Invariance and ontology in relativistic physics

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1 Introduction

Several physicists and philosophers have claimed that Einstein's theory of
relativity favors an ontology of four-dimensional objects. Witness Sir Arthur
Eddington's classic account of the explanatory role that these objects play:

> [A]n observer on the earth sees and measures an oblong block; an
> observer on another star contemplating the same block finds it to
> be a cube. Shall we say that the oblong block is the real thing,
> and that the other observer must correct his measures to make
> allowance for his motion? All the appearances are accounted
> for if the real object is the four-dimensional, and the observers
> are merely measuring different three dimensional appearances or
> sections; and it seems impossible to doubt that this is the true
> explanation. (Eddington, 1920, p 181)

A suggestive picture, but what exactly is the argument from relativity theory to the fundamentality of four-dimensional objects? According to Yuri
Balashov, "an object viewed as a 4d being is relativistically invariant in a
sense in which its 3d parts are not" (Balashov, 1999, p 659).[1] In a similar
vein, Thomas Sattig claims that

> ... there is a permanent shape standing behind the different three-
> dimensional shapes of the object, namely, an invariant four-dimensional

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[1]Balashov's claim is contested by Davidson (2013), who argues that four-dimensional
objects themselves fail to be relativistically invariant. Balashov (2014) and Calosi (2015)
claim that Davidson's reasoning is faulty. I won't enter into the details of that debate,
but my conclusions would seem to favor Davidson.

shape, rendering the various three-dimensional shapes different perspectival representations of the single invariant shape. (Sattig, 2015, p 220)

More recently, Thomas Hofweber and Marc Lange reassert that the real things are four-dimensional:

> The spacetime interval, as a frame-invariant fact, is the reality, whereas the facts related by the coordinate transformations are frame-dependent facts and hence are appearances of that reality. (Hofweber and Lange, 2017, p 876)

These claims are based on a common idea: real things are invariant, while what varies are mere appearances. In application to Einstein's theory of relativity, the claim is that four-dimensional things are invariant while three-dimensional things are not.

I will argue that these claims misconstrue the situation in relativity theory. In one sense of "invariant", there are no invariant four-dimensional objects. In another other sense of "invariant", there are invariant four-dimensional objects, but there are also invariant three-dimensional objects. In neither case are four-dimensional objects invariant in some way that three-dimensional objects are not.

To be fair, I suspect that these authors have another notion of invariance in mind, a notion exemplified by the fact that the spacetime distance between events can be decomposed in many different ways into spatial and temporal components. While granting that the spacetime description is invariant in this sense, I will question whether statements about the distances between spacetime points explain the events that are observed. I will argue, instead, that the best explanation for these events is typically just the other events that precede them in time, in combination with the dynamical laws of motion. As one particular instance of this kind of explanation, I argue that the best explanation for "shape relativity" is that there are three-dimensional objects that are in motion relative to three-dimensional observers.

2   Options for ontology

The question of concern in this paper is whether we can maintain the common sense view that there are three-dimensional material objects that change over

time, or whether material objects should be conceived of as extended in time. In the wake of the twentieth-century revival of metaphysics, there have been several interesting arguments on both sides of this question. In this paper, I will be concerned primarily with the question of whether the theory of relativity adds some new data that is relevant to this debate. The going view seems to be that, yes, relativity theory tips the balance in favor of four-dimensional objects. My conclusion is that these arguments have been overstated, and that there are in fact good reasons to favor an ontology of three-dimensional objects.

It should be noted, though, that some interpreters of relativity theory might claim that the entire debate is based on a false presupposition, viz. that relativity theory tolerates an ontology of localized objects. For example, some early interpreters of relativity theory argued that it supports an ontology of "events", which are represented by the intersections of the world-lines of particles, even though the particles themselves are, strictly speaking, a fiction.[2] Other early interpreters of relativity theory claimed that it favors an ontology of fields that are spread out through space.[3]

But are there any good arguments for these claims? Are there good arguments that relativity theory does not tolerate an ontology of localized material objects? I have not seen one. Those who argue that relativity theory favors an ontology of point-events typically assume two false things: (1) the verification criterion of meaning, and (2) that general covariance entails that there is no determinate spacetime structure. As for those who see relativity theory as favoring a field-theoretic ontology, it seems that their preference is not based on an argument, but on a program to try to implement the idea of local causality via field equations. In any case, while there are no-go theorems for localized particles in relativistic *quantum* theories (see Halvorson and Clifton, 2002), I have not seen any such theorems for classical relativity theory.

I grant that relativity theory works nicely with fields and their hyperbolic equations of motion. And, of course, there is an ontological program — endorsed by Einstein — that would explain the appearance of localized objects in terms of underlying fields. But I have yet to be convinced that the field-theoretic point of view is forced upon us by relativity theory. In fact, in one

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[2]e.g. Ernst Mach was a full-blown subjective idealist, and his disciple Joseph Petzoldt believed that relativity theory seals the fate of the "mystical" concept of substance.

[3]Here I'm thinking primarily of Einstein.

simple sense, it is trivially true that relativity theory is compatible with an ontology of fundamental localized objects: the trajectory of a material object can be described as a "world tube" in a relativistic spacetime. The situation here is, once again, different from relativistic quantum theories, where there is no mathematical object that could represent the trajectory of a localized material object. I will take it for granted, then, that classical relativity theory permits an ontology of objects that are localized in space. The remaining question is whether these objects must be temporally extended.

3   Examples

A model of Einstein's General Theory of Relativity is an n-dimensional manifold M with metric tensor g. In this case, an n-dimensional material object can be represented by a collection of timelike curves in M — i.e. a "spacetime worm" that is bounded along spacelike hypersurfaces but that is unbounded in timelike directions. Of course, familiar material objects have a finite lifespan, but we can imagine them as stretched out indefinitely in time. Nothing we say here depends on whether these worms have finite or infinite temporal length.

For the most part, the issues of concern are adequately illustrated the Special Theory of Relativity, where the flat Lorentzian manifold (M, g) can be redescribed as a metric affine space of signature $(1, n - 1)$. While we are normally interested in the case of $n = 4$, it will often be convenient to illustrate issues with $n = 2$ or $n = 3$. Nothing of decisive importance for our discussion will hang on the exact dimension of space, although the interesting phenomenon of "shape relativity" only arises when space has at least two dimensions. In particular, when space is three-dimensional, then shape properties such as "...is a cube" or "...is a sphere" are reference-frame dependent.

Here and subsequently we adopt the notation of (Malament, 2009). Let V be the n-dimensional inner-product space over $\mathbb{R}$ that underlies the affine space M. As is typical, we use angle brackets to denote the inner product on V, i.e. if $u, w \in V$, then $\langle u, w \rangle \in \mathbb{R}$. We let $\|u\| = |\langle u, u \rangle|^2$ be the norm of the vector $u \in V$. Recall that a vector $u \in V$ is said to be timelike if $\langle u, u \rangle > 0$, spacelike if $\langle u, u \rangle < 0$, and null if $\langle u, u \rangle = 0$. Timelike vectors $u, w \in V$ are said to be co-oriented if $\langle u, w \rangle > 0$. We arbitrarily choose one of the two equivalence classes of co-oriented timelike vectors and designate

Figure 1: The Life of Stick
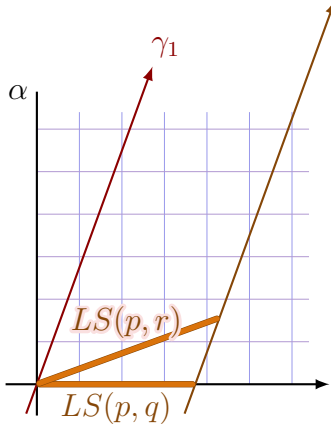
these vectors as future-oriented.

We use $+$ ambiguously for addition on $\mathbb{R}$, addition on V, and the map that takes an element $p \in M$ and a vector $u \in V$ and returns another element $p + u \in M$. For each $p, q \in M$, we let $\overrightarrow{pq}$ denote the unique vector in V such that $p + \overrightarrow{pq} = q$. For $p, q \in M$, we let $d(p, q) = \|\overrightarrow{pq}\|$, noting that d fails to have some typical properties of a distance function; e.g. the triangle inequality fails. For $p \in M$, we let $I^+(p)$ be the interior of the forward lightcone at p, i.e. the set of $q \in M$ such that $\overrightarrow{pq} \in V$ is timelike and future-oriented.

We will now look at two examples of the description of the motion of objects in Minkowski spacetime. In both cases, we pay special attention to these objects' relations to observers who are moving relative to them.

3.1 Stick

Suppose that there is an observer, Alice, located at $p \in M$, and whose four-momentum is a unit vector $u \in V$. Let α be the line in M generated by the vector u, and which represents Alice's idealized worldline if she is supposed always to exist. It is normally assumed (although the exact meaning of this claim is disputed) that two events $q, r \in M$ are "simultaneous for Alice" just in case both lie in one of her hyperplanes of simultaneity, i.e. a subset $s + u^\perp \subseteq M$ where s lies on α. Here $u^\perp$ is just the set of all vectors in V orthogonal to u.

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Suppose that there is an infinitely thin stick, named "Stick", in motion relative to Alice. (I will subsequently ignore the dimensions of space that are orthogonal to Stick; and I will speak of Stick as a three-dimensional object even though it really is just one-dimensional.) We assume that every particle that composes Stick has the same state of inertial motion, represented by the four-momentum vector $w \in V$. Suppose that Stick's leftmost edge intersects Alice's trajectory at p (see Figure 1), and let q be the point where the rightmost edge of the stick intersects the spacelike surface $p + u^{\perp}$. The bounding line of Stick on the left side is $\gamma_1 := \{p + aw : a \in \mathbb{R}\}$ and the bounding line of Stick on the right side is $\gamma_2 := \{q + aw : a \in \mathbb{R}\}$.

Suppose that Bob is an observer standing on the left hand side of stick and with the same state of motion. Then at p, Bob's simultaneity surface is $p + w^{\perp}$. Let r be the point where Bob's simultaneity surface intersects the right-hand extremity of Stick. Thus, it is typical to say that the line segment

$$L(p, r) \; = \; \{p + a\overrightarrow{pr} : a \in [0, 1]\},$$

represents Stick-for-Bob when he is located at p (although the intersection of $L(p, r)$ with the past lightcone of p is just p itself). Similarly, it is typical to say that the line segment

$$LS(p, q) \; = \; \{p + a\overrightarrow{pq} : a \in [0, 1]\},$$

represents Stick-for-Alice when she is located at p.

Recall that the **proper length** of an object is the length of that object in its own reference frame. Since we have assumed that Bob has the same state of motion as Stick, the proper length of Stick is equal to the length of its intersection with Bob's simultaneity surface $p + w^{\perp}$, i.e. the length of the line segment $L(p, r)$. Recall, moreover, that the proper length of an object is longer than its length in any other reference frame. In the present case, we have $\overrightarrow{pq} = \overrightarrow{pr} - bw$ for some $b > 0$, and since $\overrightarrow{pq}$ and $\overrightarrow{pr}$ are spacelike, and since $\langle v, \overrightarrow{pr} \rangle = 0$, it follows that $\|\overrightarrow{pq}\|^2 = \|\overrightarrow{pr}\|^2 - b^2$. Therefore, $\|\overrightarrow{pq}\| < \|\overrightarrow{pr}\|$.

Both Balashov and Sattig say explicitly that four-dimensional objects are invariant in a sense that three-dimensional objects are not. But what do they mean by invariance? In order to make progress on this, let's recall the definition of a symmetry of Minkowski spacetime. If V is the n-dimensional vector space on which M is based, then an isomorphism $\Phi : V \to V$ is a bijection that preserves the inner product. Recall that an automorphism of

Minkowski spacetime is an bijection $\varphi : M \to M$ such that hat

$$\overrightarrow{\varphi(p)\varphi(q)} = \Phi(\overrightarrow{pq}), \qquad \forall p, q \in M,$$

for some fixed isomorphism Φ of V. The set $\mathcal{P}$ of all such automorphisms is called the **Poincaré group**. (It is clear that these automorphisms form a group under composition.) We say that $\varphi \in \mathcal{P}$ is a **translation** if the corresponding vector space morphism $\Phi : V \to V$ is the identity; in this case, there is a unique vector $u \in V$ such that $\varphi(p) = p + u$ for all $p \in M$. We say that $\varphi \in \mathcal{P}$ is a **Lorentz transformation** based at $p \in M$ just in case $\varphi(p) = p$; in this case, $\varphi(p+u) = p + \Phi(u)$, for all $u \in V$. Among the Lorentz transformations there are two special cases: pure boosts and pure rotations. A pure rotation φ is characterized by the fact that the corresponding vector space morphism $\Phi : V \to V$ fixes a timelike vector $u \in V$.

In the description we have just given of Poincaré transformations, there are no coordinates in sight, and so there is no question about "active" versus "passive" transformations. There is also no sense in which these Poincaré transformations represent dynamical processes. For example, a translation $\varphi(p) = p + u$ doesn't represent God or anyone else moving the universe in the u direction. So what exactly is the significance of Poincaré symmetry? We should keep this question in mind when we run into claims to the effect that invariance is a necessary condition for reality.

One practical function of Poincaré symmetry is enabling an observer to translate descriptions that are true in another observer's context to descriptions that are true in his context. For example, suppose that Alice is located at p, and that Bob is located at $q = \varphi(p) = p + u$. Suppose, moreover, that Alice describes an object C as located at $w \in V$ from her location, and as having four-momentum vector $v \in V$. (For those of us with the God's eye view of Minkowski spacetime, we would say that C is located at $p + w \in M$.) Then Bob can translate Alice's description via the pullback map $\varphi^* : V \to V$ that is defined by

$$\varphi^*(w) = w + \overrightarrow{qp},$$

on position vectors, and that is the identity on four-momentum vectors.[4] In particular, Bob describes C as located at $\varphi^*(w) = w + \overrightarrow{qp}$.

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[4]It's overkill to call this a "pullback" map in SR, where the manifold is flat. But in GR, the pullback is non-trivial and is typically path-dependent.

The translation from one context to another is more interesting in the case where the two contexts are related by a Lorentz boost. (But the temptation is stronger here to think of the symmetry as involving an actual change, i.e. as a process by which one changes velocity.) Let $\varphi : M \to M$ be the Lorentz boost based at $p \in M$ that takes Alice's four-momentum vector u to Bob's four-momentum vector w, i.e. φ is generated by an isomorphism $\Phi : V \to V$ such that $\Phi(u) = w$ and $\Phi(u_2) = w_2$, where u_2 and w_2 are unit vectors in $u^\perp$ and $w^\perp$ respectively. There is a sense in which this Lorentz transformation converts Bob's description of Stick to Alice's, but we should disambiguate two different roles the Lorentz transformation plays.

In the typical way of speaking, Stick-for-Bob is the intersection of S with Bob's simultaneity hypersurface:

$$S_b \;=\; S \cap (p + w^\perp) \;=\; L(p, r).$$

Similarly, Stick-for-Alice is the intersection of S with her simultaneity hypersurface:

$$S_a \;=\; S \cap (p + u^\perp) \;=\; L(p, q).$$

While φ maps Alice's simultaneity surface to Bob's, it does not map the four-dimensional object S to itself, and so $\varphi(S_a) \neq S_b$. In fact, we shouldn't have expected φ to map S_a to S_b, since φ preserves lengths, while S_b is longer than S_a.

There is another sense, however, in which the Lorentz transformation does map S_a to S_b. Any set Γ of timelike lines in M determines "initial data" on the hypersurface $p + w^\perp$, viz. positions (points of intersection) and a velocities (angles of incidence). The transformation φ then defines a "semantic" function φ^* that takes the initial data on Bob's hypersurface to the corresponding initial data on Alice's hypersurface. Let $Th_a(\Gamma)$ be the set of sentences of Euclidean geometry that are made true by the points of intersection of Γ with Alice's hypersurface. Then φ determines a translation from Alice's theory $Th_a(\Gamma)$ to Bob's theory $Th_b(\Gamma)$. In fact, it can be shown that this translation is an equivalence in a precise sense. In other words, a Lorentz transformation determines an equivalence between the description in Alice's frame of reference and the description in Bob's frame of reference.

One might worry that the translation from Alice's description to Bob's cannot be an equivalence since it takes the statement "Stick's length is ℓ_a"

to a statement "Stick's length is ℓ_b", where $\ell_b \neq \ell_a$. But theoretical equivalences do not necessarily preserve assignments of numbers. For example, the conversion from Celsius to Fahrenheit takes the statement "the freezing point is zero degrees" to "the freezing point is thirty two degrees." It does follow, however, that the equivalence between Alice's description and Bob's is not a full Euclidean equivalence. This point will become even more clear in the next example, where we see that this translation does not preserve the predicate "...is an equilateral triangle".

We are left with the puzzle of how the geometrical objects S_a, S_b and S are supposed to represent objects in the physical world. Each of S_a and S_b is spacelike and connected, and so satisfies the minimal requirement for representing a common-sense material object. But the standard way of speaking about these things would have us say that S_a and S_b represent the same stick. How can that be when these regions overlap in a single point p, and when the events that compose S_a and S_b might have different properties? E.g. if the stick is composed of LED lights that change color, then a point in S_b might be red while its "counterpart" in S_a is blue. (By the "counterpart" $s' \in S_a$ of $s \in S_b$ I mean the point of intersection of $\{s + \lambda w : \lambda \in \mathbb{R}\}$ with S_a. Since s is in the timelike future of s', unless $s = p = s'$, the character of the event s can differ from, and be dynamically explained by, the character of the event s'.) The four-dimensionalist cuts the Gordian knot by saying that S_a and S_b do not represent material objects, but appearances of the four-dimensional object S.

What might Alice and Bob themselves say about the relation between S_a and S_b? The first question is whether Alice should grant that S_b represents a material object and whether Bob should grant that S_a represents a material object. Or should Alice be willing to say that S_b represents a "material object for Bob", while it does not represent a "material object for Alice"? Or could Alice say that S_b is just a different representation of the material object she represents as S_a? In fact, there is a prior question that needs to be considered: should Alice consider S_a to represent a material object? Of course, the four-dimensionalist answers no to all of these questions. But there is also a reason why a three-dimensionalist might deny that Alice should take S_a to represent a material object — in particular, if the three-dimensionalist maintains that the "object in itself" lives in its own frame of reference (so to speak).

I will say one more thing to illustrate this suggestion, and I will come back to it in the final section. If the stick is actually an object, and not just a

collection of accidentally co-moving particles, then we might suppose that its particles are interacting with each other and so synchronized in some general sense.[5] Suppose, for example, that the particles in the stick are correlated so that they change color simultaneously in their common reference frame. Then, in its own reference frame, the stick will consist of a collection of particles with the same color. But S_a represents an assemblage of particles with different colors, and so it lacks the cohesion that Stick has in its own reference frame. We can even imagine a case where Stick is a creature that is alive only if all of the lights show the same color. In that case, Stick is alive in its own reference frame, but not in Alice's!

A more realistic description might have that the explanation that the stick is a cohesive unity is that there are various forces between the particles that compose it. The question then is whether these inter-particle forces pick out, in some sense, the reference frame of the stick. I don't pretend to answer that question here, but I simply want to point out that there might be physical reasons for treating some slicing of a four-dimensional tube into three-dimensional sections as privileged over other slicings of that tube.

3.2 Triangle

Now consider a set of three parallel timelike lines $\gamma_1, \gamma_2, \gamma_3$. In other words, the three lines are generated by a common tangent vector $w \in V$. For the sake of visualization, imagine that each of these three lines represents the trajectory of a blue light. Thus, at each time, an observer will see three blue lights forming a triangle. (Note, however, that "x sees a triangle" is not actually part of the evidence basis of relativity theory.) Let $\Sigma \subseteq M$ be some spacelike hypersurface in the foliation determined by w, and for each $i = 1, 2, 3$, let p_i be the intersection of γ_i with Σ. Suppose that the points are equidistant from each other so that they form an equilateral triangle. Let

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[5]I offer no criterion here for objecthood, and I take no stand on the question of mereological universalism. I do suspect, however, that we normally have good reasons — sometimes grounded in basic physics — for calling some collections of particles (or some excitations of fields) as "material objects" while denying others that title. For example, one reason I call my iPhone an object is because its macroscopic parts maintain relatively stable spatial relations to each other. In contrast, the mereological sum of my water bottle and iPhone tends to change its spatial configuration in rather unpredicable ways.

There is another entire issue about how to make sense of *rigid* objects in relativity theory — what's typically called "Born rigidity". I take it, however, that physicists' concerns about rigidity are somewhat orthogonal to metaphysicians' concerns about objecthood.

p be the barycenter of the points p_1, p_2, p_3, and suppose that Bob is on a trajectory that remains on this barycenter through time. Since Σ is also a simultaneity hypersurface for Bob, it would be typical to say that the three lights appear to Bob to form an equilateral triangle.

Now suppose that Alice also passes through point p, but suppose that her four-momentum vector u is tilted towards the midpoint between p_2 and p_3. For example, we could take $u = w + a(\overrightarrow{p_1p_2} + \overrightarrow{p_1p_3})$ for some $a \in (0,1)$. For $i = 1, 2, 3$, let p_i' be the intersection of γ_i with $q + v^\perp$. It's easy to see then that $d(p_2', p_3') = d(p_2, p_3)$, but due to Lorentz contraction,

$$d(p_1', p_2') = d(p_1', p_3') < d(p_1, p_2) = d(p_1, p_3).$$

In other words, Alice doesn't see an equilateral triangle, but a triangle with one side longer than the other two. To make a sharper point of it: the sentence "there is an equilateral triangle" is true in Bob's frame of reference but false in Alice's frame of reference. Note, however, that this discussion of what Alice and Bob "see" is metaphorical, and is certainly not one of the original phenomena that Einstein sought to explain via the Special Theory of Relativity.

Now let S be the set of all points in M that are in the convex hull of the lines $\gamma_1, \gamma_2, \gamma_3$, i.e. the area swept out by the triangle $\langle p_1, p_2, p_3 \rangle$ over time. Let S_a be the intersection of S with Alice's simultaneity surface at p, and let S_b be the intersection of S with Bob's simultaneity surface $p + w^\perp$. Let $\varphi : M \to M$ be the Lorentz transformation determined by the orthogonal map Φ that transforms Alice's context (p, u) to Bob's context (p, w). This example displays all the same issues as the previous example, but now with the added complication that there is "shape relativity". While the intersection of S with Bob's hypersurface is an equilateral triangle, the intersection of S with Alice's hypersurface is not equilateral. Thus, while φ implements an "equivalence" between Alice's descriptions and Bob's, it's not an equivalence that preserves everything we might have assumed to be intrinsic to the relevant objects. For a real-life example, an object might be a three-dimensional sphere in one observer's reference frame, but not a sphere in another observer's reference frame. What's more, neither of these observers can, in general, lay claim to having a preferred description of the object "in itself".

The four-dimensionalist has a neat solution to this puzzle: the object in itself is the four-dimensional extended thing, and various observers can refer their varying three-dimensional appearances to this invariant four-dimensional

thing. But in what sense exactly is it invariant? In the next section, I show that the four-dimensional object is *not* invariant in the sense of being unmoved by Lorentz transformations.

4   No object unmoved

In this section we show that no non-trivial subset of Minkowski spacetime is Lorentz invariant, and *a fortiori*, no four-dimensional objects are Lorentz invariant. This result shows that if "relativistically invariant" means "unmoved by Lorentz transformations", then there are no relativistically invariant four-dimensional objects.

We begin with a couple of lemmas that establish (the well-known fact) that the Lorentz group acts transitively on Minkowski spacetime.

Lemma 1. *Let $p \in M$, let $m \in \mathbb{R}$, and let H_m be the forward hyperbola at distance m from p:*

$$H_m \;=\; I^+(p) \cap \{q \in M : d(p,q) = m\}.$$

Then for any $q, r \in H_m$, there is a Lorentz transformation Λ based at p such that $\Lambda q = r$. A similar result holds for constant-distance hyperbolae in the spacelike complement of p.

Proof. We show first that if $u, w \in V$ are co-oriented timelike vectors such that $\|u\| = \|w\| = 1$, then there is an isomorphism $\Phi : V \to V$ such that $\Phi(u) = w$. Indeed, $u^\perp$ has an orthonormal basis $\{u_2, \ldots, u_n\}$ of spacelike vectors, and $w^\perp$ has an orthonormal basis $\{w_2, \ldots, w_n\}$ of spacelike vectors. The map Φ may be defined by setting $\Phi(u) = w$ and $\Phi(u_i) = w_i$, for $i = 2, \ldots, n$, and then extending linearly.

Now let $q, r \in H_m$, that is, $\|\overrightarrow{pq}\| = \|\overrightarrow{pr}\| = m$. Thus, $u \equiv m^{-1}\overrightarrow{pq}$ and $w \equiv m^{-1}\overrightarrow{pr}$ are co-oriented timelike vectors. By the argument above, there is an isomorphism $\Phi : V \to V$ such that $\Phi(u) = w$. Define the Lorentz transformation $\varphi : M \to M$ by setting $\varphi(p + v) = p + \Phi(v)$, for all $v \in V$. It then follows that

$$\varphi(q) = \varphi(p + \overrightarrow{pq}) = p + \Phi(\overrightarrow{pr}) = p + mw = r.$$

$\square$

Lemma 2. *For any two points $p, q \in M$, there is a Lorentz transformation Λ such that $\Lambda p = q$.*

Proof. Let $p, q \in M$. We show that there are Lorentz transformations Λ_1 and Λ_2 such that $\Lambda_1 p = \Lambda_2 q$. The result then follows for $\Lambda = \Lambda_2^{-1}\Lambda_1$.

Consider first the case that p and q are spacelike related. In this case, we let Λ_2 be the identity. Now let w be a past-oriented timelike vector that is orthogonal to $\overrightarrow{pq}$ and such that $\|w\| = \|\overrightarrow{pq}\|$. If we let $s = p + \frac{1}{2}\overrightarrow{pq} + w$, then

$$\overrightarrow{ps} = w + \tfrac{1}{2}\overrightarrow{pq}, \qquad \overrightarrow{qs} = w - \tfrac{1}{2}\overrightarrow{pq}.$$

Since $\langle \overrightarrow{pq}, w \rangle = 0$, it follows that

$$\langle \overrightarrow{ps}, \overrightarrow{ps} \rangle = \langle \overrightarrow{qs}, \overrightarrow{qs} \rangle = \|w\|^2 - \tfrac{1}{4}\|\overrightarrow{pq}\|^2 > 0.$$

Thus, p and q are in the forward lightcone of s, and at the same distance. It follows from Lemma 1 that there is a Lorentz boost Λ_1 based at s such that $\Lambda_1 p = q$.

Now consider the general case. For any $p, q \in M$, there is an $r \in M$ that is spacelike related to both p and q. By the first part of this proof, there are Lorentz transformations Λ_1 and Λ_2 such that $\Lambda_1 p = r$ and $\Lambda_2 q = r$. $\qquad \square$

The previous lemma entails that no non-trivial subsets of Minkowski spacetime are invariant under all Lorentz boosts.

Proposition 3. *Let M be Minkowski spacetime and let $O \subseteq M$. If O is invariant under Lorentz transformations then either $O = M$ or $O = \emptyset$.*

Proof. Suppose that O is invariant and non-empty. Let $p \in O$. Now let q be an arbitrary element in M. By the previous lemma, there is a Lorentz transformation Λ such that $\Lambda p = q$. Since O is invariant, $q \in O$. Since q was arbitrary, it follows that $O = M$. $\qquad \square$

The observant reader might note that there are non-trivial subsets of Minkowski spacetime that *are* invariant under all Lorentz transformations based at some particular point $p \in M$. For example, the forward hyperbola H_m is invariant under Lorentz transformations based at p. Similarly, if we imagine a rod of finite length in the spacelike complement of p, and allow it to be uniformly accelerated both forwards and backwards in time, then the resulting subset O is invariant under Lorentz boosts based at p. But

these facts should hardly be of comfort to the four-dimensionalist, since (a) these contrived subsets do not represent typical material objects, and more importantly, (b) if using invariance to detect "reality", there is no reason to restrict to Lorentz transformations based at some particular point. (As to the first point, see the Appendix for a proof that if O is invariant under Lorentz boosts based at p, then O is spread out infinitely in space.) If invariance is to be used as a criterion of reality, then it should be invariance with respect to all of the relevant symmetries. In fact, the symmetry group of Minkowski spacetime is actually the Poincaré group, which includes not only Lorentz transformations but also translations; and it's obvious that only trivial subsets of Minkowski spacetime are left invariant by all translations. Therefore, if invariance means "unmoved by symmetries" then there are no invariant subsets of Minkowski spacetime.

5 Three-dimensional invariant objects

I have shown that four-dimensional shapes are *not* relativistically invariant — if by "relativistically invariant" we mean "unmoved by Lorentz transformations". However, I could be accused of misconstruing what the four-dimensionalist means by "invariant". In particular, "X is relativistically invariant" isn't supposed to mean that "the mathematical object that represents X is unmoved by Lorentz transformations" but something in the fashion of "X is described in a Lorentz-invariant fashion". The notion of a Lorentz-invariant description does make perfect sense, as I will now explain. (But then I will show that three-dimensional objects also have such descriptions.)

Consider first the simpler case of Euclidean-invariant shapes. For example, a sphere S of radius 1 can be described by the equation:

$$S \;=\; \{\vec{a} \in \mathbb{R}^3 : \|\vec{a}\| = 1\},$$

where $\|\vec{a}\|$ is the length of the vector $\vec{a}$. Of course, S is not invariant in the sense of "unmoved by Euclidean symmetries". For example, if $\vec{v} \neq 0$ then the translation $T(\vec{x}) = \vec{x} + \vec{v}$, for all $\vec{x} \in \mathbb{R}^3$, does not leave S invariant. Similarly, a Euclidean rotation based at $(1, 0, 0)$ moves S from its original place.

But the fact that S is moved by Euclidean transformations does not contradict the fact that S is a "geometric object" in Euclidean space. What do

we mean by this? Intuitively, we can imagine two people, say Alice and Bob, who describe things in Euclidean space using different coordinate systems — for example, Alice might use Cartesian coordinates while Bob uses polar coordinates. Or Alice and Bob might set the origin in different places. However, regardless of the coordinate systems they use, and regardless of which coordinates they use to describe the sphere S, Alice and Bob can agree on the sentence "there is a sphere of radius 1". It is this claim that is invariant between Alice and Bob's descriptions of the contents of space.

Here is one way that we can make that idea precise. The predicate "X is a sphere of radius ℓ" can be regimented as follows:

$$\phi(X) \equiv \exists \vec{a} \in \mathbb{R}^3, \forall \vec{b} \in \mathbb{R}^3 \left(\vec{b} \in X \leftrightarrow \|\vec{b} - \vec{a}\| \leq \ell \right).$$

This definition shows that ϕ is a Euclidean-invariant property:

For any Euclidean transformation T, $\phi(X)$ iff $\phi(TX)$.

In other words, while TX does not typically occupy the same region of $\mathbb{R}^3$ as X did, it is guaranteed that TX and X have exactly the same geometric properties. The result here is an instance of the well-known fact that explicitly definable properties (i.e. those definable via the syntax of a theory) are implicitly definable (i.e. invariant under symmetries of the models of the theory).

So now let us return to the case of interest: the role of invariance in Minkowski spacetime. We have seen that there are no interesting invariant objects in the sense of subsets X of M such that $\Lambda X = X$ for each Lorentz transformation Λ. But now we can see that "$\Lambda X = X$" is not what the four-dimensionalist intended by saying that four-dimensional objects are invariant. What he means is something like the following:

(R) Reality is what can be described, in a Lorentz-invariant way, as happening inside Minkowski spacetime.

What's more, the four-dimensionalist correctly points out that four-dimensional configurations can be described in a Lorentz invariant fashion. For example, let $\phi(X)$ be the predicate "X is a timelike line". While no timelike line is left unmoved by all Lorentz transformations, the property ϕ is invariant in the sense that $\phi(X)$ iff $\phi(\Lambda X)$ for all Lorentz transformations Λ.

We could now go on producing descriptions of happenings in Minkowski spacetime that are invariant under all Lorentz transformations. For example,

we could formulate a predicate $\psi(X)$ which means that X is a spacetime worm. Or we could formulate a relation $\theta(X, Y)$ which means that X and Y are timelike lines that intersect in one point. If we carried on in this fashion, then we could produce a set Γ of sentences that describes, in an invariant fashion, what is going on in Minkowski spacetime. Is this not a vindication of the four-dimensionalist's intuition?

Not yet. The first problem is that almost every subset of Minkowski spacetime represents a geometric object in this sense — i.e. is definable up to Poincaré transformation by some predicate of the language of Minkowski geometry. To see the point, consider again the Euclidean case. Being an open sphere is definable, as is being an open disc (of dimension two), and being an open interval (of dimension one). If we start taking conjunctions and disjunctions, and describing the distance relationship between the centers of the spheres (or discs or intervals), then we can describe arbitrarily complex shapes in an invariant fashion. In fact, the procedure I'm suggesting would basically reconstruct the Borel hierarchy, permitting the definition of more and more complex types of subsets of $\mathbb{R}^3$.

An analogous procedure can be carried out in Minkowski spacetime, showing that arbitrarily complex subsets are definable. But here there is a more particular problem for the four-dimensionalist: these definable subsets are not exclusively four-dimensional, not even the most simply definable of them. Consider, for example, the predicate:

> X is a spacelike stick of length ℓ

which picks out a class of three-dimensional objects. To see that this predicate has a simple formulation in the language of Minkowski geometry, note that it can be paraphrased as:

> There is a timelike vector u such that X is a line segment of spacetime length ℓ contained in $u^\perp$.

Each clause here has a simple definition in terms of the spacetime metric, e.g., "u is timelike" means that $\langle u, u \rangle > 0$. If we use $\beta(X)$ to denote this predicate, then $\beta(X)$ is invariant in the sense that for each Poincaré transformation T, $\beta(X)$ iff $\beta(TX)$. It follows that "x is a three-dimensional spacelike stick" is a geometric property of no less integrity than "x is a four-dimensional spacelike worm". Similarly, the geometric properties of three-dimensional objects are invariant in the same sense that the geometric properties of four-dimensional

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objects are invariant. If there is a sense in which four-dimensional objects are invariant but three-dimensional objects are not, then it must be something different than being describable by (invariant) predicates of Minkowski geometry.

6   Decomposing

I have been playing devil's advocate to the four-dimensionalists' claim of invariance, but I must admit that they do invoke an interesting feature of the relativistic spacetime description. In particular, suppose that Alice and Bob are two observers in relative motion to each other, and suppose that p and q are distinct events (e.g. the flashing of two light bulbs). As is well-known from various Einsteinian thought-experiments, Alice and Bob might make different judgments about the temporal lag between p and q. For example, Alice might judge that p occurs before q, while Bob might judge that p and q occur simultaneously. What's more, if Alice and Bob agree that the speed of light is c in both of their reference frames, then they will also come to different conclusions about the distance between p and q. In particular, Bob will judge that p and q are further apart in space than Alice judges them to be. (But take note here: Alice might take exception to the claim that there is a spatial distance between p and q. I return to this point below.) Nonetheless, if Alice and Bob apply the recipe for computing the spacetime interval between p and q in terms of their temporal and spatial differences, then they will derive the same number. i.e. the spacetime interval is an "invariant" of their descriptions. Does this not mean, as Hofweber and Lange claim, that "the spacetime distance between p and q is ℓ" is a fundamental fact, and that claims about the spatial and temporal distances between p and q are derivative facts?

Let's write $t(x, p, q)$ for the temporal distance between p and q relative to observer x, and $s(x, p, q)$ for the spatial distance between p and q relative to x. Since the temporal and spatial distances depend only on an observer's state of motion, i.e. her four-velocity vector $u \in V$, we can take x to range over unit vectors in V. The precise definitions of temporal and spatial distance are then given by:

$$t(u, p, q) \;=\; \|P_u(\overrightarrow{pq})|, \qquad s(u, p, q) \;=\; \|P_{u^\perp}(\overrightarrow{pq})\|,$$

where $P_u : V \rightarrow V$ is the projection onto u, and $P_{u^\perp}$ is the projection

onto the spacelike subspace $u^\perp$. Since u is timelike, $u^\perp$ is spacelike, and a straightforward calculation yields

$$d(p, q) \;=\; \|\overrightarrow{pq}\| \;=\; \sqrt{t(x, p, q)^2 - s(x, p, q)^2}.$$

This last equation is the Minkowskian version of the Pythagorean theorem: the length of a vector squared is the square of its timelike component minus the square of its spacelike component. The nuance here is that a vector has different decompositions into timelike and spacelike components.

Hofweber and Lange would have us think of the spatial and temporal distances between p and q as "appearances relative to a subject (and her state of motion)" while the spacetime interval between p and q is an absolute fact that explains the various appearances. This description is, however, a bit misleading — and we should ask whether a more accurate description destroys the force of Hofweber and Lange's argument. First of all, in what sense is a spatial distance between p and q an "appearance"? Consider the case of Alice, with velocity vector u such that q lies in the future of $p + u^\perp$, which means that p will enter Alice's past lightcone before q does. Thus, Alice does not think of p and q as separated in space (at a time), and there is no sense in which she "sees" p and q as lying at some distance from each other.

Of course, we could imagine that Alice lays out a ruler, that p leaves a mark on the ruler, and that q later happens at distance $s(u, p, q)$ from where p happened. In this case, Alice might want to explain why q occurred at this place on the ruler. But would Alice be satisfied by the explanation that q has a certain spacetime distance from p? It strikes me that Alice would not be satisfied with such an explanation, which has the flavor of other fatalist explanations of the form: "it was destined to be that way from all eternity." What we see here is that the perceptual analogy — invoked by Balashov, Hofweber, and Lange — breaks down. The appearance of a static visual form can often be explained by appeal to some complex physical object or state of affairs that produces — via dynamical equations of motion — that form. But in the relativistic case, it is not a static visual form that is to be explained; it is a constellation of events happening in different places and at different times. If Alice wants to know why the light flash coincided with a certain location on the ruler, then I suspect she would not find it informative to be told that the event and ruler occupy such and such places in the spacetime manifold. I suspect that Alice would prefer to have a dynamical explanation

of what processes occurred in the past and that led to the light being placed where it was and to its flashing at the time it did.

This last point reveals a further ambiguity in Hofweber and Lange's argument: what is it that is to be explained, and who is requesting the explanation? I see two possibilities here. The first possibility is that Alice wants an explanation of why events occurred where and when they did. As discussed above, I think that Alice would be wholly unsatisfied to be told that these events occurred where and when they did because that is where they are in spacetime. (At the very least, this explanation is dangerously close to being circular.) The second possibility is that we (Hofweber, Lange, myself, other philosophers) want an explanation of why Alice's measures of spatial and temporal distances are systematically coordinated with those of other observers. Upon reflection, it seems obvious that it is this fact — viz. the coordination between different measures of spatial and temporal distance — that Hofweber and Lange want to be explained, and that they fault Fine's fragmentalism for not being able to explain.

Here we reach some deep philosophical waters. It is no longer a first-order question of how some physical phenomenon is to be explained, but a higher-order question of how to explain that there is a systematic coordination between different (correct) descriptions of reality. At present, I will leave open the possibility — advocated by Hofweber and Lange — that relativity theory does, in fact, provide an "absolute conception of reality" that explains the various frame-dependent conceptions. I do want to point out, however, that one can grant the validity of Inference to the Best Explanation (IBE) for physical phenomena, while denying the unrestricted call for an *Aufhebung* of correct descriptions in some higher and more objective description of the underlying reality.[6]

For example, a German-English dictionary provides a set of rules — we might say "transformation laws" — that allows us to translate correct German descriptions to correct English descriptions. (I'm setting aside Quinean worries that there is no such thing as a correct translation.) Let's call these rules the "Laurenz transformations". Now, I take it that it would sound odd

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[6]Ted Sider seems to endorse the demand for *Aufhebung*: "To support a claim of equivalence between a pair of theories, stated in a pair of languages . . . we brought in a third language, a language in which mass is described in a unit-free way. . . . This third, more fundamental, language gave us a perspective on the fundamental facts, a perspective from which the first two theories could be seen as getting at the very same facts." (Sider, 2020, p 187).

to ask for a theory of the world that explains the Laurenz transformations. At the very least, it would display a kind of cultural chauvinism if, for example, a French linguist offered an explanation (in French of course!) of the nature of English and German speakers, and their relations to the world, and from which the Laurenz transformations can be derived. Or to put the matter in more serious terms, to assume that the Laurenz transformations can be justified by appeal to a more fundamental theory is to assume a solution to the good old-fashioned categorio-centric predicament: that we can find an absolute description of the world that explains the relations between the relative descriptions.

If we don't demand that translations between natural languages be justified by appeal to some language-neutral theory then why should we demand that translations between frame-relative descriptions be justified by appeal to some frame-neutral theory?

7 Three-dimensional explanations

I have suggested that Hofweber and Lange want primarily to explain the systematic relations between correct descriptions. Balashov, in contrast, places more emphasis on explaining the systematic relations between three-dimensional appearances over time. His explanation, of course, is that the sequence of three-dimensional appearances witnessed by one observer are the successive sections of a single four-dimensional object. I have one criticism of the proposed explanation, and I then offer an alternative explanation — both of the succession of appearances, and of the systematic relations between correct descriptions.

Four-dimensional objects have one notable defect *qua* explanatia: they are timeless, and so they neither move nor change. They do not act upon other material objects, at least in any normal sense of the word, nor are they acted upon by other material objects. In fact, four-dimensional objects do not even satisfy minimal necessary conditions for being material objects.

> Kant's criterion for objecthood: if x is a material object, then x can move.[7]

> Einstein's criterion for objecthood: if x is a material object, then

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[7]See e.g. (McLear, 2018).

> x can act upon other material objects, and can be acted upon by other material objects.[8]

Since four-dimensional objects are not dynamical, they cannot explain in a dynamical sense. Is there some other sense in which they are supposed to explain phenomena?

Perhaps the four-dimensionalist will reply that his objects can move — at least in the attenuated sense that they could have occupied some different region of spacetime than they do. Imagine, for example, two four-dimensional possible worlds (e.g. models of Special Relativity). In the first world M_1 the spacetime worms α and β intersect at some time and then bounce off of each other. In the second world M_2 these worms remain parallel. We could then truthfully say of the first model that the trajectory of α causes the trajectory of β to deviate.

I think that this suggestion is worth pursuing. Indeed, it seems generally possible to convert our ordinary three-dimensionalist ways of speaking into four-dimensionalist counterparts by a sort of translation (enabled by dynamical laws) that takes initial data to four-dimensional solutions. However, I think the four-dimensionalist must admit that his use of *causal* language derives its intelligibility from such a translation. If our models did not have time parameters and dynamical development, then talk of one thing causing another would be metaphorical at best. Suppose, for example, that W_1 and W_2 are purely geometrical possible worlds — say models of three-dimensional Euclidean geometry. If we have the same scenario as above, where two lines "bounce off each other" in one world but are parallel in the other world, then there is no immediate reason to speak of those lines as trajectories, much less as trajectories representing a causal interaction.

In the sorts of explanations we are familiar with from everyday life, explanans and explanandum stand in spatiotemporal relations to each other, and are connected to each other by laws of action and reaction. What's more, standard invocations of IBE move between events and entities of roughly the same ontological kind. For example, if I find a mess in the kitchen upon arriving home, then I might infer that my son arrived home early from school. That's because my son moves and applies forces to things, resulting in certain characteristic physical changes.

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[8]This idea was one of the reasons that Einstein found Special Relativity to be inadequate and in need of replacement by a theory in which the metric of spacetime can be acted upon by physical objects. See e.g. (Brown and Pooley, 2006).

Now, analytic philosophers have not been shy to extend IBE so that it licenses inferences to entities of a more transcendent kind from phenomena of more mundane kind. Consider, for example, the infamous indispensability argument to the effect that the success of science (or, to make it more mundane, the technological applicability of science) is best explained by the existence of numbers, sets, and other Platonic entities. I won't take issue here with these kinds of transcendental arguments; but it doesn't seem to me that Balashov intends the IBE to four-dimensional entities to be of this kind. It seems, instead, that Balashov intends the inference from phenomena to four-dimensional objects to be an ordinary scientific inference, like the inference to the existence of Neptune to explain deviations in the orbit of Uranus. But in that case, the non-dynamical nature of four-dimensional objects makes it unclear how these objects are supposed to "explain" the appearances. They cannot cause the appearances in the same way that a flagpole causes a shadow, since there is no dynamical process by which they cast "shadows" on three-dimensional hyperplanes.

But let us grant that we need *some* explanation for why a single object can appear different ways in different reference frames — or, to be more accurate, why a single object admits of different correct descriptions relative to different reference frames.[9] If the four-dimensionalist explanation were the only one on offer, then it would be the best explanation — no matter how much it stretches the concept of explanation. However, there is a three-dimensionalist explanation of the appearances in various reference frames (at least for inertial objects): the intrinsic properties of an object are those that it has in its own rest frame, and these properties explain those it appears to have in other reference frames. (I am tempted always to put "appears" in scare quotes, because it is not the visual appearance that needs to be explained, but why a certain description of the object is correct.)

One might worry that it is blatantly arbitrary to prefer the description of

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[9]Here I disagree with Rovelli's claim that relativity theory teaches us that objects have no intrinsic properties (see Rovelli, 1996; Rovelli, 2022). Rovelli claims that "x has property ϕ relative to frame F" is fundamental; and I think he would simply refuse the demand of Balashov, Hofweber, Lange, et al. to explain such relations in terms of intrinsic properties of objects. (I'm not sure whether the explainers are required to be "intrinsic", or "monadic", or yet something else. The methodological point, however, is clear: Balashov et al. find it theoretically unsatisfactory to ascribe multiple relative properties to an object without identifying a common cause for these properties.) To be clear, Rovelli has argued that a frame of reference is simply a material object, so that his view is an ontic relationalism rather than an epistemic relativism.

an object in its own reference frame. But there are at least two reasons why that description is privileged — an external reason and an internal reason. The external reason is that the description of an object X relative to its own reference frame is not viciously relative. Since X is the thing to be described, we do not add a further layer of subjectivity by saying that X should be described in the context that is (objectively) picked out by its own physical state.[10] The internal reason is that the description of X relative to its own reference frame is the unique "maximizing" description of X in the following sense: the dimensions of X are maximized in its own reference frame. To take a simple example, let X be a cube whose sides have proper length 1, i.e. length in the reference frame of X. The volume of X in its own reference frame is 1; but in any other reference frame, one or more of the sides of the cube will be contracted, and it will be correctly described as a cube with volume less than 1. Thus, there is an asymmetry between an object's size in its own reference frame and its size in any other reference frame: it achieves its maximum value in its own reference frame, and we can consider its shape in any other reference frame to be a "compressed image" of this maximally extended shape.

I'm not saying that I have an apriori reason for thinking that the intrinsic properties of a three-dimensional object are those that maximize its dimensions. The important point is just that the shape of an object in its own reference frame — i.e. its "proper shape" — can ground explanations of its shape in any other reference frame. What's more, the inference from the shape of X in its own reference frame to the shape of X in some other reference frame is a standard dynamical explanation (possibly in the spirit of Lorentz). To the question "why does X appears as it does in frame F?" we answer "because X has certain properties in its rest frame, and because it is moving relative to F". The reason that X appears "as it is in itself" in its own reference frame is because it is not moving relative to its own frame. I leave for another occasion the question of whether Lorentz contraction is properly dynamical — in the sense of involving forces and genuine physical contractions — or whether it should be thought of as a merely relational change between the object and a frame of reference.

There is also the question of whether this kind of explanation generalizes to the General Theory of Relativity, where there is no longer a notion of inertial objects that determine (global) inertial reference frames. While the

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[10]While no object is absolutely at rest, every object is at rest relative to itself.

typical approach of analytic philosophers has been to double down on the hunt for absolute realities (represented by coordinate-free, or geometric, objects), I would suggest that objects' intrinsic properties are those that they have in their own local reference frames. But we are now getting to a place where the physics is extremely complex, and the answers to these questions can make a crucial difference for how one tries to build the next theory. Further discussion of these issues will have to wait for another occasion.

8   More decomposing

There is yet one other way in which the four-dimensional description of an object might be said to be "invariant" in a way that the three-dimensional descriptions are not. As before, let Alice and Bob be two inertial observers who are in motion relative to each other, and consider some ordinary object such as the book on my desk. Then the following can occur:

1. There is no three-dimensional spatial region X that both Alice and Bob believe to be occupied by the book.

2. There is a single four-dimensional spacetime region Y that both Alice and Bob believe to be occupied by all of the temporal stages of the book.

Is this not a clear sense in which the four-dimensional description (the region Y) is invariant between Alice and Bob, while the three-dimensional descriptions vary?

I admit that this scenario does suggest that four-dimensional regions play a privileged explanatory role. There is just one spacetime worm Y, but Alice decomposes it as $\bigcup_{t \in \mathbb{R}} A_t$ while Bob decomposes it as $\bigcup_{t \in \mathbb{R}} B_t$. Clearly, the four-dimensional object is what is invariant between their descriptions.

However, there are some complications that make the case less than fully clear-cut. First, if Alice and Bob give three-dimensional explanations, then do their explanations really disagree with each other? I can see at least two ways in which they could reconcile their three-dimensional explanations without passing to a higher dimension. First, Alice and Bob could both give up the idea that the object ever "lives inside" their own spatial hypersurfaces — i.e. they could say that the object really lives inside the hypersurfaces that are orthogonal to its own velocity vector. Granted, this proposal is not

obvious consistent with standard presentism, as Alice and Bob would say that the object exists, but that it is never present at a single time in their own frames of reference.

Another possibility is that the disagreement between Alice and Bob is resolved by the fact that regions of Minkowski spacetime are not the objects themselves, but are just mathematical representations thereof. The "disagreement" between Alice and Bob is that represent the object by different three-dimensional regions of Minkowski spacetime: Alice represents its lifespan by the stack $\{A_t : t \in \mathbb{R}\}$ and Bob represents its lifespan by the disjoint stack $\{B_t : t \in \mathbb{R}\}$. But couldn't Alice and Bob come to think that their two descriptions are equivalent, in the precise sense that there is a translation that uniformly converts each A_t to some $B_{f(t)}$ and vice versa? Doesn't the existence of this translation show that Alice and Bob don't really disagree after all?

The four-dimensionalist might repeat now that Alice and Bob's descriptions cannot be equivalent since, e.g., the B_t describe an equilateral triangle while the A_t describe a triangle with one side longer than the other two. But that reply presupposes that Euclidean shape is an intrinsic feature of three-dimensional objects, and that "equivalent descriptions" would have to agree on shape. What if one of the lessons of relativity theory is that material objects don't have determinate Euclidean shapes — they only have that geometric property that is equivalently described by all the different observers? That might sound like a radical idea, but consider an analogy: one can imagine a world where objects do not have metric properties, but only topological properties. In such a world, a single object might be correctly desribed either as a coffee cup or as a doghnut. In this case, the underlying reality behind the descriptions is the unique (up to homeomorphism) topological space S^1, the unit circle in the plane — but human beings don't have a way of describing this object without adding specificity that is not in the thing itself.

9   Conclusion

It is nearly the consensus view among philosophers and physicists that relativity theory pushes us away from the common sense view of three-dimensional objects changing in time, and towards an ontology of four-dimensional, eternal objects. When philosophers have tried to give rigorous arguments for this

intuitive idea, then they have frequently appealed to a notion of relativistic invariance. But they have been less than fully clear on what "invariance" means. I have tried to clarify the notion, and this clarification shows that it does not do the work that it has been supposed to do. As best I can tell, relativity theory does not favor the view that reality, at the fundamental level, consists of four-dimensional objects. Relativity theory does offer lessons about the nature of material objects, or — to put it in the formal mode — about how we should describe these objects. But we have more work to do to understand what these lessons are.

Appendix

In this appendix we show that if $O \subseteq M$ is invariant under Lorentz boosts based at p, then an arbitrary observer will judge O to have infinite spatial extension. This claim could be made precise in a few different ways, e.g.:

1. For each spacelike hypersurface Σ passing through p, the projection of O onto Σ is non-compact.

2. Each foliation has a spacelike hypersurface Σ such that $\Sigma \cap O$ is non-compact.

3. In each reference frame, the spatial coordinate values of O are unbounded.

For an arbitrary subset $O \subseteq M$, these three conditions are not equivalent. However, they are equivalent for the kinds of subsets that might be taken to represent a material object's trajectory through time, e.g. if O consists of a family of non-intersecting timelike curves. Fortunately, the point we are trying to make doesn't depend on the precise details: the region of spacetime swept out by a material object is not invariant under the Lorentz boosts at a fixed base point.

Proposition 4. *Let $p \in M$ and let O be a subset of M that contains some point besides p. If O is invariant under Lorentz boosts based at p, then for each spacelike hypersurface Σ passing through p, the projection of O onto Σ is unbounded.*

Before we begin the proof, recall that any spacelike hypersurface Σ in M is a Cauchy surface, i.e. each inextendible timelike curve in M intersects

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Σ in a single point. To see why that's true, suppose that $\Sigma = p + u^\perp$, where u is a unit timelike vector. Suppose first that γ is a lightlike line, i.e. $\gamma = \{q + av : a \in \mathbb{R}\}$, where v is a non-zero null vector. Since $\langle u, v \rangle \neq 0$, there is some $a \in \mathbb{R}$ such that $\langle u, \overrightarrow{pq} + av \rangle = 0$. Thus, $q + av = p + (\overrightarrow{pq} + av)$ lies in $p + u^\perp$. It follows that every lightlike line intersects Σ. Now if γ is an inextendible timelike curve, and if $q \in \gamma$, then each lightlike line through q intersects Σ. If we trace γ in the direction of these intersections, then the distance to Σ approaches zero. Using the completeness property of the real numbers, it follows that γ intersects Σ.

Proof. The geometry of the proof is intuitive: for a point $q \in M$ that is spacelike related to p, there is a one-parameter group Λ_θ of Lorentz boosts based at p such that $H := \{\Lambda_\theta(q) : \theta \in \mathbb{R}\}$ is a hyperbola in the spacelike complement of p. Furthermore, H projects onto an infinite ray in any spacelike hypersurface passing through p. Now we proceed to spell out the details.

Suppose first that $w = \overrightarrow{pq}$ is spacelike. Let u be a timelike vector such that $\langle w, u \rangle = 0$ and $\|u\| = \|w\|$, and for each $\theta \in \mathbb{R}$, let

$$w_\theta = (\sinh \theta)u + (\cosh \theta)w.$$

In this case,

$$\langle w_\theta, w_\theta \rangle = (\sinh^2 \theta)\|u\|^2 - (\cosh^2 \theta)\|w\|^2 = -(\cos^2 \theta - \sin^2 \theta)\|w\|^2 = -\|w\|^2.$$

It follows that w_θ is spacelike and $\|w_\theta\| = \|w\|$. Following the procedure of Lemma 1, we let $\Lambda_\theta : V \to V$ be an isomorphism that maps w to w_θ, an we let $\varphi_\theta : M \to M$ be the Lorentz boost defined by $\varphi_\theta(p + w) = p + \Lambda_\theta(w)$.

We now show that any two points on the hyperbola H are timelike related. Since $w_\theta - w_{\theta'} = \Lambda'_\theta(w_{\theta - \theta'} - w)$, for any $\theta, \theta' \in \mathbb{R}$, it suffices to show that $w_\theta - w$ is timelike for any $\theta \in \mathbb{R}$. We calculate:

$$
\begin{aligned}
\langle w_\theta - w, w_\theta - w \rangle &= \Big\langle (\cosh \theta - 1)w + (\sinh \theta)u, (\cosh \theta - 1)w + (\sinh \theta)u \Big\rangle \\
&= \sinh^2 \theta \|u\|^2 - (\cosh \theta - 1)^2 \|w\|^2 \\
&= \left[\sinh^2 \theta - (\cosh \theta - 1)^2 \right] \|u\|^2.
\end{aligned}
$$

To finish the argument, we need to show that for an arbitrary spacelike hyperplane $E \subseteq V$, the projection of $H = \{w_\theta : \theta \in \mathbb{R}\}$ onto E is an infinite ray. Let $P_E : V \to V$ be the projection onto E, and let P_w be the projection

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onto w. Since $P_w(w_\theta) = (\cosh\theta)w$, it follows that $P_w[H]$ is an infinite ray. Since $p+E$ is a Cauchy surface and H is an inextendible timelike curve, they intersect in some point

$$w_{\theta_0} = (\sinh\theta_0)u + (\cosh\theta_0)v.$$

There is an isomorphism $\Phi : V \to V$ such that $\Phi(w) = w_{\theta_0}$. We claim now that for each w_θ,

$$\Phi P_w(w_\theta) = P_E \Phi(w_\theta).$$

$\square$

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