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NON-ADDITIVE BELIEFS IN SOLVABLE GAMES

ABSTRACT. This paper studies how the introduction of non-additive probabilities (capacities) affects the solvability of strategic games.

KEY WORDS: Solvability, Non-additive probabilities, Strategic games, Nash equlibrium

1. INTRODUCTION

Solvability eliminates the problem of coordinating on a specific Nash equilibrium of a strategic game and, therefore, constitutes a very desirable, albeit rare property. Hence it is of interest to study the effect of a new solution concept on solvability. The purpose of this note is to investigate how the introduction of non-additive probabilities affects the solvability of strategic games. In many cases, it turns out, solvability of games is destroyed, if one allows for non-additive probabilities. Although strict dominance is preserved, dominance solvability is not. A noteworthy exception are two-by-two constant-sum games.

The introduction of non-additive probabilities into game theory is motivated by three facts: First and foremost, recent advances in decision theory and the desire to assess the relevance of this progress for game theory. Second, the promise of novel or original explanations of certain phenomena. Third, a possible uneasiness with the traditional concept of a Nash equilibrium in mixed strategies.

The development of decision models with non-additive probabilities originated from casual empiricism as well as laboratory experiments suggesting that revealed preferences frequently fail to fit into the expected utility framework. The assumption of expected utility maximization is the combination of two assumptions:

1. Uncertainty can be described as **risk**, i.e. by means of lotteries (probability distributions, additive probability measures) over uncertain outcomes.

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2. Agents maximize a **linear** functional on lotteries.

Keeping 1 and giving up 2 allows for **non-linear functionals** on lotteries. See Machina (1982), references therein, and followers thereof. See also Quiggin's (1982) anticipated utility model and its derivatives, known as expected utility models with rank dependent probabilities. Abolishing 1, while maintaining a modified version of 2, leads to **non-additive probabilities** à la Gilboa and Schmeidler (Gilboa, 1987; Schmeidler, 1989).

Novel decision theories may have merit *per se*, but for economists and game theorists, the litmus test lies in economic and game theoretical applications of these models. In that judgement, the author agrees with and follows up on Dow and Werlang (1994) and Eichberger and Kelsey (2000) whose work is similarly motivated.[1] In their papers as in the present note, uncertainty is modelled by means of **capacities**, i.e. non-additive probability measures in the sense of Gilboa and Schmeidler. Both Dow and Werlang (1994) and Eichberger and Kelsey (2000) investigate "Nash equilibria under uncertainty", meaning Nash equilibria in capacities rather than Nash equilibria in pure or mixed strategies. Some of the contributions of Eichberger and Kelsey (2000) will be discussed in Section 3. Dow and Werlang (1994) show that the concept of Nash equilibrium in capacities allows for a richer set of equilibrium predictions than Nash equilibrium in mixed strategies. The more general equilibrium concept can only add to the equilibrium set; it never eliminates any of the traditional Nash equilibria. Thus the explanation of additional phenomena is obtained at the expense of less specific predictions and, therefore, is a mixed blessing. For good reason, the game theoretical literature on equilibrium refinements and equilibrium selection has expended considerable effort to progress in the opposite direction: to reduce the set of eligible equilibrium outcomes.

After identification of a potential Pandora's Box, getting more knowledgeable about the contents of that box becomes a high priority task. One *desideratum modestum* is that adoption of the concept of Nash equilibrium in capacities does not alter — or at least not affect in an essential way — the equilibrium set of games traditionally deemed "solvable", because there is no or little ambiguity concerning conventional equilibrium outcomes or payoffs of these games. Example 1 in Dow and Werlang (1994) is evidence to the contrary. That example has a unique Nash equilibrium in the conventional sense which happens to be in pure strategies and can be determined by iterative elimination of strictly dominated strategies. Thus the example represents one of the rare cases of strict dominance solvability. Nonetheless, there is an additional — significantly different and intuitively appealing — Nash equilibrium in capacities. It turns out that the capacities used in the new equilibrium are “simple capacities”.[2] Hence the more modest requirement: that Nash equilibria in simple capacities should not add any outcomes to the conventional equilibrium set: is not fulfilled either. It turns out that even in finite two-person zero-sum games, Nash equilibria in mixed strategies and Nash equilibria in simple capacities can yield significantly different equilibrium predictions. Solvability in the classical sense is lost. It remains an open question if solvability is preserved for all finite two-person zero-sum games with a unique pair of prudent strategies which happens to be a saddle-point, hence coincides with the unique equilibrium point in mixed strategies. For special subclasses of these games, the two-by-two games, an affirmative answer will be given. We also establish that in order to break up a finitely repeated Prisoners’ Dilemma, one has to resort to non-simple capacities. It is further shown that strict dominance is preserved. This result allows for non-simple capacities.

The relevant concepts are introduced in the next section. The results are presented and discussed in Section 3. Some related issues are mentioned in the final section.

2. CONCEPTS

2.1. *Non-Additive Probabilities (Capacities)*

Let N be a finite set. The **characteristic function of a TU-game with player set** N is a mapping $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$. A characteristic function v is called

0-**normalized**, if $v(\{n\}) = 0$ for all $n \in N$.

1-**normalized**, if $v(N) = 1$.

monotone, if $v(X) \leqslant v(Y)$ for $X \subseteq Y \subseteq N$.

convex or supermodular, if $v(X) + v(Y) \leqslant v(X \cup Y) + v(X \cap Y)$ for any $X, Y \subseteq N$.

concave or submodular, if $v(X) + v(Y) \geqslant v(X \cup Y) + v(X \cap Y)$ for any $X, Y \subseteq N$.

superadditive, if $v(X) + v(Y) \leqslant v(X \cup Y)$ for any disjoint $X, Y \subseteq N$.

subadditive, if $v(X) + v(Y) \geqslant v(X \cup Y)$ for any disjoint $X, Y \subseteq N$.

additive or **inessential**, if $v(X) = \sum_{n \in X} v(\{n\})$ for all $X \subseteq N$.

A **non-additive probabilitity measure**, **capacity** or **fuzzy measure** on N is a monotone and 1-normalized characteristic function v. In decision theory and game theory, convexity of capacities is often assumed and interpreted as uncertainty aversion. All other terms and properties appear in cooperative game theory as well as the mathematical literature on capacities. See Denneberg (1994a,b) for the latter. Additive capacities are the traditional additive probability measures.[3] Besides common formal definitions, I — like the literature at large — had been unaware of too many trades of ideas between cooperative game theory and the theory of capacities. One exception is Rosenmüller (1982) who refers to Rosenmüller and Weidner (1973, 1974). Another exception is Hendon et al. (1994) where explicit use is made of results from cooperative game theory. Intriguingly enough, I learnt from Peter Wakker that he [Wakker (1987)] has devised and utilized algorithms to translate results from the theory of decisions under uncertainty into theorems in the theory of cooperative games and vice versa.

2.2. *Leading Examples*

Here we present the leading examples of capacities. These include the two extreme cases, additive capacities on the one hand and complete ignorance on the other hand. Simple capacities are convex combinations of the extreme cases.

1. Any *additive probability measure* π on N is a capacity. It satisfies the convexity condition with equality. We reserve the

symbols π,σ, and τ for additive probability measures alias additive capacities alias mixed strategies.

2. *"Complete ignorance"* can be expressed by means of the capacity ω on N attaining values $\omega(N) = 1$ and $\omega(X) = 0$ for $X \subseteq N$, $X \neq N$. Throughout this paper, the symbol ω is reserved for capacities describing complete ignorance.
3. A **simple capacity** $v_{c,\pi}$ on N is a convex combination of an additive capacity π (given weight c) and the complete ignorance capacity ω (weighted with $1 - c$). I.e. $c \in [0, 1]$ and $v_{c,\pi} = c \cdot \pi + (1 - c) \cdot \omega$ or
$v_{c,\pi}(X) = c \cdot \pi(X) + (1 - c) \cdot \omega(X)$ for all $X \subseteq N$.
The weight c can be readily interpreted as **degree of confidence** and the weight $1 - c$ can be interpreted as **degree of uncertainty aversion**. In the sequel, the notation $v[c, \pi]$ will often prove convenient.

2.3. *Nash Equilibria in Capacities*

The definition of a Nash equilibrium in capacities presupposes that two more concepts are well defined:

- The support of a capacity v, denoted *supp* v.
- The integral or expected value of a function
$f : N \longrightarrow \mathbb{R}$ with respect to a capacity v,
denoted $\int f dv$ or $\int f(n)v(dn)$.

Let us assume for the moment that both supports and integrals with respect to capacities are well defined. Let us consider a finite 2-player game in normal form

$$\Gamma = (I; N^1, N^2; u^1, u^2)$$

with player set $I = \{1, 2\}$, pure strategy sets N^1 and N^2 and payoff functions u^1 and u^2. Only the case $|N^1| \geqslant 2$, $|N^2| \geqslant 2$ is of interest. For $i \in I$, let $-i$ denote i's opponent, the only other player in I. Further let for $i \in I$,

- $\mathcal{C}^i$ denote the set of capacities on N^i;
- $\mathcal{SC}^i$ denote the set of simple capacities on N^i;
- $\mathcal{M}^i$ denote the set of mixed strategies on N^i.

For $i \in I$, $v^{-i} \in \mathcal{C}^{-i}$, and $a^i \in N^i$, put

$$U^i(a^i, v^{-i}) \equiv \int u^i(a^i, \cdot) dv^{-i},$$

the expected payoff for i from playing a^i, if the other player $-i$ is believed to behave in an uncertain way described by v^{-i}. This view extends the Harsanyi–Aumann interpretation of mixed strategies as beliefs rather than actual randomizations to the case of capacities. Aumann (1987) considers "randomness as an expression of ignorance". He refers to Harsanyi (1973) as "the first to break away from the idea of explicit randomization".

DEFINITION. A **Nash Equilibrium under Knightian Uncertainty** or **Nash Equilibrium in Capacities** is a pair $(v^1, v^2) \in \mathcal{C}^1 \times \mathcal{C}^2$ for which there exist respective supports *supp* v^1 and *supp* v^2 such that for $i \in I$:

$$s^i \in \mathit{supp}\, v^i \Longrightarrow s^i \in \operatorname*{argmax}_{a^i \in N^i} U^i(a^i, v^{-i}).$$

Notice that the v^i merely reflect mutually consistent beliefs in accordance with the Harsanyi–Aumann interpretation of an equilibrium in mixed strategies. It is not required that v^i be a best response against v^{-i} in any sense. The only requirement is that the support of v^i only contain best responses (in pure strategies) against v^{-i}. An immediate consequence of this crucial detail is that one can replace $\mathcal{C}^1 \times \mathcal{C}^2$ in the above definition

— by $\mathcal{S}\mathcal{C}^1 \times \mathcal{S}\mathcal{C}^2$ so as to define a **Nash Equilibrium in Simple Capacities** which is a special case of a Nash equilibrium in capacities;

— by $\mathcal{M}^1 \times \mathcal{M}^2$ to define a **Nash Equilibrium in Mixed Strategies** which is also a Nash equilibrium in simple capacities and a special case of a Nash equilibrium in capacities;

— by $N^1 \times N^2$ so as to define a **Nash Equilibrium in Pure Strategies** which corresponds to a degenerate case of a Nash equilibrium in mixed strategies.

It remains to specify supports and integrals with regard to capacities. To this end, let v be a capacity on the set N. Let $\subset$ denote proper set inclusion and ^{c} denote complements in N. There are at least three plausible suggestions to generalize the uncontroversial concept of a support for additive probability measures to a notion of **support** for arbitrary capacities.

1st DEF.: $X \subseteq N$ is a support of v, if $v(X) = 1$ and $Y \subset X$ implies $v(Y) < 1$.

2nd DEF.: $X \subseteq N$ is a support of v, if $v(X^c) = 0$ and $Y \subset X$ implies $v(Y^c) > 0$.

3rd DEF.: $X \subseteq N$ is the support of v, if $X = \{n \in N | v(\{n\}) > 0\}$.

For pragmatic reasons, Dow and Werlang (1994) have opted for the second definition. With the first definition, the support might be "too large"; in particular, their Example 3 of breaking down backward induction seems to break down. For the sake of comparison and unified treatment, Eichberger and Kelsey (2000) and I side with them and *adopt the 2nd definition of a support*.

As for a suitable definition of $\int f dv$, the **Choquet integral** is ubiquitous and appears to be universally accepted. I follow this common practice. To this end, let N be a finite set, v be a capacity on N, and $f : N \longrightarrow \mathbb{R}$ attain values $x_1 > x_2 \ldots > x_K$. Set $T_0 = \emptyset$. For $k = 1, \ldots, K$, set

$Q_k = \{n \in N | f(n) = x_k\}$ and

$T_k = \bigcup_{\ell=1}^{k} Q_\ell = \{n \in N | f(n) \geqslant x_k\}$. Then the Choquet integral of f with respect to v is defined as

$$\int f dv = \sum_{k=1}^{K} x_k \cdot [v(T_k) - v(T_{k-1})]$$
$$= v(T_K) \cdot x_K + \sum_{k=1}^{K-1} v(T_k) \cdot (x_k - x_{k+1}).$$

Notice that for an additive v, $v(T_k) - v(T_{k-1}) = v(Q_k)$ so that one obtains ordinary integrals in the sense of Riemann or Lebesgue. For axiomatizations of Choquet Expected Utility (CEU) see Gilboa (1987), Schmeidler (1989), and Sarin and Wakker (1992). Hougaard and Keiding (1996) provide an axiomatization of preferences represented by means of the Sugeno integral and conclude that the latter is of limited use for decision theory and, consequently, for game theory.

2.4. *Weak Convergence of Capacities*

As is well known, the topology of weak convergence on the space of Borel probability measures on a compact metric space renders the space of Borel probability measures itself compact and metrizable (Parthasarathy, 1967, Theorem 6.4). A suitable metric is the so-called Prohorov metric (Billingsley, 1968, pp. 236–238). Here we endow the space of capacities on a finite set with a natural distance that induces the topology of weak convergence of capacities. The distance allows to measure deviation from additivity. For simple capacities, the deviation from additivity is proportional to the degree of uncertainty aversion [Fact 2]. Therefore, the new concept may prove useful in the future, although the application it was meant for is no longer in the paper.

For a finite set N with $|N| > 1$, we can extend the notion of weak convergence from $\mathcal{M}$, the set of additive probability measures on N to $\mathcal{C}$, the set of capacities on N. Furthermore, let $\mathcal{SC}$ denote the set of simple capacities on N and $\mathcal{F}$ the set of functions $f : N \longrightarrow \mathbb{R}$. Next consider a sequence v_t, $t \in \mathbb{N}$, in $\mathcal{C}$ and a point $v \in \mathcal{C}$.

DEFINITION . The sequence v_t, $t \in \mathbb{N}$, is called **weakly convergent to** v, if $\int f dv_t \longrightarrow \int f dv$ for all $f \in \mathcal{F}$.

Since capacities are normalized, we can omit the values $v(\emptyset) = 0$ and $v(N) = 1$ and identify a capacity with a vector in $\mathbb{R}^{\mathcal{N}}$ where $\mathcal{N} = 2^N \setminus \{\emptyset, N\}$. Then $\mathcal{C}$ is a convex and compact subset of $\mathbb{R}^{\mathcal{N}}_+$. With the canonical embedding, $\mathcal{M}$ is a closed convex subset of $\mathcal{C}$. Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^{\mathcal{N}}$. This norm induces the topology of weak convergence on $\mathcal{C}$ and, a fortiori, on $\mathcal{M}$. Namely:

FACT 1. *For a sequence v_t, $t \in \mathbb{N}$, in $\mathcal{C}$ and $v \in \mathcal{C}$, v_t is weakly convergent to v if and only if v_t is norm convergent to v.*

That norm convergence implies weak convergence follows immediately from the definition of the Choquet integral and the fact that norm convergence of v_t to v implies $v_t(X) \longrightarrow v(X)$ for all $X \subseteq N$. Conversely, let v_t be weakly convergent to v. To establish norm convergence, it suffices to show componentwise convergence, i.e. $v_t(X) \longrightarrow v(X)$ for all $X \in \mathcal{N}$. So let $X \in \mathcal{N}$. Define $f \in \mathcal{F}$ by $f(n) = 1$ for $n \in X$ and $f(n) = 0$ for $n \notin X$. Then in the definition of the Choquet integral, $K = 2, x_1 = 1, x_2 = 0$. Consequently, $\int f dv_t = 1 \cdot [v_t(X) - v_t(\emptyset)] + 0 \cdot [v_t(N) - v_t(X)] = v_t(X)$ and similarly, $\int f dv = v(X)$. Weak convergence of v_t to v implies $\int f dv_t \longrightarrow \int f dv$. Hence $v_t(X) \longrightarrow v(X)$.

For arbitrary $v \in \mathcal{C}$, **deviation from additivity** can be measured by $\text{dist}(v, \mathcal{M})$. Obviously, $\text{dist}(\sigma, \mathcal{M}) = 0$ for $\sigma \in \mathcal{M}$. Since $\mathcal{M}$ is compact and convex, there exists for each $v \in \mathcal{C}$ a unique $\sigma \in \mathcal{M}$ with $\text{dist}(v, \sigma) = \text{dist}(v, \mathcal{M})$. Specifically, for a simple capacity $v_{c,\pi} = c \cdot \pi + (1-c) \cdot \omega$, $\text{dist}(v_{c,\pi}, \pi) = \|\pi - v_{c,\pi}\| = \|\pi - c \cdot \pi\| = (1-c) \cdot \|\pi\|$. As a rule, however, π is not the point in $\mathcal{M}$ that is closest to $c \cdot \pi$ or $v_{c,\pi}$. Rather, with $m = \min_{\sigma \in \mathcal{M}} \|\sigma\|$:

FACT 2. $\text{dist}(v_{c,\pi}, \mathcal{M}) = (1 - c) \cdot m$ *for* $v_{c,\pi} \in \mathcal{SC}$.

That is the distance of a simple capacity from the set of additive probability measures is proportional to the degree of uncertainty aversion. To see this, write $\sigma = (\sigma_k)_{k \in N}$, when $\sigma \in \mathcal{M}$ is treated as a $|N|$-dimensional probability vector. Then consider the problem $\min_{\sigma \in \mathcal{M}} \|\sigma - c \cdot \pi\|^2$ which amounts to

$$\min_{\sigma \in \mathbb{R}_+^N} \sum_{X \in \mathcal{N}} \left(\sum_{k \in X} (\sigma_k - c \cdot \pi_k) \right)^2 \quad \text{subject to} \quad \sum_{k \in N} \sigma_k = 1.$$

The first order conditions are

$$\sum_{X \ni k} \sum_{x \in X} (\sigma_x - c \cdot \pi_x) = \lambda \text{ for all } k.$$

Therefore, the unique solution is

$$\sigma = (1 - c) \cdot \tau + c \cdot \pi$$

where $\tau = (1/|N|, 1/|N|, \ldots, 1/|N|)$ is the probability vector that solves the problem $\min_{\sigma \in \mathcal{M}} \|\sigma\|^2$. Consequently,

$$\begin{aligned} \text{dist}(v_{c,\pi}, \mathcal{M}) &= \text{dist}(c \cdot \pi, \mathcal{M}) = \|\sigma - c \cdot \pi\| \\ &= (1 - c) \cdot \|\tau\| = (1 - c) \cdot m. \end{aligned}$$

So $\text{dist}(v_{c,\pi}, \mathcal{M}) = (1-c) \cdot m$ as asserted. Finally, as asserted $\text{dist}(c \cdot \pi, \mathcal{M}) < (1 - c) \cdot \|\pi\|$, except when $\pi = \tau$.

The question remains if one can say anything beyond Fact 2. For an arbitrary capacity ν, one can define the **degree of ambiguity**,

$$\alpha(\nu) = \max_{X \in \mathcal{N}} \quad [1 - (\nu(X) + \nu(N \setminus X))].$$

For simple capacities, this is simply the degree of uncertainty aversion. Let us consider the norm $\langle \cdot \rangle$ on $\mathbb{R}^{\mathcal{N}}$ given by

$$\langle v \rangle = \max_{X \in \mathcal{N}} |v(X)| + |v(N \setminus X)|$$

for $v \in \mathbb{R}^{\mathcal{N}}$ and denote the corresponding distance by DIST. Then for any $\sigma \in \mathcal{M}$, $\nu \in \mathcal{C}$ and $X \in \mathcal{N}$,

$$\begin{aligned} 1 - (\nu(X) + \nu(N \setminus X)) &= \sigma(X) + \sigma(N \setminus X) \\ &\quad - (\nu(X) + \nu(N \setminus X)) \\ &\leqslant |(\sigma - \nu)(X)| + |(\sigma - \nu)(N \setminus X)|, \end{aligned}$$

hence $\alpha(\nu) \leqslant \mathrm{DIST}\,(\nu, \mathcal{M})$. Since all norms on $\mathbb{R}^{\mathcal{N}}$ are equivalent, there exists a $\rho > 0$ with $\alpha(\nu) \leqslant \rho \cdot \mathrm{dist}\,(\nu, \mathcal{M})$ for all $\nu \in \mathcal{C}$.

3. RESULTS

Since Nash equilibria in mixed strategies are particular Nash equilibria under uncertainty, existence is not an issue. The problem is rather a mega-multiplicity of Nash equilibria in capacities. This holds even true, when attention is restricted to Nash equilibria in simple capacities!

3.1. *Restriction to Simple Capacities*

For the remainder of this note, except in subsection 3.5, only simple capacities will be considered. This restriction strengthens our negative findings in subsection 3.3 and weakens our positive results in subsection 3.4 whereas our findings in subsections 3.5 and 3.6 are not affected by it. We continue to work with a finite 2-person game in normal form Γ. As observed before, a Nash equilibrium in simple capacities is in fact a particular Nash equilibrium in capacities. One of the virtues of a simple capacity $v^i[c^i, \pi^i] \in \mathcal{SC}^i$ is that it has a unique support, denoted *supp* $v^i[c^i, \pi^i]$, except possibly for $c^i = 0$.

More specifically, for definitions 2 and 3, $supp\, v^i[c^i, \pi^i] = supp\, \pi^i$, if $c^i > 0$. If $c^i = 0$, each of the singletons $\{a^i\}$, $a^i \in N^i$, is a support of $v^i[c^i, \pi^i] = \omega^i$ according to our (second) definition.[4] Another potential advantage of simple capacities is that one can easily define the product of simple capacities and a Nash equilibrium in simple capacities for games with more than two players. An additional virtue of a simple capacity $v^i[c^i, \pi^i]$ is convexity.[5] Finally, integrals w.r.t. a simple capacity assume a special form, namely

$$U^i(a^i, v^{-i}[c^{-i}, \pi^{-i}]) = c^{-i}\int u^i(a^i, \cdot)d\pi^{-i} + (1-c^{-i}) \cdot \min_{s^{-i}\in N^{-i}} u^i(a^i, s^{-i}). \quad (1)$$

Inspection of the proof of the Theorem in Dow and Werlang (1994) reveals that in general, there exists a continuum of Nash equilibria in simple capacities for a game Γ.

PROPOSITION 1. *For any* $(c^1, c^2) \in [0, 1] \times [0, 1]$, *there exists* $(\pi^1, \pi^2) \in \mathcal{M}^1 \times \mathcal{M}^2$ *such that* $(v^1[c^1, \pi^1], v^2[c^2, \pi^2])$ *constitutes a Nash equilibrium in simple capacities.*

Given the mega-multiplicity of equilibria suggested by Proposition 1, the question is whether the differences are just “nominal” or rather “real”. After all, the capacity $v^i[c^i, \pi^i]$ constituting part of an equilibrium is merely a belief held by player $-i$ about strategies played by player i. As for strategies actually played, this could be any pair $(a^1, a^2) \in supp\, v^1[c^1, \pi^1] \times supp\, v^2[c^2, \pi^2]$. Therefore, as far as equilibrium predictions are concerned, $supp\, v^1[c^1, \pi^1]$ and $supp\, v^2[c^2, \pi^2]$ are the relevant components.

3.2. *Solvable Games*

Traditionally, a non-generic subclass of games has been distinguished as “solvable”, because they allow concise equilibrium predictions.

DEFINITION . A finite 2-player game Γ is called **solvable**, if:

(a) If (σ^1, σ^2) and (π^1, π^2) are Nash equilibria in mixed strategies of Γ, then (σ^1, π^2) and (π^1, σ^2) are also Nash equilibria in mixed strategies of Γ.

(b) If (σ^1, σ^2) and (π^1, π^2) are two Nash equilibria in mixed strategies of Γ, then they both yield the same equilibrium expected payoff pair.

The **interchangeability** condition (a) is tantamount to the original definition of solvability in Nash (1951). The **payoff equivalence** condition (b) is an additional requirement, reinforcing the idea that in a solvable game a player may pick, without harm, any mixed strategy that is part of a Nash equilibrium. Solvability in this more stringent sense is satisfied by every game with a unique Nash equilibrium in mixed strategies. It is also satisfied by each 2-player zero-sum game. Condition (a) implies another condition which is sometimes listed as part of the definition of solvability and can be used to detect violations of solvability, namely existence of non-empty subsets S^1 of N^1 and S^2 of N^2 such that:

(c) If $(\sigma^1, \sigma^2) \in \mathcal{M}^1 \times \mathcal{M}^2$ is a Nash equilibrium in mixed strategies of Γ, then $supp\,\sigma^i \subseteq S^i$ for $i \in I$.

(d) There is a Nash equilibrium in mixed strategies of Γ, $(\sigma^1, \sigma^2) \in \mathcal{M}^1 \times \mathcal{M}^2$ with $supp\,\sigma^i = S^i$ for $i \in I$.

The concept of **dominance solvability** forwarded by Moulin (1979) requires that successive (iterated) elimination of dominated strategies leads to a non-empty subset of E of $N^1 \times N^2$. E consists of Nash equilibria of Γ and within the set E, the analogues of the solvability conditions (a) and (b) hold true.

3.3. *Loss of Solvability*

The appeal of solvable games rests on the fact that in these games, the inherent coordination problem associated with the concept of Nash equilibrium is absent. Since Dow and Werlang show that, in general, the concept of Nash equilibrium in (simple) capacities allows for a richer set of equilibrium predictions than Nash equilibrium in mixed strategies, the question arises if solvability is preserved as one passes from the set of Nash equilibria in mixed strategies to the potentially larger set of Nash equilibria in simple capacities. Viewed from a new angle, Example 1 of Dow and Werlang (1994) presents a game Γ with the following properties. First of all, Γ has a unique Nash equilibrium in mixed strategies which can be determined through iterative elimination of strictly dominated strategies and, consequently, is a Nash equilibrium in pure strategies, i.e. of the form $(s^1, s^2) \in N^1 \times N^2$. Hence Γ is solvable and dominance solvable with $S^i = \{s^i\}$ for $i = 1, 2$. Secondly, there is another Nash equilibrium in simple capacities, $(v^1[c^1, \pi^1], v^2[c^2, \pi^2])$ with *supp* $v^1[c^1, \pi^1] \cap S^1 = \emptyset$, $t^1 \in$ *supp* $v^1[c^1, \pi^1]$ not a best response against s^2, and different equilibrium payoffs. Thus solvability is lost in that the interchangeability condition (a) as well as the payoff equivalence condition (b) are violated. Let us reproduce that example with the focus on solvability.

EXAMPLE 1. The players are row player 1 with strategy set $N^1 = \{u, d\}$ and column player 2 with strategy set $N^2 = \{a, b\}$. In the bi-matrix listing the payoffs (Figure 1), the first component in each field refers to player 1 whereas the second component refers to player 2. The number α is positive. The number ϵ is supposed to be very small while positive. All that matters for our purposes is that $0 < \epsilon < 1$ holds.

	a	b
u	$10, 10$	$-10, 10-\alpha$
d	$10-\epsilon, 10$	$10-\epsilon, 10-\alpha$

Figure 1.

Since b is a strictly dominated strategy for player 2, it is eliminated in the first step. Against a, d is strictly dominated by u and eliminated in the second step. Hence the game's unique Nash equililibrium in pure and mixed strategies $(s^1, s^2) = (u, a)$ is reached after two rounds of elimination of strictly dominated strategies. So the game is both solvable and dominance solvable. The corresponding supports are $S^1 = \{u\}$, $S^2 = \{a\}$.

Yet the game is no longer solvable when beliefs can assume the form of simple capacities. Namely, let $\delta_d^1 \in \mathcal{M}^1$ denote the unit mass on d and $\delta_a^2 \in \mathcal{M}^2$ denote the unit mass on a. Set $v_*^2 = v^2[1-\epsilon, \delta_a^2]$. One obtains

$U^1(u, v_*^2) = (1-\epsilon) \cdot 10 + \epsilon \cdot (-10) = 10 - 20\epsilon,$
$U^1(d, v_*^2) = 10 - \epsilon > U^1(u, v_*^2),$

$U^2(\delta_d^1, a) > U^2(\delta_d^1, b)$, and
supp $\delta_d^1 = \{d\}$, *supp* $v^2[1-\epsilon, \delta_a^2] = \{a\}$.

Moreover, $U^1(d, v_*^2) = u^1(d, a) < u^1(u, a)$. Therefore, as asserted earlier,

(i) (δ_d^1, v_*^2) is an additional Nash equilibrium in simple capacities of Γ;
(ii) *supp* $\delta_d^1 \cap S^1 = \emptyset$;
(iii) d or δ_d^1 is not a best response against a;
(iv) for player 1, both the expected payoff $U^1(d, v_*^2)$ and the actual payoff $u^1(d, a)$ in the additional equilibrium are less than the payoff $u^1(u, a)$ in the original equilibrium (u, a).

Example 1 illustrates how the introduction of simple capacities as possible beliefs can affect the solvability of finite two-person games in normal form. From the very beginning of game theory, finite two-person zero-sum games have been regarded as the quintessential solvable games. It turns out that even solvability of these games can be affected by the introduction of simple capacities as possible beliefs. This is demonstrated by

EXAMPLE 2. The players are row player 1 with strategy set $N^1 = \{u, m, d\}$ and column player 2 with strategy set $N^2 = \{L, M, R\}$. In the bi-matrix listing the payoffs (Figure 2), the first component in each field refers to player 1 whereas the second component refers to player 2. The large numbers in the payoff matrix are chosen to render the argument extremely transparent. They also serve to illustrate that the bounds e^1 and e^2 in Proposition 2 can be large.

	L	M	R
u	$2^{33}, -2^{33}$	$-2^{33}, 2^{33}$	$1, -1$
m	$-2^{33}, 2^{33}$	$2^{33}, -2^{33}$	$1, -1$
d	$-1, 1$	$-1, 1$	$1, -1$

Figure 2.

In this game, the unique **prudent strategy pair** is (d, R) which, however, is **not a saddle point**. The unique Nash equilibrium in mixed strategies of this game is (σ^1, σ^2) with σ^1 given by the probability vector $(\sigma^1(u), \sigma^1(m), \sigma^1(d)) = (1/2, 1/2, 0)$ and σ^2 given by the probability vector $(\sigma^2(L), \sigma^2(M), \sigma^2(R)) = (1/2, 1/2, 0)$. The expected equilibrium payoff pair is $(0, 0)$ and the equilibrium supports are $S^1 = \{u, m\}$ and $S^2 = \{L, M\}$.

Now let $\delta_d^1 \in \mathcal{M}^1$ denote the unit mass on d and $\delta_R^2 \in \mathcal{M}^2$ denote the unit mass on R. Let $c^1, c^2 \in (0, 1)$ and consider

$v_*^1 = v^1[c^1, \delta_d^1]$,
$v_*^2 = v^2[c^2, \delta_R^2]$.

Then:

$U^1(u, v_*^2) = c^2 - (1 - c^2) \cdot 2^{33}$,
$U^1(m, v_*^2) = c^2 - (1 - c^2) \cdot 2^{33}$,
$U^1(d, v_*^2) = c^2 - (1 - c^2)$.

Hence d is the unique best response against v_*^2 and *supp* $v_*^1 = \{d\}$. Furthermore,

$U^2(v_*^1, L) = c^1 - (1 - c^1) \cdot 2^{33}$,
$U^2(v_*^1, M) = c^1 - (1 - c^1) \cdot 2^{33}$,
$U^2(v_*^1, R) = -1$.

Hence R is the unique best reponse against v_*^1, provided $-1 > c^1 - (1 - c^1) \cdot 2^{33}$ or $2^{33} - 1 > c^1(2^{33} + 1)$. Now for $c^1 \leqslant 1 - 2^{-32}$, $2^{33} - 1 > c^1(2^{33} + 1)$ where $2^{-32} \approx 2.3 \cdot 10^{-10}$. Moreover, *supp* $v_*^2 = \{R\}$. Therefore, there exists c^1 very close to 1 such that (v_*^1, v_*^2) constitutes a Nash equilibrium in simple capacities. The corresponding supports satisfy *supp* $v_*^i \cap S^i = \emptyset$ for $i = 1, 2$. The expected payoffs are $(2c^2 - 1, -1)$ and the actual payoffs are $(1, -1)$ in contrast with the mixed strategy equilibrium payoff pair $(0, 0)$. Furthermore, (σ^1, v_*^2) and (v_*^1, σ^2) are not Nash equilibria in simple capacities. Thus once again, solvability has been lost in a significant way: The counter-parts of (a) and (b) are violated.

3.4. *Preservation of Solvability*

Eichberger and Kelsey (2000; Proposition 4.1) show that, for low degrees of confidence c^1 and c^2, Nash equilibria in simple capacities induce maxmin (prudent) play. In the present context, this amounts to the following:

PROPOSITION 2. *There exist* $e^1, e^2 \in (0,1]$ *such that:*
If $(v^1, v^2) = (v^1[c^1, \pi^1], v^2[c^2, \pi^2])$ *is any equilibrium in simple capacities with* $(\pi^1, \pi^2) \in \mathcal{M}^1 \times \mathcal{M}^2$ *and* $(c^1, c^2) \in [0, e^1] \times [0, e^2]$ *for* $i = 1, 2$*, then for* $i \in I$*:*

$$s^i \in \operatorname*{argmax}_{a^i \in N^i} U^i(a^i, v^{-i}) \Longrightarrow s^i \in \operatorname*{argmax}_{a^i \in N^i} \min_{s^{-i} \in N^{-i}} u^i(a^i, s^{-i}).$$

In Example 2, c^2 can be chosen anywhere in $[0, 1)$ while $e^1 = 1 - 2^{-32}$ will do. So Example 2 conforms to Proposition 2 with large e^1 and e^2. What is more important, a change in equilibrium supports and equilibrium payoffs can be observed so that classical solvability does not persist, once simple capacities are admitted as equilibrium beliefs. It remains to be seen if solvability is preserved for finite two-person zero-sum games with a unique pair of prudent strategies which happens to be a saddle-point, hence coincides with the unique equilibrium point in mixed strategies. A partial answer is obtained as a corollary to Proposition 5.1 in Eichberger and Kelsey (2000). Some more notation is helpful for presenting their result.

For $i \in I$, $a^i, b^i \in N^i$, we say a^i **dominates** or **tops** b^i and write $a^i \top b^i$, if
$u^i(a^i, s^{-i}) \geqslant u^i(b^i, s^{-i})$ for all $s^{-i} \in N^{-i}$ and
$u^i(a^i, s^{-i}) > u^i(b^i, s^{-i})$ for some $s^{-i} \in N^{-i}$.
For $i \in I$, we define $D^i \subseteq N^i$ by

$$D^i \equiv \{b^i \in N^i | a^i \top b^i \text{ for some } a^i \in N^i\}$$

and $V^i : N^i \longrightarrow \mathbb{R}$ by

$$V^i(a^i) \equiv \min_{s^{-i} \in N^{-i}} u^i(a^i, s^{-i}).$$

V^i is called **separating**, if it is injective (1:1).[6] D^i is the set of dominated strategies of player i.

PROPOSITION 3 (Eichberger & Kelsey). *Suppose* V^1 *and* V^2 *are separating. If* $(v^1, v^2) = (v^1[c^1, \pi^1], v^2[c^2, \pi^2])$ *is any equilibrium in simple capacities with* $(\pi^1, \pi^2) \in \mathcal{M}^1 \times \mathcal{M}^2$ *and* $(c^1, c^2) \in (0, 1)^2$*, then for* $i \in I$*:*

$$s^i \in \operatorname{supp} v^i \Longrightarrow s^i \notin D^i.$$

Among other things, Proposition 3 implies that in a finitely repeated Prisoners' Dilemma game, a Nash equilibrium in simple capacities does not yield cooperation in early stages of the game. Dow and Werlang's (1994) Example 3 exhibits cooperation in the first stage of a twice repeated Prisoners' Dilemma game. This effect occurs in a Nash equilibrium in capacities that resorts to complex capacities. Yet this outcome cannot be achieved if beliefs are restricted to simple capacities. In contrast, preliminary results by Dow, Orioli, and Werlang (1996) indicate that the introduction of simple capacities can alter the outcome of a centipede game to the effect that backward induction breaks down. Remarkably, the backward induction outcome is restored when the first mover is sufficiently uncertainty averse: To any given high enough uncertainty aversion of the first mover corresponds a unique Nash equilibrium in simple capacities in which the first mover takes the amount on the table immediately.

As a direct consequence of Proposition 3, one obtains

COROLLARY 1. *Let* Γ *be a finite two-person zero-sum game. Suppose* V^1 *and* V^2 *are separating and for the unique maximizers* a_*^1 *and* a_*^2 *of* V^1 *and* V^2*, respectively, the following two conditions hold:*

(i) (a_*^1, a_*^2) *is a saddle point of* Γ.

(ii) $D^i = N^i \setminus \{a_*^i\}$ *for* $i \in I$.

Then for any Nash equilibrium in simple capacities of Γ,
$(v^1, v^2) = (v^1[c^1, \pi^1], v^2[c^2, \pi^2])$ *with* $(\pi^1, \pi^2) \in \mathcal{M}^1 \times \mathcal{M}^2$ *and* $(c^1, c^2) \in (0, 1]^2$*:*

$$\operatorname{supp} v^1 = \{a_*^1\} \textit{ and } \operatorname{supp} v^2 = \{a_*^2\}.$$

Condition (i) guarantees that the unique maxmin (prudent) strategy pair (a_*^1, a_*^2) constitutes the only Nash equilibrium in mixed strategies of Γ. Condition (ii) renders Proposition 3 immediately applicable. Hence, under further restrictions on finite two-person zero-sum games, solvability is preserved when beliefs assume the form of simple capacities.

	$C2$	$D2$
$C1$	$a, -a$	$b, -b$
$D1$	$c, -c$	$0, 0$

Figure 3.

3.5. *Preservation of Strict Dominance*

As pointed out by one of the referees, strict dominance is preserved when beliefs become non-additive and even non-simple. Intriguingly enough, this does not imply that dominance solvability is preserved as evidenced by Example 1.

PROPOSITION 4. *Let $a^i_* \in N^i$ be a strictly dominant strategy for player i. Then $U(a^i_*, v^{-i}) > U(a^i, v^{-i})$ for all $a^i \neq a^i_*$ and $v^{-i} \in \mathcal{C}^{-i}$.*

Proof. Since strategy sets are finite, there exists $\Delta > 0$ such that $u^i(a^i_*, a^{-i}) > \Delta + u^i(a^i, a^{-i})$ for all $a^i \neq a^i_*$ and $a^{-i} \in N^{-i}$. Hence, for all $a^i \neq a^i_*$ and $v^{-i} \in \mathcal{C}^{-i}$,

$$U(a^i_*, v^{-i}) = \int u^i(a^i_*, \cdot)dv^{-i} \geqslant \int [\Delta + u^i(a^i, \cdot)]dv^{-i}$$
$$= \Delta + \int u^i(a^i, \cdot)dv^{-i} > U(a^i, v^{-i}).$$

□

3.6. 2×2 *Zero-Sum Games*

In the very special instance of 2×2 zero-sum games, an exhaustive case by case study is possible which will demonstrate that solvability (in a somewhat wider sense) persists in these games. Such a game has player set $I = \{1, 2\}$. Each player has action set $N^i = \{Ci, Di\}$ and the payoff bi-matrix can be assumed to be of the form of Figure 3.

Suppose V^1 and V^2 are separating and have unique maximizers a^1_* and a^2_*, respectively. Furthermore, suppose that (a^1_*, a^2_*) is a saddle

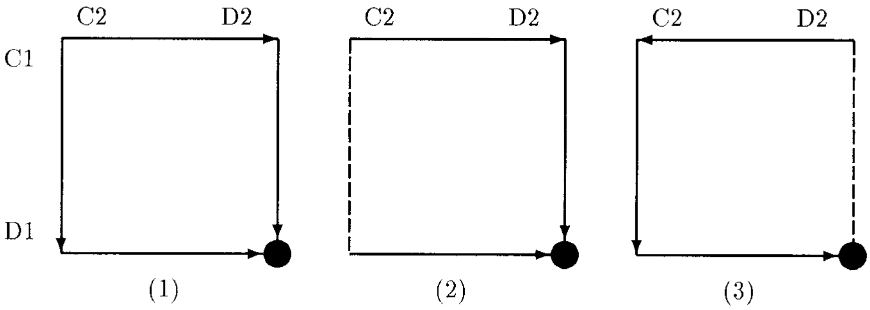

Figure 4.

point (*the* saddle point) of this game. Without loss of generality, assume $(a^1_*, a^2_*) = (D1, D2)$. There are, up to symmetry, three prototypes of games with these properties. The ordinal preferences for the three prototypes (1) – (3) are depicted in Figure 4. (An arrow indicates a unique best response. A dashed line indicates indifference. A fat dot locates the saddle point.)

Prototype (1): Since player i has strictly dominant strategy Di, his equilibrium strategies are of the form $v^i = v^i[c^i, \delta^i_{Di}]$ with corresponding support $\{Di\}$. Essentially, solvability is preserved.

Prototype (2): Since player 2 has strictly dominant strategy $D2$, his equilibrium strategies are of the form $v^2 = v^2[c^2, \delta^2_{D2}]$ with corresponding support $\{D2\}$.

Since $0 > b$ and $a = c > 0$, $D1$ is the unique best response against δ^2_{D2} and also unique best response against ω^2. Hence $D1$ is the unique best response against an equilibrium strategy of 2. Thus 1's equilibrium strategies are of the form $v^1 = v^1[c^1, \delta^1_{D1}]$ with corresponding support $\{D1\}$. Essentially, solvability is preserved.

Prototype (3): We get $c > a$ and $b = 0$ from player 1's payoffs and $b > a$ and $c > 0$ from player 2's payoffs. Hence
$c > 0 = b > a$ and $-c < 0 = -b < -a$.

Against ω^2, the unique best response is $D1$.

Against π^2, the unique best response is $D1$ — unless $\pi^2 = \delta^2_{D2}$ in which case $C1$ is also a best response. So

$$C1 \text{ is best response } \iff v^2 = \delta^2_{D2}.$$

It remains to be seen, if δ^2_{D2} is a best response against any $v^1 \in \mathcal{SC}^1$ with $C1 \in \mathit{supp}\ v^1$. Let us consider such a v^1, i.e. $v^1 = v^1[c^1, \pi^1]$

with $\pi^1 \neq \delta^1_{D1}$. Set $d^1 = c^1\pi^1(C1)$. Then

$$Eu^2[v^1, C2] = d^1 \cdot (-a) + (1 - d^1) \cdot (-c)$$
$$\text{and} \quad Eu^2[v^1, D2] = 0.$$

Therefore, $D2$ is best response $\Leftrightarrow$ $d^1 \leqslant \frac{c}{c-a}$. Hence for $d^1 \leqslant \frac{c}{c-a}$, (v^1, δ^2_{D2}) is an additional equilibrium with $C1 \in$ *supp* v^1. Strictly speaking, solvability is lost. However, the additional equilibria are payoff equivalent to the original equilibrium $(D1, D2)$, with expected payoffs and actual payoffs equal to $(0, 0)$. In that sense, solvability is restored.

4. FINAL REMARKS

As the modelling device of non-additive probabilities gets transplanted from decision theory into game theory, new conceptual and technical challenges appear. This paper points out tradeoffs between additional explanations and solvability of games. Conflicting interests of this sort are not new to game theory. The literature on equilibrium selection and a host of refinements aims at narrowing the solution set (fewer phenomena explained, more accurate predictions). Examples of criteria employed are payoff dominance, risk dominance, stability, robustness, evolutionary stability, forward induction, backward induction, perfection, sequentiality, properness, divineness, intuitiveness, renegotiation-proofness, stationarity, symmetry, cheap talk, and many others. Another strand of literature allows for a broadening of the solution set (more phenomena explained, vaguer predictions). The expanded equilibrium concepts allow for mixed strategies, correlated strategies, rationalizable sets, curb sets, craziness, cheap talk again, and so forth and finally, non-additive beliefs. On a priori grounds, one would not expect that players hold arbitrary beliefs. Future research should and, most likely, will try to reconcile the two competing branches of game theoretical investigation. Specifically, the concept of non-additive beliefs could be investigated in conjunction with some of the standard equilibrium refinement concepts. Rationalizability with non-additive beliefs is explored in the working paper version of Ghirardato and Le Breton (2000). Moreover, belief formation is a widely unexplored, urgent subject of inquiry, even in traditional game theory — despite recent advances in the learning literature. Mukerji (1997) contributes to the epistemic foundations of non-additive beliefs.

A novel issue in game theory that is absent from decision theory, is the appropriate definition of the support of a capacity. This issue deserves further scrutiny. Ryan (1988a,b) is concerned with the previously overlooked problem of support expansion: Dempster-Shafer updating can add states to the support. He forwards several definitions of a support in addition to the ones given above. Another crucial issue which arises in games with more than two players and is absent from decision theory, is the appropriate definition of a product capacity. Hendon et al. (1996) deal with the definition of the product of two capacities in general and of two belief functions in particular. Ghirardato (1997) addresses the definition of the product of two capacities, the role of independence and the validity of Fubini's theorem. See also section 5 of Ben-Porath, Gilboa, and Schmeidler (1997).

There are other pressing issues regarding game-theoretic models with non-additive probabilities. But first, it should be emphasized that the Choquet Expected Utility (CEU) theory of the current paper is competing, both in decision and game theory, with the multiple priors model of Gilboa and Schmeidler (1989). For further elaborations on the latter, see Kelsey (1994). Recent decision theoretic investigations by Klibanoff (1996b) and Nehring (1999) show discernible differences between the two competing approaches. Contemporary work in decision theory by Ghiaradato (1994b), Ghirardato and Marinacci (1997), Mukerji (1997), and Nehring (1999) wrestles with the proper definition of ambiguity, ambiguity aversion and revealed unambiguous events, e.g. Most of the earlier literature on decisions under uncertainty is summarized in Camerer and Weber (1992) and Kelsey and Quiggin (1992).

It should not be surprising then that the definition of a Nash equilibrium in capacities given here and its variations found elsewhere do not encompass all relevant game theoretical models with non-additive probabilities. Epstein (1997), Klibanoff (1996a), Lo (1996, 1999), Marinacci (2000) and Ritzberger (1996) rank among the several important contributions not mentioned before. In the context of the multiple priors model, Klibanoff (1996a) and Lo (1996) have studied games where players are uncertainty averse, but in addition are allowed to randomize. They conclude that under certain conditions, players may exhibit preference for randomization over pure strategies between which they are indifferent. Eichberger and Kelsey (1996b) argue that preference for randomization is displayed in an Anscombe–Aumann framework whereas it is unlikely in the decision theoretical framework à la Savage.

The proper choice of Choquet integral in a multi-stage game appears to be related to another crucial modelling choice one has to make: how to update non-additive capacities. Basic research has been performed by Denneberg (1994a), Gilboa and Schmeidler (1993), Jaffrey (1992), Sundberg and Wagner (1992), Wasserman and Kadane (1990), among others. The focus of some of this analysis lies on concave capacities whereas convex capacities are much more interesting for economic and game theoretic modelling purposes. In this respect, recent findings by Eichberger and Kelsey (1996c) are most intriguing. They show that as a rule, the Choquet expected utility class of preferences as axiomatized by Gilboa (1987), Schmeidler (1989), and Sarin and Wakker (1992), is not closed under dynamically consistent updating in the sense of Machina (1989). More recent progress on conditional capacities is reported in Denneberg (1995). Lehrer (1996) proposes an elegant geometric approach to conditional expectations for non-additive probabilities in analogy to the treatment of the additive case.

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NOTES

1. See also Groes et al. (1998). For specific economic applications, see Ben-Porath, Gilboa, and Schmeidler (1997), Dow and Werlang (1992a,b), Ghirardato (1994a), Eichberger and Kelsey (1995, 1966a), Mukerji (1998).
2. A simple capacity is defined as a convex combination of risk and complete ignorance. The precise formal definition will be given in Section 2.
3. In the language of cooperative game theory, the habitual term for these capacities would be "inessential". The author leaves it to the reader whether to adopt such language in the present context.
4. With respect to the first definition, the support of ω^i would be N^i. With respect to the third definition, the support of ω^i would be the empty set.
5. If $|N^i| = 2$, every convex capacity on N^i is a simple capacity — and vice versa. Belief functions or lower probabilities as studied by Hendon et al. (1994) and Groes et al. (1998) form another distinguished class of convex capacities. Many of the properties of simple capacities generalize to the class of E-capacities studied by Eichberger and Kelsey (1999).
6. For generic payoff matrices, V^i is separating. However, with respect to economic or other applications, many interesting games have non-generic payoff matrices.

REFERENCES

Aumann, R.J. (1987), Correlated Equilibrium as an Expression of Bayesian Rationality, *Econometrica* 55: 1–18.

Ben-Porath, E., Gilboa, I. and Schmeidler, D. (1997), On the Measurement of Inequality under Uncertainty, *Journal of Economic Theory* 75: 194–204.

Billingsley, P. (1968), *Convergence of Probability Measures*, John Wiley & Sons, Inc., New York.

Camerer, C. and Weber, M. (1992), Recent Developments in Modeling Preferences: Uncertainty and Ambiguity, *Journal of Risk and Uncertainty* 5: 325–370.

Denneberg, D. (1994a), Conditioning (Updating) Non-Additive Probability Measures, *Annals of Operations Research* 52: 21–42.

Denneberg, D. (1994b), *Non-Additive Measure and Integral*, Kluwer, Dordrecht.

Denneberg, D. (1995), *Extensions of a Measurable Space and Linear Representation of the Choquet Integral*, Universität Bremen, Mathematische Arbeitspapiere Nr. 45.

Dow, J., and Werlang, S.R.C. (1992a), Uncertainty Aversion, Risk Aversion and the Optimal Choice of Portfolio, *Econometrica* 60: 197–204.

Dow, J. and Werlang, S.R.C. (1992b), Excess Volatility of Stock Prices and Knightian Uncertainty, *European Economic Review* 36: 631–638.

Dow, J. and Werlang, S.R.C. (1994), Equilibrium under Knightian Uncertainty: Breaking Down Backward Induction, *Journal of Economic Theory* 64: 305–324.

Dow, J., Orioli, S. and Werlang, S.R.C. (1996), The Centipede Game under Knightian Uncertainty, mimeo.

Eichberger, J. and Kelsey, D. (1995), Signalling Games with Uncertainty, mimeo.

Eichberger, J. and Kelsey, D. (1996a), Free Riders Do Not Like Uncertainty, mimeo.

Eichberger, J., and Kelsey, D. (1996b), Uncertainty Aversion and Preference for Randomisation, *Journal of Economic Theory* 71: 31–43.

Eichberger, J., and Kelsey, D. (1996c), Uncertainty Aversion and Dynamic Consistency, *International Economic Review* 37: 625–640.

Eichberger, J. and Kelsey, D. (1999), E-Capacities and the Ellsberg Paradox, *Theory and Decision* 46: 107–140.

Eichberger, J. and Kelsey, D. (2000), Non-Additive Beliefs and Strategic Equilibria, *Games and Economic Behavior* 30: 183–215.

Epstein, L.G. (1997), Preference, Rationalizability and Equilibrium, *Journal of Economic Theory* 73: 1–29.

Ghirardato, P. (1994a), Agency Theory with Non-Additive Uncertainty, mimeo.

Ghirardato, P. (1994b), Coping with Ignorance: Unforeseen Contingencies and Non-Additive Uncertainty, mimeo.

Ghirardato, P. (1997), On Independence for Non-Additive Measures, with a Fubini Theorem, *Journal of Economic Theory* 73: 261–291.

Ghirardato, P., and Le Breton, M. (2000), Choquet Rationalizability, *Journal of Economic Theory* 90: 277–285.

Ghirardato, P., and Marinacci, M. (1997), Ambiguity Made Precise: A Comparative Foundation and Some Implications, California Institute of Technology, Social Science Working Paper 1026.

Gilboa, I. (1987), Expected Utility with Purely Subjective Non-Additive Probabilities, *Journal of Mathematical Economics* 16: 65–88.

Gilboa, I. and Schmeidler, D. (1989), Maxmin Expected Utility with a Non-Unique Prior, *Journal of Mathematical Economics* 18: 141-153.

Gilboa, I. and Schmeidler, D. (1993), Updating Ambiguous Beliefs, *Journal of Economic Theory* 59: 33–49.

Groes, E., Jacobsen, H.J., Sloth, B. and Tranæs, T. (1998), Nash Equilibrium in Lower Probabilities, *Theory and Decision* 44: 37–66.

Harsanyi, J.C. (1973), Games with Randomly Disturbed Payoffs: A New Rationale for Mixed Strategy Equilibrium Points, *International Journal of Game Theory* 2: 1–23.

Hendon, E., Jacobsen, H.J., Sloth, B. and Tranæs, T. (1994), Expected Utility with Lower Probabilities, *Journal of Risk and Uncertainty* 8: 197–216.

Hendon, E., Jacobsen, H.J., Sloth, B. and Tranæs, T. (1996), The Product of Capacities and Belief Functions, *Mathematical Social Sciences* 32: 177–202.
Hougaard, J.L. and Keiding, H. (1996), Representation of Preferences on Fuzzy Measures by a Fuzzy Integral, *Mathematical Social Sciences* 31: 1–17.
Jaffrey, Y. (1992) Bayesian Updating and Belief Functions, *IEEE Transactions on Systems, Man and Cybernetics* 22: 1144–1152.
Kelsey, D. (1994), Maxmin Expected Utility and Weight of Evidence, *Oxford Economic Papers* 46: 425–444.
Kelsey, D., and Quiggin, J. (1992), Theories of Choice under Ignorance and Uncertainty, *Journal of Economic Surveys* 6: 133–153.
Klibanoff, P. (1996a), Uncertainty, Decision, and Normal Form Games, mimeo.
Klibanoff, P. (1996b), Characterizing Uncertainty Aversion Through Preference for Mixture, mimeo, forthcoming in *Social Choice and Welfare*.
Lehrer, E. (1996), An Approach to the Conditional Expectation with Non-Additive Probabilities, mimeo.
Lo, K.C. (1996), Equilibrium in Beliefs Under Uncertainty, *Journal of Economic Theory* 71: 443–484.
Lo, K.C. (1999), Extensive Form Games with Uncertainty Averse Players, *Games and Economic Behavior* 28: 256–270.
Machina, M. (1982), Expected Utility Analysis Without the Independence Axiom, *Econometrica* 50: 277–323.
Machina, M. (1989), Dynamic Consistency and Non-Expected Utility Models of Choice Under Uncertainty, *Journal of Economic Literature* 27: 1622–1668.
Marinacci, M. (2000), Ambiguous Games, *Games and Economic Behavior* 31: 191–219.
Moulin, H. (1979), Dominance Solvable Voting Schemes, *Econometrica* 47: 1337–1351.
Mukerji, S. (1997), Understanding the Non-Additive Probability Decision Model, *Economic Theory* 9: 23–46.
Mukerji, S. (1998), Ambiguity and Incompleteness of Contractual Form, *American Economic Review* 88: 1207–1231.
Nash, J. (1951), Non-Cooperative Games, *Annals of Mathematics* 54: 286–295.
Nehring, K. (1999), Capacities and Probabilistic Beliefs: A Precarious Coexistence, *Mathematical Social Sciences* 38: 197–213.
Nehring, K. (1999), Preference for Flexibility in a Savage Framework, *Econometrica* 67: 101–119.
Parthasarathy, K.R. (1967), *Probability Measures on Metric Spaces*, Academic Press: New York and London.
Quiggin, J. (1982), A Theory of Anticipated Utility, *Journal of Economic Behavior and Organization* 3: 323–343.
Ritzberger, K. (1996), On Games under Expected Utility with Rank Dependent Probabilities, *Theory and Decision* 40: 1–27.
Rosenmüller, J. (1982), L.P.-Games with Sufficiently Many Players, *International Journal of Game Theory* 11: 129–149.

Rosenmüller, J. and Weidner, H.G. (1973), A Class of Extreme Convex Set Functions with Finite Carrier, *Advances in Mathematics* 10: 1–38.

Rosenmüller, J. and Weidner, H.G. (1974), Extreme Convex Set Functions with Finite Carrier: General Theory, *Discrete Mathematics* 10: 343–382.

Ryan, M.J. (1998a), Supports and the Updating of Capacities, University of Auckland.

Ryan, M. (1998b), A Refinement of Dempster-Shafer Equilibrium, University of Auckland.

Sarin, R. and Wakker, P. (1992), A Simple Axiomatisation of Non-Additive Expected Utility, *Econometrica* 60: 1255–1272.

Schmeidler, D. (1989), Subjective Probability and Expected Utility without Additivity, *Econometrica* 57: 571–587.

Sundberg, C. and Wagner, C. (1992), Generalized Finite Differences and Bayesian Conditioning of Choquet Capacities, *Advances in Applied Mathematics* 13: 262–272.

Wakker, P. (1987), From Decision Making under Uncertainty to Game Theory, pp. 163-180, in H.J.M. Peters and O.J. Vrieze (eds.), *Surveys of Game Theory and Related Topics*, CWI Tract 39, Centre for Mathematics and Computer Science, Amsterdam.

Wasserman, L.A., and Kadane, J.B. (1990), Bayes' Theorem for Choquet Capacities, *Annals of Statistics* 18: 1328–1339.

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