On the nature of continuous physical quantities in classical and quantum mechanics

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Abstract

Within the traditional Hilbert space formalism of quantum mechanics, it is not possible to describe a particle as possessing, simultaneously, a sharp position value and a sharp momentum value. Is it possible, though, to describe a particle as possessing just a sharp position value (or just a sharp momentum value)? Some, such as Teller [21], have thought that the answer to this question is No — that the status of individual continuous quantities is very different in quantum mechanics than in classical mechanics. On the contrary, I shall show that the same subtle issues arise with respect to continuous quantities in classical and quantum mechanics; and that it is, after all, possible to describe a particle as possessing a sharp position value without altering the standard formalism of quantum mechanics.

1   Introduction

Consider the following experimental setup: We have a source of quantum-mechanical particles (e.g. electrons), a fluorescent screen, and a barrier between the source and the screen which has one or more openings. The barrier, with its openings, prepares an ensemble of particles whose state, upon arrival at the screen, is given by some wavefunction ψ. Let's assume for simplicity that the screen is one-dimensional and infinite in both directions (i.e., represented by $\mathbb{R}$). The screen should be thought of as a measuring device for the

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position of particles in the ensemble, and we know that the pattern of shading on the screen will be distributed in accordance with the squared modulus $|\psi|^2$ of the wavefunction. For any individual particle in the ensemble, and any region S on the screen, $\int_S |\psi|^2$ gives the probability that the screen will flash inside S when the particle hits the screen. It would be very natural then to attempt to interpret the probability distribution $|\psi|^2$ as a measure of our ignorance about the precise state of individual particles in the ensemble. That is, it is natural to think that each particle in the ensemble does in fact have a "sure-fire" (probability 1) disposition to be found at a certain point $\lambda \in \mathbb{R}$, and probabilities arise out of our ignorance about precisely which point λ that is.

It is clear that the traditional interpretive obstacles in quantum theory do not prevent us from thinking this way of this measurement. For one, since we are not asking how a particle gets from the source to the screen, interference effects play no role. Second, since we are in a position measurement context (i.e., we are ignoring other observables such as momentum), there can be no question of the no-hidden-variables theorems presenting any obstacles to our thinking of $|\psi|^2$ as a mixture of pure position states.

However, there remains a serious difficulty for supplying an ignorance interpretation of $|\psi|^2$: There does not appear to be any object in the standard Hilbert space formalism of quantum mechanics (QM) which could represent the state of an individual particle in the ensemble. According to Teller [21], then, attempting to give an ignorance interpretation to $|\psi|^2$ amounts to questioning the "descriptive completeness" of quantum theory:

> "...if imprecise or imperfectly defined values of a continuous quantity are to be explained in terms of assumed underlying states with exact point values, these assumed underlying states cannot be described within the theory, and the theory is descriptively incomplete." (p. 347)

But since QM is descriptively complete, $|\psi|^2$ should not be interpreted as a measure of our ignorance of the precise state of the individual particles.

Now it is certainly true that no *vector* in Hilbert space can be taken to describe the state of an individual particle with a sharp location. This is just the familiar point that operators with continuous spectra, like position, have no eigenvectors. Furthermore, Gleason's Theorem [11] entails that any (pure) σ-additive probability measure on the logic of quantum propositions (i.e., lattice of subspaces of a Hilbert space) is given by a vector ψ, via the

standard formula $E \mapsto \langle \psi | E\psi \rangle$. I believe, however, that the σ-additivity assumption in Gleason's theorem is too restrictive. Indeed, I will argue that if we required σ-additivity of the states of a *classical* mechanical system, then we would similarly have to conclude — in direct contradiction to what we know to be true — that there can be no states assigning sharp values to continuous quantities in that case as well. In fact, I will show that in both cases it is possible to assign sharp values to continuous quantities using finitely-additive ("singular") states. I will also demonstrate that difficulties which arise when using singular states in conjunction with the traditional von Neumann account of ideal measurement are resolved when we adopt a more realistic account of measurement.

2   Observables, properties, and states

It is well-known that there are methods of extending the Hilbert space formalism of QM so that it includes "generalized eigenstates" for continuous spectrum observables (e.g., the rigged Hilbert space formalism). However, these constructions are not needed for my purposes: Any continuous spectrum observable is associated with a Boolean algebra of spectral projections, and a 2-valued homomorphism on this Boolean algebra gives a generalized eigenstate. Furthermore, such homomorphisms should not be considered any less a part of the Hilbert space formalism in the continuous case than they are in the discrete case, where they just happen to be represented by proper eigenvectors.

2.1   Preliminaries

I begin by establishing the terminology I will be using in this paper. Let $\mathcal{B}$ be an arbitrary σ-complete Boolean algebra, with additive identity $\mathbf{0}$ and multiplicative identity $\mathbf{I}$. We say that ω is a *state* of $\mathcal{B}$ just in case ω is a mapping of $\mathcal{B}$ into $[0,1]$ such that $\omega(\mathbf{I}) = 1$ and

$$\omega(A \vee B) = \omega(A) + \omega(B), \tag{1}$$

for each disjoint $A, B \in \mathcal{B}$, i.e., $A \wedge B = \mathbf{0}$. Clearly, the set of states on $\mathcal{B}$ is a convex set. If ω is an extreme point among the states on $\mathcal{B}$, we say that ω is a *pure* state. That is, ω is pure just in case: If $\omega = a\rho + (1-a)\tau$, for some $a \in (0,1)$ and states ρ, τ on $\mathcal{B}$, then $\omega = \rho = \tau$. The state ω is pure if and

only if it is a Boolean homomorphism of $\mathcal{B}$ onto $\{0,1\}$. If ω is not pure, we say that ω is *mixed*. We say that a state ω is *normal* just in case

$$\omega\left(\vee_{i=1}^{\infty}A_i\right) = \sum_{i=1}^{\infty}\omega(A_i), \tag{2}$$

for each countable pairwise disjoint sequence $\{A_i\}$ in $\mathcal{B}$. If ω is not normal, we say that ω is *singular*. Recall that a family $\mathcal{F}$ of elements in $\mathcal{B}$ is called a *filter* just in case: (i) $\emptyset \notin \mathcal{F}$, (ii) $A \wedge B \in \mathcal{F}$, when $A, B \in \mathcal{F}$, and (iii) if $A \in \mathcal{F}$ and $A \leq B$, then $B \in \mathcal{F}$. If, furthermore, either $A \in \mathcal{F}$ or $\neg A \in \mathcal{F}$ for all $A \in \mathcal{B}$, then $\mathcal{F}$ is said to be an *ultrafilter*. Each ultrafilter $\mathcal{U}$ on a Boolean algebra $\mathcal{B}$ is the preimage of 1 under some pure state ω of $\mathcal{B}$, and ω is normal if and only if $\mathcal{U}$ is closed under countable meets. For any non-zero element $A \in \mathcal{B}$, the set $\mathcal{U}_A = \{B \in \mathcal{B} : A \leq B\}$ is a filter, and is an ultrafilter if A is an atom in $\mathcal{B}$. In this case, $\mathcal{U}_A$ is called the *principal ultrafilter* generated by A. It is easy to see then that principal ultrafilters give rise to normal pure states.

The discussion of this paper focuses on two Boolean algebras: The σ-complete Boolean algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$, and its quotient $\mathcal{B}(\mathbb{R})/\mathcal{N}$ by the ideal of Lebesgue measure zero sets. Recall that $\mathcal{B}(\mathbb{R})$ is the (Boolean) σ-algebra generated by the open subsets of $\mathbb{R}$. The logical operations on $\mathcal{B}(\mathbb{R})$ are just the set-theoretic operations $\cap$ (intersection), $\cup$ (union), and $\sim$ (complement in $\mathbb{R}$). Clearly, $\mathcal{B}(\mathbb{R})$ is purely atomic; i.e., for each $S \in \mathcal{B}(\mathbb{R})$ there is some atomic element $A \in \mathcal{B}(\mathbb{R})$ (i.e., a singleton set) such that $A \subseteq S$. Thus, by the considerations adduced above, $\mathcal{B}(\mathbb{R})$ has an abundance of pure normal states.

Let $\mathcal{N}$ denote the family of Lebesgue measure zero sets in $\mathcal{B}(\mathbb{R})$. We define an equivalence relation $\approx$ on $\mathcal{B}(\mathbb{R})$ by setting $S_1 \approx S_2$ just in case $S_1 \triangle S_2 \in \mathcal{N}$, where $S_1 \triangle S_2 = (S_1 \backslash S_2) \cup (S_2 \backslash S_1)$. For each $S \in \mathcal{B}(\mathbb{R})$, let $\pi(S) = \{S' \in \mathcal{B}(\mathbb{R}) : S' \approx S\}$, and let

$$\mathcal{B}(\mathbb{R})/\mathcal{N} = \{\pi(S) : S \in \mathcal{B}(\mathbb{R})\}. \tag{3}$$

For any countable family $\{\pi(S_i)\}$ of elements in $\mathcal{B}(\mathbb{R})/\mathcal{N}$, we define,

$$\vee_{i=1}^{\infty}\pi(S_i) = \pi\left(\cup_{i=1}^{\infty}S_i\right). \tag{4}$$

(This is well-defined since $\mathcal{N}$ is a σ-ideal.) For $\pi(S) \in \mathcal{B}(\mathbb{R})/\mathcal{N}$, we set $\neg\pi(S) = \pi(\sim S)$. It then follows that $\langle \mathcal{B}(\mathbb{R})/\mathcal{N}, \vee, \neg \rangle$ is a σ-complete Boolean

algebra. It is also possible to show — although I do not prove it here — that $\mathcal{B}(\mathbb{R})/\mathcal{N}$ has *no* atoms; i.e., for each nonzero $A \in \mathcal{B}(\mathbb{R})/\mathcal{N}$, there is some nonzero $B \in \mathcal{B}(\mathbb{R})/\mathcal{N}$ such that $B < A$. This, in fact, is equivalent to the nonexistence of pure normal states for $\mathcal{B}(\mathbb{R})/\mathcal{N}$, which I do prove below in Proposition 1.

2.2 The position observable in QM

It is well-known that no wavefunctions ψ in a Hilbert space give a probability density $|\psi|^2$ that is focused at one point. In this section, I will show that this well-known fact may be described in purely logical terms as the nonexistence of pure normal states on the Boolean algebra $\mathcal{B}(\mathbb{R})/\mathcal{N}$.

The state space of quantum mechanics is obtained by requiring an irreducible representation of the canonical commutation relation (CCR):

$$[\widehat{Q}, \widehat{P}] = i\hbar I, \tag{5}$$

by operators $\widehat{Q}$, $\widehat{P}$ on some Hilbert space $\mathcal{H}$. In its integrated (Weyl) form, there is a unique (up to unitary isomorphism) representation of the CCR on the Hilbert space $L_2(\mathbb{R})$ of equivalence classes of square integrable functions from $\mathbb{R}$ into $\mathbb{C}$ [20, 23]. (Two functions are equivalent just in case they agree except possibly on a set of measure zero.) In this representation, the position operator $\widehat{Q}$ is defined on a dense subset of $L_2(\mathbb{R})$ by the equation

$$(\widehat{Q}\psi)(q) = q \cdot \psi(q). \tag{6}$$

A "proposition" about the position of the particle is then given by a projection operator which is a function of $\widehat{Q}$. In particular, for each $S \in \mathcal{B}(\mathbb{R})$, we define an operator $E(S)$ on $L_2(\mathbb{R})$ by the equation

$$(E(S)\psi)(q) = \chi_S(q) \cdot \psi(q). \tag{7}$$

The set $\mathcal{O}_Q = \{E(S) : S \in \mathcal{B}(\mathbb{R})\}$ is a σ-complete Boolean algebra of projection operators on the Hilbert space $\mathcal{H}$, and E is a Boolean σ-homomorphism from $\mathcal{B}(\mathbb{R})$ onto $\mathcal{O}_Q$. Moreover, $E(S) = \mathbf{0}$ if and only if S has Lebesgue measure zero. Thus, the kernel of the mapping E is $\mathcal{N}$, and (by the first isomorphism theorem for rings) $\mathcal{O}_Q$ is isomorphic to the quotient Boolean algebra $\mathcal{B}(\mathbb{R})/\mathcal{N}$. ($E$ is usually called the spectral measure for $\widehat{Q}$.)

Now, every wavefunction $\psi \in \mathcal{H}$ gives rise to a normal state ρ_ψ on $\mathcal{O}_Q$ in the standard way:

$$\rho_\psi(E(S)) := \langle \psi | E(S) \psi \rangle = \int_{\mathbb{R}} \chi_S(q) |\psi(q)|^2 dq = \int_S |\psi(q)|^2 dq. \qquad (8)$$

However, it follows from purely logical considerations that there cannot be normal "eigenstates" for $\widehat{Q}$.

Proposition 1. *There are no pure normal states of $\mathcal{B}(\mathbb{R})/\mathcal{N}$.*

Proof. Let ω be a pure state of $\mathcal{B}(\mathbb{R})/\mathcal{N}$, and let $\mathcal{U} = \omega^{-1}(1)$ be the corresponding ultrafilter. (Case 1) Suppose that $\pi(S) \notin \mathcal{U}$ for all compact subsets S of $\mathbb{R}$. For each $n \in \mathbb{N}$, let $A_n = [-n, n]$. Then, $\pi(A_n) \notin \mathcal{U}$ and since $\mathcal{U}$ is an ultrafilter, $\pi(\sim A_n) = \neg \pi(A_n) \in \mathcal{U}$. However,

$$\wedge_{n=1}^\infty \pi(\sim A_n) = \pi \left(\cap_{n=1}^\infty \sim A_n \right) = \mathbf{0}, \qquad (9)$$

and therefore $\mathcal{U}$ is not closed under countable meets. (Case 2) If $\pi(S) \in \mathcal{U}$ for some compact set S in $\mathbb{R}$, then $\mathcal{U}$ contains $\pi(\mathcal{F}_\lambda)$ where $\mathcal{F}_\lambda$ is the family of open neighborhoods of some point $\lambda \in \mathbb{R}$. For each $n \in \mathbb{N}$, let

$$B_n = \left(\lambda - n^{-1}, \lambda + n^{-1} \right). \qquad (10)$$

Then, $\pi(B_n) \in \mathcal{U}$ for each $n \in \mathbb{N}$. However,

$$\wedge_{n=1}^\infty \pi(B_n) = \pi \left(\cap_{n=1}^\infty B_n \right) = \pi \left(\{\lambda\} \right) = \mathbf{0}, \qquad (11)$$

since $\{\lambda\}$ has Lebesgue measure zero. Thus, $\wedge_{n=1}^\infty \pi(B_n) \notin \mathcal{U}$ and $\mathcal{U}$ is again not closed under countable meets. $\qquad \square$

Thus, assuming that quantum theory is descriptively complete, and that quantum states must be σ-additive, $|\psi|^2$ can not be interpreted as a measure of our ignorance of the true (pure) state of individual particles in the ensemble. Some philosophers draw the moral from this that quantum theory is teaching us something new about the nature of continuous quantities, namely that we should not think or talk about continuous quantities possessing point values, nor should we think or talk about statistical states as distributions over point-valued states. For example, Fine [8] argues that we should think of statistical state ρ_ψ as a map that assigns a *set*, rather than a numerical value, to $\widehat{Q}$. In particular,

$$\rho_\psi(\widehat{Q}) := \bigcap \{ S \, : \, \rho_\psi(E(S)) = 1 \} . \qquad (12)$$

Of course, $\rho_\psi(\widehat{Q})$ will never be a singleton, and — perhaps contrary to our intuitions — the statement that the particle lies in S is a maximally specific description of the location of the particle.

Teller [21] agrees with Fine that a statistical state should be interpreted not as a mixture of value states, but as assigning a "partless" set value. Nonetheless, Teller does recognize that this description of the location of the particle can be misleading: it suggests — contrary to the predictions of quantum theory — that no matter where we look in S, we will find a "piece" of the particle. Teller suggests, however, that we should think of the property "being located in S" as including,

> "a collection of dispositions or potentialities to manifest a more refined position of the same nature whenever a more refined measurement interaction takes place." (p. 357)

Thus, Teller seems to be claiming that the probability distribution $|\psi|^2$ itself gives a maximally specific description of an individual element in the ensemble.

Teller claims, moreover, that this dual property/disposition nature of position in QM is "nothing exceptional;" in fact, it is just like the color white which is "a manifest property and at the same time an array of dispositions" (ibid). This analogy to color, however, seems to show exactly what leaves us uneasy about Teller's proposal. In the case of color (e.g., white), we can explain an object's dispositions to appear certain ways in certain contexts in terms of categorical properties of the object (viz., the chemical composition of its surface). On the other hand, Teller is claiming that a quantum-mechanical particle must have irreducibly probabilistic dispositions with no categorical basis (apart from, perhaps, ψ itself, which is difficult to see as anything other than a catalog of those dispositions). While quantum mechanics does present us with cases where we cannot fulfill classical ideals of explanation, I do not think Teller's account of position is one of these cases.

2.3 The ignorance interpretation in classical mechanics

According to Teller, these irreducibly probabilistic dispositions are a peculiar feature of quantum mechanics. Unlike the quantum case, the probabilities

in classical mechanics may be thought of as measures of our ignorance of the precise state:

> "In the context of classical physics the outcome of an inexact measurement may be described as a probability distribution over exact values, and we know how to state what this means: There is a probability (interpreted as a relative frequency, subjective degree of belief, or propensity) for the value of the measured quantity to have one or another of the exact values in the support of the distribution." (p. 352)

Let us attempt, then, to give a mathematically rigorous description of this purported distinction between the classical and quantum cases.

A classical particle moving in one dimension has a *phase space* $\mathbb{R}^2$, with one coordinate for the position Q of the particle, and the other coordinate for the momentum $P = m \cdot (dQ/dt)$ of the particle. We may think of Borel subsets of $\mathbb{R}^2$ as representing statements that ascribe properties to the particle. (For background material, see Chap. 1 of Ref. 22.) A dynamical variable is represented by a measurable, real-valued function F on phase space. For example, the position of the particle is represented by the function $F((q, p)) = q$. For any Borel set S in $\mathbb{R}$, $F^{-1}(S)$ consists precisely of those "states" (i.e., points in phase space) for which the value of F lies in S. F^{-1} (considered as a set mapping) is in fact a Boolean σ-homomorphism from $\mathcal{B}(\mathbb{R})$ onto some Boolean subalgebra $\mathcal{B}_F$ of $\mathcal{B}(\mathbb{R}^2)$. For example, the disjunctive proposition *"The value of F lies either in S_1 or in S_2"* is represented formally by

$$F^{-1}(S_1 \cup S_2) = F^{-1}(S_1) \cup F^{-1}(S_2). \tag{13}$$

Consider now the completely classical example of a person (or machine) Bob throwing darts at a (one-dimensional) dartboard. Since we are ignorant of the factors that influence each individual throw, we assign a probability density ρ to the points on the dartboard (represented here by $\mathbb{R}$). Now, just as in the quantum-mechanical case, the distribution ρ on $\mathbb{R}$ defines a mixed normal state on $\mathcal{B}_F$ by means of the equation

$$\rho(F^{-1}(S)) = \int_{\mathbb{R}} \chi_S(q)\rho(q) \, dq = \int_S \rho(q) \, dq. \tag{14}$$

Any particular dart in the "ensemble" should, however, be characterized by a *pure* state on $\mathcal{B}_F$. (i.e., there is some point $\lambda \in \mathbb{R}$ such that the dart has

probability 1 of hitting that point). And, in fact, everything works out fine in this case since there *are* pure normal states of $\mathcal{B}_F$. Indeed, the atoms in $\mathcal{B}_F$ are just lines in $\mathbb{R}^2$ on which F is constant: $A_\lambda = \{(q, p) : F((q, p)) = \lambda\}$. Since each atom in a Boolean algebra gives rise to a pure normal state, there is a normal pure state ω_λ of $\mathcal{B}_F$ defined explicitly by the condition that $\omega_\lambda(S) = 1$ if and only if $A_\lambda \subseteq S$. Conversely, if ω is a pure normal state of $\mathcal{B}_F$, then $\omega = \omega_\lambda$ for some $\lambda \in \mathbb{R}$.[1] Thus, the pure normal states on $\mathcal{B}_F$ are in one-to-one correspondence with the sets on which F takes a constant value. Finally, it is easy to see that the distribution ρ does admit interpretation as a measure of ignorance over the pure normal states, namely $\rho = \int \omega_\lambda d\rho$.

However, before we grant that there is a fundamental difference here between the quantum and classical cases, let's think a bit more carefully about the interpretation of the mathematical objects we've been dealing with. In particular, an element $F^{-1}(S) \in \mathcal{B}_F$ is interpreted as a statement that *ascribes a property* to the particle, namely the statement *"The particle is located in S."* Now take an element $E(S) \in \mathcal{O}_Q$. Is it the quantum equivalent of the classical statement $F^{-1}(S)$; i.e., an ascription of a property *"located in S"*? If we interpret $E(S)$ this way, then it is trivially true that a quantum particle cannot have a sharp location:

> Let $\lambda \in \mathbb{R}$. Then $E(\{\lambda\}) = \mathbf{0}$ since $\{\lambda\}$ has Lebesgue measure zero. Thus, the proposition *"The particle is located at λ"* is a contradiction. *QED*

Thus the property ascription interpretation of elements of $\mathcal{O}_Q$ rules out our being able to describe particles with sharp locations. However, the property ascription interpretation is not the traditional interpretation of projection operators in quantum theory. According to the traditional interpretation, the projection operators are *observables* or *experimental propositions*. This is sometimes made precise by saying that $E(S)$ represents the statement:

> *"A measurement of $\widehat{Q}$ will yield a value lying in S."*

Under this interpretation, $E(\{\lambda\}) = \mathbf{0}$ does *not* entail that a particle cannot be located at λ; it simply means that it is false (in fact, a contradiction) to say that any measurement of $\widehat{Q}$ will yield the precise value λ. (This,

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[1]First show that $\omega(F^{-1}(S)) = 1$ for some compact set S. Standard topological arguments then show that $\omega(F^{-1}(V)) = 1$ for every open neighborhood V or some point $\lambda \in \mathbb{R}$. Now use the fact that $\{\lambda\}$ is the intersection of a countable family of open sets.

for example, may be the result of the fact that precise measurements of $\widehat{Q}$ are impossible.) But now, if $\mathcal{O}_Q$ consists of observables, while $\mathcal{B}_F$ consists of property ascriptions, differences in the state spaces of the two logics do not necessarily reflect differences in the descriptive resources of classical and quantum mechanics.

The distinction between observables (experimental propositions) and property ascriptions should not be drawn only in quantum physics. For example, although $F^{-1}(\lambda) \in \mathcal{B}_F$ is a classical property ascription, it is doubtful that any classical experiment ever determines that the value of the quantity F is the *exact* real number λ; and thus the proposition "Val$(F) = \lambda$" is not really an experimental proposition — even in the classical case. The same intuition is expressed by Birkhoff and von Neumann [1] who claim that, it would be absurd,

> "to call an "experimental proposition," the assertion that the angular momentum (in radians per second) of the earth around the sun was at a particular instant a rational number!" (p. 825)

Indeed, what experiment would we perform in order to determine that it was a rational number?

There will certainly be a number of different ways to give a mathematically precise account of the notion of approximate measurement. Here, though, I will just explain briefly von Neumann's [24, 595-598] account, without trying to argue positively for its merits. I wish simply to show that according to von Neumann's account of approximate measurement in *classical* physics, the logic of classical experimental propositions (for one continuous quantity) is mathematically identical to $\mathcal{O}_Q$.

The fact that measurements are of arbitrarily good, but always finite, precision is given formal statement by von Neumann as follows:

> For each $n \in \mathbb{N}$, the value space Ω of F may be partitioned into a finite or countably infinite family of measurable sets $\{N_i^{(n)} : i = 1, 2, \ldots\}$, of which each two have measure-zero overlap. We may do this so that the partition $\{N_i^{(n+1)}\}$ is strictly finer than $\{N_i^{(n)}\}$ for each n, and if we let δ_n denote the maximum diameter of the sets in $\{N_i^{(n)}\}$, then $\delta_n \to 0$. We say that some procedure is a F-measurement of n-th level precision just in case it determines the $i \in \mathbb{N}$ such that Val$(F) \in N_i^{(n)}$. Then, F-measurements of all finite levels of precision are possible.

With this as the basic principle of measurement, von Neumann proves: For any set $S \in \mathbb{R}$, we can determine via measurement (up to an arbitrary level of confidence) whether $\text{Val}(F) \in S$, if and only if S has non-vanishing Lebesgue measure. Accordingly, it is not possible to determine with any good level of confidence that $\text{Val}(F)$ lies in a Lebesgue measure zero set — in agreement with the intuition that "$\text{Val}(F) = \lambda$" and "$\text{Val}(F)$ *is a rational number*" are not experimental propositions. Furthermore, if two subsets $S_1, S_2 \in \mathcal{B}(\mathbb{R})$ agree except possibly on a set of measure zero, then there is no way to determine that $\text{Val}(F) \in S_1 \backslash S_2$ or that $\text{Val}(F) \in S_2 \backslash S_1$. Thus, two elements $S_1, S_2 \in \mathcal{B}_F$ define the same classical observable just in case $S_1 \triangle S_2 = (S_1 \backslash S_2) \cup (S_2 \backslash S_1) \in \mathcal{N}$. But then the logic of classical observables $\mathcal{O}_F$ for the quantity F is *also* the quotient Boolean algebra $\mathcal{B}(\mathbb{R})/\mathcal{N}$. (See p. 825 of Ref. 1.)

Now, Teller claims that,

> "...if we believe that systems possess exact values for continuous quantities, classical theory contains the descriptive resources for attributing such values to the system, whether or not measurements are taken to be imprecise in some sense. Quantum mechanics has no such descriptive resource" (p. 352).

Apparently, then, Teller endorses the claim that classical mechanics does have states assigning exact values to a continuous quantity, even after we take into account the fact that absolutely precise measurements are not possible. However, when we take that into account, the logic of classical experimental propositions $\mathcal{O}_F$ is *isomorphic* to the logic of quantum experimental propositions $\mathcal{O}_Q$. Moreover, if we require quantum states to be σ-additive (thereby depriving quantum theory of the resources to describe exact values), we should also require classical states to be σ-additive — which would also deprive classical theory of the resources to describe exact values. But this conclusion cannot be right: Classical theory does already have these resources to describe exact values, namely points in phase space. The difficulty, then, is not with a lack of resources; but rather with the connection between the theoretical description of point values (viz., pure normal states on $\mathcal{B}_F$) and the "surface" description given by states on $\mathcal{O}_F$. I will argue in the next section that the solution to this difficulty is that the "hidden states" give rise to finitely-additive (singular) states on $\mathcal{O}_F$.

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2.4 Singular states

Let $\mathcal{S}$ be any family of elements in $\mathcal{B}(\mathbb{R})/\mathcal{N}$ with the finite meet property. That is, for any finite family $\{A_1, \ldots, A_n\} \subseteq \mathcal{S}$, we have

$$A_1 \wedge \cdots \wedge A_n \neq \mathbf{0}. \tag{15}$$

Then, the Ultrafilter Extension Theorem [19, p. 339] entails that there is some 2-valued homomorphism ω on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ such that $\omega(A) = 1$ for all $A \in \mathcal{S}$. In particular, $\mathcal{B}(\mathbb{R})/\mathcal{N}$ *does* have pure states, even though they are not σ-additive. Furthermore, any probability distribution ρ on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ can be written explicitly as a mixture of pure states; i.e., there is some measure μ on the pure state space of $\mathcal{B}(\mathbb{R})/\mathcal{N}$ such that

$$\rho = \int \omega_\alpha \, d\mu(\alpha) \,. \tag{16}$$

Thus, the distribution ρ does admit an ignorance interpretation; viz., it measures our ignorance of the (singular) state which describes an individual particle in the ensemble.

The pure states on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ should be thought of as "surface states." That is, each pure state on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ gives a consistent catalog of sure-fire responses of the system to all possible measurements. I will now argue that for each $\lambda \in \mathbb{R}$, there is at least one surface state whose predictions can be interpreted as consistent with the fact that $\mathrm{Val}(F) = \lambda$, and inconsistent with the fact that $\mathrm{Val}(F) = \xi$ for any $\xi \neq \lambda$.

Consider again the dart throwing example. According to the von Neumann measurement theory, there is no experiment which would tell us the real number λ at which the dart lands. However, we can rule out segments of the dart board. In particular, choose some open segment V along the line. Then, if the system responds 0 to the measurement $\pi(V)$, we are entitled to conclude that the dart did not land at any point in the segment V. In other words, a necessary condition for the value of F to be λ in the state ω is that:

$$\omega(\pi(V)) = 1, \text{ for all } V \in \mathcal{F}_\lambda \qquad (*)$$

where $\mathcal{F}_\lambda$ is the family of open neighborhoods of λ. Suppose, conversely, that ω satisfies $(*)$ with respect to some point $\lambda \in \mathbb{R}$. (If $(*)$ holds, we say that the state ω converges to the point λ.) Then, for any other point ξ along the line, there is some open neighborhood W of ξ which excludes λ. A

measurement of $\pi(W)$ in the state ω then shows that the dart did not land at ξ. Thus, when ω converges to λ, the only interpretation consistent with the von Neumann theory of measurement is that $\text{Val}(F) = \lambda$; that is, the "hidden" state of the system is given by some phase space point along the line $\{(q,p) : F((q,p)) = \lambda\}$.

Now consider again some mixed state ρ on $\mathcal{B}(\mathbb{R})/\mathcal{N}$. Then, ρ is a mixture of pure states as in Eq. 16, and each pure state assigns some definite value to F. (We do have to be sure that ρ falls off sufficiently quickly at infinity. See Sec. 3 of Ref. 12.) Thus, this validates the idea that in classical mechanics, the distribution ρ represents the "...probability for the value of the measured quantity to have one or another of the exact values in the support of the distribution."

However, everything we have just said about singular states of $\mathcal{B}(\mathbb{R})/\mathcal{N}$ was entirely neutral as to whether we were discussing classical or quantum mechanics. Thus, a wavefunction $|\psi|^2$ may also be thought of as representing a probability distribution for $\widehat{Q}$ to have one or another of the exact values in the support of $|\psi|^2$.[2] According to Teller, though, even if there were some mathematically acceptable method for describing sharp positions in QM, doing so would not be *physically* acceptable:

> "Such an extended physical theory would describe systems having totally indeterminate momentum, not even highly localized to any finite interval. Such systems would have an infinite expectation value for their kinetic energy" (p. 353).

This argument, however, is not a criticism of singular states *per se*. Since the energy observable is an unbounded operator, there are vectors in the Hilbert space that are not in the domain of this observable, and so these vectors too will assign "infinite kinetic energy" to the system. Thus, following Teller's reasoning here to its logical conclusion, not every (unit) vector in the Hilbert space gives a physically realizable state; rather, only those vectors in the domain of the energy observable give rise to physically realizable states. But

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[2] According to Clifton [7, p. 32], the Bohm theory may be thought of as interpreting each quantum state as a mixture of singular states which assign definite values to $\widehat{Q}$. It remains to be seen, however, the extent to which singular states accurately represent the hidden variables of the Bohm theory, including their dynamics. More generally, singular states are used in [12] to generalize Bub and Clifton's [2, 3] classification scheme for hidden variable theories (which, in its original form, applied only to the case of finite-dimensional Hilbert spaces).

the same reasoning should also apply to vectors that are not in the domain of some other physically relevant observable (such as $\widehat{Q}$). And once we've cleansed the Hilbert space of vectors not in the domains of all physically relevant observables, would anything be left?

Besides these obvious retorts, it is not completely clear what is meant by Teller's talk about assigning infinite expectation values. What we do know — in the case of pure (singular) states of $\mathcal{O}_Q$ — is that these states cannot be thought of as assigning some finite expectation values to the energy observable. (Precisely: Each pure, convergent state on $\mathcal{O}_Q$ gives rise to a mixed state on the logic of experimental propositions about energy, and measure one of the pure states in its integral decomposition do not converge to a finite value. See Ref. 12, Section 3.) It may be mathematically convenient, then, to adjoin ∞ to the normal range of values, but we should *not* say that these states assign "∞" to the energy observable. Rather, we should think of this infinity — as we do of the infinities related to singularities in General Relativity — as indicating the limits of the descriptive capabilities of our theory. And if, with Teller, we take the theory to be descriptively complete, then there is nothing more to describe about energy in these situations.

3 Potential difficulties with singular states

In the previous section, I have argued that singular states permit us to give an ignorance interpretation of the probability distributions associated with individual continuous quantities in QM. I imagine my reader may still not be convinced, though, of the acceptability of singular states. In this section, I will confirm that there are indeed difficulties with these singular states. In the final section, however, I will show that these difficulties can be remedied by taking a more modest view of the measurements that can be made on a continuous quantity.

It would be quite natural to think that if we pick a point $\lambda \in \mathbb{R}$, it determines uniquely a pure state ω on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ that converges to λ. There is, however, a difficulty. Suppose we divide the screen (or dartboard) into two halves $A = \{q : q > \lambda\}$ and $B = \{q : q < \lambda\}$. Then,

$$\pi(A) \vee \pi(B) = \pi(A) \vee \pi(B) \vee \pi(\{\lambda\}) = \pi(\mathbb{R}) = \mathbf{I}, \qquad (17)$$

since $\{\lambda\}$ has Lebesgue measure zero. Thus,

$$\omega(\pi(A)) + \omega(\pi(B)) = 1, \qquad (18)$$

while either $\omega(\pi(A)) = 0$ or $\omega(\pi(B)) = 0$. Thus, any pure state ω must assign 1 to either $\pi(A)$ or $\pi(B)$, but not to both. What would be the right answer for a state in which the particle is located at λ?

Actually, it is easy to show that both value assignments are consistent with the particle being located at λ. Indeed, let $\mathcal{S}_1 = \mathcal{F}_\lambda \cup \{\pi(A)\}$ and let $\mathcal{S}_2 = \mathcal{F}_\lambda \cup \{\pi(B)\}$. Since both $\mathcal{S}_1$ and $\mathcal{S}_2$ have the finite meet property, there is an ultrafilter $\mathcal{U}_1$ which contains $\mathcal{S}_1$ and an ultrafilter $\mathcal{U}_2$ which contains $\mathcal{S}_2$. Since $\mathcal{U}_1$ and $\mathcal{U}_2$ both converge to λ, both should be interpreted as surface manifestations of some "hidden" state in which the particle is located at λ. On the other hand, the particle's being in some hidden state (λ, p), gives no information concerning its disposition to respond to a measurement of $\pi(A)$.

The apparent mismatch between hidden states and surface states is, in fact, quite severe.

Proposition 2. *For each $\lambda \in \mathbb{R}$, there are at least $\aleph_0$ distinct ultrafilters on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ that converge to λ.*

In order to prove this proposition, we will first require a lemma.

Lemma 1. *Suppose that $\{B_j : j \in J\}$ is a family of open subsets of $\mathbb{R}$ such that $B_i \cap B_j = \emptyset$ when $i \neq j$ and such that $\lambda \in \mathrm{clo}(B_j)$ for all $j \in J$. Then, there is a family $\{\mathcal{U}_j : j \in J\}$ of distinct ultrafilters on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ such that $\pi(\mathcal{F}_\lambda) \subseteq \mathcal{U}_j$ for all $j \in J$.*

Proof. Let $\{B_j : j \in J\}$ be given as above. For each $j \in J$, let

$$\mathcal{S}_j = \pi(\mathcal{F}_\lambda) \cup \{\pi(B_j)\}. \tag{19}$$

From the Ultrafilter Extension Theorem, $\mathcal{S}_j$ is contained in a Boolean ultrafilter $\mathcal{U}_j$ if and only if the meet of any finite collection of elements in $\mathcal{S}_j$ is nonzero. Let $\{\pi(U_1), \pi(U_2), \ldots, \pi(U_n)\} \subseteq \mathcal{S}_j$. If $U_i \in \mathcal{F}_\lambda$ for each i, then $\cap_{i=1}^n U_i$ is an open set that contains λ. If $U_1 = B_j$, and $U_i \in \mathcal{F}_\lambda$ for $i \geq 2$, then $\cap_{i=2}^n U_i$ is an open set that contains λ, and since $\lambda \in \mathrm{clo}(B_j)$, $\cap_{i=1}^n U_i$ is a nonempty open set. In either case, the open set $O := \cap_{i=1}^n U_i$ is nonempty. Since Lebesgue measure does not vanish on any nonempty open set, we have

$$\wedge_{i=1}^n \pi(U_i) = \pi\left(\cap_{i=1}^n U_i\right) = \pi(O) \neq 0. \tag{20}$$

Thus, there is an ultrafilter $\mathcal{U}_j$ that contains $\mathcal{S}_j$. Since $\pi(B_i) \wedge \pi(B_j) = 0$ when $i \neq j$, it follows that $\mathcal{U}_i \neq \mathcal{U}_j$ when $i \neq j$. $\qquad\square$

Proof of the proposition. We will consider the case where $\lambda = 0$. In light of the lemma, it will suffice to construct a family $\{B_n : n \in \mathbb{N}\}$ of open subsets of $\mathbb{R}$ such that $B_n \cap B_m = \emptyset$ when $n \neq m$ and such that $0 \in \mathrm{clo}(B_n)$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N} \cup \{0\}$, let $a_n = 2^{-n}$. For each $n, m \in \mathbb{N}$, let $b_{nm} = (a_n + a_{n-1})/2^m$, and let

$$A_m = \bigcup_{n=1}^{\infty} (a_n, b_{nm}). \tag{21}$$

Note that $A_m \supset A_{m+1}$ for all $m \in \mathbb{N}$, and in fact $A_m - A_{m+1}$ has nonempty interior. Thus, if we set

$$B_m = \mathrm{int}(A_m - A_{m+1}), \tag{22}$$

then B_m is a nonempty open set. We must show that $B_i \cap B_j = \emptyset$ when $i \neq j$ and that $0 \in \mathrm{clo}(B_i)$ for all i. Suppose that $i \neq j$. We may assume that $i + 1 \leq j$, in which case $A_{i+1} \supseteq A_j$, and

$$B_i \cap B_j \subseteq (A_i - A_{i+1}) \cap (A_j - A_{j+1}) \subseteq A_j - A_{i+1} = \emptyset. \tag{23}$$

It is easy to verify that the sequence $\{b_{nm} : n \in \mathbb{N}\}$ is in $\mathrm{clo}(B_m)$ for each m. But $\lim_n b_{nm} = 0$ and since $\mathrm{clo}(\mathrm{clo}(B_m)) = \mathrm{clo}(B_m)$, it follows that $0 \in \mathrm{clo}(B_m)$. $\qquad\square$

Thus, even though we are able to explain probabilistic dispositions as measures of ignorance of sure-fire dispositions, we are not able to further reduce the latter to categorical location properties. So, one could still think that attributing categorical location properties is objectionable on the grounds that these attributions are explanatorily bankrupt.

There is also a serious practical obstacle for using singular states as the explanans for a reduction of probabilistic dispositions; namely, none of these singular states can be explicitly defined. By saying that these singular states cannot be "explicitly defined," I mean to contrast this with the pure normal states of $\mathcal{B}(\mathbb{R})$ given by principal ultrafilters, as well as with mixed normal states on $\mathcal{B}(\mathbb{R})/\mathcal{N}$. First of all, it is quite trivial to give an explicit definition of a principal ultrafilter on $\mathcal{B}(\mathbb{R})$. Indeed, once I choose some $\lambda \in \mathbb{R}$, then (trivially) if you give me a Borel set S, then I can tell you whether or not S is in the principal ultrafilter generated by $\{\lambda\}$. We have also seen that

any quantum-mechanical wavefunction ψ defines a mixed normal state ρ_ψ on $\mathcal{B}(\mathbb{R})/\mathcal{N}$ by means of the formula

$$\rho_\psi(\pi(S)) = \int_S |\psi(q)|^2 dq. \tag{24}$$

Since it is possible to explicitly define a wavefunction ψ, Eq. 24 gives an explicit recipe for computing the value $\rho_\psi(\pi(S))$ for any Borel set S.

On the other hand, although we "know" that there are ultrafilters (i.e., pure states) on $\mathcal{B}(\mathbb{R})/\mathcal{N}$, we do not know this because someone has constructed an example of such an ultrafilter. In order to obtain an ultrafilter, we note that a certain (explicitly defined) family $\mathcal{S}$ of elements in $\mathcal{B}(\mathbb{R})/\mathcal{N}$ has the finite meet property, and then we invoke the Ultrafilter Extension Theorem to extend $\mathcal{S}$ to an ultrafilter $\mathcal{U}$. This extension procedure is a paradigm example of nonconstructive mathematics: We are told that there is some pure state ω on $\mathcal{B}(\mathbb{R})/\mathcal{N}$, but we are not given a recipe for determining the value $\omega(A)$ for an arbitrary element $A \in \mathcal{B}(\mathbb{R})/\mathcal{N}$.

A strong case can be made that the sometimes imprecise distinction between "giving an explicit example" and "proving existence (nonconstructively)," corresponds to a precise distinction in the strength of the set-theoretic axioms used to prove the existence of the object in question (see Ref. 19). In particular, many concrete mathematical objects (such as the Real numbers [17, Appendix]) are constructed using the Principle of Recursive Constructions:

> (PRC) *Suppose X is a non-empty set, and let a function G be given from the set of finite sequences in X into $\mathcal{P}(X)\backslash\{\emptyset\}$. Then there exists exactly one function $F : \mathbb{N} \mapsto X$ such that $F(0) \in G(\emptyset)$ and $F(n) \in G(F(0), \ldots, F(n-1))$ for all $n > 0$.*

PRC is equivalent in ZF to the axiom of Dependent Choices [14, p. 147]:

> (DC) *If R is a binary relation on a nonempty set X such that for every $x \in X$ there exists a $y \in X$ with $\langle x, y\rangle \in R$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that $\langle x_n, x_{n+1}\rangle \in R$ for every $n \in \mathbb{N}$.*

DC is a very weak form of the Axiom of Choice: ZF+DC entails the Axiom of Countable Choice, but ZF+DC does not entail the Ultrafilter Extension Theorem, nor does it entail any of the "paradoxical" consequences of ZF+AC

(e.g. the Banach-Tarski paradox) [19, Chap. 6]. In fact, a good case can be made that *applied* mathematics makes use *only* of those objects in the ZF+DC universe [26]. For example, all theorems of classical (19th century) analysis, and even all theorems of the traditional Hilbert space formalism of quantum mechanics can be proved in ZF+DC [10].

Following [19], let us say that a mathematical object is *intangible* just in case that object exists in the ZF+AC universe, but that object cannot be proved to exist in the ZF+DC universe. For example, a free ultrafilter on $\mathbb{N}$ (i.e., an ultrafilter containing all subsets of $\mathbb{N}$ whose complements are finite) is an intangible [18]. We now show that pure states of $\mathcal{B}(\mathbb{R})/\mathcal{N}$ are also intangibles.

Proposition 3. *It cannot be proved in ZF+DC that there is a pure state of* $\mathcal{B}(\mathbb{R})/\mathcal{N}$.

Proof. Let WUF (weak-ultrafilter principle) denote the statement that there is a free ultrafilter on $\mathbb{N}$, and let PS denote the statement that there is a pure state on $\mathcal{B}(\mathbb{R})/\mathcal{N}$. We must show that ZF+DC+¬PS is consistent. Since ZF+DC+¬WUF is consistent [18], it will suffice to show that ZF+DC⊢ (PS→WUF). Thus, suppose that PS holds; i.e., there is an ultrafilter $\mathcal{U}$ on $\mathcal{B}(\mathbb{R})/\mathcal{N}$. By Prop. 1, $\mathcal{U}$ is not closed under countable meets; i.e., there is a family $\{P_n : n \in \mathbb{N}\}$ of elements of $\mathcal{B}(\mathbb{R})/\mathcal{N}$ such that $P_n \in \mathcal{U}$ for all $n \in \mathbb{N}$, but $\wedge_{n \in \mathbb{N}} P_n \notin \mathcal{U}$. (It's important to note that Prop. 1 uses no choice axiom stronger than DC.) Define a family $\mathcal{J}$ of subsets of $\mathbb{N}$ by

$$J \in \mathcal{J} \quad \Longleftrightarrow \quad \wedge_{n \in J} P_n \notin \mathcal{U}. \tag{25}$$

It is easily verified that $\mathcal{J}$ is an ultrafilter in $\mathcal{P}(\mathbb{N})$ that contains all subsets of $\mathbb{N}$ whose complements are finite. □

4 The logic of unsharp experimental propositions

I have argued that it is possible to give an ignorance interpretation of the probabilities associated with a continuous quantity in quantum mechanics. In order to do so, however, I had to make use of singular states, which turn out to have some pretty odd properties that militate against their playing a serious role in explaining a particle's probabilistic dispositions. Some might attribute

the oddities encountered in the last section to the fact that we permitted ourselves to use states that are not σ-additive. On the contrary, I will now argue that the difficulty is not with the states but with the observables: We were trying to account for too many measurement outcomes.

If we take $\mathcal{B}(\mathbb{R})/\mathcal{N}$ to be in one-to-one correspondence with possible experiments, then Prop. 2 shows that there are experiments whose outcomes could distinguish two states, both of which correspond to one point $\lambda \in \mathbb{R}$. If, however, these distinguishing experiments cannot in fact be performed, then these two states corresponding to λ are empirically equivalent. In this section, I will show how to make this idea precise using the fact that real experimental questions are always "unsharp."

Consider again a person, Bob, throwing darts at a one-dimensional dartboard. With the knowledge we have of Bob's skills, we choose a region S on the dartboard which we feel confident Bob can hit on 100 consecutive throws. Since we don't have time, though, to watch each of Bob's throws, we rig a device to keep score. In particular, each time a dart hits the board, we get a printout that records y when the dart lands in S and that records n when the dart lands outside of S. Now, after Bob has made his 100 throws, we take a look at the printout: there are 98 y's and 2 n's. Do we then conclude that Bob was not as skilled a dart thrower as we thought? Not necessarily. Perhaps it was our score-keeping device, not Bob, which failed on those two counts. Or, perhaps some environmental factor interfered with two of Bob's throws. In any case, the point is that in a real experimental situation, the measuring device itself and the environment introduce factors of uncertainty which have to be taken into account when interpreting measurement results.

To make the imperfections in the experimental setup mathematically precise within the confines of the standard Hilbert space formalism of QM, we can associate a confidence measure e with a given measuring device. (Here I follow Busch et al. [4].) If, using this measuring device, we attempt to test for the claim "the particle is in S," then what we actually measure is the "smeared" observable

$$E^e(S) := \int_{\mathbb{R}} e(q)\, E(S+q)\, dq = (\chi_S * e)(\widehat{Q}). \tag{26}$$

Here the operation $*$ is the convolution product defined by

$$(f * g)(q_0) := \int_{\mathbb{R}} f(q)\, g(q_0 - q)\, dq. \tag{27}$$

For a fixed confidence measure e, the mapping $S \mapsto E^e(S)$ is a positive operator valued (POV) measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If S_1 and S_2 are disjoint Borel sets, then we represent the disjunctive question *"Is the particle in $S_1 \cup S_2$?"* formally by

$$E^e(S_1) \oplus E^e(S_2) := E^e(S_1) + E^e(S_2). \tag{28}$$

The conjunctive question *"Is the particle in S_1 and S_2?"* is represented by the operator $E^e(S_1 \cap S_2)$. In general, however, it is not the case that

$$E^e(S_1 \cap S_2) = E^e(S_1)E^e(S_2). \tag{29}$$

In fact, the operator on the right-hand side of Eq. 29 will not generally be in the range of E^e. Finally, the question *"Is the particle in the complement of S?"* is represented formally by

$$\neg E^e(S) := E^e(\mathbb{R} \backslash S). \tag{30}$$

We obtain the standard projection valued (PV) measure E if we apply the above construction to the case in which we have absolute confidence in the accuracy of our measuring apparatus. Formally, if we let δ be the Dirac delta function, then

$$E^\delta(S) = (\chi_S * \delta)(\widehat{Q}) = \chi_S(\widehat{Q}) = E(S), \tag{31}$$

for all Borel sets S. On the other hand, if e has non-zero deviation (i.e., e is an integrable *function*), then for each Borel set S, $(\chi_S * e)$ is a uniformly continuous function [13, Prop. 3.2].

Realistically, then, the family $\mathcal{E}$ of experimental propositions about the location of the particle should consist only of elements of the form $E^e(S)$, where e is some confidence function with non-zero deviation. Let's be generous, though, and suppose that $\mathcal{E}$ contains all operators of the form $f(\widehat{Q})$ where f is a uniformly continuous function from $\mathbb{R}$ into $[0, 1]$. If we say that $A \perp B$ just in case $A + B \leq \mathbf{I}$, then we may extend the exclusive disjunction defined in Eq. 28 by setting

$$A \oplus B := A + B, \tag{32}$$

when $A \perp B$. It is easy to verify then that $(\mathcal{E}, \oplus, \mathbf{0}, \mathbf{I})$ is an *effect algebra*, or *unsharp quantum logic* [5, 6, 9]. (Recall that every Boolean algebra is an effect algebra if we set: $A \perp B$ iff. $A \wedge B = \mathbf{0}$, and $A \oplus B = A \vee B$ [9].) Thus,

in particular, for each $A \in \mathcal{E}$, there is a unique element $\neg A = \mathbf{I} - A \in \mathcal{E}$ such that $A \perp \neg A$ and $A \oplus \neg A = \mathbf{I}$. For $A, B \in \mathcal{E}$, we say that $A \leq B$ just in case there is some $C \perp A$ such that $A \oplus C = B$. A *state* on $\mathcal{E}$ is a mapping $\omega : \mathcal{E} \mapsto [0,1]$ such that $\omega(\mathbf{I}) = 1$ and

$$\omega(A \oplus B) = \omega(A) + \omega(B), \tag{33}$$

whenever $A \oplus B$ is defined. From this it follows that whenever $A, B \in \mathcal{E}$ and $A \leq B$, then $\omega(A) \leq \omega(B)$ and

$$\omega(B - A) = \omega(B) - \omega(A). \tag{34}$$

We say that a state ω of $\mathcal{E}$ is pure just in case: If $\omega = a\rho + (1-a)\tau$, for some $a \in (0,1)$, then $\omega = \rho = \tau$. Unlike the Boolean case, however, a pure state of a general effect algebra may take on any value in the interval $[0,1]$. (For example, even if Bob hits the bullseye on each try out of 100, our less than ideal score-keeper might only credit him with only, say, 98 hits.)

I began this section with the idea that there are too many observables in $\mathcal{O}_Q \cong \mathcal{B}(\mathbb{R})/\mathcal{N}$, and that we should be more modest about what can actually be measured. Let me show precisely, now, how the effect algebra $\mathcal{E}$ does represent a more modest perspective on what the *experimental* propositions about position truly are.

Both $\mathcal{E}$ and $\mathcal{O}_Q$ are subsets of the algebra $\mathcal{R}_Q$ of all operators of the form $f(\widehat{Q})$, where f is some Borel function from $\mathbb{R}$ into $\mathbb{C}$. Although the intersection of $\mathcal{E}$ and $\mathcal{O}_Q$ contains only $\mathbf{0}$ and $\mathbf{I}$, we should think of $\mathcal{E}$ as containing a much smaller set of observables than $\mathcal{O}_Q$. Indeed, any element in $\mathcal{E}$ may be uniformly approximated by linear combinations of elements in $\mathcal{O}_Q$ (by the spectral theorem). Thus, any pure state ω on $\mathcal{O}_Q$ will give rise to a unique pure state $\omega|_{\mathcal{E}}$ on $\mathcal{E}$. (This slight abuse of notation is justified by the fact that pure states on $\mathcal{O}_Q$ extend uniquely to pure states of the von Neumann algebra $\mathcal{R}_Q$.) On the other hand, since the family of uniformly continuous functions is closed under linear combinations and uniform limits, no element of $\mathcal{O}_Q$ (other than $\mathbf{0}$ and $\mathbf{I}$) can be approximated by linear combinations of elements in $\mathcal{E}$. Thus, $\mathcal{E}$ is "coarser grained" than $\mathcal{O}_Q$ in the sense that two states which give different outcomes for measurements in $\mathcal{O}_Q$ may give identical outcomes for all measurements in $\mathcal{E}$. And, indeed, the elements of $\mathcal{E}$ do not permit us to distinguish between states that correspond to a common point $\lambda \in \mathbb{R}$.

Proposition 4. *Let ω, ρ be pure states of $\mathcal{O}_Q$, both of which converge to λ. Then,*

$$\omega(f(\widehat{Q})) = \rho(f(\widehat{Q})) = f(\lambda), \tag{35}$$

for all $f(\widehat{Q}) \in \mathcal{E}$.

Proof. See Prop. 3.4 of Ref. 12. $\square$

This proposition gives us everything we want. For each $\lambda \in \mathbb{R}$, there is precisely one (explicitly defined) pure state ω_λ of $\mathcal{E}$ corresponding to λ. (Moreover, it can be shown that every "convergent" state of $\mathcal{E}$ is of this form. See Proposition 5 in the appendix.) Therefore, any statistical state ρ_ψ of $\mathcal{E}$ decomposes as an integral

$$\rho_\psi = \int_{\mathbb{R}} \omega_\lambda \, d\mu(\lambda), \tag{36}$$

which permits interpretation as a measure of our ignorance of the precise, categorical location property of individual particles in the ensemble.

One might now object that, after all, I have abandoned the standard formulation of QM for a richer formulation that permits ascription of precise values to continuous quantities. In one trivial sense, this is true: the singular states I have invoked are not typically mentioned in standard formulations of QM (e.g., von Neumann [25] or Mackey [16]). However, singular states are *implicit* in the standard formulation of QM, in the sense that they share all features of vector states which entitle the latter to be considered candidates for representing physical reality. Thus, while von Neumann and Mackey did not avail themselves of singular states, there is no reason in principal why they could not have done so.

I have also gone beyond von Neumann and Mackey by making use of positive-operator valued measures. However, since the range of a POV measure consists simply of operators acting on Hilbert space, it is not clear why one would think of this as a departure from the Hilbert-space formulation. Of course, I have departed from one popular interpretation of the formalism according to which spectral projections of any quantity can, in some contexts, represent properties of the system. Instead, I have claimed, these spectral projections should be interpreted as "experimental propositions;" and, correspondingly, arbitrary POV measures represent varying degrees of confidence we have in our measuring apparatuses.

In the case where we have less than perfect confidence in our measuring apparatus, it is clear how to relate an underlying description in which a continuous quantity possesses a sharp value with a surface description in terms of possible measurement outcomes. We must follow this strategy not only in quantum mechanics, but also in classical mechanics. On the level of individual continuous quantities, the theories are on a par.

A   Appendix

Lemma 2. *Let ω be a state of $\mathcal{E}$. Then $\omega(aA) = a\,\omega(A)$ for all $A \in \mathcal{E}$ and $a \in (0,1)$.*

Proof. Suppose first that $a = 1/2$. Then, $\omega(A) - \omega((1/2)A) = \omega((1/2)A)$ and so $\omega((1/2)A) = (1/2)\omega(A)$. It then follows easily by induction that $\omega(aA) = a\omega(A)$ when $a = 2^{-n}$ for some $n \in \mathbb{N}$.

Suppose now that a is a diadic rational; i.e., $a = m/2^n$ for some $m, n \in \mathbb{N}$, where $m < 2^n$. Note then that

$$\frac{m}{2^n}A = \underbrace{\frac{1}{2^n}A \oplus \frac{1}{2^n}A \oplus \cdots \oplus \frac{1}{2^n}A}_{m \text{ times}}, \tag{37}$$

and so

$$\omega\left(\frac{m}{2^n}A\right) = m\omega\left(\frac{1}{2^n}A\right) = \frac{m}{2^n}\omega(A). \tag{38}$$

Finally, let a be an arbitrary element in $(0,1)$. Then, $a = \lim_n a_n$ for some strictly decreasing sequence $\{a_n\} \subseteq (0,1)$ of diadic rationals. Now, for every $N \in \mathbb{N}$, there is some $M \in \mathbb{N}$ such that

$$a_nA - aA \le 2^{-N}\mathbf{I}, \tag{39}$$

for all $n \ge M$, and therefore

$$\omega\left(a_nA - aA\right) \le 2^{-N}, \tag{40}$$

for all $n \geq M$. We may also assume that $a_n - a \leq 2^{-N}$ when $n \geq M$. Thus,

$$
\begin{aligned}
|\omega(aA) - a\,\omega(A)| &= |\omega(aA) - \omega(a_n A) + a_n \omega(A) - a\,\omega(A)| & (41) \\
&\leq |\omega(aA) - \omega(a_n A)| + |a_n \omega(A) - a\,\omega(A)| & (42) \\
&\leq 2^{-N} + 2^{-N}. & (43)
\end{aligned}
$$

Since this is true for all $N \in \mathbb{N}$, it follows that $\omega(aA) = a\,\omega(A)$. $\qquad\square$

There are pure states on $\mathcal{E}$ that assign 0 to all elements of the form $E^e(S)$, where S is a compact subset of $\mathbb{R}$. (That such states exist can be seen from the proof of Proposition 5 below.) When S is compact, then the function $f := (\chi_S * e)$ vanishes at infinity; i.e., for each $\epsilon > 0$, there in a $N \in \mathbb{N}$ such that $f(x) < \epsilon$ when $|x| > N$ [13, Prop. 3.2]. This motivates the following definition.

Definition. Let ω be a pure state of $\mathcal{E}$. We say that ω converges just in case $\omega(f(\widehat{Q})) > 0$ for some f that vanishes at infinity.

Proposition 5. *Let ω be a convergent pure state of $\mathcal{E}$. Then, $\omega(f(\widehat{Q})) = f(\lambda)$ for some $\lambda \in \mathbb{R}$.*

Proof. Recall that the Stone-Cech compactification $\beta\mathbb{R}$ of $\mathbb{R}$ is the unique compact Hausdorff space such that every bounded continuous function $f : \mathbb{R} \mapsto \mathbb{C}$ can be extended uniquely to a continuous function $\overline{f} : \beta\mathbb{R} \mapsto \mathbb{C}$. Let $BUC(\mathbb{R})$ denote the set of bounded, uniformly continuous functions from $\mathbb{R}$ into $\mathbb{C}$. Since a uniform limit of elements in $BUC(\mathbb{R})$ is again in $BUC(\mathbb{R})$, it follows that there is a unique compact Hausdorff space $\gamma\mathbb{R}$ such that every $f \in BUC(\mathbb{R})$ has a unique extension to a continuous function $\overline{f} : \gamma\mathbb{R} \mapsto \mathbb{C}$. We may refer to $\gamma\mathbb{R}$ as the uniform compactification of $\mathbb{R}$. Let $C(\gamma\mathbb{R})$ denote the set of continuous functions from $\gamma\mathbb{R}$ into $\mathbb{C}$. Thus, there is an algebraic isomorphism from $BUC(\mathbb{R})$ onto $C(\gamma\mathbb{R})$ and we may identify $\mathcal{E}$ with the family of functions in $C(\gamma\mathbb{R})$ with range $[0, 1]$.

We show that every (pure) state ω of $\mathcal{E}$ has a unique extension to a (pure) state $\widehat{\omega}$ of the C^*-algebra $C(\gamma\mathbb{R})$. We may then appeal to the result that any pure state $\widehat{\omega}$ on $C(\gamma\mathbb{R})$ is of the form $\widehat{\omega}(f) = f(x_0)$ for some $x_0 \in \gamma\mathbb{R}$ [15, Corollary 3.4.2].

Let ω be a state of $\mathcal{E}$. Let $C^+(\gamma\mathbb{R})$ denote the set of functions from $\gamma\mathbb{R}$ into $[0, +\infty)$. If $f \in C^+(\gamma\mathbb{R})$, then $f/\|f\| \in \mathcal{E}$ and we may define

$$
\widehat{\omega}(f) := \|f\| \cdot \omega\left(f/\|f\|\right). \tag{44}
$$

Using Lemma 2, it is easy to see that $\widehat{\omega}$ is homogenous with respect to positive real numbers, and it is straightforward to verify that $\widehat{\omega}$ is additive over positive functions. Thus, $\widehat{\omega}$ is an additive function from $C^+(\gamma\mathbb{R})$ into $[0,\infty)$. It follows then that $\widehat{\omega}$ extends uniquely to a linear mapping from $C(\gamma\mathbb{R})$ into $\mathbb{C}$ [19, Prop. 11.55].

Suppose now that ω is a pure state of $\mathcal{E}$, and let $\widehat{\omega}$ be the unique extension of ω to $C(\gamma\mathbb{R})$ as defined above. Suppose that $\widehat{\omega} = a\rho + (1-a)\tau$, where ρ, τ are states of $C(\gamma\mathbb{R})$ and $a \in (0,1)$. Then $\rho\,|_{\mathcal{E}}$ and $\tau\,|_{\mathcal{E}}$ are states of $\mathcal{E}$, and $\omega = a\rho\,|_{\mathcal{E}} + (1-a)\tau\,|_{\mathcal{E}}$. However, since ω is a pure state of $\mathcal{E}$, we have $\omega = \rho\,|_{\mathcal{E}} = \tau\,|_{\mathcal{E}}$ and since extensions are unique, it follows that $\widehat{\omega} = \rho = \tau$. Thus, $\widehat{\omega}$ is a pure state of $C(\gamma\mathbb{R})$.

Thus, there is some $x_0 \in \gamma\mathbb{R}$ such that $\widehat{\omega}(f) = f(x_0)$ for each $f \in C(\gamma\mathbb{R})$. We show that if $x_0 \in \gamma\mathbb{R}\backslash\mathbb{R}$, then the state ω does not converge. Indeed, since $\mathbb{R}$ is dense in $\gamma\mathbb{R}$, there is a net $(x_a)_{a\in\mathbb{A}}$ in $\mathbb{R}$ such that $x_a \to x_0$. Let $f \in \mathcal{E}$ be a function that vanishes at infinity. We show that $\overline{f}(x_0) = 0$. Let $\epsilon > 0$ be given. Then, there is an $N \in \mathbb{N}$ such that $\overline{f}(x) < \epsilon$ for all $x \in \mathbb{R}$ with $|x| \geq N$. However, since the set $[-N, N]$ is compact in $\gamma\mathbb{R}$ and $x_0 \notin \mathbb{R}$, there is some $b \in \mathbb{A}$ such that $x_a \notin [-N, N]$ for all $a \geq b$. Since $f(x_0) = \lim f(x_a)$, $f(x_0) < \epsilon$. Since this is true for any $\epsilon > 0$, it follows that $f(x_0) = 0$ and $\omega(f) = 0$. $\qquad\square$

References

[1] Birkhoff, G. and von Neumann, J.: 1936, The logic of quantum mechanics, *Annals of Mathematics* **37**: 823–843.

[2] Bub, J.: 1997, *Interpreting the Quantum World*, Cambridge University Press.

[3] Bub, J., and R. Clifton.: 1996, A uniqueness theory for 'no-collapse' interpretations of quantum mechanics, *Studies in the History and Philosophy of Modern Physics* **27**: 181–219.

[4] Busch, P., Grabowski, M., and Lahti, P.J.: 1997, *Operational Quantum Physics*, Springer, NY.

[5] Dalla Chiara, M.L.: 1994, Unsharp quantum logics, *International Journal of Theorerical Physics* **34**: 1331–1336.

[6] Dalla Chiara, M.L. and Giutini, R.: 1994, Partial and unsharp quantum logics, *Foundations of Physics* **24**: 1161–1177.

[7] Clifton, R.: 1999, Beables in algebraic quantum theory, pp. 12–44 of Jeremy Butterfield and Constantine Pagonis (eds.) *From Physics to Philosophy*, Cambridge University Press.

[8] Fine, Arthur: 1971, Probability in quantum mechanics and in other statistical theories, pp. 79–92 of M. Bunge (ed.) *Problems in the Foundations of Physics*, Springer, NY.

[9] Foulis, D.J., and Bennett, M.K.: 1994, Effect algebras and unsharp quantum logics, *Foundations of Physics* **24**: 1331–1352.

[10] Garnir, H.G., DeWilde M., and Schmets, J.: 1968, *Analyse Fonctionnelle: Théorie constructive des espaces linéaires à semi-normes.* Birkhäuser Verlag, Basel.

[11] Gleason, A.M.: 1957, Measures on the closed subspaces of a Hilbert space, *Journal of Mathematics and Mechanics* **6**(6): 885-893.

[12] Halvorson, H. and Clifton, R.: 1999, Maximal beable subalgebras of quantum mechanical observables, *International Journal of Theoretical Physics* **38**: 2441–2484.

[13] Hirsch, F. and Lacombe, G.: 1999, *Elements of Functional Analysis.* Springer, NY.

[14] Just, W. and Weese, M.: 1991, *Discovering Modern Set Theory*, Providence, RI, American Mathematical Society.

[15] Kadison, R. and Ringrose J.: 1997, *Fundamentals of the Theory of Operator Algebras*, American Mathematical Society, Providence, RI.

[16] Mackey, G.W.: 1968, *Mathematical Foundations of Quantum Mechanics*, W.A. Benjamin, NY.

[17] Moschovakis, Y.N.: 1994, *Notes on Set Theory*, Springer, NY.

[18] Pincus, D. and Solovay, R.M.: 1977, Definability of measures and ultrafilters, *Journal of Symbolic Logic* **42**: 179–190.

[19] Schechter, E.: 1997, *Handbook of Analysis and its Foundations*, Academic Press, NY.

[20] Summers, S.J.: (forthcoming), On the Stone-von Neumann uniqueness theorem and its ramifications, in M. Redei and M. Stoelzner (eds.) *John von Neumann and the Foundations of Quantum Mechanics*, Kluwer, Dordrecht.

[21] Teller, Paul: 1979, Quantum mechanics and the nature of continuous physical quantities, *Journal of Philosophy* LXXVI: 345–361.

[22] Varadarajan, V.S.: 1985, *Geometry of Quantum Theory*, Springer, NY.

[23] von Neumann, J.: 1931, Die Eindeutigkeit der Schrödingerschen Operatoren, *Mathematische Annalen* **104**: 570–578.

[24] von Neumann, J.: 1932, Operatorenmethoden in der klassischen Mechanik, *Annals of Mathematics* **33**: 595–598.

[25] von Neumann, J.: 1932, *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin.

[26] Wright, J.D.M.: 1977, Functional Analysis for the practical man, pp. 283–290 of K.-D. Bierstedt and B. Fuchssteiner (eds.) *Functional Analysis: Surveys and Recent Results*, North-Holland, Amsterdam.