*Necessities and Necessary Truths: A Prolegomenon to the Use of Modal Logic in the Analysis of Intensional Notions**

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In philosophical logic necessity is usually conceived as a sentential operator rather than a predicate. An intensional sentential operator does not allow one to express quantified statements such as 'There are necessary a posteriori propositions' or 'All laws of physics are necessary' in first-order logic in a straightforward way, while they are readily formalised if necessity is formalised by a predicate. Replacing the operator conception of necessity by the predicate conception, however, causes various problems and forces one to reject many philosophical accounts involving necessity that are based on the use of operator modal logic. We argue that the expressive power of the predicate account can be restored if a truth predicate is added to the language of first-order modal logic, because the predicate 'is necessary' can then be replaced by 'is necessarily true'. We prove a result showing that this substitution is technically feasible. To this end we provide partial possible-worlds-semantics for the language with a predicate of necessity and perform the reduction of necessities to necessary truths. The technique applies also to many other intensional notions that have been analysed by means of modal operators.

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1 Predicates versus sentential operators of necessity

Philosophy abounds with quantified claims involving necessity and other intensional notions. Examples are 'Some a posteriori propositions are necessary', 'All theorems of Peano arithmetic are analytic' or 'There is a necessary fact that implies all necessary facts.' Modal logic has been considered to be the proper formal framework for analysing statements involving necessity and other intensional notions, although in modal logic these quantified sentences cannot be readily formalised. For the sentential operator $\Box$ of modal logic does not allow for a direct formalisation of predicate phrases such as 'is necessary'; it only provides a direct formal counterpart to the adverbial modifier 'necessarily' (and some other modifiers of sentences). Thus quantified claims cannot be expressed in the framework of first-order logic augmented by a sentential operator in a straightforward way: the expressions $\exists x(\text{Aposteriori}(x) \land \Box x)$, $\forall x(\text{PA-Theorem}(x) \rightarrow \Box x)$ and $\exists x(\Box x \land \forall y(\Box y \rightarrow \text{Imp}(x, y)))$, where $\Box$ stands for the respective intensional notion, are not formalisations of the above quantified statements, simply because they are not even well formed in the language of modal logic: modal operators have to be combined with *formulae* not individual variables.

Obviously the mentioned formal expressions can be proclaimed sentences by declaring $\Box$ a *predicate*. On the predicate account of necessity one can quantify over the objects of which necessity is predicated. We shall use the term 'propositions' for these objects without committing ourselves to a particular view on their nature until more specific assumptions on the ontology of these entities are actually needed.

The predicate and the operator conceptions differ also on sentences without quantification. Here the predicate view forces a parsing of those sentences that is different from the parsing familiar from modal logic and its operator conception of necessity. In English these sentences often contain 'that'-phrases. The operator and the predicate view force two different parsings of these sentences: on the predicate view, 'that'-phrases as in 'It is necessary that gold has atomic number 79' are taken as singular terms denoting propositions. Consequently, the sentence is analysed in the following way:

$$\underbrace{\text{It is necessary}}_{\text{modal predicate}}\ \underbrace{\text{that gold has atomic number 79}}_{\text{singular term}}.$$

On the operator approach, in contrast, necessity is treated as an adverbial modifier of the sentence 'gold has atomic number 79':

$\underbrace{\text{It is necessary that}}_{\text{sentential operator}}\ \underbrace{\text{gold has atomic number 79}}_{\text{sentence}}.$

Since on this parsing 'that'-clauses are not conceived as singular terms, the 'that'-clauses cannot be superseded by individual variables which could be quantified. In this way the quantification problem is a consequence of the operator analysis of sentences with 'that'-clauses.

At any rate the quantification problem seems favour the predicate view. The operator conception seems to cripple the expressive power of the language: quantified statements, which play a crucial role in philosophical discussions, are not expressible without further ploys, if necessity is to be treated as a sentential operator.

2 The benefits of the operator conception

Despite the quantification problem, the operator view of intensional notions, however, has at least certain prima facie advantages over the predicate conception.

The foremost reason to defend the operator conception of necessity and other intensional notions against arguments from the quantification problem is the aim to salvage those parts of philosophy of language and philosophical logic that rely on modal logic. A large number of theories in various areas of philosophy rely on the use of intensional logics. The aim to justify these applications of modal logic together with its possible-worlds-semantics in the analysis of intensional notions provides a strong motivation for elaborating an answer to the quantification problem.

There are further reasons to defend the operator view. If necessity is treated as a predicate it needs to apply to certain objects, at least if one assumes referential semantics as we do. In addition to necessity one will have to deal with further intensional notions, and it is far from being clear whether these different intensional notions should be conceived of as predicates of the same objects as, for instance, metaphysical necessity. Belief, for example, will apply to objects that are more fine grained than the objects of which metaphysical necessity is predicated.[1] If the objects of belief are different from the objects that are necessary, then it seems difficult to understand sentences such as

Kurt's arithmetical beliefs are necessary.

[1] Bealer (1982) discusses the problems of 'granularity' from the perspective of a proponent of the predicate view.

If Kurt's beliefs belong to a category of objects that is disjoint from the category of objects that can be necessary and Kurt has arithmetical beliefs, then the above sentence will be false.

This ontological problem is not to be confused with general nominalistic qualms about propositions and other *abstracta* as objects of necessity and contents of propositional attitudes; the problem we address remains even if these general ontological qualms are put aside.

It also has been argued that treating 'that'-clauses as singular terms denoting objects like propositions is implausible, because they cannot be substituted *salva veritate* by coextensive singular terms. If the 'that'-clause in 'It is necessary that gold has atomic number 79' by the coextensive term 'the proposition that gold has atomic number 79', the clearly ungrammatical expression 'It is necessary the proposition that gold has atomic number 79' is obtained. This obstacle for the predicate view may not be unsurmountable, but on the operator view the problem does not arise at all, because 'that'-clauses are not conceived as singular terms.[2]

Another advantage of the operator treatment of intensional notions consists in the avoidance of paradoxes. Montague's (1963) paradox is the most prominent of them. Since operators do not allow for diagonalisation, that is, self-reference, difficulties arising from self-reference are avoided on the operator account.[3]

Far less foreseeable inconsistencies may arise from the interaction of modal predicates with each other or with still other notions. While the paradoxes of necessity, belief, truth and other notions taken separately have received some attention, the paradoxes arising from the interaction of such predicates have been largely ignored. Halbach (2002, 2006, 2008), Horsten and Leitgeb (2001) and Niebergall (1991) give examples of inconsistencies arising from the interplay of

[2]This kind of problem is especially virulent with propositional attitude reports. See, e.g., (Künne 2003), (Moltmann 2003), (Schiffer 1992), (Schiffer 2003), (Rosefeldt 2007) for recent accounts. For a proposed solution of the problem see also (King 2002).

[3]As a referee correctly pointed out, the full force of the predicate conception is not needed. Grim (1993), for instance, has shown that the Paradox of the Knower can be obtained in context where knowledge is treated as an operator. Grim's observations show that the operator/predicate distinction in the present paper is very rough: one can have the effects of diagonalisation in an operator setting. It is also possible to restrict a language with a predicate of necessity in such a way that the quantification problems arise. Belnap's and Gupta's (1993, p. 240) language $\mathcal{DL}^-$ provides an example of a language with such a restriction. Thus we should not talk simpliciter about the predicate conception, but rather about the predicate conception with diagonalisation and full quantification over the objects, and necessity. Similarly the operator conception excludes means of diagonalisation. Making all this precise is far from trivial. At any rate, the required assumptions for our reduction are made explicit in our formal account.

such notions.[4]

We do not claim that the mentioned advantages of the operator approach are decisive, but we feel that they warrant an attempt to solve the problems of the operator conception of necessity. In order to enjoy all these advantages of the operator account, one has to find a solution to the quantification problem. If philosophers relying on the operator view, modal logic, and possible-worlds-semantics cannot meet this challenge, they cannot account for what they are saying when they claim that there are a priori beliefs that are not necessary. The importance and relevance of modal metaphysics is threatened by the quantification problem: if possible worlds cannot provide an analysis of the philosophical modal idiom, it is hardly of any value to the philosopher.

Our reduction of necessity to necessary truth, however, is only *one* aspect of a more thorough defense of the operator approach to necessity and other intensional notions; and by no means do we take our reduction to be a conclusive defense of the operator view. But it shows how one of the most obvious problems of the operator conception can be tackled in a precise way.

In the following we shall concentrate on the notion of necessity without being specific about the sort or kind of necessity. The following proposal is intended to apply to various kinds of necessity. Indeed our proposal for necessity should be seen as paradigmatic: the suggested method may be applied to many other notions that are usually analysed in terms of modal operators. As should be obvious throughout, we do not employ very specific assumptions on necessity other than the plausibility of possible-worlds-semantics. At the end of the paper we shall address some problems concerning the application of our method to the analysis of propositional attitudes.

3 Dodging the quantification argument

Various devices have been proposed in order to restore the expressive power required for formalising quantified statements. According to one strategy, quantifiers are shifted into the metalanguage: 'All theorems of Peano arithmetic are necessary', for instance, is formalised as the *scheme* $\mathrm{Thm}(\ulcorner\varphi\urcorner) \rightarrow \Box\varphi$. While this or some similar strategies may be feasible in simple cases, it falters at existentially quantified statements such as 'Some a posteriori propositions are necessary' or sentences with more quantifiers such as 'There is a necessary fact that implies all necessary facts.'

[4]Naturally there are even fewer proposals to resolve these inconsistencies. Horsten (1998) and Leitgeb (2006) provide such proposals.

'Substitutional' quantifiers of a certain type and other non-standard quantifiers have been invoked. This approach may be complementary to the account investigated in this paper, in the sense of being intertranslatable with our approach; we do not pursue it here.[5]

Several other proposed solutions of the quantification problem rely on a truth predicate. If truth is conceived as a predicate of sentences and the predicate 'x is necessary' is replaced by

('necessarily' followed by x) is true

quantification can be expressed.

Kripke (1975, p. 713, fn. 33) recommends this approach, that is, he prefers 'true necessity' over 'necessary truth' and emphasises that this order is especially preferable if propositional attitudes rather than necessity are considered. Kripke's remark is probably motivated by the possiblity that one can believe something without believing it to be true, perhaps because the subject lacks an appropriate concept of truth. The 'necessary truth'-version discussed in the present paper has–among other–the advantage of being more natural. We shall discuss Kripke's observation in another paper.[6]

We shall not compare these various proposals but rather dwell on an account that is similar to the last mentioned: on this account, only the order of the operator of necessity and the truth predicate is reversed: the predicate phrase 'x is necessary' is substituted by 'x is necessarily true'. The latter phrase incorporates a truth predicate but only the adverb 'necessarily' for necessity, which is formalised in modal logic as an operator. If this substitution preserves meaning (or at least validity in any given model), a necessity predicate can be reduced to a necessity operator in the presence of a truth predicate.

Although this reduction seems to be an obvious way around the quantification problem, we are not aware of any more detailed account of this reduction.[7] In

[5]'Substitutional' quantification is in certain contexts tantamount to truth (see, e.g., McGee 2000). Kripke's (1976) account of substitutional quantification relies heavily on his (1975) theory of truth. Thus the approach proposed below may be only a variant of solutions relying on substitutional quantification. Therefore a solution involving this sort of quantification may turn out to be a merely notational variant of the solutions relying on the availability of a truth predicate.

[6]Tim Williamson has pointed out that even in the case of necessity 'true necessity' might be preferable for the reduction of the predicate to the operator, because the addition of an actuality operator or predicate could cause problems for our reduction, which are avoided on the 'true necessity' account. We think that an operator of actuality can also be incorporated into our account, if certain assumptions on the interaction of this operator with truth are made. As this subtle issue requires a lengthier discussion, we do not enter this issue any further.

[7]Some authors, e.g., Kripke (1975, p. 713), mention it in passing.

this paper we shall concentrate on the technical feasibility of this approach. For we think that before one compares the various proposals for restoring the expressive power of a language with a necessity predicate and before one invokes more sophisticated philosophical arguments, one should first give precise accounts of the various proposals and check whether they are not already doomed by merely 'technical' difficulties. We shall effect this for the reduction of 'is necessary' to 'is necessarily true' in the present paper.

To this end we shall have to prove that if a sentence containing a necessity predicate is true in a structure then the sentence is still true in the structure, if the necessity predicate has been replaced by 'is necessarily true'. This strategy for eliminating the necessity predicate presupposes that we have notions of truth in a structure for sentences of the source language, that is, of some base language augmented by a necessity predicate, and for the target language, that is, the expansion of the base language by a necessity operator and by a truth predicate. We shall define such notions for the source and the target language, where structures are conceived as possible-worlds models. The semantics for the necessity operator is provided by the usual possible-worlds account of modal logic; interpreting the necessity predicate and the truth predicate is a more challenging task because of the paradoxes.

There are some questions about how precisely the substitution of 'is necessary' by 'is necessarily true' is to be defined: will one have to substitute the predicate of necessity in 'that'-sentences (or within quotes in the scope of the necessity predicate)? We consider the following example:

> The proposition that 0=0 is necessary is itself necessary.

The substitution of the second occurrence of 'is necessary' only yields the following result:

> The proposition that 0=0 is necessary is necessarily true.

Thus if we did not substitute into 'that' sentences, we should require a truth predicate that applies to propositions (expressed by a sentence) containing the necessity predicate. The need for such a truth predicate would undermine the whole project: The problems with semantics for a necessity predicate motivated the whole reduction. If we had to provide a truth predicate for propositions with the necessity predicate all these problems would be reintroduced through the backdoor.

The following paraphrase of the example does not involve such a truth predicate:

> It is necessarily true that 0=0 is necessarily true.

This implies also that the reduction does not require only a substitution of the necessity predicate in the following sentence:

> All propositional tautologies (in the language with the necessity predicate) are necessary.

This universal claim pertains also to tautologies that contain the necessity predicate. We want to translate the claim by a sentence equivalent to the following:

> All propositional tautologies (in the language with the necessity predicate) are necessarily true after the necessity predicate has been replaced by 'is necessarily true' in them.[8]

This 'thorough' substitution creates a technical problem: One might try to define the substitution recursively: first the necessity predicate is substituted in sentences where the necessity predicate is applied only to sentences that do not themselves contain a necessity predicate;[9] in the next step the translation is defined for those sentences applying to sentences involving one more iteration of necessity. This approach, however, cannot succeed, because on this recursive approach one will not define the result of the substitution of 'ungrounded' sentences like

A1 Sentence A1 is necessary.

A2 Propositional tautologies are necessary.

A1 and A2 do not apply only to sentences with a definite upper limit on the iterations of necessity: they are ungrounded.[10] A1 predicates necessity of itself; thus a recursive translation scheme of the kind just outlined does not apply to A1. A2 is not self-referential in the same sense as A1, but it attributes necessity to all propositional tautologies irrespective of the number of iterations of necessity in them. Some propositional tautologies contain A2 as a subsentence.

We do not want to imply that A1 and A2 are unproblematic at the semantical level. But for the time being, we do not commit ourselves to any claim about their

[8]'replaced' means here that not only used occurrences need to be replaced. The exact notion of substitution will be made precise below. The exact formulation also depends on one's ontological framework. Thus the sentence has to be reformulated according to one's preferred theory of propositions. If propositions are conceived as sentences, the above sentence makes sense. If propositions are conceived as language independent objects, the example sentence will need to be reformulated.

[9]Here we apply the predicate of necessity to sentences. Alternatively we could apply it to propositions conceived as objects expressed by sentences. In this case one would have to replace 'sentence x is necessary' by 'the proposition expressed by sentence x is necessary.' We hope the reader will let us get away with this simplified account, even if it is in conflict with his metaphysical persuasion.

[10]For a precise account of groundedness and self-reference see (Leitgeb 2005).

semantic status; we only seek to define a translation function. A1 and A2 show that it is not feasible to define a translation function recursively in the indicated way. One might propose to define the translation function only on 'unproblematic' sentences, for instance, only on the grounded sentences.

Such a restriction, however, can be avoided: in the formal development of the account we shall show how to obtain a translation function on *all* sentences with a necessity predicate whether they are grounded or not. We shall define a translation function with the desired properties by invoking the Recursion Theorem in section 7.

Using a truth predicate in order to eliminate one for necessity seems to replace one evil with another. Truth needs to apply to certain objects, truth is the chief example of a paradoxical notion: so if one eliminates the predicate of necessity by using a truth predicate, one merely seems to shift problems from one notion to the other.

However, we think that this solution of the quantification problem — if it is actually feasible — would yield remarkable benefits. Although it cannot be used to remove all ontological commitments (because objects of truth will still be required), it allows one to concentrate all ontological problems on the truth predicate. Thus the problem of different granularities of propositions is solved: propositions are no longer needed as objects of modal notions, such as necessity and perhaps others which are intensional; only objects to which truth can be attributed are required.

The elimination of the necessity predicate and other predicates of intensional notions, also solves the paradoxes of modality. Of course, this does not solve the paradoxes of truth, but all efforts to solve the paradoxes can be concentrated on the truth predicate. Moreover, paradoxes arising from the interaction of modal notions are definitely resolved, because after the elimination of all the predicates for modal notions, only the truth predicate remains.

At any rate the proposed defense of the operator view allows for a division of labour: philosophers relying on modal logic can carry on within their framework freed from the threat of the quantification problem, while modal metaphysics retains its relevance for the analysis of modal notions. Thus our account of how a predicate of necessity can be eliminated by an operator of necessity becomes a prolegomenon to any metaphysics of modality based on the operator conception of modality.

In the main part of this paper we shall elaborate on the technical execution of the elimination of the necessity predicate by a sentential operator and a truth predicate by providing the technical framework for this elimination. There may be various less technical issues that threaten the feasibility of this reduction. We shall not address them in this paper. Furthermore we leave the evaluation of the whole reduction and its comparison to alternative answers to the quantification problem to a future paper. At any rate, we think of the technical elaboration as a service to many philosophers who have frequently answered the challenge of the quantification problem by gesturing at this reduction without ever providing any evidence that the proposed elimination of a necessity predicate is actually feasible.[11]

4 Outline of the strategy

As announced in the previous section, we provide a translation from a language $\mathcal{L}_N$ with a necessity predicate N into a language with a necessity operator $\Box$ and a truth predicate Tr. We shall then show that this translation preserves truth in any given structure. More precisely, if a sentence containing the necessity predicate is true in a given structure, then its translation, i.e., the result of replacing the necessity predicate with the corresponding operator plus the truth predicate, will also be true in the structure. As announced above, we shall employ possible-worlds-semantics as structures. Possible-worlds-semantics offers the most popular and probably the most natural method for obtaining semantics for necessity. Moreover the reduction was motivated by the desire to justify the use of possible-worlds-semantics. The possible-worlds structures we propose will contain various worlds and each of these worlds will provide an interpretation of the non-logical vocabulary of the base language, that is, the language without the predicates for necessity and truth, and without the necessity operator.

The development of semantics for both a predicate of necessity and a predicate of truth is impeded by the paradoxes. We have to decide which solution of the paradoxes to adopt in the sequel. We shall settle for Kripke's (1975) fixed-point semantics with the Strong Kleene evaluation scheme. More precisely, we shall opt for the smallest fixed point of Kripke's construction.

There may be good reasons to adopt other policies on the paradoxes, but we feel that our policy is a safe one: it will not declare a sentence true whose truth is doubtful; more technically speaking, in Kripke's semantics no sentence will come out as true whose truth is not supported by the truth of non-semantic sentences

[11] A referee hinted at Belnap's and Gupta's (1993, Chapter 6E) reduction of a predicate of necessity to an operator. Belnap and Gupta, however, consider a heavily restricted language with a predicate of necessity that suffers from the quantification problem. They are aware of this restriction and their account may be seen as an exploration of the extent to which a predicate can be reduced to an operator in the absence of auxiliary devices such as a truth predicate.

(see Yablo 1982 and Leitgeb 2005). Thus we are fairly confident about the soundness of this solution to the paradoxes in the sense that we expect all sentences that are true at Kripke's minimal fixed-point model to be true also at an intuitive level. We shall prove that the reduction of a necessity predicate to an operator of necessity goes through at least for those unproblematic sentences.

We now describe the possible-worlds structures that will serve as models for both kinds of languages.

We fix a non-empty set W of worlds together with an accessibility relation $R \subseteq W \times W$. Each single world $w \in W$ is identified with a model (in the model-theoretic sense) for the base language $\mathcal{L}$, which is the language without the necessity predicate N, the necessity operator $\Box$, or the truth predicate Tr.[12]

The language $\mathcal{L}$ should be a primitive recursive first-order language that provides means to talk about objects of which necessity and truth can be predicated. For simplicity we assume that the language $\mathcal{L}$ contains a closed term $\ulcorner e \urcorner$ for any expression e of the language $\mathcal{L}_N$ and the language $\mathcal{L}_\Box^{\mathrm{Tr}}$, that is, $\mathcal{L}$ plus the modal operator $\Box$ and the truth predicate. Every world contains appropriate designata for the terms $\ulcorner e \urcorner$ in its domain. In particular, if e_1 and e_2 are different expressions, the designata of $\ulcorner e_1 \urcorner$ and $\ulcorner e_2 \urcorner$ ought to be different objects. One may use numerical codes of expressions for this purpose. In order to keep our notation simple, we identify the expressions e with these objects. The worlds may contain any objects beyond the expressions e. Moreover, we assume that all primitive recursive syntactical operations are expressible in $\mathcal{L}$. Consequently, the expressions e are necessarily existing entities, because they must be elements of every possible world.[13]

In the next section we formally define the truth of a sentence of $\mathcal{L}_N$ at a possible world $w \in W$; the truth of a sentence of $\mathcal{L}_\Box^{\mathrm{Tr}}$ will then be defined in the following section. Of course only the necessity and the truth predicate generate problems, whilst the semantics of the modal operator $\Box$ is straightforward and completely standard.

[12]We have identified worlds with models in order to keep our exposition as simple as possible. We could use most other conceptions of worlds, whether they are realist or 'ersatzist' or still something else, as long as the worlds provide truth conditions for sentences in $\mathcal{L}$.

[13]These assumptions may seem too specific: it seems that we assume that certain objects occur in all worlds and we seem to reject thereby conceptions of worlds in David Lewis' style where the domains of all worlds are pairwise disjoint. However, we could use counterparts of these expressions in all possible worlds. In contrast to ordinary objects it seems less contentious to assume that the counterpart relation is transitive on the set of expressions (or propositions).

5 Possible-worlds-semantics for a language with a predicate of necessity

5.1 Partially interpreted predicates and the Strong Kleene scheme

In this section we generalise Kripke's theory to possible-worlds-semantics in order to define the truth of sentences of $\mathcal{L}_N$ at a world $w \in W$.[14]

The language is obtained by adding a unary predicate N to a base language $\mathcal{L}$.[15] We shall impose further restriction on $\mathcal{L}$ only when needed.

First we define the well known Strong Kleene evaluation scheme for sentences. The Strong Kleene scheme is conservative in the sense that it will only assign a classical truth value (true or false) to a sentence, if this truth value is already fixed 'classically'. For instance, a conjunction only will be declared true if both conjuncts are true; and a conjunction will only be declared false if at least one conjunct is false. We think that this strategy is sensible if one thinks of sentences lacking a classical truth as semantically underdetermined in our semantical framework.

If w is a (classical) model (possibly one of the worlds in W) for the base language $\mathcal{L}$ (i.e., for the language without an operator or predicate for necessity or a truth predicate), the pair (w, S) is a partial model for $\mathcal{L}_N$ if S is a set of sentences of $\mathcal{L}_N$. We define a satisfaction relation $\models^{\mathrm{SK}}$ between partial models, formulae of $\mathcal{L}_N$ and variable assignments a over w. A variable assignment is a function from the individual variables of $\mathcal{L}$ into the domain of w. $(w, S) \models^{\mathrm{SK}} \varphi[a]$ will obtain if and only if the assignment a satisfies the formula φ, if the necessity predicate N is given the extension S and all other non-logical vocabulary is interpreted according to the model w.

In the first two clauses P may be any predicate symbol of $\mathcal{L}$ and $t_1 \dots t_n$ may be arbitrary terms. a is a variable assignment.

(i) $(w, S) \models^{\mathrm{SK}} Pt_1 \dots t_n\,[a]$ iff the model w satisfies $Pt_1 \dots t_n$ under a in the usual sense.

(ii) $(w, S) \models^{\mathrm{SK}} \neg Pt_1 \dots t_n\,[a]$ iff the model w does not satisfy $Pt_1 \dots t_n$ under a in the usual sense.

For the necessity predicate N, which is not in $\mathcal{L}$ but only in $\mathcal{L}_N$, two special clauses are employed: N is assigned the partial extension/anti-extension pair $S = (S^+, S^-)$.

[14] In this paper we are dealing with at least three different notions of truth: in the present section and the following we define metatheoretic notion of truth for sentences of the language $\mathcal{L}_N$ and of the language $\mathcal{L}_\Box^{\mathrm{Tr}}$, respectively. The latter language contains a truth predicate Tr in the object language, which is the third notion of truth.

[15] Later we propose to extend our approach to de re-modality where the necessity predicate is conceived as a binary predicate applying to formulae and variable assignments.

N will apply to all sentences in S^+ and $\neg N$ will apply to all sentences in S^-, but there may be sentences to which neither applies.

(iii) $(w, S) \models^{SK} Nt\ [a]$ iff the value of the term t under the assignment is a $\mathcal{L}_N$-sentence φ in S^+.

(iv) $(w, S) \models^{SK} \neg Nt\ [a]$ iff the value of the term t under the assignment is not a $\mathcal{L}_N$-sentence or a $\mathcal{L}_N$-sentence φ in S^-. Volker: changed clause

(v) $(w, S) \models^{SK} \neg\neg\varphi[a]$ iff $(w, S) \models^{SK} \varphi[a]$.

(vi) $(w, S) \models^{SK} (\varphi \wedge \psi)[a]$ iff $(w, S) \models^{SK} \varphi[a]$ and $(w, S) \models^{SK} \psi[a]$.

(vii) $(w, S) \models^{SK} \neg(\varphi \wedge \psi)[a]$ iff $(w, S) \models^{SK} \neg\varphi[a]$ or $(w, S) \models^{SK} \neg\psi[a]$.

(viii) $(w, S) \models^{SK} \forall x\varphi[a]$ iff for all variable assignments c differing from a only in the value of x $(w, S) \models^{SK} \varphi[c]$.

(ix) $(w, S) \models^{SK} \neg\forall x\varphi[a]$ iff for at least one variable assignment c differing from a only in the value of x $(w, S) \models^{SK} \neg\varphi[c]$.

As usual we simply skip the assignment and write $(w, S) \models^{SK} \varphi$ instead of $(w, S) \models^{SK} \varphi[a]$, when φ does not contain free variables. If no assignment is provided, the notation $(w, S) \models^{SK} \varphi$ implies that φ is a sentence.

In this definition no reference was made to the accessibility relation R between worlds. Thus $(w, S) \models^{SK} \varphi$ means that the sentence φ is true at world w, if the necessity predicate N is given the extension S^+ and antiextension S^-. Of course, S may be any pair of sets, and only in the next subsection shall we show how to specify a sensible S.

If both φ is neither in S^+ nor S^-, then neither $(w, S) \models^{SK} N\ulcorner\varphi\urcorner$ nor $(w, S) \models^{SK} \neg N\ulcorner\varphi\urcorner$ will obtain; in this case $N\ulcorner\varphi\urcorner$ is said to lack a truth value. The logic for the N-free fragment $\mathcal{L}$ is classical logic; only sentences involving the partial predicate N may lack a truth value.

5.2 Interpreting the necessity predicate

In order to provide an appropriate interpretation of the necessity predicate N, we consider functions f assigning a pair of sets of $\mathcal{L}_N$-sentences to every world in W. We call such f *evaluation functions* for the language $\mathcal{L}_N$. These evaluation functions for $\mathcal{L}_N$ fix an interpretation (extension/antiextension pair) of the necessity predicate at every world. We need to find evaluation functions that respect the fundamental idea of possible-worlds-semantics as far as possible: $N\ulcorner\varphi\urcorner$ should be true at a world w if and only if φ is true at all worlds accessible from w, and $\neg N\ulcorner\varphi\urcorner$ should be true at w if and only if $\neg\varphi$ is true at a world accessible from w.

In order to obtain an interpretation satisfying this requirement, we define an operator Γ_N on evaluation functions for $\mathcal{L}_N$. The result $\Gamma_N(f)$ of applying the operator to a given evaluation function f is defined by fixing the value of $\Gamma_N(f)$ for each world:

$$\begin{aligned} (\Gamma_N(f))(w)^+ &:= \{\varphi \in \mathcal{L}_N : \text{for all } v\ (wRv \Rightarrow (v, f(v)) \models^{\text{SK}} \varphi)\} \\ (\Gamma_N(f))(w)^- &:= \{\varphi \in \mathcal{L}_N : \text{there exists } v\ (wRv \Rightarrow (v, f(v)) \models^{\text{SK}} \neg\varphi)\} \end{aligned} \tag{1}$$

Thus $\Gamma_N(f)$ yields, if applied to a world w, both (a) the set $(\Gamma_N(f))(w)^+$ as the extension of N at w, namely the set of all sentences that are true at all worlds v accessible to w, when N itself is assigned at all worlds v the extension/anti-extension $f(v)$; as well as (b) the set $(\Gamma_N(f))(w)^-$ of sentences false at some world accessible to w, as the anti-extension of N at w.

We define an ordering on the set of all evaluation functions: $f \leq g$ obtains if and only if for all $w \in W$ we have both $f(w)^+ \subseteq g(w)^+$ and $f(w)^- \subseteq g(w)^-$. The operator Γ_N is monotone with respect to this ordering, that is, $f \leq g$ implies $\Gamma_N(f) \leq \Gamma_N(g)$. The monotonicity follows from the definition of $\models^{\text{SK}}$. The monotonicity of the Strong Kleene scheme plays a crucial rôle in Kripke's theory of truth (see, e.g. McGee 1991). Here we have merely generalised Kripke's approach to more worlds.

The monotonicity of Γ_N implies the existence of fixed points of Γ_N. That is, there will be evaluation functions f satisfying $\Gamma_N(f) = f$. Such functions interpret N in the desired way: if f is a fixed point of Γ_N, the following obtains for all sentences φ of $\mathcal{L}_N$ and all worlds $w \in W$:

$$\begin{aligned} (w, f(w)) \models^{\text{SK}} N\ulcorner\varphi\urcorner \quad &\text{iff} \quad \text{for all } v \text{ with } wRv:\ (v, f(v)) \models^{\text{SK}} \varphi \\ (w, f(w)) \models^{\text{SK}} \neg N\ulcorner\varphi\urcorner \quad &\text{iff} \quad \text{there exists } v \text{ with } wRv:\ (v, f(v)) \models^{\text{SK}} \neg\varphi \end{aligned} \tag{2}$$

In other words, using the interpretation $f(w)$ for N, $N\ulcorner\varphi\urcorner$ will be true at a world w if and only if φ is true at all worlds that can be seen from w, and $\neg N\ulcorner\varphi\urcorner$ will be true if and only if $\neg\varphi$ is true at some world accessible from w.[16]

In some cases one will be able to assign a classical interpretation to N, as our present approach does not *force* truth value gaps for all possible $\langle W, R\rangle$.

[16]Kripke's (1975) theory of truth is a special case of the account we have just presented: if there is only one world w, which can see itself then (2) can be simplified to the following equivalence:

$$(w, f(w)) \models^{\text{SK}} N\ulcorner\varphi\urcorner \text{ iff } (w, f(w)) \models^{\text{SK}} \varphi$$

Thus N will be the truth predicate as in Kripke's theory.

More precisely, depending on W and R there may be a fixed point g such that $(w, g(w)) \models^{\mathrm{SK}} \varphi$ or $(w, g(w)) \models^{\mathrm{SK}} \neg\varphi$ obtains for every sentence φ of $\mathcal{L}_N$ and for every world $w \in W$.

Halbach, Leitgeb, and Welch (2003) have specified conditions on frames that allow for an interpretation of N without truth value gaps or gluts. The interpretations yielding classical logic for the language with N will be fixed points of Γ_N, though not minimal fixed points. It is also there shown that R cannot be reflexive on W, if the frame allows for a classical interpretation of N. Such frames are similar to those of provability logic (cf Boolos (1993)).

Since we definitely want to consider reflexive frames, it will be inevitable in some frames that some sentences lack truth values (at some or all worlds). So we do not follow Halbach, Leitgeb, and Welch (2003) in focusing on frames that allow for fixed points of Γ_N that yield classical logic.

However even frames that force truth value gaps, as reflexive frames do, will generally allow for many fixed points of Γ_N (except for some trivial cases). We do not want to repeat the discussion surrounding Kripke's (1975) theory, and we shall focus in the following on the *minimal* fixed point. This is the evaluation function f_N with $f_N \leq g$ for all fixed points g of Γ_N. f_N can be obtained by iterated applications of Γ_N to the empty evaluation function, that is, the valuation that assigns a pair of empty sets as extension/anti-extension to every world. As desired, this interpretation will assign the correct truth values to all grounded sentences.

6 Possible-worlds-semantics for a language with an operator of necessity and with a truth predicate

We now pass on to the target language $\mathcal{L}_{\Box}^{\mathrm{Tr}}$ containing a truth predicate Tr and an *operator* $\Box$ of necessity instead of a necessity predicate. As we shall be dealing with de re-modality only later, we rule out de re-necessity at the syntactic level: $\Box\varphi$ is only a well formed formula of $\mathcal{L}_{\Box}^{\mathrm{Tr}}$, if φ does not contain any free variables.

In this language the truth predicate will again be interpreted according to the Strong Kleene evaluation scheme. The interpretation will be defined in a way very similar to that of N, by an inductive definition. The underlying idea here is that $\mathrm{Tr}\ulcorner\varphi\urcorner$ should be true at a world w if and only if φ is true at that world w. In contrast to (2) this clause does not directly refer to R and possible worlds other than w. However, the other possible worlds enter the scene through the operator $\Box$, whose interpretation is fixed by the satisfaction relation $\models_{\Box}^{\mathrm{SK}}$.

Since $\mathcal{L}_{\Box}^{\mathrm{Tr}}$ incorporates the modal operator $\Box$, Strong Kleene logic has to be extended to intensional logic. Thus whether a sentence is true at a world does not only depend on w and the interpretation of Tr at that world but also on the interpretation of Tr at other worlds. This interpretation is provided by an evaluation function for the language $\mathcal{L}_{\Box}^{\mathrm{Tr}}$, that is, a function assigning a set of $\mathcal{L}_{\Box}^{\mathrm{Tr}}$-sentences to every world $w \in W$.

In contrast to the necessity predicate N, the operator $\Box$ of necessity does not introduce truth value gaps: $\Box\varphi$ may lack a truth value only if φ already lacks a truth value at some accessible world.

6.1 Partial modal logic

We define a relation $\models_{\Box}^{\mathrm{SK}}$ between pairs of worlds w and evaluation functions f for $\mathcal{L}_{\Box}^{\mathrm{Tr}}$, formulae of $\mathcal{L}_{\Box}^{\mathrm{Tr}}$ and variable assignments. As before, W and R remain fixed and the definition of the satisfaction relation $(w, f) \models_{\Box}^{\mathrm{SK}}$ depends on both.

(i) $(w, f) \models_{\Box}^{\mathrm{SK}} Pt_1 \ldots t_n\,[a]$ iff the model w satisfies $Pt_1 \ldots t_n$ under a in the usual sense. Here again P may be any predicate symbol of $\mathcal{L}$ and $t_1 \ldots t_n$ may be any terms

(ii) $(w, f) \models_{\Box}^{\mathrm{SK}} \neg Pt_1 \ldots t_n\,[a]$ iff the model w does not satisfy $Pt_1 \ldots t_n$ under a in the usual sense

(iii) $(w, f) \models_{\Box}^{\mathrm{SK}} \mathrm{Tr}t[a]$ iff the value of the term t under the assignment is a sentence $\varphi \in f(w)$

(iv) $(w, f) \models_{\Box}^{\mathrm{SK}} \neg\mathrm{Tr}t\,[a]$ iff the value of the term t under the assignment is not an $\mathcal{L}_{\Box}^{\mathrm{Tr}}$-sentence or it is a sentence φ whose negation is in $f(w)$ Volker: changed

By (iii) the extension of the truth predicate at a world w is given by the evaluation function f, while the antiextension is the set of all sentences whose negation is in $f(w)$. (v)–(ix) are analogous to the clauses in the definition of $\models^{\mathrm{SK}}$; they capture the Strong Kleene scheme.

(v) $(w, f) \models_{\Box}^{\mathrm{SK}} \neg\neg\varphi[a]$ iff $(w, f) \models_{\Box}^{\mathrm{SK}} \varphi[a]$

(vi) $(w, f) \models_{\Box}^{\mathrm{SK}} (\varphi \wedge \psi)[a]$ iff $(w, f) \models_{\Box}^{\mathrm{SK}} \varphi[a]$ and $(w, f) \models_{\Box}^{\mathrm{SK}} \psi[a]$

(vii) $(w, f) \models_{\Box}^{\mathrm{SK}} \neg(\varphi \wedge \psi)[a]$ iff $(w, f) \models_{\Box}^{\mathrm{SK}} \neg\varphi[a]$ or $(w, f) \models_{\Box}^{\mathrm{SK}} \neg\psi[a]$

(viii) $(w, f) \models_{\Box}^{\mathrm{SK}} \forall x\varphi[a]$ iff for all c differing from a only in the value of x $(w, f) \models_{\Box}^{\mathrm{SK}} \varphi[c]$

(ix) $(w, f) \models^{\mathrm{SK}}_{\Box} \neg\forall x\varphi[a]$ iff for at least one c differing from a only in the value of x $(w, f) \models^{\mathrm{SK}}_{\Box} \neg\varphi[c]$

The last two clauses (x) and (xi) are standard in propositional modal logic. The syntax of $\mathcal{L}^{\mathrm{Tr}}_{\Box}$ forces φ to be a sentence. Thus no variable assignments are needed.

(x) $(w, f) \models^{\mathrm{SK}}_{\Box} \Box\varphi$ iff (for all v: wRv implies $(v, f) \models^{\mathrm{SK}}_{\Box} \varphi$)

(xi) $(w, f) \models^{\mathrm{SK}}_{\Box} \neg\Box\varphi$ iff (there is at least one v such that wRv and $(v, f) \models^{\mathrm{SK}}_{\Box} \neg\varphi$)

The only deviation from classical modal logic arises from the possibility that φ could be neither true nor false at some worlds accessible from w. If φ is false at one of those worlds $\Box\varphi$ will be false at w (that is, $\neg\Box\varphi$ will be true at w). If φ is true at all worlds accessible from w, $\Box\varphi$ will be true at w. Only if φ lacks a truth value at some world or worlds accessible from w but is true at all other worlds accessible from w, will $\Box\varphi$ lack a truth value at w.

6.2 *Interpreting the truth predicate*

The ordering on the evaluation functions for $\mathcal{L}^{\mathrm{Tr}}_{\Box}$ is defined in a simiar same way as for the evaluation function for $\mathcal{L}_N$: $f \leq g$ obtains if and only if for all $w \in W$ $f(w) \subseteq g(w)$.

Again we define a function on the evaluation functions for the language $\mathcal{L}^{\mathrm{Tr}}_{\Box}$. This is the so called 'Kripke jump' parametrised by possible worlds:

$$(\Gamma_{\mathrm{Tr}}(f))(w) \ := \ \{\varphi \in \mathcal{L}^{\mathrm{Tr}}_{\Box} : \ (w, f) \models^{\mathrm{SK}}_{\Box} \varphi\} \tag{3}$$

Thus the evaluation function $\Gamma_{\mathrm{Tr}}(f)$ assigns to w the set of all sentences true at w if Tr is interpreted by $f(w)$.

Also Γ_{Tr} is monotone, that is, $f \leq g$ implies $\Gamma_{\mathrm{Tr}}(f) \leq \Gamma_{\mathrm{Tr}}(g)$. Consequently there are fixed points of Γ_{Tr}. If f is a fixed point of Γ_{Tr}, the following equivalence holds for all sentences φ of $\mathcal{L}^{\mathrm{Tr}}_{\Box}$:

$$(w, f) \models^{\mathrm{SK}}_{\Box} \mathrm{Tr}\ulcorner\varphi\urcorner \text{ iff } (w, f) \models^{\mathrm{SK}}_{\Box} \varphi \tag{4}$$

The minimal fixed point f_{Tr} of Γ_{Tr} is obtained by iterated application of Γ_{Tr} to the minimal or 'empty' evaluation function, that is, the function g with $g(w) = \varnothing$ for all worlds $w \in W$.

7 The reduction

After expounding semantics for both the languages $\mathcal{L}_N$ (or at least for its grounded fragment) and $\mathcal{L}_\Box^{\mathrm{Tr}}$, we specify a translation $^\circ$ from $\mathcal{L}_N$ to $\mathcal{L}_\Box^{\mathrm{Tr}}$ that replaces 'is necessary' by 'is necessarily true', and we show that this translation preserves truth at a world.

We make now further assumptions on the language $\mathcal{L}$ and its interpretation at all worlds. We assume that all primitive recursive operations on syntactical objects, i.e., on codes of such objects, can be expressed in $\mathcal{L}$. Thus $\mathcal{L}$ must contain predicate and/or function expressions that are interpreted at any world w in W in the appropriate way. We shall use this assumption when applying the Recursion Theorem (see Rogers 1967 Ch 11. Thm I) for primitive recursive functions. The use of the Recursion Theorem, however, should not convey the impression that the proposed reduction relies on an arithmetical framework. The main ingredient in the proof of the Recursion Theorem is diagonalisation, which is available in languages that can express the basic syntactic operations of substitution and forming names of expressions. In any case the reduction is carried out at the metatheoretic level.

First we define a binary function $h(x,y)$ by primitive recursion such that the following equation obtains for all natural numbers e and n:

$$h(e,n) := \begin{cases} \varphi & \text{if } n \text{ is an atomic sentence } \varphi \text{ of } \mathcal{L}, \\ \Box \mathrm{Tr}\, E(e,s) & \text{if } n \text{ is } Ns \text{ for some term } s \\ \neg h(e,\varphi) & \text{if } n \text{ is } \neg\varphi, \\ h(e,\varphi) \wedge h(e,\psi) & \text{if } n \text{ is } (\varphi \wedge \psi), \\ h(e,\varphi) \vee h(e,\psi) & \text{if } n \text{ is } (\varphi \vee \psi), \\ \forall x\, h(e,\varphi) & \text{if } n \text{ is } \forall x\, \varphi, \\ \exists x\, h(e,\varphi) & \text{if } n \text{ is } \exists x\, \varphi, \\ 0 & \text{else.} \end{cases}$$

In accordance with our identification of expressions of the language with their codes, we understand also syntactic operations as operations on the codes. Thus, for instance, we can apply $\neg$ to $h(e,\varphi)$ in line 3. We assume in the following that 0 is not the code of a sentence; we want to ensure that objects that are not sentences of $\mathcal{L}_N$ are translated into objects that are not sentences of $\mathcal{L}_\Box^{\mathrm{Tr}}$.

Volker: added sentence here; changed last clause above

The notation in the second line of this definition needs some explanation. There is a universal binary primitive recursive evaluation function $\mathcal{E}(x,y)$ that gives, when the index number (i.e. code) e of a primitive recursive function $f = \{e\}$ is substituted for x, the value $f(n) = \{e\}(n)$ of the argument n under the function

f, when n is substituted for y. Since our language $\mathcal{L}$ contains the language of arithmetic, this universal evaluation function can be expressed in $\mathcal{L}$. For the sake of a more perspicuous presentation we assume that $\mathcal{L}$ contains a binary function symbol $E(x, y)$ for this universal function. If no such function symbol is available, it can be circumscribed using a suitable formula.

Now we apply the Recursion Theorem to the function $h(x, y)$ to show that there is an index, i.e. a natural number, e_0, of a primitive recursive function that satisfies the following equation:

$$\{e_0\}(n) = h(e_0, n)$$

Now let us abbreviate by x° the function $\{e_0\}(x)$ and we shall show that we have obtained a translation function with the desired properties.

Lemma 1 *For all formulae φ of the language $\mathcal{L}_N$ the following obtains:*

$$\varphi^\circ := \begin{cases} \varphi & \text{if } \varphi \in \mathcal{L} \text{ is atomic,} \\ \Box \mathrm{Tr}\, s^\bullet & \text{if } \varphi = Ns \text{ for some term } s \\ \neg\psi^\circ & \text{if } \varphi = \neg\psi, \\ \psi^\circ \wedge \chi^\circ & \text{if } \varphi = (\psi \wedge \chi), \\ \psi^\circ \vee \chi^\circ & \text{if } \varphi = (\psi \vee \chi), \\ \forall x\, \psi^\circ & \text{if } \varphi = \forall x\, \psi, \\ \exists x\, \psi^\circ & \text{if } \varphi = \exists x\, \psi. \end{cases}$$

$^\bullet$ *stands for some function expression of $\mathcal{L}$ representing the primitive recursive function $^\circ$. Thus $s^\bullet$ is short for $E(e_0, s)$.*

We illustrate the manner of operation of the translation function by considering the sentence $N\ulcorner N\ulcorner 0 = 0\urcorner\urcorner$. The lemma shows that the translation function $^\circ$ yields the desired translation if applied to $N\ulcorner N\ulcorner 0 = 0\urcorner\urcorner$, because $(N\ulcorner N\ulcorner 0 = 0\urcorner\urcorner)^\circ$ is the sentence $\Box\mathrm{Tr}\ulcorner N\ulcorner 0 = 0\urcorner\urcorner^\bullet$, which is equivalent to the desired formula $\Box\mathrm{Tr}\ulcorner(N\ulcorner 0 = 0\urcorner)^\circ\urcorner$, which is $\Box\mathrm{Tr}\ulcorner(\Box\mathrm{Tr}\ulcorner 0 = 0\urcorner^\bullet)\urcorner$. Of course, $\ulcorner 0 = 0\urcorner^\bullet = \ulcorner 0 = 0\urcorner$ is true at any world and so $(N\ulcorner N\ulcorner 0 = 0\urcorner\urcorner)^\circ$ is equivalent to $\Box\mathrm{Tr}\ulcorner(\Box\mathrm{Tr}\ulcorner 0 = 0\urcorner)\urcorner$.

We are now ready to show that the translation function preserves truth in the minimal fixed points. Recall that f_N was defined as the minimal fixed point of Γ_N and f_{Tr} as the minimal fixed point of Γ_{Tr}.

Theorem 2 *For all worlds $w \in W$ and sentences φ of $\mathcal{L}_N$ the following implication obtains:*

$$(w, f_N(w)) \models^{\mathrm{SK}} \varphi \quad \textit{implies} \quad (w, f_{\mathrm{Tr}}) \models^{\mathrm{SK}}_{\Box} \varphi^\circ$$

We started from some structure incorporating possible worlds W and an accessibility relation R between the worlds. Then we argued that the sentences of $\mathcal{L}_N$ that are unquestionably true (in the sense of being grounded) at an arbitrary world w in this structure are those that are true under the strong Kleene scheme when the necessity predicate N is interpreted by $f_N(w)$. The sentences of $\mathcal{L}_\Box^{\mathrm{Tr}}$ that are unquestionably true at w in this structure based on W and R are those that are true under the usual possible-worlds-semantics (extended to the Strong Kleene) where the truth predicate is interpreted in Kripke's (1975) fashion by the minimal fixed point model. Finally, Theorem 2 shows that if we replace 'is necessary' by 'is necessarily true' in those sentences of $\mathcal{L}_N$ unquestionably true in the structure, then we obtain unquestionably true sentences of $\mathcal{L}_\Box^{\mathrm{Tr}}$. Thus, the necessity predicate can be replaced by an operator for necessity and a truth predicate.

7.1 Proof of Theorem 2

Throughout the proof the set W of worlds and the accessibility relation R on the worlds are kept fixed. We use standard techniques and notation for positive inductive definitions (see, e.g. Moschovakis 1974).

The minimal evaluation function f_0 for $\mathcal{L}_N$ is the function that assigns the empty set to every world $w \in W$. The minimal fixed point evaluation function f_N can be obtained by iterated application of the operator Γ_N to f_0. More formally, there will be an ordinal λ with the following property:

$$f_N = \Gamma_N^{\lambda}(f_0) = \Gamma_N^{\lambda+1}(f_0)$$

An upper index α in Γ^α indicates the number of applications of the operator Γ_N. The second equation asserts that after stage λ applying Γ_N does not add anything. The magnitude of λ depends on W and R.

Similarly the minimal fixed point f_{Tr} of Γ_{Tr} may be obtained by applying Γ_{Tr} sufficiently often to the minimal evaluation function f_0. Thus there will be an ordinal κ such that the following equation obtains:[17]

$$f_{\mathrm{Tr}} = \Gamma_{\mathrm{Tr}}^{\kappa}(f_0) = \Gamma_{\mathrm{Tr}}^{\kappa+1}(f_0)$$

We shall now show by transfinite induction on α the following implication for all assignments a:

$$\text{If } (w, \Gamma_N^{\alpha}(f_0)(w)) \models^{\mathrm{SK}} \psi[a], \text{ then } (w, \Gamma_{\mathrm{Tr}}^{\alpha}(f_0)) \models_\Box^{\mathrm{SK}} \psi^\circ[a] \tag{5}$$

[17] The minimal closure ordinals for both operations Γ_N and Γ_{Tr} may differ. If R is empty, that is if no world can see any world, for instance, Γ_N will reach a fixed point already after its first application.

The proof involves a side induction on the complexity of φ and its negation $\neg\varphi$ emulating the inductive definition of $\models^{\mathrm{SK}}$.

In the case $\alpha = 0$ the extensions of N and Tr are empty, as $\Gamma_N^0(f_0) = \Gamma_{\mathrm{Tr}}^0(f_0) = f_0$. Thus if $\varphi = Nt$ for some term t, we shall have $(w, \Gamma_N^\alpha(f_0)(w)) \not\models^{\mathrm{SK}} Nt[a]$ and $(w, \Gamma_N^\alpha(f_0)(w)) \not\models^{\mathrm{SK}} \neg Nt[a]$. For atomic φ in $\mathcal{L}$ and complex φ the claim in the case $\alpha = 0$ is also easily established.

The case $\alpha = \beta + 1$ is the most interesting. Assume $(w, \Gamma_N^{\beta+1}(f_0)(w)) \models^{\mathrm{SK}} Nt[a]$. Then the value of the term t under the assignment a must be some sentence χ such that for all v with wRv the following obtains:

$$(v, \Gamma_N^{\beta}(f_0)(v)) \models^{\mathrm{SK}} \chi$$

By induction hypothesis this implies for all v accessible from w:

$$(v, \Gamma_{\mathrm{Tr}}^{\beta}(f_0)) \models_{\Box}^{\mathrm{SK}} \chi^\circ$$

Using the definition (3) of Γ_{Tr} we conclude:

$$(v, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_{\Box}^{\mathrm{SK}} \mathrm{Tr}\ulcorner\chi^\circ\urcorner$$

Since the value of the term t under a is the sentence χ and $^\bullet$ expresses $^\circ$ in $\mathcal{L}$, we arrive at the following:

$$(v, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_{\Box}^{\mathrm{SK}} \mathrm{Tr}\, t^\bullet[a]$$

This holds for all worlds v such that wRv. Clause (x) of the definition of $\models_{\Box}^{\mathrm{SK}}$ of section 6.1 allows us to return to the world w:

$$(w, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_{\Box}^{\mathrm{SK}} \Box\mathrm{Tr}\, t^\bullet[a]$$

By the definition of $^\circ$, $\Box\mathrm{Tr}t^\bullet$ is the formula $(Nt)^\circ$ and therefore we arrive at the desired conclusion:

$$(w, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_{\Box}^{\mathrm{SK}} (Nt)^\circ[a]$$

If φ is of the from $\neg Nt$, the claim is established in a similar manner except that one has to take into account the case that the value of t under a fails to be a sentence of $\mathcal{L}_N$. If φ is a complex formula, the claim can be proved in a straightforward way, because the inductive clauses in the definition of $\models^{\mathrm{SK}}$ and $\models_{\Box}^{\mathrm{SK}}$ are analogous.

Now assume $(w, \Gamma_N^{\beta+1}(f_0)(w)) \models^{\mathrm{SK}} \neg Nt[a]$. Then two cases may obtain: First, the value of the term t under the assignment a is not a sentence of $\mathcal{L}_N$ or it is some sentence χ.

Volker: changes from here in case t isn't a sentence

First case. If the value of the term t under the assignment a is not a sentence of $\mathcal{L}_N$, then the value of $t^\bullet$ is 0 and thus not a sentence of $\mathcal{L}_\Box^{\mathrm{Tr}}$. Thus we have also

$$(v, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_\Box^{\mathrm{SK}} \neg \mathrm{Tr}\, t^\bullet[a]$$

Second case. The value of the term t under the assignment a is some sentence χ. Then there is some v with wRv for the following obtains:

Volker: from here I have made no changes

$$(v, \Gamma_N^{\beta}(f_0)(v)) \models^{\mathrm{SK}} \neg\chi$$

By induction hypothesis (and clause 3 of Lemma 1) this implies for some v accessible from w:

$$(v, \Gamma_{\mathrm{Tr}}^{\beta}(f_0)) \models_\Box^{\mathrm{SK}} \neg\chi^\circ$$

Using now in addition to the definition (3) of Γ_{Tr}, (iv) of the definition of satisfaction $\models_\Box^{\mathrm{SK}}$ we conclude:

$$(v, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_\Box^{\mathrm{SK}} \neg\mathrm{Tr}\ulcorner\chi^\circ\urcorner$$

Just as before, since the value of the term t under a is the sentence χ and $^\bullet$ expresses $^\circ$ in $\mathcal{L}$, we arrive at the following:

$$(v, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_\Box^{\mathrm{SK}} \neg \mathrm{Tr}\, t^\bullet[a]$$

Now clause (xi) of the definition of $\models_\Box^{\mathrm{SK}}$, (and again using the definition of $^\circ$ which implies that $\neg\Box\mathrm{Tr}t^\bullet$ is the formula $(\neg Nt)^\circ$) yields the conclusion at world w:

$$(w, \Gamma_{\mathrm{Tr}}^{\beta+1}(f_0)) \models_\Box^{\mathrm{SK}} (\neg Nt)^\circ[a]$$

If φ is a complex formula, the claim can be proved in a straightforward way, because the inductive clauses in the definition of $\models^{\mathrm{SK}}$ and $\models_\Box^{\mathrm{SK}}$ are analogous.

At limit stages, that is, if μ is a limit ordinal, $\Gamma_N^{\mu}(f_0)$ is the union $\bigcup_{\gamma<\mu}\Gamma_N^{\gamma}(f_0)$. In this case the claim is also easily established by an induction on the complexity of the formula and its negation.

8 Extensions of the main result

In this section we sketch some extensions and variations of our main result Theorem 2.

8.1 de re-modality

So far we have dealt only with de dicto-modalities. We showed how to reduce de dicto-necessity conceived as a predicate to de dicto-necessity conceived as an operator and a truth predicate. The restriction to de dicto-modality has facilitated the presentation of the technicalities and the restriction to de dicto-necessity is sensible in the case of conceptual necessity or analyticity, but in order to deal with notions like metaphysical necessity we have to apply our reductive strategy also to de re-modalities. In the languages with a modal operator we shall have to allow quantifiers to bind variables that are in the scope of the necessity operator $\Box$.

Canonical techniques for providing semantics for quantified modal logic, that is, for a necessity operator are well known. There are, however, deeper disagreements on how to model de re-necessity, if necessity is treated as a predicate.

On one account necessity is still treated as a unary predicate of propositions. In order to express de re-necessity, it is assumed that properties can be described in the language that yield propositions when applied to objects. The value of the property applied to certain objects is of course independent from how these objects are described. Thus properties are conceived as functions sending objects to propositions.

Alternatively one can adapt Tarski's proposal for truth and conceive necessity as a binary predicate applying to formulae (or properties) and variable assignments (see, for instance, Belnap and Gupta 1993, p. 237). We think that there are decisive arguments in favour of this approach, as the former encounters certain problems (as those in Bealer 1993 and Williamson 2002), which can be overcome by the latter (see Halbach and Sturm 2004). The details of a formal treatment of de re-necessity in such framework are somewhat intricate and will not be covered in the present paper.

Without specifying the details of such an account, we claim that such a binary necessity predicate of necessity can still be reduced to an operator of necessity (that can be applied to formulae with free variables) and a binary satisfaction predicate. Basically the same techniques can be applied as in our exposition, although the technical details become more involved.

8.2 Adding a truth predicate to $\mathcal{L}_N$

On the account presented so far, only the language $\mathcal{L}_\Box^{\mathrm{Tr}}$ contains a truth predicate, while we did not assume that the language $\mathcal{L}_N$ with the necessity predicate contains a truth predicate. The advocate of the predicate approach to necessity, however, might want to avail himself to a truth predicate along with the necessity predicate. Therefore the question arises whether our reduction of the necessity predicate is still feasible if a truth predicate is added to $\mathcal{L}_N$. This new truth predicate differs from the truth predicate of $\mathcal{L}_\Box^{\mathrm{Tr}}$ (the language with the operator $\Box$) in that it also applies to sentences containing the necessity predicate; but this new truth predicate will not apply to sentences containing the operator $\Box$.

In order to obtain semantics for $\mathcal{L}_N$ expanded in this way, we have to define extensions for the necessity predicate and the truth predicate simultaneously by transfinite induction using a combination of the techniques of sections 5.2 and 6.2.

Although details become somewhat intricate, one still can define a translation function and prove a version analogous to our main result Theorem 2. The translation function cannot leave the truth predicate unmodified, because the truth predicate of $\mathcal{L}_N$ applies to sentences of $\mathcal{L}_N$ expanded by the truth predicate, while the truth predicate of $\mathcal{L}_\Box^{\mathrm{Tr}}$ applies to sentences containing the operator $\Box$ of necessity. This difficulty can be overcome by translating also sentences to which the truth predicate is applied.

At any rate our approach can be adapted to situations where the source language with the necessity predicate already contains a truth predicate. Thus the reduction of a necessity predicate to an operator of necessity is not achieved by depriving the source language of a truth predicate.

8.3 Inverting the translation

In order to show that being necessary is equivalent to being necessarily true, one would have to provide also a translation from $\mathcal{L}_\Box^{\mathrm{Tr}}$ into $\mathcal{L}_N$, at least for sentences where the truth predicate *only* occurs directly succeeding the operator $\Box$.

In general, a reduction of $\mathcal{L}_\Box^{\mathrm{Tr}}$ to $\mathcal{L}_N$, however, is not possible, at least not without imposing further conditions imposed on the worlds and their accessibility relation. The necessity operator can easily be reduced to the necessity predicate (see Schweizer 1992): a sentence of the form 'necessarily A' is simply replaced by

> 'A' is necessary.

In many cases, however, it is not possible to define the truth predicate in terms of a necessity predicate. For instance, if there is only one world and the accessibility relation is empty, then the truth predicate is still non-trivial. In contrast, the necessity predicate is trivial, because it will apply to every sentence of $\mathcal{L}_N$ due to the fact that all sentences of $\mathcal{L}_N$ are true at all accessible worlds.

If there is a single world that can see itself, a reduction of $\mathcal{L}_\Box^{\mathrm{Tr}}$ to $\mathcal{L}_N$ is feasible, as the extensions of the truth predicate of $\mathcal{L}_\Box^{\mathrm{Tr}}$ and the necessity predicate of $\mathcal{L}_N$

will coincide: necessity is truth at all accessible worlds and truth is truth at the actual world. Since in this case the actual and the accessible worlds coincide, truth and necessity coincide.

Apart from such pathological cases truth will not be definable in terms of a necessity predicate; and it would be strange if (actual) truth were definable in terms of metaphysical necessity, for instance.

The intricacies of inverting the translation may provide a justification for treating truth as a predicate, while necessity and possibly other intensional notions are conceived as operators only: if one is prepared to accept a predicate of necessity or truth, but if one wants at the same time as few predicates as possible, one should probably opt for a truth predicate and not for a predicate of necessity. We admit, however, that our account offers only some indications for a justification of this asymmetric treatment of truth and necessity. More arguments would be needed for a full justification. In particular, we have not yet said much about other intensional notions than necessity and about other aspects of the reduction than the purely technical feasibility we have focused on.

8.4 Alternate evaluation schemes

On the semantics offered in section 5 a sentence like 'All propositional tautologies are necessary.' lacks a truth value at any world: if λ is the liar sentence formulated with the necessity rather than the truth predicate, then under the Strong Kleene scheme the disjunction $\lambda \vee \neg\lambda$ will lack a truth value, because λ lacks a truth value.

We chose the Strong Kleene evaluation scheme because it yields the correct truth values for at least the grounded sentences whose semantical status are hardly questionable. It is less obvious that $\lambda \vee \neg\lambda$ should be necessary when λ and its negation are neither true nor false. There has been an extensive discussion of the merits of the various evaluation schemes. The Strong Kleene scheme and supervaluations emerged from it as the most suitable candidates (see McGee 1991 and Belnap and Gupta 1993). In the case of necessity supervaluations seem even more attractive than in the case of truth, because we do not expect compositional semantics from necessity; and the lack of compositional features renders supervaluations less attractive in the case of truth. Here we do not delve into this discussion, but rather attend to the effects of replacing the Strong Kleene scheme with supervaluations in our formal development.

In section 5 only the relation $\models^{\mathrm{SK}}$ has to be redefined in a suitable way. If then the relation $\models^{\mathrm{SK}}_{\Box}$ of section 6 is also reformulated to accord to the supervaluational approach, the main result Theorem 2 still holds, although the proof become slightly more complicated because of the more involved definitions of the relations corresponding to $\models^{\mathrm{SK}}$ and $\models^{\mathrm{SK}}_{\Box}$ in supervaluations.

But it is far from clear whether one actually should redefine $\models^{\mathrm{SK}}_{\Box}$ in terms of supervaluations, because this will base the properties of the truth predicate on the supervaluational approach; and we remarked above that supervaluations are less attractive for truth than for necessity. If therefore the definition of $\models^{\mathrm{SK}}_{\Box}$ is kept, so that the Strong Kleene scheme is retained for truth while supervaluations is applied to the necessity predicate, the main result Theorem 2 fails: $N\ulcorner\lambda \vee \neg\lambda\urcorner$ will be true at all worlds, whereas $\Box\mathrm{Tr}\ulcorner\lambda \vee \neg\lambda\urcorner$ will not generally be valid.

We conclude that we are not at liberty to choose evaluation scheme for necessity and truth independently, if we want to adhere to the reduction of necessity to necessary truth.

8.5 Application to intensional notions other than necessity

So far we have been talking about necessity without being very specific about the kind of necessity. This is on purpose: our approach is supposed to apply to various kinds of necessity such as metaphysical necessity, physical necessity and analyticity. As we have indicated already, our account is not confined to alethic modalities: we think that our account can shed some light also on other notions.

For instance, our methods can be applied to the operators of temporal logic. Horsten and Leitgeb (2001) have shown that natural assumptions on predicates for future and past truth lead to inconsistency. Replacing them by operators is natural and reduces the inconsistency to the truth-theoretic paradoxes. In the case of temporal logic rephrasing the predicates corresponding to the operators for future and past by the operators and a truth predicate is also very natural: in English we usually use a truth predicate in order to express quantification in such contexts: 'x will be true' is preferable to 'is future'.

Other notions, however, are less amenable to our approach. We mention here only some problems without going into details or suggesting solutions. Propositional attitudes, for instance, pose some special problems. Our strategy would involve a substitution of

I believe x

by

I believe that (the proposition expressed by) x is true

or something similar; and one might argue that it is possible to believe something without believing it to be true. As pointed out earlier, reversing the order of the necessity operator and the truth predicate in the translation may resolve the problem (see the remark on Kripke 1975, p. 713, fn. 33 above).

However, we should aim at a uniform approach to intensional notions, mainly because the relations between modalities are most easily investigated in a framework where the various modal notions are treated syntactically in an analogous way.

Our strategy is even less applicable to deontic modalities, We think that this is not a major drawback, as deontic modalities have a special status in various respects. Taken as predicates, they seem to apply to completely different objects: propositions are hardly obligatory. The objects that are obligatory or permissible (if there are any) are not candidates for being believed or being necessary. Therefore there is no need for a framework that renders alethic and deontic modalities directly comparable.

9 Conclusion

Our reduction of necessity to necessary truth is only *one* aspect of a more thorough defense of the operator approach to necessity and other intensional notions; and by no means do we take our reduction to be a conclusive defense of the operator view.

In this paper we have only tried to show how to overcome a certain technical obstacle for the reduction of a predicate to an operator and a truth predicate can be overcome. There may be further technical and philosophical considerations that could pose problems for the reduction considered here. But we hope we have provided techniques that can be used in further discussions of the reducibility of predicates to operators of necessity and other notions.

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