let $X \equiv Q_1 \widehat{-} Q_2$, the relative position of two particles moving in one-dimension; and, similarly, we let $P \equiv P_1 \widehat{+} P_2$, their total momentum. Since ρ assigns a definite (nonzero) relative position to the particles, and assigns them a definite (zero) total momentum, knowing the value of Q_1 in state ρ allows one to predict with certainty the value of Q_2, and similarly for D_1 and D_2. Thus we have the conditions employed by EPR, in conjunction with their reality criterion, to argue for the simultaneous determinacy of Q_2 and D_2.

We pause to note a technical difficulty—a feature of the EPR state ρ, not present in the singlet spin state version—that EPR do not address. Since $[Q_2, P] = [D_2, X] = i\hbar I$, Corollary 3.8 dictates that neither Q_2 nor D_2 has beable status for ρ—or for any state obtained from ρ after a measurement on particle 1 is performed. Thus any argument which purports to establish the existence of simultaneous definite values in state ρ for Q_2 and D_2 is necessarily suspect. In fact, since $[Q_1, P] = -[D_1, X] = i\hbar I$, Corollary 3.8 also dictates that the probability of obtaining a value in any finite interval of the real line for Q_1 or D_1 is always zero in the state ρ. This blocks the use of EPR's reality criterion, which first requires that either Q_1 or D_1 is measured and a finite value obtained. However, a natural way to overcome this obstacle is simply to understand EPR as setting out to establish that all *bounded* Borel functions of both Q_2 and D_2 have simultaneous reality in the state ρ.

Unfortunately, we have been unable to confirm that ρ must dictate strict correlations between *arbitrary* Borel functions of Q_1 and Q_2 (or D_1 and D_2).

Open Problem. Let $\mathcal{H} \equiv L_2(\mathbb{R}^2)$, and let E_1, E_2, F_1 and F_2 denote, respectively, the projection-valued measures for Q_1, Q_2, D_1 and D_2. For any pure state ω of $\mathfrak{L}(\mathcal{H})$, we say that ω is *completely EPR-correlated* just in case $\omega(E_2(\mathtt{C} - \ell)E_1(\mathtt{C})) = \omega(E_1(\mathtt{C}))$ and $\omega(F_1(\mathtt{C})F_2(\mathtt{C})) = \omega(F_1(\mathtt{C}))$ for all Borel subsets $\mathtt{C}$ of $\mathbb{R}$.

(i) Are there completely EPR-correlated states?

(ii) Must ρ (as defined above) be completely EPR-correlated?

On the other hand, we shall shortly establish (Lemma 4.13 below) that ρ strictly correlates the bounded uniformly continuous (BUC) functions of Q_1 with those of Q_2, and the BUC functions of D_1 with D_2. Let $C^*(Q_i)$ denote the C^*-algebra of all BUC functions of Q_i, and similarly for $C^*(D_i)$. Then we shall take as the object of the EPR argument the establishment (at a minimum) of the simultaneous reality of $C^*(Q_2)$ and $C^*(D_2)$ in ρ. Should the answer to (ii) above be positive, EPR's reality criterion would entitle them to substitute $W^*(Q_2)$ and $W^*(D_2)$ for $C^*(Q_2)$ and $C^*(D_2)$; but, then the same substitution would apply to Bohr's reply. Note also that, since ρ is not ultraweakly continuous, such a substitution is not automatically warranted.

Let $W^*(Q_1, Q_2)$ be the abelian von Neumann algebra generated by Q_1 and Q_2. (The reader should note that everything we subsequently establish about Q_1 and Q_2 in the state ρ follows, by symmetry, for D_1 and D_2 as well.) Since $W^*(Q_1, Q_2)$ is abelian, we may represent it as the space of continuous functions on the set S of pure states of $W^*(Q_1, Q_2)$. Moreover, $\rho|_{W^*(Q_1, Q_2)}$ may be represented as a probability measure μ_ρ on S.

Remark 4.10. Let R be a (possibly unbounded) self-adjoint operator. Recall that for any $f \in \mathcal{B}(\mathbb{R})$, the operator $f(R)$ is canonically constructed by employing the representation of R as a function $\phi_0(R)$ in the space $\mathcal{N}(\mathsf{S}_0)$ of unbounded functions on the set S_0 of pure states of $W^*(R)$. Suppose that $\mathfrak{V}$ is another abelian von Neumann algebra such that $R\,\eta\,\mathfrak{V}$, and let S_1 be the set of pure states of $\mathfrak{V}$. Then, R is represented by a function $\phi_1(R)$ in $\mathcal{N}(\mathsf{S}_1)$ In such a case, we may mimic the canonical construction in order to obtain "functions" of R in $\mathfrak{V}$. Fortunately, we are guaranteed that, whether we perform the construction relative to $W^*(R)$ or relative to $\mathfrak{V}$, there can be no ambiguity concerning the resulting operator $f(R)$ [23, Remark 5.6.28]. This fact will be important for what follows, since we will be concerned with functional relationships between X, Q_1 and Q_2. In this case, all three operators are affiliated with the abelian von Neumann algebra $W^*(Q_1, Q_2)$.

In an abuse of notation (which will be justified in what follows) let $\widetilde{g}$ denote a Borel subset of S. Then, we know that there is a *unique* clopen set h such that $h \triangle \widetilde{g}$ is meager,

where $h \triangle \widetilde{g} = (h - \widetilde{g}) \cup (\widetilde{g} - h)$. Let f be another clopen set such that $\widetilde{g} \subseteq f$. Then, noting that $h \cap f$ is also clopen, it follows from an elementary set-theoretic argument that $h \subseteq f$. Using this fact, we may establish the following lemma:

Lemma 4.11. *Let* $\mathsf{L} \equiv \{\omega \in \mathsf{S} : \phi(X)(\omega) = \ell\}$. *Then,* $\mu_\rho(\mathsf{L}) = 1$.

Proof. Fix $n \in \mathbb{N}$ and let f be the characteristic function of the clopen set

$$\mathsf{S}_n \equiv \left[\phi(X)^{-1}(\ell - n^{-1}, \ell + n^{-1})\right]^{-}. \tag{4.41}$$

By the construction of ρ, we have $\rho(E(\mathsf{C}_n)) = 1$ where E is the spectral-measure for X and $\mathsf{C}_n \equiv (\ell - n^{-1}, \ell + n^{-1}) \subseteq \mathbb{R}$ (cf. Lemma 3.1.(ii)). In other words, $\int h \, d\mu_\rho = 1$, where $h = \phi(E(\mathsf{C}_n))$. Recall that h is defined to be the unique closest continuous function to $\widetilde{g}$, where $\widetilde{g}(\omega) = 1$ if $\phi(X)(\omega) \in (\ell - n^{-1}, \ell + n^{-1})$ and $\widetilde{g}(\omega) = 0$ otherwise. However, $\widetilde{g} \leq h$, for if $\widetilde{g}(\omega) = 1$, then $\omega \in \phi(X)^{-1}(\ell - n^{-1}, \ell + n^{-1})$. Now applying the considerations prior to this lemma (identifying sets with their characteristic functions), we have $h \leq f$ and $\int f \, d\mu_\rho = 1$. That is, $\mu_\rho(\mathsf{S}_n) = 1$ for all $n \in \mathbb{N}$. Moreover, since $\mathsf{S}_n \supseteq \mathsf{S}_{n+1}$ for all n, $\mu_\rho(\cap \mathsf{S}_n) = \lim_n \mu_\rho(\mathsf{S}_n) = 1$. Since $\cap_{n \in \mathbb{N}} \mathsf{S}_n \subseteq \mathsf{L}$, it follows that $\mu_\rho(\mathsf{L}) = 1$. $\qquad \square$

Lemma 4.12. *Let* Z *and* Z' *be the closed, nowhere dense subsets of* S *at which* $\phi(Q_1)$ *and* $\phi(Q_2)$, *respectively, are not defined. Then* L *is the disjoint union of* $\mathsf{L} \cap (\mathsf{Z} \cap \mathsf{Z}')$ *and* $\mathsf{L} \backslash (\mathsf{Z} \cup \mathsf{Z}')$.

Proof. Let Z'' be the set of points in S at which $\phi(Q_1) \widehat{-} \phi(Q_2)$ is not defined, and suppose that $\omega \in \mathsf{L} \cap \mathsf{Z} \subseteq (\mathsf{S} \backslash \mathsf{Z}'') \cap \mathsf{Z}$. Since $\mathsf{Z} \cup \mathsf{Z}' \cup \mathsf{Z}''$ is closed and nowhere dense, there is a net $(\tau_a) \subseteq \mathsf{S} \backslash (\mathsf{Z} \cup \mathsf{Z}' \cup \mathsf{Z}'')$ such that $\tau_a \to \omega$. Using the fact that $\phi(Q_1 \widehat{-} Q_2)(\tau_a) = \phi(Q_1)(\tau_a) - \phi(Q_2)(\tau_a)$ for each τ_a (since $\tau_a \in \mathsf{S} \backslash (\mathsf{Z} \cup \mathsf{Z}')$), and the fact that $\lim_a |\phi(Q_1)(\tau_a)| = \infty$, it follows that $\lim_a |\phi(Q_2)(\tau_a)| = \infty$, and thus $\omega \in \mathsf{Z}'$. A similar argument shows that if $\omega \in \mathsf{L} \cap \mathsf{Z}'$, then $\omega \in \mathsf{Z}$. $\qquad \square$

Lemma 4.13. *For any* $A \in C^*(Q_2)$, *there is an* $A' \in C^*(Q_1)$ *such that* $\omega(A') = \omega(A)$ *for all* $\omega \in \mathsf{L}$ *(and* A' *may be chosen so that* $\mathrm{sp}(A') = \mathrm{sp}(A)$).

Proof. Let $A \in C^*(Q_2)$. Then, $A = f(Q_2)$, for some BUC function f on $\mathbb{R}$. Let $A' \equiv g(Q_1)$, where $g(x) = f(x - \ell)$, $(x \in \mathbb{R})$, and let h_1 and h_2 be the unique continuous functions on S corresponding, respectively, to $\widetilde{f}$ and $\widetilde{g}$. (Clearly, h_1 and h_2 have identical range and it follows from [23, Prop. 5.6.20] that $\mathrm{sp}(A) = \mathrm{sp}(A')$.)

We now show that that $\widetilde{f}|_{\mathsf{L}} = \widetilde{g}|_{\mathsf{L}}$. For this, let $\omega \in \mathsf{L}$. (Case 1a) Suppose that $\omega \in \mathsf{L} - (\mathsf{Z} \cup \mathsf{Z}')$. Then $\ell = \phi(Q_1) \widehat{-} \phi(Q_2)(\omega) = \phi(Q_1)(\omega) - \phi(Q_2)(\omega)$ and $\phi(Q_1)(\omega) - \ell = \phi(Q_2)(\omega)$. Therefore,

$$\widetilde{g}(\omega) \;=\; g(\phi(Q_1)(\omega)) \;=\; f(\phi(Q_1)(\omega) - \ell) \;=\; f(\phi(Q_2)(\omega)) \;=\; \widetilde{f}(\omega). \tag{4.42}$$

(Case 1b) Suppose that $\omega \in \mathsf{L} \cap (\mathsf{Z} \cap \mathsf{Z}')$. Then, by definition, $\widetilde{f}(\omega) = \widetilde{g}(\omega) = 0$, since $\phi(Q_1)$ and $\phi(Q_2)$ are not defined at ω.

We now show that $h_1|_{\mathsf{L}} = h_2|_{\mathsf{L}}$. (Case 2a) Suppose that $\omega \in \mathsf{L} - (\mathsf{Z} \cup \mathsf{Z}')$. By Case 1a, it will be sufficient to show that $h_1(\omega) = \widetilde{f}(\omega)$ and $h_2(\omega) = \widetilde{g}(\omega)$. In order to establish this, note that $\widetilde{f}$ and $\widetilde{g}$ are continuous on $\mathsf{S} - (\mathsf{Z} \cup \mathsf{Z}')$ (since each is the composition of two continuous functions). Moreover, by definition, $\widetilde{f}$ may not disagree with the continuous function h_1 on any open set (in S), and $\widetilde{g}$ may not disagree with the continuous function h_2 on any open set (in S). Therefore, $h_1|_{\mathsf{S}-(\mathsf{Z}\cup\mathsf{Z}')} = \widetilde{f}|_{\mathsf{S}-(\mathsf{Z}\cup\mathsf{Z}')}$ and $h_2|_{\mathsf{S}-(\mathsf{Z}\cup\mathsf{Z}')} = \widetilde{g}|_{\mathsf{S}-(\mathsf{Z}\cup\mathsf{Z}')}$.

(Case 2b) Suppose that $\omega \in \mathsf{L} \cap (\mathsf{Z} \cap \mathsf{Z}')$. Since $\mathsf{Z} \cup \mathsf{Z}'$ is nowhere dense, there is a net $(\tau_a) \subseteq \mathsf{S} \backslash (\mathsf{Z} \cup \mathsf{Z}')$ such that $\tau_a \to \omega$. For each a,

$$\left| h_1(\omega) - h_2(\omega) \right| = \left| h_1(\omega) - h_1(\tau_a) + h_1(\tau_a) - h_2(\tau_a) + h_2(\tau_a) - h_2(\omega) \right| \tag{4.43}$$

$$\leq \left| h_1(\omega) - h_1(\tau_a) \right| + \left| h_1(\tau_a) - h_2(\tau_a) \right| + \left| h_2(\tau_a) - h_2(\omega) \right|. \tag{4.44}$$

If we take the limit over a of the RHS of (4.44), the first and third terms go to zero since h_1 and h_2 are continuous. Thus, $|h_1(\omega) - h_2(\omega)| \leq \lim_a |h_1(\tau_a) - h_2(\tau_a)|$.

If we let $x_a \equiv \phi(Q_2)(\tau_a)$ and $y_a \equiv \phi(Q_1)(\tau_a) - \ell$, then it follows from the continuity of f that $h_1(\tau_a) = f(x_a)$ and $h_2(\tau_a) = f(y_a)$ (see Case 2a). Thus, $|h_1(\omega) - h_2(\omega)| \leq \lim_a |f(x_a) - f(y_a)|$. Now, using the fact that $\phi(Q_1) \widehat{-} \phi(Q_2)$ is continuous at ω, and $(\tau_a) \subseteq \mathsf{S} \backslash (\mathsf{Z} \cup \mathsf{Z}')$, we have

$$\ell = [\phi(Q_1)\widehat{-}\phi(Q_2)](\omega) = \lim_a [\phi(Q_1)\widehat{-}\phi(Q_2)](\tau_a) = \lim_a [\phi(Q_1)(\tau_a) - \phi(Q_2)(\tau_a)], \tag{4.45}$$

and $\lim_a |x_a - y_a| = 0$. But this, in conjunction with the fact that f is uniformly continuous entails that $\lim_a |f(x_a) - f(y_a)| = 0$. Therefore, $h_1(\omega) = h_2(\omega)$. $\qquad \square$

Remark 4.14. Lemma 4.13, and its analogue for D_1 and D_2, is necessary for EPR to be able to use ρ to argue for the simultaneous determinacy of $C^*(Q_2)$ and $C^*(D_2)$. The reader will note that this lemma cannot be straightforwardly modified for the case where $A = f(Q_1)$ for any f in $\mathcal{B}(\mathbb{R})$ or in $\mathcal{C}(\mathbb{R})$ (cf. the open problem above). On the one hand, the assumption of continuity is needed to show that $h_1(\omega) = h_2(\omega)$ when $\omega \in \mathsf{L} - (\mathsf{Z} \cup \mathsf{Z}')$. On the other hand, the assumption of *uniform* continuity is needed to show that $h_1(\omega) = h_2(\omega)$ when $\omega \in \mathsf{L} \cap (\mathsf{Z} \cap \mathsf{Z}')$.

Lastly, we turn to Bohr's reply. As in our earlier analysis of the spin version of EPR's argument, we need to consider the effect of an ideal measurement of Q_1 that strictly correlates its values to those of an apparatus, initially in a ground state ω_0, with a pointer observable R satisfying $\mathrm{sp}(R) = \mathrm{sp}(Q_1)$. The final post-measurement state of apparatus and particles will have the form $(\omega_0 \otimes \rho)_U$, where the unitary evolution effecting the measurement correlation satisfies $[U, Q_1] = 0$ (consistent with the measurement being ideal). Observing

the registered value for any element of $C^*(R)$, the value of the corresponding element of $C^*(Q_1)$ may then be inferred. If we again understand Bohr's reply in terms of selecting the appropriate maximal $\mathfrak{R}$-beable algebra for this measurement context, he can be seen (modulo our remarks at the end of last section) as endorsing the attribution of determinate values to the elements in $\mathfrak{B}(C^*(R), (\omega_0 \otimes \rho)_U)$. The following proposition, however, shows that we may, without loss of generality, take $C^*(Q_1)$ as the family of privilged observables rather than $C^*(R)$.

Proposition 4.15. *Let $\mathcal{G}_0$ be a Hilbert space isomorphic to $\mathcal{G}_1$ and let R be a self-adjoint operator acting on $\mathcal{G}_0$. Let ω_0 be a pure state of $\mathfrak{L}(\mathcal{G}_0)$. Suppose that U is a unitary operator in $\mathfrak{L}(\mathcal{G}_0 \otimes \mathcal{G}_1 \otimes \mathcal{G}_2)$ such that $[U, I_0 \otimes A] = 0$ for all $A \in \mathfrak{L}(\mathcal{G}_1 \otimes \mathcal{G}_2)$. Let $\omega \equiv \omega_0 \otimes \rho$ and let $\hat{\omega} \equiv \omega_U$. Suppose that*

$$\hat{\omega}\big(E(\mathtt{C}) \otimes E(\mathtt{C}) \otimes I_2\big) \;=\; \hat{\omega}\big(E(\mathtt{C}) \otimes I_1 \otimes I_2\big) \;=\; \hat{\omega}\big(I_0 \otimes E(\mathtt{C}) \otimes I_2\big), \qquad (4.46)$$

for all Borel subsets $\mathtt{C}$ of $\mathbb{R}$. (Here E is the spectral measure for R.) Let $\mathfrak{R}$ be any C^*-subalgebra of $W^*(R)$. Then, for any $A \in \mathfrak{L}(\mathcal{G}_2)$, $I_{01} \otimes A \in \mathfrak{B}[\mathfrak{R} \otimes I_{12}, \hat{\omega}]$ if and only if $I_1 \otimes A \in \mathfrak{B}[\mathfrak{R} \otimes I_2, \rho]$.

Proof. We show first that $I_{01} \otimes A \in \mathfrak{B}[\mathfrak{R} \otimes I_1 \otimes I_2, \hat{\omega}]$ if and only if $I_{01} \otimes A \in \mathfrak{B}[I_0 \otimes \mathfrak{R} \otimes I_2, \omega]$. For this, let $(\pi_{\hat{\omega}}, \mathcal{H}_{\hat{\omega}}, x_{\hat{\omega}})$ be the GNS representation of $\mathfrak{L}(\mathcal{G}_0 \otimes \mathcal{G}_1 \otimes \mathcal{G}_2)$ induced by the state $\hat{\omega}$. Then, it follows from Eq. 4.46 that $\pi_{\hat{\omega}}(E(\mathtt{C}) \otimes I_1 \otimes I_2)x_{\hat{\omega}} = \pi_{\hat{\omega}}(I_0 \otimes E(\mathtt{C}) \otimes I_2)x_{\hat{\omega}}$ for any Borel set $\mathtt{C}$. From this it follows that $\pi_{\hat{\omega}}(f(R) \otimes I_1 \otimes I_2)x_{\hat{\omega}} = \pi_{\hat{\omega}}(I_0 \otimes f(R) \otimes I_2)x_{\hat{\omega}}$ for any Borel function f on $\mathrm{sp}(R)$. Thus, if we let $\mathcal{T} \equiv [\pi_{\hat{\omega}}(\mathfrak{R} \otimes I_1 \otimes I_2)x_{\hat{\omega}}] \subseteq \mathcal{H}_{\hat{\omega}}$, then

$$\mathcal{T} = [\pi_{\hat{\omega}}(I_0 \otimes \mathfrak{R} \otimes I_2)x_{\hat{\omega}}]. \qquad (4.47)$$

Let $(\pi_{\omega}, \mathcal{H}_{\omega}, x_{\omega})$ be the GNS triple for $\mathfrak{L}(\mathcal{G}_0 \otimes \mathcal{G}_1 \otimes \mathcal{G}_2)$ induced by the state ω. Let $\mathcal{S} \equiv [\pi_{\omega}(I_0 \otimes \mathfrak{R} \otimes I_2)x_{\omega}]$. For any $A \in \mathfrak{L}(\mathcal{G}_2)$, $I_{01} \otimes A \in [\mathfrak{L}(\mathcal{G}_0 \otimes \mathcal{G}_1) \otimes I]'$. Thus, $[\pi_{\omega}(I_{01} \otimes A), \pi_{\omega}(B \otimes I_2)]x_{\omega} = 0$ for any $B \in \mathfrak{R} \otimes I_1 \otimes I_2$, or for any $B \in I_0 \otimes \mathfrak{R} \otimes I_2$. Thus, we need only show that $\pi_{\hat{\omega}}(I_{01} \otimes A)x_{\hat{\omega}} \in \mathcal{T}$ if and only if $\pi_{\omega}(I_{01} \otimes A)x_{\omega} \in \mathcal{S}$.

Now, for any $B \in \mathfrak{L}(\mathcal{G}_0 \otimes \mathcal{G}_1 \otimes \mathcal{G}_2)$,

$$\omega(B) \;=\; \hat{\omega}(UBU^*) \;=\; \langle \pi_{\hat{\omega}}(B)\pi_{\hat{\omega}}(U)^*x_{\hat{\omega}}, \pi_{\hat{\omega}}(U)^*x_{\hat{\omega}}\rangle. \qquad (4.48)$$

Thus, by the uniqueness of the GNS representation, $(\pi_{\hat{\omega}}, \mathcal{H}_{\hat{\omega}}, \pi_{\hat{\omega}}(U)^*x_{\hat{\omega}})$ is unitarily equivalent to $(\pi_{\omega}, \mathcal{H}_{\omega}, x_{\omega})$. That is, there is a unitary operator $V : \mathcal{H}_{\omega} \to \mathcal{H}_{\hat{\omega}}$ such that $Vx_{\omega} = \pi_{\hat{\omega}}(U)^*x_{\hat{\omega}}$ and $V^*\pi_{\hat{\omega}}(B)V = \pi_{\omega}(B)$ for all $B \in \mathfrak{L}(\mathcal{G}_0 \otimes \mathcal{G}_1 \otimes \mathcal{G}_2)$. Now, $\pi_{\omega}(I_{01} \otimes A)x_{\omega} \in \mathcal{S}$ if and only if $V\pi_{\omega}(I_{01} \otimes A)x_{\omega} \in V\mathcal{S} = [V\pi_{\omega}(I_0 \otimes \mathfrak{R} \otimes I_2)x_{\omega}]$, if and only if $\pi_{\hat{\omega}}(I_{01} \otimes A)\pi_{\hat{\omega}}(U)^*x_{\hat{\omega}} \in [\pi_{\hat{\omega}}(I_0 \otimes \mathfrak{R} \otimes I_2)\pi_{\hat{\omega}}(U)^*x_{\hat{\omega}}]$. However, by assumption, U commutes

with $I_{01} \otimes A$ and with all elements from $I_0 \otimes \mathfrak{R} \otimes I_2$. Thus, $\pi_\omega(I_{01} \otimes A)x_\omega \in \mathcal{S}$ if and only if $\pi_{\hat\omega}(I_{01} \otimes A)x_{\hat\omega} \in [\pi_{\hat\omega}(I_0 \otimes \mathfrak{R} \otimes I_2)x_{\hat\omega}] = \mathcal{T}$, where the final equality follows from Eq. 4.47.

To complete the proof, it must be shown that for any $A \in \mathfrak{L}(\mathcal{G}_2)$, $I_{01} \otimes A \in \mathfrak{B}[I_0 \otimes \mathfrak{R} \otimes I_2, \omega_0 \otimes \rho]$ if and only if $I_1 \otimes A \in \mathfrak{B}[\mathfrak{R} \otimes I_2, \rho]$. However, this follows immediately from the fact that the GNS representation for $\omega_0 \otimes \rho$ is unitarily equivalent to the tensor product of the GNS representations for ω_0 and ρ. $\qquad\square$

In combination with the previous result, our final proposition below establishes, in direct analogy to the spin case, that $\mathfrak{B}(C^*(R), (\omega_0 \otimes \rho)_U)$ contains all BUC functions of Q_2 but *not* of D_2.

Proposition 4.16.

    (i) $C^*(Q_2) \subseteq \mathfrak{B}(C^*(Q_1), \rho)$.
    (ii) $C^*(D_2) \nsubseteq \mathfrak{B}(C^*(Q_1), \rho)$.

Proof. (i) Since $C^*(Q_2)$ is a C^*-algebra, it will be sufficient to show that for every unitary element $A \in C^*(Q_2)$, $A \in \mathfrak{B}(C^*(Q_1), \rho)$. Moreover, since $\pi_\rho(A)$ commutes with all elements in $\pi_\rho(C^*(Q_1))$, the result would follow from Prop. 4.8.(i) if we could show that $\pi_\rho(A)x_\rho \in \mathcal{S}$. We proceed to show this.

From Lemma 4.13, there is a unitary $A' \in C^*(Q_1)$ such that $\omega(A) = \omega(A')$, for all $\omega \in \mathsf{L}$. Since each $\omega \in \mathsf{L}$ is dispersion-free, it follows that $\omega((A')^*A) = \overline{\omega(A')}\omega(A) = |\omega(A)|^2 = 1$. Thus,

$$\rho((A')^*A) = \int_{\mathsf{S}} \omega_s((A')^*A)d\mu_\rho(s) = \int_{\mathsf{L}} \omega_s((A')^*A)d\mu_\rho(s) = \mu_\rho(\mathsf{L}) = 1, \qquad (4.49)$$

where we have used Lemma 4.11 in the second and final equalities. From (4.49) it follows that, in the GNS representation (of $\mathfrak{L}(\mathcal{H})$) for ρ, $\langle \pi_\rho((A')^*A)x_\rho, x_\rho \rangle = 1$. Hence, we may use the fact that π_ρ is a $*$-homomorphism in combination with the Cauchy-Schwarz inequality to conclude that $\pi_\rho(A)x_\rho = c\pi_\rho(A')x_\rho$, for some $c \in \mathbb{C}^1$. In particular, $\pi_\rho(A)x_\rho \in [\pi_\rho(C^*(Q_1))x_\rho] = \mathcal{S}$, as we wished to show.

(ii) Since $\mathfrak{B}(C^*(Q_1), \rho)$ is beable for ρ, it has a dispersion-free state ω. We show that this entails that $W_s \equiv e^{isD_2} \notin \mathfrak{B}(C^*(Q_1), \rho)$ for all $s \neq 0$. In order to see this, note first that $U_t \equiv e^{itQ_2} \in C^*(Q_2) \subseteq \mathfrak{B}(C^*(Q_1), \rho)$, for all $t \in \mathbb{R}$, since e^{itx} is uniformly continuous on $\mathbb{R}$. Suppose, for reductio ad absurdem, that $W_s \in \mathfrak{B}(C^*(Q_1), \rho)$ for some $s \neq 0$. Since Q_2 and D_2 satisfy the Weyl-form of the CCR we have $U_tW_s = e^{ist}W_sU_t$ for all $t \in \mathbb{R}$, and $\omega(U_tW_s) = e^{ist}\omega(W_sU_t)$ for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ such that $st \neq n\pi$ for any $n \in 2\mathbb{Z}$. Since ω is dispersion-free on U_t, it follows that $\omega(U_t)\omega(W_s) = e^{ist}\omega(W_s)\omega(U_t)$. Moreover, $\omega(U_t) \neq 0$ and $\omega(W_s) \neq 0$ since ω must assign each unitary operator a value

in its spectrum (Prop. 1.1.(ii)). Thus, we have $e^{ist} = 1$ contrary to our assumption that $st \neq n\pi$ for any $n \in 2\mathbb{Z}$. Therefore $W_s \notin \mathfrak{B}(C^*(Q_1), \rho)$ when $s \neq 0$. $\qquad\qquad\square$

By symmetry of reasoning, if we suppose that the BUC functions of D_1 are actually measured in the original EPR experiment, instead of those of Q_1, it will become legitimate to regard all BUC functions of D_2, but *not* of Q_2, as having determinate values. And, as in the spin case, one has no grounds within the Copenhagen interpretation (so reconstructed) for asserting that both $C^*(Q_2)$ and $C^*(D_2)$ are determinate in state ρ relative to any fixed measurement context for particle 1.

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