NEUTRO-BCK-ALGEBRA

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Abstract

This paper introduces the novel concept of Neutro-BCK-algebra. In Neutro-BCK-algebra, the outcome of any given two elements under an underlying operation (neutro-sophication procedure) has three cases, such as: appurtenance, non-appurtenance, or indeterminate. While for an axiom: equal, non-equal, or indeterminate. This study investigates the Neutro-BCK-algebra and shows that Neutro-BCK-algebra are different from BCK-algebra. The notation of Neutro-BCK-algebra generates a new concept of NeutroPoset and Neutro-Hass-diagram for NeutroPosets. Finally, we consider an instance of applications of the Neutro-BCK-algebra.
Keywords: Neutro-BCK-algebra, NeutroPoset, Neutro-Hass diagram.

1 Introduction

Neutrosophy, as a newly-born science, is a branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. It can be defined as the incidence of the application of a law, an operation, an axiom, an idea, a conceptual accredited construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making it intelligible. Neutrosophic Sets and Systems international journal (which is in Scopus and Web of Science) is a tool for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics, neutrosophic measure, neutrosophic integral, and so on, studies that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc. Recently, Florentin Smarandache [2019] generalized the classical Algebraic Structures to NeutroAlgebraic Structures NeutroAlgebras) and AntiAlgebraic Structures (AntiAlgebras) and he proved that the NeutroAlgebra is a generalization of Partial Algebra.[7] He considered $< A >$ as an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, he split the nonempty space and worked onto three regions two opposite ones corresponding to $< A >$ and $< antiA >$, and one corresponding to neutral (indeterminate) $< neutA >$ (also denoted $< neutroA >$) between the opposites, regions that may or may not be disjoint - depending on the application, but they are exhaustive (their union equals the whole space). A NeutroAlgebra is an algebra which has at least one NeutroOperation operation that is well-defined (also called inner-defined) for some elements, indeterminate for others, and outer-defined for the others or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A Partial Algebra is an algebra that has at least one partial operation (well-defined for some elements, and indeterminate for other elements), and all its axioms are classical (i.e., the axioms are true for all elements). Through a theorem he proved that NeutroAlgebra is a generalization of Partial Algebra, and examples of NeutroAlgebras that are not partial algebras were given. Also, the NeutroFunction and NeutroOperation were introduced.[7]

Regarding these points, we now introduce the concept of Neutro-BCK-algebras based on axioms of BCK-algebras, but having a different outcome. In the system of BCK-algebras, the operation is totally well-defined for any two given elements, but in Neutro-BCK-algebras its outcome may be well-defined, outer-defined, or indeterminate. Any BCK-algebra is a system which considers that all its axioms are true; but we weaken the conditions that the axioms are not necessarily totally true, but also partially false, and partially indeterminate. So, one of our main motivation is a weak coverage of the classical axioms of BCK-algebras. This causes new partially ordered relations on a non-empty set, such as NeutroPosets and Neutro-Hass Dia-

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grams. Indeed Neutro-Hass Diagrams of NeutroPosets contain relations between elements in the set that are true, false, or indeterminate.

2 Preliminaries

In this section, we recall some definitions and results from,[7] which are needed throughout the paper.

Let $n \in \mathbb{N}$. Then an n-ary operation $\circ : X^n \to Y$ is called a NeutroOperation if it has $x \in X^n$ for which $\circ(x)$ is well-defined (degree of truth (T)), $x \in X^n$ for which $\circ(x)$ is indeterminate (degree of indeterminacy (I)), and $x \in X^n$ for which $\circ(x)$ is outer-defined (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (total, or classical) Operation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary AntiOperation. Again, in this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general. In this regards, for any given set X, we classify n-ary operation on X^n by (i); (**classical**) **Operation** is an operation well-defined for all set's elements, (ii); **NeutroOperation** is an operation partially well-defined, partially indeterminate, and partially outer-defined on the given set and (iii); **AntiOperation** is an operation outer-defined for all set's elements.

Moreover, we have (i); a (**classical**) **Axiom** defined on a non-empty set is an axiom that is totally true (i.e. true for all set's elements), (ii); **NeutroAxiom (or neutrosophic axiom)** defined on a non-empty set is an axiom that is true for some elements (degree of true = T), indeterminate for other elements (degree of indeterminacy = I), and false for the other elements (degree of falsehood = F), where T, I, F are in $[0, 1]$ and (T, I, F) is different from $(1, 0, 0)$ i.e., different from totally true axiom, or classical Axiom and (T, I, F) is different from $(0, 0, 1)$ i.e., different from totally false axiom, or AntiAxiom. (iii); an **AntiAxiom** of type $\mathcal{C}$ defined on a non-empty set is an axiom that is false for all set's elements.

Based on the above definitions, there is a classification of algebras as follows. Let X be a non-empty set and $\mathcal{O}$ be a family of binary operations on X. Then $(A, \mathcal{O})$ is called

(i) a (**classical**) **Algebra** of type $\mathcal{C}$, if $\mathcal{O}$ is the set of all total Operations (i.e. well-defined for all set's elements) and $(A, \mathcal{O})$ is satisfied by (classical) Axioms of type $\mathcal{C}$(true for all set's elements).

(ii) a **NeutroAlgebra (or neutro-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one NeutroOperation (or NeutroFunction), or $(A, \mathcal{O})$ is satisfied by at least one NeutroAxiom of type $\mathcal{C}$ that is referred to the set's (partial-, neutro-, or total-) operations or axioms;

(iii) an **AntiAlgebra (or anti-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one AntiOperation or $(A, \mathcal{O})$ is satisfied by at least one AntiAxiom of type $\mathcal{C}$.

3 Neutro-BCK-algebra

3.1 Concept of Neutro-BCK-algebra

In this section, we introduce several concepts suc has: Neutro-BCK-algebra, Neutro-BCK-algebra of type 5, NeutroPoset and Neutro-Hass Diagram and investigate the properties of these concepts.

Definition 3.1. [2] Let X be a non-empty set with a binary operation "$*$" and a constant "0". Then, $(X, *, 0)$ is called a *BCK-algebra* if it satisfies the following conditions:

$(BCI\text{-}1)$ $((x * y) * (x * z)) * (z * y) = 0$,

$(BCI\text{-}2)$ $(x * (x * y)) * y = 0$,

$(BCI\text{-}3)$ $x * x = 0$,

$(BCI\text{-}4)$ $x * y = 0$ and $y * x = 0$ imply $x = y$,

$(BCK\text{-}5)$ $0 * x = 0$.

Now, we define Neutro-BCK-algebras as follows.

Definition 3.2. Let X be a non-empty set, $0 \in X$ be a constant and "$*$" be a binary operation on X. An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a Neutro-BCK-algebra, if it satisfies at least one of the following NeutroAxioms (while the others are classical BCK-axioms):

$(NBCI\text{-}1)$ $\big(\exists\, x, y, z \in X$ such that $((x*y)*(x*z))*(z*y)=0)\big)$ and $\big(\exists\, x, y, z \in X$ such that $((x*y)*(x*z))*(z*y) \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}2)$ $\big(\exists\, x, y \in X$ such that $(x*(x*y))*y=0\big)$ and $\big(\exists\, x, y \in X$ such that $(x*(x*y))*y \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}3)$ $\big(\exists\, x \in X$ such that $x*x=0\big)$ and $\big(\exists\, x \in X$ such that $x*x \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}4)$ $\big(\exists\, x, y \in X$, such that if $x*y=y*x=0$, we have $x=y\big)$ and $\big(\exists\, x, y \in X$, such that if $x*y=y*x=0$, we have $x \neq y\ \big)$;

$(NBCK\text{-}5)$ $\big(\exists\, x \in X$ such that $0*x=0\big)$ and $\big(\exists\, x \in X$ such that $0*x \neq 0$ or indeterminate $\big)$. Each above NeutroAxiom has a degree of equality (T), degree of non-equality (F), and degree of indeterminacy (I), where $(T, I, F) \notin (1,0,0), (0,0,1)$.

If $(X, *, 0)$ is a NeutroAlgebra and satisfies the conditions $(NBCI\text{-}1)$ to $(NBCI\text{-}4)$ and $(NBCK\text{-}5)$, then we will call it is a Neutro-BCK-algebra of type 5 (i.e. it satisfies 5 NeutroAxioms).

Example 3.3. Let $X=\mathbb{Z}$. Then
(i) $(X, *, 0)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x*y=x-y+xy$.
(ii) $(X, *, 1)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x*y=xy$.
(iii) $(X, *, 1)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x*y=\begin{cases}1 & \text{if } x \text{ an even}\\ xy & \text{if } x \text{ an odd}\end{cases}$.

Let $X \neq \emptyset$ be a finite set. We denote $\mathcal{N}_{BCK}(X)$ and $\mathcal{N}_{NBCK}(X)$ by the set of all Neutro-BCK-algebras and Neutro-BCK-algebras of type 5 that are constructed on X, respectively.

Theorem 3.4. *Let $(X, *, 0)$ be a Neutro BCK-algebra. Then*

(i) *If $|X|=1$, then $(X, *, 0)$ is a trivial BCK-algebra.*

(ii) *If $|X|=2$, then $|\mathcal{N}_{BCK}(X)|=2$ and $|\mathcal{N}_{NBCK}(X)|=\infty$.*

(iii) *If $|X|=3$, then there exists $\emptyset \neq Y \subseteq X$, such that $(Y, *', 0)$ is a nontrivial or trivial BCK-algebra.*

Proof. We consider only the cases $(ii), (iii)$, because the others are immediate.
(ii) Let $X=\{0, x\}$. Then we have 2 trivial Neutro-BCK-algebras $(X, *_1), (X, *_2)$ and an infinite number of trivial Neutro-BCK-algebras of type 5 $(X, *, 0)$ in Table 1, where $w \notin X$.
(iii) Let $X=\{0, x, y\}$. Now consider $Y=\{0, x\}$ and define a Neutro-BCK-algebra $(X, *', 0)$ in Table 1. Clearly $(Y, *', 0)$ is a non-trivial BCK-algebra. If $Y=\{0\}$, it is a trivial BCK-algebra. $\square$

Table 1: Neutro-BCK-algebras of order 2

$*_1$	0	x
0	0	x
x	0	x

,

$*_2$	0	x
0	x	0
x	x	0

,

$*$	0	x
0	x	0
x	w	0

and

$*'$	0	x	y
0	0	0	y
x	x	0	0
y	0	y	x

.

Theorem 3.5. *Every BCK-algebra, can be extended to a Neutro-BCK-algebra.*

Proof. Let $(X, *, 0)$ be a BCK-algebra and $\alpha \notin X$, and U be the universe of discourse that strictly includes $X \cup \alpha$. For all $x, y \in X \cup \{\alpha\}$, define $*_\alpha$ on $X \cup \{\alpha\}$ by $x *_\alpha y = x*y$ where, $x, y \in X$ and if $\alpha \in \{x, y\}$, define $x *_\alpha y$ as indeterminate or $x *_\alpha y \notin X \cup \alpha$. Then $(X \cup \{\alpha\}, *_\alpha, 0)$ is a Neutro-BCK-algebra. $\square$

Example 3.6. Let $X=\{0,1,2,3,4,5\}$. Consider Table 3.
Then
(i) If $a=0$, then $(X, *_1, 0)$ is a Neutro-BCK-algebra and if $a=1$, then $(X \setminus \{3,4,5\}, *_1, 0)$ is a BCK-algebra.
(ii) $(X, *_2, 0)$ is a Neutro-BCK-algebra and $(X \setminus \{4,5\}, *_2, 0)$ is a BCK-algebra.
(iii) If $s=t=y=z=0, w=3$, then $(X, *_3, 0)$ is a Neutro-BCK-algebra and for $s=t=1, y=2, z=3$, $(X \setminus \{5\}, *_3, 0)$ is a BCK-algebra. If $s=t=y=z=0, w=\sqrt{2}$, then $(X, *_3, 0)$ is a Neutro-BCK-algebra of type 5 where $s, t \in \{0,1\}, x \in \{4,5\}, y \in \{2,0\}, z \in \{3,0\}$ and $w \in \{3, \sqrt{2}\}$.

Table 2: Neutro-BCK-algebras and Neutro-BCK-algebra of type 5

$*_1$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	a	2	0	3
2	2	2	0	0	2	0
3	3	0	1	2	0	5
4	0	4	0	1	4	0
5	4	0	1	0	2	3

,

$*_2$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	0	0	0	5
2	2	1	0	0	5	0
3	3	2	1	0	0	2
4	0	1	0	4	1	2
5	5	0	4	0	0	x

and

$*_3$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	t	0	s	0
2	2	2	0	y	0	3
3	3	1	3	0	z	0
4	4	4	4	4	0	1
5	0	2	0	2	0	w

,

Remark 3.7. In Neutro-BCK-algebra $(X, *_3, 0)$, which is defined as in Example 3.6, we have $(1, 5) \in\leq$ and $(5, 0) \in\leq$, but $(1, 0) \notin\leq$, where $(x, y) \in\leq$ means $x *_3 y = 0$. Thus $\leq$, necessarily, is not a transitive relation. So we have the following definition of neutro-partially ordered relation on Neutro-BCK-algebra.

Definition 3.8. Let X be a non-empty set and R be a binary relation on X. Then R is called a

(i) neutro-reflexive, if $\exists\, x \in X$ such that $(x, x) \in R$ (degree of truth T), and $\exists\, x \in X$ such that (x, x) is indeterminate (degree of indeterminacy I), and $\exists\, x \in X$ such that $(x, x) \notin R$ (degree of falsehood F);

(ii) neutro-antisymmetric, if $\exists\, x, y \in X$ such that $(x, y) \in R$ and $(x, y) \in R$ imply that $x = y$ (degree of truth T), and $\exists\, x, y \in X$ such that (x, y) or (y, x) are indeterminate in R (degree of indeterminacy I), and $\exists\, x, y \in X$ such that $(x, y) \in R$ and $(y, x) \in R$ imply that $x \neq y$ (degree of falsehood F);

(iii) neutro-transitive, if $\exists\, x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$ imply that $(x, z) \in R$ (degree of truth T),and $\exists\, x, y, z \in X$ such that (x, y) or (y, z) are indeterminate in R (degree of indeterminacy I), and $\exists\, x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$, but $(x, z) \notin R$ (degree of falsehood F). In all above neutro-axioms $(T, I, F) \notin (1, 0, 0), (0, 0, 1)$.

(iv) neutro-partially ordered binary relation, if the relation satisfies at least one of the above neutro-axioms neutro-reflexivity, neutro-antisymmetry, neutro-transitivity, while the others (if any) are among the classical axioms reflexivity, antisymmetry, transitivity.

If R is a neutro-partially ordered relation on X, we will call (X, R) by neutro-poset. We will denote, the related diagram with to neutro-poset (X, R) by neutro-Hass diagram.

We define binary relations " $\leq_1, \leq_2$ " on X by $\big(x \leq_1 y$ if or only if $x * y = 0$ or $x \leq_1 x\,\big)$ and $\big(x \leq_2 y$ if and only if ($x * y \neq 0$ or indeterminate) or $x \leq_2 x\,\big)$. So we have the following theorem.

Theorem 3.9. *An algebra $(X, *, 0)$ is a Neutro-BCK-algebra if and only if it satisfies the following conditions:*

$(NBCI\text{-}1')$ $\big(\exists\, x, y \in X$ *such that* $((x*y)*(x*z)) \leq_1 (z*y)\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $((x*y)*(x*z)) \leq_2 (z*y)\big)$,

$(NBCI\text{-}2')$ $\big(\exists\, x, y \in X$ *such that* $(x * (x * y)) \leq_1 y\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $(x * (x * y)) \leq_2 y\big)$,

$(NBCI\text{-}3')$ $\big(\exists\, x, y \in X$ *such that* $x \leq_1 x\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $x \leq_2 x\big)$,

$(NBCI\text{-}4')$ $\big(\forall\, x, y \in X$, *if* $x \leq_1 y$ *and* $y \leq_1 x$, *we get* $x = y\big)$ *and* $\big(\forall\, x, y \in X$, *if* $x \leq_2 y$ *and* $y \leq_2 x$, *we get* $x = y\big)$,

$(NBCK\text{-}5')$ $\big(\exists\, x, y \in X$ *such that* $0 \leq_1 x\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $0 \leq_2 x\big)$.

Proof. Let $(X, *, 0)$ be a Neutro-BCK-algebra. We prove only the item $(NBCI\text{-}1')$, other items are similar to. Since $(X, *, 0)$ be a Neutro-BCK-algebra, $\big(\exists\, x, y \in X$ such that $(x*(x*y))*y = 0\big)$ and $\big(\exists\, x, y \in X$ such that $(x * (x * y)) * y \neq 0$ or indeterminate$\big)$. By definition, $\big(\exists\, x, y \in X$ such that $((x * y) * (x * z)) \leq_1 (z * y)\big)$ and $\big(\exists\, x, y \in X$ such that $((x * y) * (x * z)) \leq_2 (z * y)\big)$. Conversely, let the items $(NBCI\text{-}1')$ to $(NBCI\text{-}4')$ and $(NBCK\text{-}4')$. Just prove $(NBCI\text{-}1)$ and other items are similar to. Since $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_1 (z*y)\big)$ and $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_2 (z*y)\big)$, we get that $\big(\exists\, x, y \in X$ such that $((x * y) * (x * z)) * (z * y) = 0)\big)$ and $\big(\exists\, x, y \in X$ such that $((x * y) * (x * z)) * (z * y) \neq 0$ or indeterminate $\big)$. $\square$

Let $(X, *, 0)$ be a Neutro-BCK algebra. Define binary relation $\leq$ on X, by $x \leq y$ if and only $x \leq_1 y$ and $y \leq_2 x$. So we have the following results.

Theorem 3.10. *Let $(X, *, 0)$ be a Neutro-BCK algebra and $x, y, z \in X$. Then*

(i) *if $x \neq y$ and $x \leq y$, then $y \leq x$;*

(ii) *$\leq$ is a reflexive and symmetric relation on X;*

(iii) *$\leq$ is a neutro-transitive algebra relation on X.*

Proof. (i) Let $x \neq y \in X$ and $x \leq y$. If $y \leq x$, by definition we obtain $(x * y = y * x = 0)$ and $(x * y = y * x \neq 0)$ and so $x = y$.

$(ii), (iii)$ It is clear by item (i) and Remark 3.7.

(iii) It is obtained by (ii). □

Corollary 3.11. *Let $(X, *, 0)$ be a Neutro-BCK algebra. Then $(X, *, 0, \leq_1), (X, *, 0, \leq_2)$ and $(X, *, 0, \leq)$ are neutro-posets.*

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BCK-algebras, where $X_1 \cap X_2 = \emptyset$. For some $x, y \in X$, define an operation $*$ as follows: $x * y = \begin{cases} x *_1 y & \text{if if } x, y \in X_1 \setminus X_2 \\ x *_2 y & \text{if if } x, y \in X_2 \setminus X_1 \\ 0_1 & \text{if if } x \in X_1, y \in X_2 \\ 0_2 & \text{if if } x \in X_2, y \in X_1 \end{cases}$, where $0_1 * 0_2 = 0_2$ and $0_2 * 0_1 = 0_1$.

Theorem 3.12. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BCK-algebras, where $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. Then*

(i) *$(X, *, 0_1)$ is a Neutro-BCK-algebra;*

(ii) *$(X, *, 0_2)$ is a Neutro-BCK-algebra;*

Proof. (i) We only prove $(NBCI\text{-}4)$. Let $x * y = 0_1$. It follows that $x \in X_1$ and $y \in X_2$ or $x, y \in X_1$. If $x, y \in X_1$, because $(X_1, *_1, 0_1)$ is a BCK-algebra, $y * x = 0_1$ implies that $x = y$. But for $x \in X_1$ and $y \in X_2$, we have $y * x \neq 0_1$ so $(NBCI\text{-}4)$ is valid in any cases. Other items are clear.

(ii) It is similar to item (i). □

Example 3.13. Let $X_1 = \{a, b\}$ and $X_2 = \{w, x, y, z\}$. Then $(X_1, *, a)$ and $(X_2, *, w)$ are BCK-algebras. So by Theorem 3.12, $(X_1 \cup X_1, *, a)$ and $(X_1 \cup X_1, *, w)$ are Neutro-BCK-neutralgebras in Table 3.

Table 3: BCK-algebras and Neutro-BCK-algebra

$*$	a	b	w	x	y	z
a	a	a	w	a	a	a
b	b	a	a	a	a	a
w	a	w	w	w	w	w
x	w	w	x	w	w	w
y	w	w	y	x	w	w
z	w	w	z	x	x	w

.

Corollary 3.14. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BCK-algebras. Then*

(i) *$(X, *, 0_1, \leq_1), (X, *, 0_2, \leq_2)$ and $(X, *, 0_2, \leq_2)$ are posets.*

(ii) *$(X, *, 0_1, \leq_2), (X, *, 0_2, \leq_1)$ are neutro-posets.*

Example 3.15. Consider the Neutro-BCK-algebra in Example 3.13. Then we have neutro-posets $(X, *, w, \leq_1), (X, *, a, \leq_2)$ and $(X, *, 0_2, \leq)$ in Table 4, where $-$ means that elements are not comparable and I means that are indeterminates.

Definition 3.16. Let $(X, *, 0)$ be a Neutro-BCK-algebra, $\theta \in X$ and $Y \subseteq X$. Then

Table 4: neutro-posets

$\leq_1$	a	b	w	x	y	z
a	a	$-$	a	x	y	z
b	$-$	b	w	x	y	z
w	a	w	w	w	w	w
x	x	x	w	x	x	x
y	y	y	w	x	y	y
z	z	z	w	x	y	z

,

$\leq_2$	a	b	w	x	y	z
a	a	b	a	x	y	z
b	b	b	w	x	y	z
w	a	w	w	I	I	I
x	x	x	I	x	I	I
y	y	y	I	I	y	I
z	z	z	I	I	I	z

,

$\leq$	a	b	w	x	y	z
a	a	a	w	a	a	a
b	a	b	b	b	b	b
w	w	b	w	$-$	$-$	$-$
x	a	b	$-$	x	$-$	$-$
y	a	b	$-$	$-$	y	$-$
z	a	b	$-$	$-$	$-$	z

.

(i) Y is called a Neutro-BCK-subalgebra, if (1) $0 \in Y$, (2) for all $x, y \in Y$, we have $x * y \in Y$, (3) satisfies in conditions ($NBCI$-3), ($NBCI$-4) and ($NBCK$-5).

(ii) $\theta \in X$ is called a source element, if it is a minimum or maximum element in neutro-Hass diagram of $(X, *, 0)$.

Theorem 3.17. *Let $(X, *, 0)$ be a Neutro-BCK-algebra and $Y \subseteq X$. If Y is a Neutro-BCK-subalgebra of X, then*

(i) *$(Y, *, 0)$ is a Neutro-BCK-algebra.*

(ii) *X is a Neutro-BCK-subalgebra of X.*

Proof. They are clear. □

Corollary 3.18. *Let $(X, *, 0)$ be a Neutro-BCK-algebra and $|X| = n$. Then there exist $m \leq n$ and $x_1, x_2, \ldots, x_m \in X$ such that $(\{0, x_1, x_2, \ldots, x_m\}, *, 0)$ is a Neutro-BCK-algebra of X.*

Theorem 3.19. *Let X be a non-empty set. Then there exists a binary operation "$\bullet$" on X and $0 \in X$ such that*

(i) *$(X, \bullet, x_0)$ is a Neutro-BCK-algebra.*

(ii) *For all $\emptyset \neq Y \subseteq X$, $Y \cup \{x_0\}$ is a Neutro-BCK-subalgebra of X.*

(iii) *If X is a countable set, then in neutro-Hass diagram $(X, \bullet, x_0)$, we have $|Maximal(X)| = 1$ and $Minimal(X) = |X| - 1(|X|$ is cardinal of $X)$.*

(iv) *neutro-Hass diagram $(X, \bullet, x_0)$ has a source element.*

Proof. Let $x, y \in X$. Fixed $x_0 \in X$ and define $x * y = y$.
(i) Some modulations show that $(X, *, x_0)$ is a Neutro-BCK-algebra.
(ii) By Theorem 3.4 and definition, it is clear.
(iii) Let $X = \{x_0, x_1, x_2, x_3, \ldots\}$. Then by Corollary 3.11, $(X, \leq, x_0)$ is a neutro-poset and so has a neutro-Hass diagram as Figure 1. □

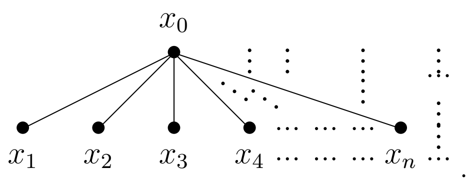

Figure 1: neutro-Hass diagram $(X, \leq, x_0)$ with source x_0.

Theorem 3.20. *Let $(X, \leq_X)$ be a chain. Then*

(i) *there exists $*_X$ on X and $0 \in X$ such that $(X, *_X, 0)$ is a Neutro-BCK-algebra.*

(ii) *for all $x, y \in X$, we have $x \leq y$ if and only if $y \leq_X x$.*

(iii) *In neutro-Hass diagram $(X, \bullet, x_0)$, 0 is source element.*

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there exists $*_X$ on X and $0 \in X$ such that $(X, *_X, 0)$ is a Neutro-BCK-algebra.

Proof. Let $0, x, y \in X$, where $0 = Min(X)$.

(i) Define $x *_X y = \begin{cases} x \vee y & \text{if } x \leq_X y \\ x \wedge y & \text{otherwise} \end{cases}$. Some modulations show that $(X, *_X, 0)$ is a Neutro-BCK-algebra.

(ii) Let $x, y \in X$. Clearly $x * x = x$, then by definition $x \leq y$ if and only if $x * y = 0$ and $y * x \neq 0$ if and only if $y = 0$ if and only if $y \leq_X x$.

(iii) By item (ii), we get the neutro-Hass diagram $(X, \leq_X, 0)$ in Figure 1, so 0 is source element. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras, where $X_1 \cap X_2 = \emptyset$. Define $*$ on $X_1 \cup X_2$, by $x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1 \setminus X_2 \\ x *_2 y & \text{if } x, y \in X_2 \setminus X_1 \\ y & \text{otherwise} \end{cases}$.

Theorem 3.21. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Then*

(i) $(X_1 \cup X_2, *, 0_1)$ *is a Neutro-BCK-algebra.*

(ii) $(X_1 \cup X_2, *, 0_2)$ *is a Neutro-BCK-algebra.*

Proof. It is obvious. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Define $*$ on $X_1 \times X_2$, by $(x, y) * (x', y') = (x *_1 x', y *_2 y')$, where $(x, y), (x', y') \in X_1 \times X_2$.

Theorem 3.22. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Then $(X_1 \times X_2, *, (0_1, 0_2))$ is a Neutro-BCK-algebra.*

Proof. We prove only the item $(NBCI$-4). Let $(x, y), (x', y') \in X_1 \times X_2$. If $(x, y) * (x', y') = (x', y') * (x, y) = (0_1, 0_2)$, then $(x *_1 x', y *_2 y') = (0_1, 0_2)$ and $(x' *_1 x, y' *_2 y) = (0_2, 0_1)$. It follows that $(x, y) = (x', y')$. In a similar way, $(x, y) * (x', y') = (x', y') * (x, y) \neq (0_1, 0_2)$, we get that $(x, y) = (x', y')$. Thus, $(X_1 \times X_2, *, (0_1, 0_2))$ is a Neutro-BCK-algebra. □

3.2 Application of Neutro-BCK-algebra

In this subsection, we describe some applications of Neutro-BCK-algebra.

In the following example, we describe some applications of Neutro-BCK-algebra. We discuss applications of Neutro-BCK-algebra for studying the competition along with algorithms. The Neutro-BCK-algebra has many utilizations in different areas, where we connect Neutro-BCK-algebra to other sciences such as economics, computer sciences and other engineering sciences. We present an example of application of Neutro-BCK-algebra in COVID-19.

Example 3.23. (COVID-19) Let $X = \{a = China, b = Italy, c = USA, d = Spain, e = Germany, f = Iran\}$ be a set of top six COVID-19 affected countries. There are many relations between the countries of the world. Suppose $*$ is one of relations on X which is described in Table 5. This relation can be economic impact, political influence, scientific impact or other chasses. For example $x * y = z$, means that the country z influences the relationship $*$ from country x to country y. Clearly $(X, *, China)$ is a Neutro-BCK-algebra.

Table 5: Neutro-BCK-algebra

$*$	$China$	$Italy$	USA	$Spain$	$Germany$	$Iran$
$China$	$China$	$Iran$	$Spain$	$Germany$	$Italy$	USA
$Italy$	$China$	$Italy$	$Iran$	$Germany$	$Spain$	$Germany$
USA	$China$	$Italy$	USA	USA	$Iran$	$Iran$
$Spain$	$China$	$China$	$China$	$Spain$	USA	$Italy$
$Germany$	$China$	$Germany$	$Italy$	$Spain$	$Germany$	$Italy$
$Iran$	$China$	$Spain$	USA	USA	$China$	$Iran$

.

And so we obtain neutro-Hass diagram as Figure 2. Applying Figure 2, we obtain that China is main source of COVID-19 to top five affected countries and Iran, Spain, Italy are indeterminated countries in COVID-19 affection together, USA effects Spain and Germany effects Iran.

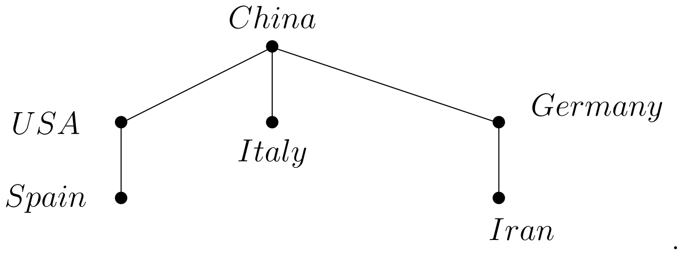

Figure 2: neutro-Hass diagram $(X, *, China)$ associated to infected COVID-19 .

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