Einstein, incompleteness, and the epistemic view of quantum states

Nicholas Harrigan

QOLS, Blackett Laboratory, Imperial College London,

Prince Consort Road, London SW7 2BW, United Kingdom

Robert W. Spekkens

Department of Applied Mathematics and Theoretical Physics,

University of Cambridge, Cambridge CB3 0WA, United Kingdom

Abstract

Does the quantum state represent reality or our knowledge of reality? In making this distinction precise, we are led to a novel classification of hidden variable models of quantum theory. We show that representatives of each class can be found among existing constructions for two-dimensional Hilbert spaces. Our approach also provides a fruitful new perspective on arguments for the non-locality and incompleteness of quantum theory. Specifically, we show that for models wherein the quantum state has the status of something real, the failure of locality can be established through an argument considerably more straightforward than Bell's theorem. The historical significance of this result becomes evident when one recognizes that the same reasoning is present in Einstein's preferred argument for incompleteness, which dates back to 1935. This fact suggests that Einstein was seeking not just *any* completion of quantum theory, but one wherein quantum states are solely representative of our knowledge. Our hypothesis is supported by an analysis of Einstein's attempts to clarify his views on quantum theory and the circumstance of his otherwise puzzling abandonment of an even simpler argument for incompleteness from 1927.

I. INTRODUCTION

We explore a distinction among hidden variable models of quantum theory that has hitherto not been sufficiently emphasized, namely, whether the quantum state is considered to be ontic or epistemic. We call a hidden variable model *ψ-ontic* if every complete physical state or *ontic state* [1] in the theory is consistent with only one pure quantum state; we call it *ψ-epistemic* if there exist ontic states that are consistent with more than one pure quantum state. In *ψ*-ontic models, distinct quantum states correspond to disjoint probability distributions over the space of ontic states, whereas in *ψ*-epistemic models, there exist distinct quantum states that correspond to overlapping probability distributions. Only in the latter case can the quantum state be considered to be truly epistemic, that is, a representation of an observer's knowledge of reality rather than reality itself. This distinction will be made rigorous further on.

It is interesting to note that, to the authors' knowledge, all mathematically explicit hidden variable models proposed to date are *ψ*-ontic (with the exception of a proposal by Kochen and Specker [2] that only works for a two-dimensional Hilbert space and which we will discuss further on).[1] The study of *ψ*-epistemic hidden variable models is the path less traveled in the hidden variable research program. This is unfortunate given that recent work has shown how useful the assumption of hidden variables can be for explaining a variety of quantum phenomena *if* one adopts a *ψ*-epistemic approach [3–6].

It will be useful for us to contrast hidden variable models with the interpretation that takes the quantum state alone to be a complete description of reality. We call the latter the *ψ-complete* view, although it is sometimes referred to as the *orthodox* interpretation[2].

Arguments against the *ψ*-complete view and in favor of hidden variables have a long history. Among the most famous are those that were provided by Einstein. Although he did not use the term 'hidden variable interpretation', it is generally agreed that such an

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[1] Subtleties pertaining to Nelson's mechanics and unconventional takes on the deBroglie-Bohm interpretation will also be discussed in due course.

[2] Note that while Bohr argued for the completeness of the quantum state, he did so within the context of an instrumentalist rather than a realist approach and consequently his view is not the one that we are interested in examining here. Despite this, the realist *ψ*-complete view we have in mind does approximate well the views of many researchers today who identify themselves as proponents of the Copenhagen interpretation.

interpretation captures his approach. Indeed, Einstein had attempted to construct a hidden variable model of his own (although ultimately he did not publish this work) [7, 8]. One of the questions we address in this article is whether Einstein favored either of the two sorts of hidden variable theories we have outlined above: ψ-ontic or ψ-epistemic. Experts in the quantum foundations community have long recognized that Einstein had already shown a failure of locality for the ψ-complete view with a very simple argument at the Solvay conference in 1927 [9]. It is also well-known in such circles that a slightly more complicated argument given in 1935 — one appearing in his correspondence with Schrödinger, not the Einstein-Podolsky-Rosen paper — provided yet another way to see that locality was ruled out for the ψ-complete view[3] [11, 12]. What is not typically recognized, and which we show explicitly here, is that the latter argument was actually strong enough to also rule out locality for ψ-ontic hidden variable theories. In other words, Einstein showed that not only is locality inconsistent with ψ being a complete description of reality, it is also inconsistent with ψ being ontic, that is, inconsistent with the notion that ψ represents reality even in an incomplete sense. Einstein thus provided an argument for the epistemic character of ψ based on locality.

Fuchs has previously argued in favor of this conclusion. In his words, "[Einstein] was the first person to say in absolutely unambiguous terms why the quantum state should be viewed as information [...]. His argument was simply that a quantum-state assignment for a system can be forced to go one way or the other by interacting with a part of the world that should have no causal connection with the system of interest." [13]. One of the main goals of the present article is to lend further support to this thesis by clarifying the relevant concepts and by undertaking a more detailed exploration of Einstein's writings. We also investigate the implications of our analysis for the history of incompleteness and nonlocality arguments in quantum theory.

In particular, our analysis helps to shed light on an interesting puzzle regarding the evolution of Einstein's arguments for incompleteness.

The argument Einstein gave at the 1927 Solvay conference requires only a single measurement to be performed, whereas from 1935 onwards he adopted an argument using a measurement chosen from two possibilities. Why did Einstein complicate the argument

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[3] Borrowing a phrase from Asher Peres [10], these facts are "well known to those who know things well".

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in this way? Indeed, as has been noted by many authors, this complication was actually detrimental to the effectiveness of the argument, given that most of the criticisms directed against the two-measurement form of the argument (Bohr's included) focus upon his use of counterfactual reasoning, an avenue that is not available as a response to the 1927 version [14–18].

The notion that Einstein introduced this two-measurement complication in order to simultaneously beat the uncertainty principle is plausible for the Einstein-Podolsky Rosen (EPR) paper, which *does* take aim at the uncertainty principle. But, as emphasized by Fine [12] and Howard [19], the EPR paper was written by Podolsky and, by Einstein's own admission, did not provide an accurate synopsis of his (Einstein's) views. In the versions of the argument that were authored by Einstein, such as those appearing in his correspondence with Schrödinger, the uncertainty principle is explicitly de-emphasized. Moreover, to the authors' knowledge, whenever Einstein summarizes his views on incompleteness in publications or in his correspondence after 1935, it is the argument appearing in his correspondence with Schrödinger, rather than the EPR argument, to which he appeals.

We suggest a different answer to the puzzle. Einstein consistently used his more complicated 1935 argument in favor of his simpler 1927 one because the extra complication bought a stronger conclusion, namely, that the quantum state is not just incomplete, but *epistemic*. We suggest that Einstein implicitly recognized this fact, even though he failed to emphasize it adequately.

Finally, our results demonstrate that one doesn't need the "big guns" of Bell's theorem [20] to rule out locality for any theories in which ψ is given ontic status; more straightforward arguments suffice. Bell's argument is only necessary to rule out locality for ψ-epistemic hidden variable theories. It is therefore surprising that the latter sort of hidden variable theory, despite being the most difficult to prove inconsistent with locality and despite being the last, historically, to have been subject to such a proof, appears to have somehow attracted the least attention, with Einstein a notable but lonely exception to the rule.

II. THE DISTINCTION BETWEEN ψ-ONTIC AND ψ-EPISTEMIC ONTOLOGICAL MODELS

A. What is an ontological model?

We begin by defining some critical notions. First is that of an ontological model of a theory. Our definition will require that the theory be formulated operationally, which is to say that the primitives of description are simply preparation and measurement procedures – lists of instructions of what to do in the lab. The goal of an operational formulation of a theory is simply to prescribe the probabilities of the outcomes of different measurements given different preparation procedures, that is, the probability $p(k|M,P)$ of obtaining outcome k in measurement M given preparation P. For instance, in an operational formulation of quantum theory, every preparation P is associated with a density operator ρ on Hilbert space, and every measurement M is associated with a positive operator valued measure (POVM) $\{E_k\}$. (In special cases, these may be associated with vectors in Hilbert space and Hermitian operators respectively.) The probability of obtaining outcome k is given by the generalized Born rule, $p(k|M,P) = \text{Tr}(\rho E_k)$.

In an ontological model of an operational theory, the primitives of description are the properties of microscopic systems. A preparation procedure is assumed to prepare a system with certain properties and a measurement procedure is assumed to reveal something about those properties. A complete specification of the properties of a system is referred to as the *ontic state* of that system, and is denoted by λ. The ontic state space is denoted by Λ. It is presumed that an observer who knows the preparation P may nonetheless have incomplete knowledge of λ. In other words, the observer may assign a non-sharp probability distribution $p(\lambda|P)$ over Λ when the preparation is known to be P. Similarly, the model may be such that the ontic state λ determines only the probability $p(k|\lambda,M)$ of different outcomes k for the measurement M. We shall refer to $p(\lambda|P)$ as an *epistemic state*, because it characterizes the observer's knowledge of the system. We shall refer to $p(k|\lambda,M)$, considered as a function of λ, as an *indicator function*. For the ontological model to reproduce the predictions of the operational theory, it must reproduce the probability of k given M and P through the formula $\int \mathrm{d}\lambda p(k|M,\lambda)p(\lambda|P) = p(k|M,P)$.

An ontological model of quantum theory is therefore defined as follows.

Definition 1 *An ontological model of operational quantum theory posits an ontic state space Λ and prescribes a probability distribution over Λ for every preparation procedure P, denoted $p(\lambda|P)$, and a probability distribution over the different outcomes k of a measurement M for every ontic state $\lambda \in \Lambda$, denoted $p(k|\lambda, M)$. Finally, for all P and M, it must satisfy,*

$$\int d\lambda p(k|M, \lambda)p(\lambda|P) = \mathrm{tr}\left(\rho E_k\right), \tag{1}$$

where ρ is the density operator associated with P and E_k is the POVM element associated with outcome k of M.

The structure of the posited Λ encodes the kind of reality envisaged by the model, while $p(\lambda|P)$ and $p(k|M, \lambda)$ specify what can be known and inferred by observers. Note that we refer to preparation and measurement procedures rather than quantum states and POVMs because we wish to allow for the possibility of contextual[4] ontological models [1].

B. Classifying ontological models of quantum theory: heuristics

An important feature of an ontological model is how it takes the quantum states describing a system to be related to the ontic states of that system. The simplest possibility is a one-to-one relation.[5] A schematic of such a model is presented in part (a) of Fig. 1, where we have represented the set of all quantum states by a one-dimensional ontic state space Λ labeled by ψ. We refer to such models as *ψ-complete* because a pure quantum state provides a complete description of reality. Many might consider this to be the 'orthodox' interpretation.

Of course, the ontological model framework also allows for the possibility that a complete description of reality may require *supplementing* the quantum state with additional variables. Such variables are commonly referred to as 'hidden' because their value is typically presumed to be unknown to someone who knows the identity of the quantum state. In such models, knowledge of ψ alone provides only an *incomplete* description of reality.

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[4] In a preparation (measurement) contextual ontological model, different preparation (measurement) procedures corresponding to the same density operator (POVM) may be assigned different epistemic states (indicator functions) by the ontological model.

[5] Note that it is because of such models, wherein nothing is hidden to one who knows the quantum state, that we adopt the term "ontological model" as opposed to "hidden variable model". Some authors might prefer to use the latter term on the grounds that a ψ-complete model is simply a trivial instance of a hidden variable model, but we feel that such a terminology would be confusing.
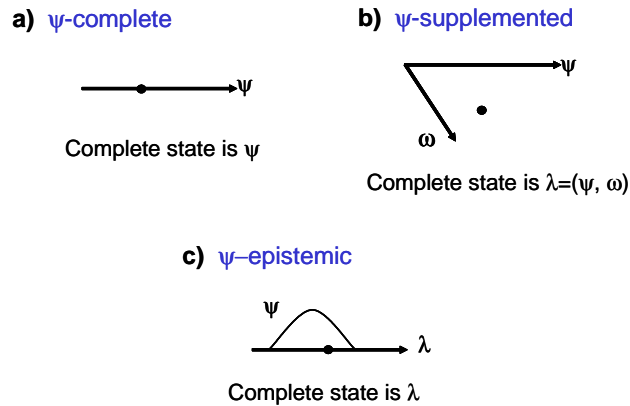
a) ψ-complete

Complete state is ψ

b) ψ-supplemented

Complete state is λ=(ψ, ω)

c) ψ–epistemic

Complete state is λ

FIG. 1: Schematic view of the ontic state space for **(a)** a ψ-complete model, **(b)** a ψ-supplemented model and **(c)** a ψ-epistemic model.

The ontic state space for such a model is schematized in part (b) of Fig. 1. Although there may be an arbitrary number of hidden variables, we indicate only a single hidden variable ω in our diagram, represented by an additional axis in the ontic state space Λ. Specification of the complete ontic configuration of a system (a point $\lambda \in \Lambda$) now requires specifying both ψ and the hidden variable ω. We refer to models wherein ψ must be supplemented by hidden variables as *ψ-supplemented*. Almost all ontological models of quantum mechanics constructed to date have fallen into this class. For example, in the conventional view of the deBroglie-Bohm interpretation [21, 22], the complete ontic state is given by ψ together with (that is, supplemented by) the positions of all particles. The ontic nature of ψ in the deBroglie-Bohm interpretation is clear from the fact that it plays the role of a pilot wave, so that distinct ψs describe physically distinct universes. Bell's 'beable' interpretations [23] and modal interpretations of quantum mechanics [24–27] also take ψ to be a sort of pilot wave and thus constitute ψ-supplemented models [6]. As another example, Belifante's survey of hidden variable theories [28] considers only ψ-supplemented models.

There is a different way in which ψ could be an incomplete description of reality: it could represent a state of incomplete knowledge about reality. In other words, it could be that ψ is not a variable in the ontic state space at all, but rather encodes a probability distribution over the ontic state space. In this case also, specifying ψ does not completely specify the

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[6] Note that another way in which to express how ψ-complete and ψ-supplemented models differ from ψ-epistemic models is that only in the former is ψ itself a beable [23].

ontic state. In such a model, a variation of ψ does not represent a variation in any physical degrees of freedom, but instead a variation in the space of possible ways of knowing about some underlying physical degrees of freedom. This is illustrated schematically in part (c) of Fig. 1. We refer to such models as ψ-epistemic.[7]

C.   Classifying ontological models of quantum theory: a more rigorous approach

It will be convenient for our purposes to provide precise definitions of ψ-complete, ψ-supplemented, and ψ-epistemic models in terms of the epistemic states that are associated with different ψ. In other words, for each model, we enquire about the probability distribution over the ontic state space that is assigned by an observer who knows that the preparation procedure is associated with the quantum state ψ. Despite appearances, this does not involve any loss of generality. For instance, although it might appear that ψ-complete models can only be defined by their ontological claims, namely, that pure quantum states are associated one-to-one with ontic states, such claims can always be re-phrased as epistemic claims, in this case, that knowing the quantum state to be ψ implies having a state of complete knowledge about the ontic state.

We now provide precise definitions of two distinctions among ontological models from which one can extract the three categories introduced in Sec. II B. The first distinction is between models that are ψ-complete and those that are not.

Definition 2 *An ontological model is ψ-**complete** if the ontic state space Λ is isomorphic to the projective Hilbert space $\mathcal{PH}$ (the space of rays of Hilbert space) and if every preparation procedure P_ψ associated in quantum theory with a given ray ψ is associated in the ontological model with a Dirac delta function centered at the ontic state λ_ψ that is isomorphic to ψ, $p(\lambda|P_\psi) = \delta(\lambda - \lambda_\psi)$.*[8]

Hence, in such models, the only feature of the preparation that is important is the pure quantum state to which it is associated. Epistemic states for a pair of preparations associated with distinct quantum states are illustrated schematically in part (a) of Fig. 2.[9]

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[7] There is, however, a subtlety in ensuring that a probability distribution associated with ψ is truly epistemic; we address this issue shortly.

[8] The Dirac delta function on Λ is defined by $\int_\Lambda \delta(\lambda - \lambda_\psi) f(\lambda) \mathrm{d}\lambda = f(\lambda_\psi)$.

[9] In the case of a mixture of pure states, one uses the associated mixture of epistemic states. For instance,

Definition 3 *If an ontological model is not ψ-complete, then it is said to be ψ-**incomplete**.*

Identifying a model as ψ-incomplete does not specify how such a failure is actually manifested. It might be that Λ is parameterized by ψ *and* by supplementary variables, or it could alternatively be that the quantum state does not parameterize the ontic states of the model at all. In order to be able to distinguish these two possible manifestations of ψ-incompleteness, we introduce a second dichotomic classification of ontological models.

Definition 4 *An ontological model is ψ-**ontic** if for any pair of preparation procedures, P_ψ and P_ϕ, associated with distinct quantum states ψ and ϕ, we have $p(\lambda|P_\psi)p(\lambda|P_\phi) = 0$ for all λ.*[10]

Hence, the epistemic states associated with distinct quantum states are completely non-overlapping in a ψ-ontic model. In other words, different quantum states pick out disjoint regions of Λ. The idea of a ψ-incomplete model that is also ψ-ontic is illustrated schematically in part (b) of Fig. 2. Here, the ontic state space is parameterized by ψ (represented by a single axis) and a supplementary hidden variable ω. The epistemic state $p(\lambda|P_\phi)$ representing a preparation procedure associated with ϕ has the form of a Dirac delta function along the ψ axis, which guarantees the disjointness property for epistemic states associated with distinct quantum states. Even if an ontological model is presented to us in a form where it is not obvious whether ψ parameterizes Λ, by verifying that the above definition is satisfied, one verifies that such a parametrization can be found.

Another useful way of thinking about ψ-ontic models is that the ontic state λ 'encodes' the quantum state ψ because a given λ is only consistent with one choice of ψ. Alternatively,

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if the preparation is of a pure state ψ_i with probability w_i, then the epistemic state is $\sum_i w_i p(\lambda|\psi_i)$. Note, however, that it is not at all clear how to deal in a ψ-complete model with *improper* mixtures, that is, mixed density operators that arise as the reduced density operator of an entangled state. This fact is often used to criticize such models.

[10] A more refined definition of the distinction would require $\int_\Lambda d\lambda \sqrt{p(\lambda|P_\psi)} \sqrt{p(\lambda|P_\phi)} = 0$, that is, the vanishing of the *classical fidelity*, rather than the product, of the probability distributions associated with any pair of distinct pure quantum states. This refinement is important for dealing with ontological models wherein the only pairs of distributions that overlap do so on a set of measure zero. Intuitively, one would not want to classify these as ψ-epistemic, but only the fidelity-based definition does justice to this intuition. This definition will not, however, be needed here.

we can see this encoding property as follows. By Bayes' theorem, one infers that any ψ-ontic model satisfies $p(P_\psi|\lambda)p(P_\phi|\lambda) = 0$ for $\psi \neq \phi$, which implies that for every λ, there exists some ψ such that $p(P_\psi|\lambda) = 1$ and $p(P_\phi|\lambda) = 0$ for all $\phi \neq \psi$.

Definition 5 *If an ontological model fails to be ψ-ontic, then it is said to be ψ-**epistemic**.*

It is worth spelling out what the failure of the ψ-ontic property entails: there exists a pair of preparation procedures, P_ψ and P_ϕ and a $\lambda \in \Lambda$ such that $p(\lambda|P_\psi)\, p(\lambda|P_\phi) \neq 0$, which is to say that the two epistemic states *do* overlap. Using Bayes' theorem we can equivalently formulate this requirement as $\exists\, P_\psi, P_\phi, \lambda : p(P_\psi|\lambda)\, p(P_\phi|\lambda) \neq 0$, which asserts that the ontic state λ is consistent with both the quantum state ψ and the quantum state ϕ. In a ψ-epistemic model, multiple distinct quantum states are consistent with the same state of reality – the ontic state λ does not encode ψ. It is in this sense that the quantum state is judged epistemic in such models. This is illustrated schematically in part (c) of Fig. 2, where the ontic state space Λ is one-dimensional, and preparations associated with distinct ψ are associated with overlapping distributions on Λ.

Some comments are in order. The reader might well be wondering why we do not admit that *any* ψ-incomplete model is 'epistemic', simply because it associates a probability distribution of nontrivial width over Λ with each quantum state. We admit that although it might be apt to say that ψ-incomplete *models* have an epistemic character, the question of interest here is whether *pure quantum states* have an epistemic character. It is for this reason that we speak of whether a model is 'ψ-epistemic' rather than simply 'epistemic'. By our definitions, ψ has an ontic character if and only if a variation of ψ implies a variation of reality and an epistemic character if and only if a variation of ψ does *not* necessarily imply a variation of reality.

For any model we can specify a ψ-complete versus ψ-incomplete and ψ-ontic versus ψ-epistemic classification. At first sight, this suggests that there will be four different types of ontological model. This impression is mistaken however; there are only three different types of model because one of the four combinations describes an empty set. Specifically, if a model is ψ-complete, then it is also ψ-ontic. This follows from the fact that if a model is ψ-complete, then $p(\lambda|P_\psi) = \delta(\lambda - \lambda_\psi)$, where λ_ψ is the ontic state isomorphic to ψ, and from the fact that $\delta(\lambda - \lambda_\psi)\,\delta(\lambda - \lambda_\phi) = 0$ for $\psi \neq \phi$.
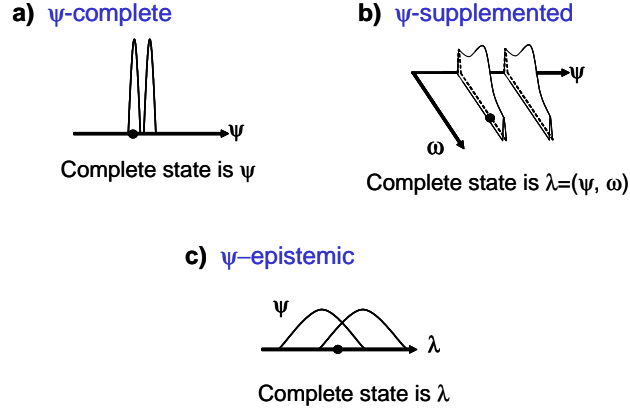
a) ψ-complete

Complete state is ψ

b) ψ-supplemented

Complete state is λ=(ψ, ω)

c) ψ–epistemic

Complete state is λ

FIG. 2: Schematic representation of how probability distributions associated with ψ are related in **(a)** a ψ-complete model, **(b)** a ψ-supplemented model and **(c)** a ψ-epistemic model. Note that the distributions over ψ in **(a)** and **(b)** are meant to be Dirac-delta functions.

The contrapositive of this implication asserts that for the quantum state to have an epistemic character, it cannot be a complete description of reality. We have therefore proven:

Lemma 6 *The following implications between properties of ontological models hold*[11]*:*

$$\psi\text{-}complete \rightarrow \psi\text{-}ontic,$$

and its negation,

$$\psi\text{-}epistemic \rightarrow \psi\text{-}incomplete. \tag{2}$$

So it is impossible for a model to be both ψ-complete and ψ-epistemic. Given Lemma 6, we can unambiguously refer to models that are ψ-complete and ψ-ontic as simply ψ-complete, and models that are ψ-incomplete and ψ-epistemic as simply ψ-epistemic. The ψ-supplemented models constitute the third category.

Definition 7 *Ontological models that are ψ-incomplete and ψ-ontic will be referred to as* **ψ-supplemented***.*

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[11] Implications such as $C_1 \rightarrow C_2$ between two classes C_1 and C_2 of ontological models should be read as 'any model in class C_1 is necessarily also in class C_2'.
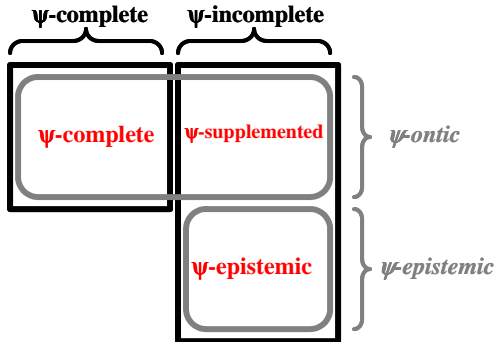
FIG. 3: Two distinctions and the three classes of ontological model that they define.

The classification of ontological models is schematized in Fig. 2. We emphasize that these are schematic representations. In particular, we note that ψ-supplemented models are not all of the type depicted in Fig. 2(b). The ontic state space $\Lambda = \{\lambda\}$ in this figure is a Cartesian product of the projective Hilbert space $\mathcal{PH} = \{\psi\}$ and a space describing the supplementary variables $\Omega = \{\omega\}$, that is, $\Lambda = \mathcal{PH} \times \Omega$. However, one can imagine models wherein this is not the case. For instance, the nature of the supplementary variables might vary with ψ. In the most general ψ-supplemented model, the relation between Λ and $\mathcal{PH}$ is as follows. By virtue of being ψ-ontic, every such model defines a (surjective) function from Λ and $\mathcal{PH}$ specifying a ψ for every λ. It is ψ-supplemented as long as this function is non-injective, so that there is at least one ψ that is the image of more than one λ (assuming Λ is a discrete set) or the image of a region of nonzero measure (assuming Λ is a measure space). Equivalently, not only must the distributions $\{p(\lambda|P_\psi) \mid \psi \in \mathcal{PH}\}$ be disjoint (as is required for the model to be ψ-ontic), there must also be at least one with support on Λ that is not confined to a singleton set or a set of measure zero.

It is worth emphasizing a few critical points before proceeding. First, the sort of thing that can be judged ψ-ontic or ψ-epistemic is an ontological model of quantum theory, not quantum theory itself. "Ontological model" is a technical term for the purposes of this paper, defined in Sec. II A. Therefore, to categorize any given interpretation of the quantum formalism as ψ-ontic or ψ-epistemic, it is first necessary to cast it into the mold of an ontological model. If an interpretation resists being so cast, then it cannot be fit into our categorization.

The first step on the road to categorization is to identify the space of ontic states. Here it is useful to bear in mind the most important property of the ontic state, encoded into Eq. (1):

it screens off the preparation from the measurement. More precisely, for all measurements, the variable that runs over the outcomes of the measurement and the variable that runs over preparation procedures are conditionally independent given the ontic state. In other words, the preparation, the ontic state of the system, and the measurement outcome must form a Markov chain.

Another important point to note is that although the definitions of ψ-complete, ψ-supplemented and ψ-epistemic refer only to the probability distributions associated with different preparations, one cannot focus on these alone when trying to construct a ψ-epistemic ontological model. The reason is that although one could *enforce* ψ-epistemicity through one's choice of distributions, one could not necessarily find positive indicator functions (which specify the probability of the outcome of a measurement given a particular ontic state) that reproduce the quantum statistics via Eq. (1).

D. Examples

We now provide examples of models that fall into each of our three classes.

1. The Beltrametti-Bugajski model

The model of Beltrametti and Bugajski [29] is essentially a thorough rendering of what most would refer to as an orthodox interpretation of quantum mechanics.[12] The ontic state space postulated by the model is precisely the projective Hilbert space, $\Lambda = \mathcal{PH}$, so that a system prepared in a quantum state ψ is associated with a sharp probability distribution[13] over Λ,

$$p\left(\lambda|\psi\right) = \delta\left(\lambda - \psi\right), \tag{3}$$

where we are using ψ interchangeably to label the Hilbert space vector and to denote the ray spanned by this vector

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[12] Note, however, that there are several versions of orthodoxy that differ in their manner of treating measurements. The Beltrametti-Bugajski model is distinguished by the fact that it fits within the framework for ontological models we have outlined.

[13] Preparations which correspond to mixed quantum states can be constructed as a convex sum of such sharp distributions

The model posits that the different possible states of reality are simply the different possible quantum states. It is therefore ψ-complete by Definition 2. It remains only to demonstrate how it reproduces the quantum statistics.

This is achieved by assuming that the probability of obtaining an outcome k of a measurement procedure M depends indeterministically on the system's ontic state λ as

$$p\left(k|M,\lambda\right) = \mathrm{tr}\left(|\lambda\rangle\langle\lambda|E_k\right), \tag{4}$$

where $|\lambda\rangle \in \mathcal{H}$ denotes the quantum state associated with $\lambda \in \mathcal{PH}$, and where $\{E_k\}$ is the POVM that quantum mechanics associates with M. It follows that,

$$\begin{aligned} \mathrm{Pr}\left(k|M,\psi\right) &= \int_\Lambda d\lambda \; p\left(k|M,\lambda\right) \; p(\lambda|\psi) \\ &= \int_\Lambda d\lambda \; \mathrm{tr}\left(|\lambda\rangle\langle\lambda|E_k\right) \; \delta\left(\lambda - \lambda_\psi\right) \tag{5} \\ &= \mathrm{tr}\left(|\psi\rangle\langle\psi|E_k\right), \tag{6} \end{aligned}$$

and so the quantum statistics are trivially reproduced.

If we restrict consideration to a system with a two dimensional Hilbert space then Λ is isomorphic to the Bloch sphere, so that the ontic states are parameterized by the Bloch vectors of unit length, which we denote by $\vec{\lambda}$. The Bloch vector associated with the Hilbert space ray ψ is denoted $\vec{\psi}$ and is defined by $|\psi\rangle\langle\psi| = \frac{1}{2}I + \frac{1}{2}\vec{\psi}\cdot\vec{\sigma}$ where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denotes the vector of Pauli matrices and I denotes the identity operator.

If we furthermore consider M to be a *projective* measurement, then it is associated with a projector-valued measure $\{|\phi\rangle\langle\phi|, |\phi^\perp\rangle\langle\phi^\perp|\}$ or equivalently, an orthonormal basis $\{|\phi\rangle, |\phi^\perp\rangle\}$. It is convenient to denote the probability of getting the ϕ outcome given ontic state $\vec{\lambda}$ simply by $p(\phi|\vec{\lambda})$. Eq. (4) simplifies to,

$$\begin{aligned} p(\phi|\vec{\lambda}) &= |\langle\phi|\lambda\rangle|^2 \tag{7} \\ &= \frac{1}{2}\left(1 + \vec{\phi}\cdot\vec{\lambda}\right). \tag{8} \end{aligned}$$

The epistemic states and indicator functions for this case of the Beltrametti-Bugajski model are illustrated schematically in Fig. 4.
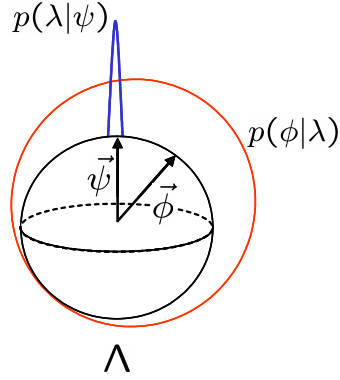
FIG. 4: Illustration of the epistemic states and indicator functions in the Beltrametti-Bugajski model.

2. The Bell-Mermin model

We now present an ontological model for a two dimensional Hilbert space that is originally due to Bell [30] and was later cast into a more intuitive form by Mermin [31].

The model employs an ontic state space Λ that is a Cartesian product of a pair of state spaces, $\Lambda = \Lambda' \times \Lambda''$. Each of Λ' and Λ'' is isomorphic to the unit sphere. It follows that there are two variables required to specify the systems total ontic state, $\vec{\lambda}' \in \Lambda'$ and $\vec{\lambda}'' \in \Lambda''$. A system prepared according to quantum state ψ is assumed to be described by a product distribution on $\Lambda' \times \Lambda''$,

$$p(\vec{\lambda}', \vec{\lambda}''|\psi) = p(\vec{\lambda}'|\psi)p(\vec{\lambda}''|\psi). \tag{9}$$

The distribution over $\vec{\lambda}'$ is a Dirac delta function centered on $\vec{\psi}$, that is, $p(\vec{\lambda}'|\psi) = \delta(\vec{\lambda}' - \vec{\psi})$. The distribution over $\vec{\lambda}'' \in \Lambda''$ is uniform over the unit sphere, $p(\vec{\lambda}''|\psi) = \frac{1}{4\pi}$, independent of ψ. These epistemic states are illustrated in Fig. 5. Consequently,

$$p(\vec{\lambda}', \vec{\lambda}''|\psi) = \frac{1}{4\pi}\delta(\vec{\lambda}' - \vec{\psi}). \tag{10}$$

Suppose now that we wish to perform a projective measurement associated with the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$. The Bell-Mermin model posits that the ϕ outcome will occur if and only if the vector $\vec{\lambda}' + \vec{\lambda}''$ has a positive inner product with the Bloch vector $\vec{\phi}$. This measurement is therefore associated with the indicator function,

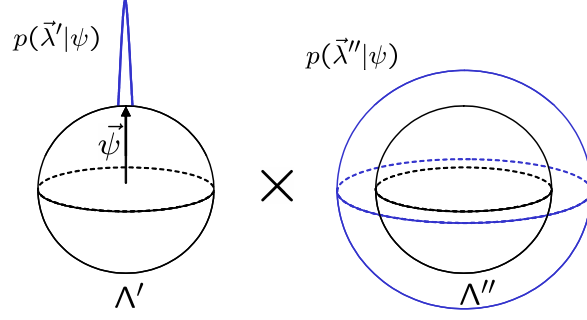$$p(\phi|\vec{\lambda}', \vec{\lambda}'') = \Theta(\vec{\phi} \cdot (\vec{\lambda}' + \vec{\lambda}'')), \tag{11}$$

FIG. 5: Illustration of the epistemic states in the Bell-Mermin model.

where Θ is the Heaviside step function defined by

$$\Theta(x) = 1 \text{ if } x > 0$$
$$= 0 \text{ if } x \leq 0.$$

The Bell-Mermin model's predictions for $p(\phi|\psi)$ [calculated as the overlap of the epistemic distributions from Eq. (10) with the indicator function of Eq. (11)] successfully reproduce the quantum mechanical Born rule,

$$
\begin{aligned}
p(\phi|\psi) &= \frac{1}{4\pi} \iint d\Lambda' d\Lambda'' \, \delta(\vec{\lambda}' - \vec{\psi}) \, \Theta(\vec{\phi} \cdot (\vec{\lambda}' + \vec{\lambda}'')) \\
&= \frac{1}{4\pi} \int d\Lambda'' \, \Theta(\vec{\phi} \cdot (\vec{\psi} + \vec{\lambda}'')) \\
&= \frac{1}{2} \left(1 + \vec{\psi} \cdot \vec{\phi}\right) \\
&= |\langle \psi|\phi \rangle|^2 .
\end{aligned}
\tag{12}
$$

We can see immediately that the Bell-Mermin model is ψ-incomplete because $\Lambda = \Lambda' \times \Lambda'' \neq \mathcal{PH}$. Furthermore,

$$
\begin{aligned}
p(\lambda|\psi)p(\lambda|\phi) &= p(\vec{\lambda}', \vec{\lambda}''|\psi)p(\vec{\lambda}', \vec{\lambda}''|\phi) \\
&= \frac{1}{16\pi^2} \delta(\vec{\lambda}' - \vec{\psi})\delta(\vec{\lambda}' - \vec{\phi}) \\
&= 0 \ \text{ if } \ \psi \neq \phi,
\end{aligned}
\tag{13}
$$

implying that the Bell-Mermin model is ψ-ontic. Recalling Definition 7, we conclude that this model falls into the class ψ-supplemented.

Because the ontic state space of this model is four dimensional, it is difficult to illustrate it in a figure. We can present the distributions over $\vec{\lambda}'$ and $\vec{\lambda}''$ on separate unit spheres, as in Fig. 5 , but the indicator functions cannot be presented in this way.

16

3. The Kochen-Specker model

As our final example, we consider a model for a two-dimensional Hilbert space due to Kochen and Specker [2]. The ontic state space Λ is taken to be the unit sphere, and a quantum state ψ is associated with the probability distribution,

$$p(\lambda|\psi) = \frac{1}{\pi}\Theta(\vec{\psi}\cdot\vec{\lambda})\,\vec{\psi}\cdot\vec{\lambda}, \tag{14}$$

where $\vec{\psi}$ is the Bloch vector corresponding to the quantum state ψ. It assigns the value $\cos\theta$ to all points an angle $\theta < \frac{\pi}{2}$ from ψ, and the value zero to points with $\theta > \frac{\pi}{2}$. This is illustrated in Fig. 6.

Upon implementing a measurement procedure M associated with a projector $|\phi\rangle\langle\phi|$ a positive outcome will occur if the ontic state $\vec{\lambda}$ of the system lies in the hemisphere centered on $\vec{\phi}$, i.e.,

$$p(\phi|\lambda) = \Theta(\vec{\phi}\cdot\vec{\lambda}). \tag{15}$$

It can be checked that the overlaps of $p(\lambda|\psi)$ and $p(\phi|\lambda)$ then reproduce the required quantum statistics,

$$\begin{aligned}
p(\phi|\psi) &= \int d\Lambda\, \frac{1}{\pi}\Theta(\vec{\psi}\cdot\vec{\lambda})\Theta(\vec{\phi}\cdot\vec{\lambda})\,\vec{\psi}\cdot\vec{\lambda} \\
&= \frac{1}{2}(1 + \vec{\psi}\cdot\vec{\phi}) \\
&= |\langle\psi|\phi\rangle|^2.
\end{aligned} \tag{16}$$

Referring to Definition 3 we see that this model is ψ-incomplete, since although Λ is isomorphic to the system's projective Hilbert space, Eq. (14) implies that the model associates *non-sharp* distributions with quantum states. Furthermore,

$$p(\lambda|\psi)p(\lambda|\phi) = \frac{1}{\pi^2}\Theta(\vec{\psi}\cdot\vec{\lambda})\,\Theta(\vec{\phi}\cdot\vec{\lambda})\,\vec{\psi}\cdot\vec{\lambda}\,\vec{\phi}\cdot\vec{\lambda},$$

is nonzero for nonorthogonal ϕ and ψ, showing, via Definition 5, that the Kochen-Specker model is ψ-epistemic.

4. Connections between the models

It is not too difficult to see that the Bell-Mermin model is simply the Beltrametti-Bugajski model supplemented by a hidden variable that uniquely determines the outcomes of all
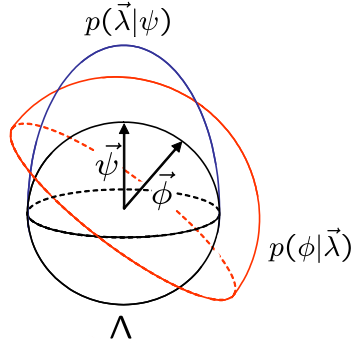
FIG. 6: Illustration of the epistemic states and indicator functions of the Kochen-Specker model.

projective measurements. We need only note that within the Bell-Mermin model, the probability of obtaining the measurement outcome ϕ given $\vec{\lambda}'$ (i.e. not conditioning on the supplementary hidden variable $\vec{\lambda}''$) is,

$$
\begin{aligned}
p(\phi|\vec{\lambda}') &= \int_\Lambda d\vec{\lambda}'' p(\phi|\vec{\lambda}', \vec{\lambda}'') p(\vec{\lambda}'') \\
&= \int_\Lambda d\vec{\lambda}'' \Theta(\vec{\phi} \cdot (\vec{\lambda}' + \vec{\lambda}'')) \frac{1}{4\pi} \\
&= \frac{1}{2}\left(1 + \vec{\phi} \cdot \vec{\lambda}'\right),
\end{aligned}
$$

which is precisely the indicator function of the Beltrametti-Bugajski model, Eq. (8). So, whereas in the Beltrametti-Bugajski model, outcomes that are not determined uniquely by $\vec{\lambda}'$ (i.e. for which $0 < p(\phi|\vec{\lambda}') < 1$) are deemed to be objectively indeterministic, in the Bell-Mermin model this indeterminism is presumed to be merely epistemic, resulting from ignorance of the value of the supplementary hidden variable $\vec{\lambda}''$. Note that although the Bell-Mermin model eliminates the objective indeterminism of the Beltrametti-Bugajski model, it pays a price in ontological economy – the dimensionality of the ontic state space is doubled.

Furthermore, there is a strong connection, previously unnoticed, between the Bell-Mermin model and the Kochen-Specker model. Although the ontic state is specified by two variables, $\vec{\lambda}'$ and $\vec{\lambda}''$, in the Bell-Mermin model, the indicator functions for projective measurements, presented in Eq. (11), depend only on $\vec{\lambda}' + \vec{\lambda}''$. It follows that if one re-parameterizes the ontic state space by the pair of vectors $\vec{u} = \vec{\lambda}' + \vec{\lambda}''$ and $\vec{v} = \vec{\lambda}' - \vec{\lambda}''$, then the indicator functions depend only on $\vec{u}$. Consequently, the only aspect of the epistemic state that is significant for calculating operational predictions is the marginal $p(\vec{u}|\psi)$. This is calculated

to be,

$$p(\vec{u}|\psi) = \int d\vec{v}\; p(\vec{u}, \vec{v}|\psi) \tag{17}$$

$$= \int d\vec{v}\frac{1}{4\pi}\delta(\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v} - \vec{\psi})$$

$$= \frac{2}{\pi}\Theta(\vec{\psi}\cdot\vec{u})\;\vec{\psi}\cdot\vec{u}.$$

But, on normalizing the vector $\vec{u}$ to lie on the unit sphere, this is precisely the form of the epistemic state posited by the Kochen-Specker model, Eq. (14), with $\vec{u}$ substituted for $\vec{\lambda}$.

It follows that the Kochen-Specker model is simply the Bell-Mermin model with the variable $\vec{v}$ eliminated, so that the variable $\vec{u}$ completely specifies the ontic state. (Reducing the ontic state space in this way leaves the empirical predictions of the model intact because these did not depend on $\vec{v}$.)

A methodological principle that is often adopted in the construction of physical theories is that one should not posit unnecessary ontological structure. Appealing to Occam's razor in the present context would lead one naturally to judge the variable $\vec{v}$ to be un-physical, akin to a gauge degree of freedom, and to thereby favor the minimalist ontological structure posited by the Kochen-Specker model over that of the Bell-Mermin model.

We see, therefore, that the price in ontological overhead that was paid by the Bell-Mermin model to eliminate objective indeterminism from the Beltrametti-Bugajski model did not need to be paid. The Kochen-Specker model renders the indeterminism epistemic without any increase in the size of the ontic state space.

It is interesting to note that starting from the orthodox model of Beltrametti and Bugajski for two dimensional Hilbert spaces, if one successively enforces (1) a principle that any indeterminism must be epistemic rather than objective, and (2) a principle that any gauge-like degrees of freedom must be eliminated as un-physical, one arrives at the ψ-epistemic model of Kochen and Specker. One is led to wonder whether such a procedure might be applied to ontological models of quantum theory in higher dimensional Hilbert spaces.

This concludes our discussion of the classification scheme for ontological models. We now turn our attention to the question of how these classes fare on the issue of locality.
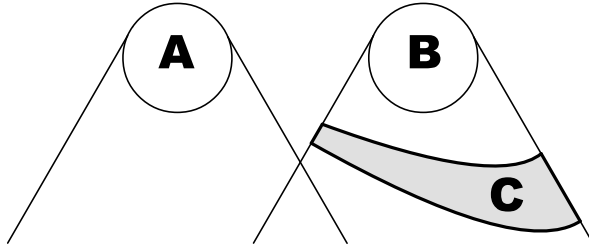
FIG. 7: Space-time regions used in Definition 9 and by Bell in [32]. A and B denote two events lying outside of each others backwards light cones, whilst C is a hypersurface intersecting the backwards lightcone of B but not A.

III. LOCALITY IN ONTOLOGICAL MODELS

A necessary component of any sensible notion of locality is separability, which we define as follows.

Definition 8 *Suppose a region R can be divided into local regions $R_1, R_2, ..., R_n$. An ontological model is said to be* separable *(denoted S) only if the ontic state space Λ_R of region R is the Cartesian product of the ontic state spaces Λ_{R_i} of the regions R_i,*

$$\Lambda_R = \Lambda_{R_1} \times \Lambda_{R_1} \times \cdots \times \Lambda_{R_n}.$$

The assumption of separability is made, for instance, by Bell when he restricts his attention to theories of *local* beables. These are variables parameterizing the ontic state space "which (unlike for example the total energy) can be assigned to some bounded space-time region" [33].

Separability is generally not considered to be a sufficient condition for locality. An additional notion of locality, famously made precise by Bell [32], appeals to the causal structure of relativistic theories. The definition makes reference to the space-time regions defined in Fig. 7. Regions A and B are presumed to be space-like separated.

Definition 9 *A separable ontological model is* locally causal *(LC) if and only if the probabilities of events in space-time region B are unaltered by specification of events in space-time region A, when one is already given a complete specification of the events in a space-time region C that screens off B from the intersection of the backward light cones of A and B.*

20

Local causality can be expressed as

$$p(B|A, \lambda_C) = p(B|\lambda_C), \tag{18}$$

where B is a proposition about events occurring in region B, λ_C is the ontic state of space-time region C (recalling that the ontic state of a system is a complete specification of the properties of that system), and A is a proposition about events in region A.

Finally, we define locality to be the conjunction of these two notions.

Definition 10 *An ontological model is* local *(L) if and only if it is separable and locally causal.*

A. ψ-ontic models of quantum theory are nonlocal

1. Are ψ-ontic models separable?

In a separable model, every ontic state of a pair of spatially separated systems can be written as an ordered pair of ontic states – one for each system. That is, if the systems are labelled A and B, then $\lambda_{AB} = (\lambda_A, \lambda_B)$. Recall that λ_A may represent a set of variables $\lambda_A = (\alpha_A, \beta_A, \dots)$, and similarly for λ_B. In a ψ-ontic model, the ontic state of any pair of systems can be given by a set of variables that includes the quantum state ψ_{AB} of the pair, and which may include additional variables which we will denote collectively by ω_{AB} (whether or not it includes these additional variables determines whether it is ψ-complete or ψ-supplemented). Thus, in a ψ-ontic model, the ontic state has the form $\lambda_{AB} = (\psi_{AB}, \omega_{AB})$. The question is: can we factorize such an ontic state as $\lambda_{AB} = (\lambda_A, \lambda_B)$? This determines whether or not a ψ-ontic model can be separable.[14]

In order to answer this question for a particular ontological model, it is necessary that the latter specify precisely the relationship between the variables it posits and regions of space-time. However, for a given ψ-ontic model, this relationship may not be clear.

The most natural answer to the question is that the factorization of the ontic state for a composite into ontic states for the components can only be achieved if the factorization of

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[14] We thank Travis Norsen for an enlightening conversation on this subject which prompted a substantial revision to our discussion of separability in ontological models.

the quantum state can be achieved, that is, only if $\psi_{AB} = \psi_A \otimes \psi_B$. For entangled states, factorization cannot be achieved and one can conclude that a ψ-ontic model is not separable. It would then follow from Definition 10 that any such model is not local. In this view,

$$\psi\text{-ontic} \implies \neg S \implies \neg L. \tag{19}$$

In this case, one needn't test whether ψ-ontic models are locally causal, given that they fail to even exhibit separability, which is a prerequisite to making sense of the notion of local causality.

There is in fact good evidence that this kind of reasoning captures Einstein's earliest misgivings about quantum theory. Already in 1926, Einstein judges Schrödinger's wave mechanics to be "altogether too primitive" [34]. Howard has argued that the significant issue for Einstein, even in those early days, was separability [35]. For instance, in order to describe multi-particle systems, Schrödinger had replaced de Broglie's waves in 3-space with waves in configuration space (he had also abandoned the notion of particle trajectories, thereby endorsing a ψ-complete view). But Einstein was dubious of this move: "The field in a many-dimensional coordinate space does not smell like something real"[36], and "If only the undulatory fields introduced there could be transplanted from the n-dimensional coordinate space to the 3 or 4 dimensional!"[37].

We have characterized Eq. (19) as a consequence of a "natural" position on the relationship between the degrees of a freedom posited by a ψ-ontic model and regions of space-time. However, alternative interpretations of this relationship might avoid this conclusion. For instance, one can imagine a ψ-supplemented model wherein (i) the supplementary variables for a composite system are all factorizable into variables for the components, and (ii) the quantum state of the composite system is presumed to "live at" one special region of space-time, say, the center of mass of the components. For such a model, there is a radical disconnect between the spatio-temporal locations of systems and the location associated with the quantum state of the collection. Such a position is certainly very bizarre and perhaps no one would espouse it, but it does not appear to be inconsistent with our definition of ψ-onticity. Therefore, although ψ-ontic models in their most natural interpretation can be judged nonlocal by dint of their failure to be separable, there is some wiggle room – some exotic interpretive strategies within which separability might be preserved.[15]

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[15] For ψ-complete rather than ψ-supplemented models, interpretative contortions that save separability have

No matter. In the next section, we shall demonstrate that even if one grants separability, all ψ-ontic models fail to be local by virtue of falling to be locally causal.

Still, this discussion of separability leads to some interesting historical questions. Why did Einstein focus on the failure of local causality rather than the failure of separability in his arguments against completeness? Why did Bell bother to establish the failure of local causality for theories that he acknowledged to be nonseparable?[16] These questions will not be the focus of our article. However, it seems to us that there remains considerable conceptual murkiness on the subject of separability in ψ-ontic ontological models, and we hope that our discussion of the matter might prompt others to consider the issue.

2. ψ-ontic models are not locally causal

We now demonstrate that there exists a very simple argument establishing that ψ-ontic models must violate locality. The argument constitutes a "nonlocality theorem" that is stronger than Einstein's 1927 argument but weaker than Bell's theorem. In the next section, we shall argue that it is in fact the content of Einstein's (non-EPR) 1935 argument for incompleteness and we shall explore what light is thereby shed on his interpretational stance. For now, however, we shall simply attempt to present the argument in the clearest possible fashion.

Consider two separated parties, Alice and Bob, who each hold one member of a pair of two-level quantum systems prepared in the maximally entangled state $|\phi^+\rangle = (|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2}$. If Alice chooses to implement a measurement M_{01} associated with the basis $\{|0\rangle, |1\rangle\}$, then depending on whether she obtains outcome 0 or 1, she updates the quantum state of Bob's system to $|0\rangle$ or $|1\rangle$ respectively (these occur with equal probability). On the other hand, if she implements a measurement $M_\pm$ associated with the basis $\{|+\rangle, |-\rangle\}$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, then she updates the quantum state of Bob's system

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even less plausibility. For if spatially separated system are generically entangled and one doesn't have any supplementary variables, then it would seem that *no* ontological claims can be made about localized regions of space, not even when the theory is applied to macroscopic objects.

[16] For instance, in Ref. [38], Bell says this of the beables upon which the probabilities of local outcomes are conditioned: "It is notable that in this argument nothing is said about the locality, or even localizability, of the variable λ. These variables could well include, for example, quantum mechanical state vectors, which have no particular localization in ordinary space-time."

to $|+\rangle$ or $|-\rangle$ depending on her outcome. Although Alice cannot control which individual pure quantum state will describe Bob's system, she can choose which of two disjoint sets, $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$, it will belong to. Schrödinger described this effect as 'steering' Bob's state [39].

This steering phenomenon allows us to prove the following theorem

Theorem 11 *Any ψ-ontic ontological model that reproduces the quantum statistics (QS-TAT) violates locality,*[17]

$$\psi\text{-}ontic \wedge QSTAT \rightarrow \neg L.$$

Proof. We prove the contrapositive, that any local model that reproduces the quantum statistics must be ψ-epistemic.

The measurements that Alice performs can be understood as 'remote preparations' of Bob's system (recall from Sec. II A that a preparation is simply a list of experimental instructions and therefore need not involve a direct interaction with the system being prepared). Denote by P_0 and P_1 the remote preparations corresponding to Alice measuring M_{01} and obtaining the 0 and 1 outcomes respectively (these preparations are associated with the states $|0\rangle$ and $|1\rangle$ of Bob's system). Let P_+ and P_- be defined similarly. Finally, denote by P_{01} the remote preparation that results from a measurement of M_{01} but wherein one does not condition on the outcome, and similarly for $P_\pm$. Given these definitions, we can infer that,

$$p(\lambda|P_{01}) = \frac{1}{2}p(\lambda|P_0) + \frac{1}{2}p(\lambda|P_1)\,, \tag{20}$$

$$p(\lambda|P_\pm) = \frac{1}{2}p(\lambda|P_+) + \frac{1}{2}p(\lambda|P_-)\,, \tag{21}$$

where λ is the ontic state of Bob's system, which is well-defined by virtue of the assumption of separability. Eqs. (20) and (21) are justified by noting that the probability one assigns to λ in the unconditioned case is simply the weighted sum of the probability one assigns in

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[17] Note that no notion of 'realism' appears in our implication. This is because there is no sense in which there is an assumption of realism that could be abandoned while salvaging locality. There *is* a notion of realism at play when we grant that experimental procedures prepare and measure properties of systems, but it is a *prerequisite* to making sense of the notion of locality. Norsen has emphasized this point [40, 41].

each of the conditioned cases, where the weights are the probabilities for each condition to hold [1].

The assumption of local causality implies that the probabilities for Bob's system being in various ontic states are independent of the measurement that Alice performs. Consequently,[18]

$$p(\lambda|P_{01}) = p(\lambda|P_{\pm}). \tag{22}$$

Multiplying together Eqs. (20) and (21) and making use of Eq. (22), we obtain,

$$4\,p\left(\lambda|P_{01}\right)^2 = p\left(\lambda|P_+\right)p\left(\lambda|P_0\right) + p\left(\lambda|P_+\right)p\left(\lambda|P_1\right)$$
$$+ p\left(\lambda|P_-\right)p\left(\lambda|P_0\right) + p\left(\lambda|P_-\right)p\left(\lambda|P_1\right). \tag{23}$$

Therefore, for any λ within the support of $p\left(\lambda|P_{01}\right)$ (a non-empty set), we must have,

$$p\left(\lambda|P_+\right)p\left(\lambda|P_0\right) + p\left(\lambda|P_+\right)p\left(\lambda|P_1\right)$$
$$+ p\left(\lambda|P_-\right)p\left(\lambda|P_0\right) + p\left(\lambda|P_-\right)p\left(\lambda|P_1\right) > 0, \tag{24}$$

which requires that at least one of the following inequalities be satisfied,

$$p\left(\lambda|P_+\right)p\left(\lambda|P_0\right) > 0,$$
$$p\left(\lambda|P_+\right)p\left(\lambda|P_1\right) > 0,$$
$$p\left(\lambda|P_-\right)p\left(\lambda|P_0\right) > 0,$$
$$p\left(\lambda|P_-\right)p\left(\lambda|P_1\right) > 0. \tag{25}$$

It follows that there exists at least one pair of distinct quantum states (either $|+\rangle, |0\rangle$ or $|+\rangle, |1\rangle$ or $|-\rangle, |0\rangle$ or $|-\rangle, |1\rangle$) such that the epistemic states associated with them are overlapping on the ontic state space. By Definition 5, we infer that the ontological model must therefore be ψ-epistemic. ∎

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[18] Note that an assumption of no superluminal signalling is not sufficient to obtain Eq. (22) because $p(\lambda|P_{01})$ and $p(\lambda|P_{\pm})$ could be different in a way that every indicator function on system B that corresponds to a possible measurement is unable to distinguish them.

IV.   REASSESSING EINSTEIN'S ARGUMENTS FOR INCOMPLETENESS

A.   The EPR incompleteness argument

It is well known that Einstein disputed the claim that the quantum state represented a complete description of reality on the grounds that such a view implied a failure of locality. Einstein's views on the matter are often assumed to be well represented by the contents of the EPR paper [42]. There is, however, strong evidence suggesting that this is far from the truth. Einstein describes his part in the paper in a letter to Schrödinger dated June 19, 1935 [43]:

> "For reasons of language this [paper] was written by Podolsky after many discussions. But still it has not come out as well as I really wanted; on the contrary, the main point was, so to speak, buried by the erudition."

Fine describes well the implications of these comments: [12].

> "I think we should take in the message of these few words: Einstein did not write the paper, Podolsky did, and somehow the central point was obscured. No doubt Podolsky (of Russian origin) would have found it natural to leave the definite article out of the title [Can quantum mechanical description be considered complete?]. Moreover the logically opaque structure of the piece is uncharacteristic of Einstein's thought and writing. There are no earlier drafts of this article among Einstein's papers and no correspondence or other evidence that I have been able to find which would settle the question as to whether Einstein saw a draft of the paper before it was published. Podolsky left Princeton for California at about the time of submission and it could well be that, authorized by Einstein, he actually composed it on his own."

A more accurate picture of Einstein's views is achieved by looking to his single-author publications and his correspondence. Although it is not widely known, Einstein presented a simple argument for incompleteness at the 1927 Solvay conference. Also, in the letter to Schrödinger that we quote above, Einstein gives his own argument for incompleteness, which makes use of a similar *gedankenexperiment* to the one described in the EPR paper, but has a significantly different logical structure.
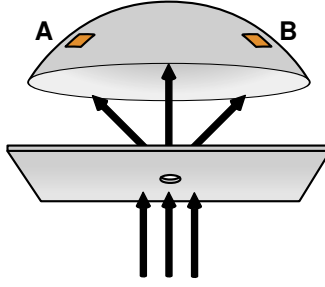
FIG. 8: Einstein's 1927 Gedankenexperiment, in which a single particle wavefunction diffracts at a small opening (bottom) before impinging upon a hemispherical detector (top). According to quantum mechanics, the probability of a double detection at two distinct regions A and B of the detector is zero.

Before turning to the details of these two arguments, we summarize the time-line of their presentation relative to EPR,

- **October 1927:** Einstein presents an incompleteness argument at the Solvay conference [9].

- **May 1935:** The EPR argument for incompleteness is published [42].

- **June 1935:** Einstein presents an incompleteness argument (one that differs substantially from the EPR argument) in his correspondence with Schrödinger [43]. (This first appears in print in March 1936 [44].) We will refer to this as Einstein's 1935 argument, not to be confused with the conceptually distinct EPR argument from the same year.

B.   Einstein's 1927 incompleteness argument

Einstein's first public argument for the incompleteness of quantum mechanics was presented during the general discussion at the 1927 Solvay conference [9]. Einstein considered a *gedankenexperiment* in which electron wave-functions are diffracted through a small opening and then impinge upon a hemispherical screen, as illustrated in Fig. 8. He noted that [45],

"The scattered wave moving towards [the screen] does not show any preferred direction. If $|\psi|^2$ were simply regarded as the probability that at a certain

27

point a given particle is found at a given time, it could happen that the *same* elementary process produces an action in *two or several* places of the screen. But the interpretation, according to which $|\psi|^2$ expresses the probability that *this* particle is found at a given point, assumes an entirely peculiar mechanism of action at a distance which prevents the wave continuously distributed in space from producing an action in *two* places on the screen."

Norsen has presented the essence of this argument in an elegant form[19] that we reproduce here [18]. Consider two points A and B on the screen and denote by 1_A and 0_A respectively the cases where there is or isn't an electron detected at A (and similarly for B). We take the initial quantum state of the electron to be of the form,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle), \qquad (26)$$

where $|A(B)\rangle$ is the quantum state that leads to an electron detection at $A(B)$. Now suppose that one considers an ontological model of the scenario, employing ontic states $\lambda \in \Lambda$. Then the probability of obtaining a simultaneous detection at both sites A and B is given by $p(1_A \wedge 1_B | \lambda) = p(1_A | \lambda) p(1_B | 1_A, \lambda)$. Suppose furthermore that the model describing these events is assumed to be local, then we can write $p(1_B | 1_A, \lambda) = p(1_B | \lambda)$ and thus $p(1_A \wedge 1_B | \lambda) = p(1_A | \lambda) p(1_B | \lambda)$. If the model is taken to satisfy ψ-completeness then $\lambda = \psi$, and we infer that,

$$p(1_A \wedge 1_B | \psi) = p(1_A | \psi) p(1_B | \psi). \qquad (27)$$

Inserting the quantum mechanical predictions $p(1_A | \psi) = p(1_B | \psi) = \frac{1}{2}$, we obtain $p(1_A \wedge 1_B | \psi) = \frac{1}{4}$, which entails a nonzero probability for simultaneous detections at both A and B, in stark contradiction with what is predicted by quantum mechanics.

Hence the logical structure of this rendition of Einstein's 1927 argument is that L $\wedge$ QSTAT $\wedge$ ψ-complete $\rightarrow$ contradiction, i.e., that,

$$\text{L} \wedge \text{QSTAT} \rightarrow \psi\text{-incomplete.} \qquad (28)$$

Note that, unlike the 1935 argument to which we shall turn in the next section, the 1927 argument cannot be used to show the failure of locality for ψ-ontic models because if ψ

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[19] The form of the argument is chosen to parallel the form of Bell's argument in order to make evident the hypocrisy of a widespread tendency among commentators to praise Bell's reasoning while rejecting Einstein's.

is supplemented with a hidden variable ω, then the complete description of the system is $\lambda = (\psi, \omega)$, and Eq. (27) is replaced by,

$$p(1_A \wedge 1_B | \psi, \omega) = p(1_A | \psi, \omega) p(1_B | \psi, \omega). \tag{29}$$

Because there is no reason to assume that $p(1_A | \psi, \omega) = p(1_A | \psi)$ nor that $p(1_B | \psi, \omega) = p(1_B | \psi)$ (conditioning on the hidden variable will in general change the probability of detection), one can no longer infer a nonzero probability for simultaneous detections at both A and B, and the contradiction is blocked.

C. Einstein's 1935 incompleteness argument

In his 1935 correspondence with Schrödinger, after noting that the EPR paper did not do justice to his views, Einstein presents a different version of the argument for incompleteness. He begins by adopting a notion of completeness that differs markedly from that of the EPR paper [43],

> "[...] one would like to say the following: ψ is correlated one-to-one with the real state of the real system. [...] If this works, then I speak of a complete description of reality by the theory. But if such an interpretation is not feasible, I call the theoretical description 'incomplete'."

It is quite clear that by 'real state of the real system', Einstein is referring to the ontic state pertaining to a system. Bearing this in mind, his definition of completeness can be identified as precisely our notion of ψ-completeness given in Definition 2. Einstein then reiterates to Schrödinger the beginning of the EPR argument, starting by considering a joint system (AB) to be prepared in an entangled state by some 'collision' between the subsystems A and B. He then emphasizes (what we would now call) the 'steering phenomenon' by noting how a choice of measurement on A can result in the subsystem B being described by one of two quantum states ψ_B or $\psi_{\underline{B}}$.

Einstein then uses this scenario to derive his preferred proof of incompleteness,

> "Now what is essential is exclusively that ψ_B and $\psi_{\underline{B}}$ are in general different from one another. I assert that this difference is incompatible with the hypothesis that

the description is correlated one-to-one with the physical reality (the real state). After the collision, the real state of (AB) consists precisely of the real state of A and the real state of B, which two states have nothing to do with one another. *The real state of B thus cannot depend upon the kind of measurement I carry out on A.* ('Separation hypothesis' from above.) But then for the same state of B there are two (in general arbitrarily many) equally justified ψ_B, which contradicts the hypothesis of a one-to-one or complete description of the real states."

Einstein is clearly presuming separability with his assertion that "the real state of (AB) consists precisely of the real state of A and the real state of B". He furthermore appeals to local causality when he asserts that "The real state of B thus cannot depend upon the kind of measurement I carry out on A", because he is ruling out the possibility of events at A having causes in the space-like separated region B.

Now, although Einstein's conclusion is nominally to deny ψ-completeness, he does so by showing that there can be many quantum states associated with the same ontic state, "for the same state of B there are two (in general arbitrarily many) equally justified ψ_B". The proof need not have taken this form. An alternative approach would have been to try to deny ψ-completeness by showing that there are many ontic states associated with the same quantum state. For our purposes, this distinction is critical because what Einstein has shown through his argument is that a variation in ψ need not correspond to a variation in the ontic state. Recalling Definition 4, we see that Einstein has established the failure of ψ-onticity! His 1935 incompleteness argument rules out ψ-onticity *en route* to ruling out ψ-completeness.

The structure of his argument, in our terminology, is:

$$\text{L} \wedge \text{QSTAT} \rightarrow \neg(\psi\text{-ontic}) \rightarrow \psi\text{-incomplete}. \tag{30}$$

But the second implication is actually a weakening of the conclusion, because among the ψ-incomplete models are some which are ψ-ontic (those we have called ψ-supplemented) and the argument is strong enough to rule these out.

Einstein would have done better, therefore, to characterize his argument as,

$$\text{L} \wedge \text{QSTAT} \rightarrow \neg(\psi\text{-ontic}),$$

which is our Theorem 11.

V. HISTORICAL IMPLICATIONS

A. A puzzle

What can we gain from this retrospective assessment of Einstein's incompleteness arguments? There is one long-standing puzzle on which it may shed new light: why did Einstein ever switch from the simple 1927 argument, which involves only a single measurement, to the 1935 argument, which involves two?

The move he made in 1935 to the two measurement argument described in Sec. IV C proved to be a permanent one. He published the argument for the first time in 1936 [44] and from this point onwards, the 1935 argument proved the mainstay of his assault on orthodox quantum theory, appearing in various writings [46, 47], most notably his own autobiographical notes [48]. In fact, there is evidence to suggest that this argument was still on Einstein's mind as late as 1954 [49].

Many commentators have noted that an EPR-style argument for incompleteness can be made even if one imagines that only a single measurement is performed [14–17]. The resulting argument is similar to Einstein's 1927 argument, although it differs insofar as it appeals to a pair of systems rather than a single particle and makes use of the EPR criterion for reality rather than the assumption of ψ-completeness. Nonetheless, the point being made by these authors is the same as the one we have just noted: having multiple possible choices of measurement is not required to reach the conclusion of incompleteness from the assumption of locality. Furthermore, the extra complication actually detracts from the argument (whether it follows the reasoning of the EPR paper or Einstein's correspondence with Schrödinger), because it introduces counterfactuals and modal logic into the game, and this is precisely where most critics, including Bohr [50], have focussed their attention. The single measurement versions of the argument are, of course, completely immune to such criticisms.

One explanation that has been offered for Einstein's move to two measurements is that one can thereby land a harder blow on the proponent of the orthodox approach by also defeating the uncertainty principle in the course of the argument. Maudlin refers to this "extra twist of the knife" as "an unnecessary bit of grandstanding (probably due to Podolsky)"[17]. Although this may be an accurate assessment of what is going on in the EPR paper, it does

not explain Einstein's post-1935 conversion to the two-measurement form of the argument. Indeed, Einstein explicitly de-emphasizes the uncertainty principle in his own writings. For instance, in his 1935 letter to Schrödinger, he remarks: "I couldn't care less[20] whether ψ_B and $\psi_{\underline{B}}$ can be understood as eigenfunctions of observables B, $\underline{B}$"."[43]

Another explanation worth considering concerns the experimental significance of the two gedankenexperiments. Although Einstein's incompleteness arguments imply a dilemma between ψ-completeness and locality, a sceptic who conceded the validity of the argument could still evade the dilemma by choosing to reject some part of quantum mechanics, specifically, those aspects that were required to reach Einstein's conclusion. To eliminate this possibility, one would have to provide experimental evidence in favor of these aspects. From this perspective, there is a significant difference between the 1927 and 1935 gedankenexperiments. In the case of the former, the measurement statistics to which Einstein appeals (perfect anti-correlation of measurements of local particle number) can also be obtained from the mixed state $\frac{1}{2}(|A\rangle\langle A| + |B\rangle\langle B|)$ rather than the pure state $(1/\sqrt{2})(|A\rangle + |B\rangle)$. It follows that the sceptic could avoid the dilemma by positing that such coherence was illusory. To convince the sceptic, further experimental data – for instance, a demonstration of coherence via interference – would be required. On the other hand, the measurement statistics of the 1935 gedankenexperiment cannot, in general, be explained under the sceptic's hypothesis (which in this case amounts to positing a separable mixed state[21]). Indeed, *any* hypothesis that takes system B to be in a mixture of pure quantum states (that are unaffected by events at A) can be ruled out by the 1935 set-up because the latter allows one to make predictions about the outcomes of incompatible measurements on B that are in violation of the uncertainty principle. This has been demonstrated by Reid in the context of the EPR scenario [51] and by Wiseman *et al.*[52] more generally. Although Wiseman has argued that this provides a reason for favoring the 1935 over the 1927 version of Einstein's incompleteness argument [53], he does not suggest that it was Einstein's reason. Indeed, this is unlikely to have been the case. Certainly, we are not aware of anything in Einstein's writings that would suggest so.[22]

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[20] "ist mir *wurst*" (emphasis in original).

[21] The term "separable" is here used in the sense of entanglement theory: a mixture of product states.

[22] Although Schrödinger had some doubts about the validity of quantum theory, these concerned whether experiments would confirm the existence of the steering phenomenon ("I am not satisfied about there

B. A possible explanation

Our analysis of Einstein's incompleteness arguments suggests a very different explanation. In Sec. IV C, we demonstrated that the 1935 argument is able to prove that both ψ-complete *and* ψ-supplemented models are incompatible with a locality assumption, leaving ψ-epistemic models as the only approach holding any hope of preserving locality. In contrast, the 1927 argument cannot achieve this stronger conclusion, as was noted in Sec. IV B. (This also follows from the fact that the deBroglie-Bohm theory constitutes a ψ-supplemented model which provides a local explanation of the 1927 thought experiment.) One can therefore understand Einstein's otherwise baffling abandonment of his 1927 incompleteness argument in favor of the more complicated 1935 one by supposing that he sought to advocate a particular kind of ontological model, namely, a ψ-epistemic one. This interpretation of events is bolstered by the fact that Einstein often followed his discussions of the incompleteness argument with an endorsement of the epistemic view of quantum states. We turn to the evidence for this claim in his papers and correspondence.

In addition to his conviction that "[...] the description afforded by quantum-mechanics is to be viewed [...] as an incomplete and indirect description of reality, that will again be replaced later by a complete and direct description." [46], Einstein specifically advocated that

> "[t]he ψ-function is to be understood as the description not of a single system
> but of an ensemble of systems." [55],

and that the meaning of the quantum state was "similar to that of the density function of classical statistical mechanics."[56]

It is not immediately obvious that this is equivalent to an epistemic interpretation of the quantum state. We argue for this equivalence on the grounds that the ensembles Einstein mentions are simply a manner of grounding talk about the probabilities that characterize an observer's knowledge. In other words, the only difference between "ensemble talk" and "epistemic talk" is that in the former, probabilities are understood as relative frequencies in

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being enough experimental evidence for that."[54]). This sentiment was a *reaction* to the 1935 form of Einstein's argument and so could not have motivated it. It is unlikely that anyone would have been sceptical of the spatial coherence assumed in Einstein's 1927 argument.

an ensemble of systems, while in the latter, they are understood as characterizations of the incomplete knowledge that an observer has of a single system when she knows the ensemble from which it was drawn. Ultimately, then, the only difference we can discern between the ensemble view and the epistemic view concerns how one speaks about probabilities, and although one can debate the merits of different conceptions of probability, we do not feel that the distinction is significant in this context, nor is there any indication of Einstein having thought so.

Indeed, in a 1937 letter to Ernst Cassirer, Einstein seems to use the two manners of characterizing his view *interchangeably* as he spells out what conclusion should be drawn from his 1935 incompleteness argument [57],

> "[...] this entire difficulty disappears if one relates ψ_2 not to an individual system but, in Born's sense, to a certain state-ensemble of material points 2. Then, however, it is clear that ψ_2 does not describe the totality of what "really" pertains to the partial system 2, rather only what we know about it in this particular case."

Einstein's endorsement of an epistemic understanding of the quantum state is also explicit elsewhere in his personal correspondence (of which relevant extracts have been conveniently collected together in essays by Fine and Howard [11, 35, 58]). For instance, in a 1945 letter to Epstein, after providing an incompleteness argument containing all the features of the one used in 1935, Einstein concludes that [59],

> "Naturally one cannot do justice to [the argument] by means of a wave function. Thus I incline to the opinion that the wave function does not (completely) describe what is real, but only a to us empirically accessible maximal knowledge regarding that which really exists [...] This is what I mean when I advance the view that quantum mechanics gives an incomplete description of the real state of affairs."

Perhaps the most explicit example occurs in a 1948 reply to Heitler, criticizing Heitler's notion that the observer plays an important role in the process of wave-function collapse, and advocating [60],

"that one conceives of the psi-function only as an incomplete description of a real state of affairs, where the incompleteness of the description is forced by the fact that observation of the state is only able to grasp part of the real factual situation. Then one can at least escape the singular conception that observation (conceived as an act of consciousness) influences the real physical state of things; the change in the psi-function through observation then does not correspond essentially to the change in a real matter of fact but rather to the alteration in *our knowledge* of this matter of fact."(emphasis in original)

The result implicit in Einstein's 1935 argument – that a ψ-epistemic interpretation of the quantum state is the only one that does not obviously lead to a failure of locality – has been superseded by Bell's theorem [20]. The latter famously demonstrates that *any* ontological model reproducing quantum statistics must violate locality (as emphasized in Refs. [40, 61]). Einstein's hopes are dashed: locality cannot be salvaged *even if* one adopts a ψ-epistemic interpretation of the quantum state. We do not dispute this. The point we wish to make is simply that the 'big guns' of Bell's theorem are *only needed to deal with ψ-epistemic models.* Any ψ-ontic model can be seen to be non-local by an argument that appeared in print as far back as 1936.

Therefore, in the 28 years between the publication of Einstein's 1935 incompleteness argument (in 1936) and the publication of Bell's theorem (in 1964), only ψ-epistemic ontological models were actually viable to those who were daring enough to defy convention and seek an interpretation that preserves locality. Why is it then that during the pre-Bell era, there was not a greater recognition among such researchers of the apparent promise of ψ-epistemic approaches vis-a-vis locality?

It seems likely to us that the distinction between ψ-supplemented and ψ-epistemic hidden variable models was simply not sufficiently clear. One searches in vain for any semblance of the distinction in Einstein's description of the alternative to the orthodox ψ-complete view during the general discussion at the 1927 Solvay conference. But nothing in what we have said would lead one to expect that Einstein had clearly understood the distinction as early as 1927. What *is* surprising is that, after 1935, Einstein seems to voice his support for an epistemic view of ψ in his papers and correspondence, and yet never bothers to articulate, nor explicitly denounce, the other way in which his bijective notion of completeness

(ψ-completeness) could fail, namely, by ψ being ontic but supplemented with additional variables.

By characterizing his 1935 argument as one that merely established the *incompleteness* of quantum theory on the assumption of locality, Einstein did it a great disservice. For in isolation, a call for the *completion* of quantum theory would naturally have led many to pursue hidden variable theories that interpreted the fundamental mathematical object of the theory, the wave function, in the same manner in which the fundamental object of other physical theories were customarily treated – as ontic. But such a strategy was known by Einstein to be unable to preserve locality. Thus it is likely that the force of Einstein's 1935 argument from locality to the epistemic interpretation of ψ was not felt simply because the argument was not sufficiently well articulated.

A proper assessment of the plausibility of these historical possibilities would require a careful reexamination of Einstein's papers and correspondence with the distinction between ψ-ontic and ψ-epistemic ontological models in mind. We hope that such a reassessment might yield further insight into the history of incompleteness and nonlocality arguments.

VI.   THE FUTURE OF ψ-EPISTEMIC MODELS

Einstein could motivate his search for a ψ-epistemic ontological model of quantum theory on the grounds that it was the only avenue left open for preserving locality. We cannot. The possibility of preserving locality has been foreclosed by Bell's theorem. However, Bell's theorem does not provide any reason for preferring a ψ-ontic approach over one that is ψ-epistemic; it is neutral on this front. Moreover, there are many new motivations (unrelated to locality) that can now be provided in favor of ψ-epistemic models. For instance, it is shown in Refs. [4, 5] that information-theoretic phenomena such as teleportation, no-cloning, the impossibility of discriminating non-orthogonal states, the information-disturbance trade-off, aspects of entanglement theory, and many others, are found to be derivable within toy theories that presume hidden variables and wherein the analogue of ψ is a state of incomplete knowledge. This interpretation of ψ is further supported by a great deal of foundational work that does not presuppose hidden variables [62–72]. ψ-epistemic ontological models are therefore deserving of more attention than they have received to date.

However, it remains unclear to what extent a ψ-epistemic ontological model of quantum

theory is even possible. Recall that the Kochen-Specker model discussed in Sec. II D 3 secured such an interpretation for pure states and projective measurements in a two-dimensional Hilbert space. But can one be found in more general cases? [23]

We here need to dispense with a possible confusion that might arise. In the same paper wherein they presented their 2d model, Kochen and Specker proceed to prove a no-go theorem for certain kinds of ontological models seeking to reproduce the predictions of quantum mechanics in 3d Hilbert spaces. One might therefore be led to the impression that Kochen and Specker rule out ψ-epistemic models for 3d Hilbert spaces. This is not the case, however, as we now clarify.

As soon as one moves to projective measurements in a Hilbert space of dimension greater than two, it is possible to define a distinction between contextual and noncontextual ontological models [1]. What Kochen and Specker showed [2] (it was also shown independently by Bell [30]) is that noncontextual ontological models cannot reproduce the predictions of quantum theory for Hilbert spaces of dimension 3 or greater. Furthermore, the notion of noncontextuality can be extended from projective measurements to nonprojective measurements, to preparations, and to transformations [1]. In all cases, one can demonstrate a negative verdict for noncontextual models of quantum theory [1]. Indeed, by moving beyond projective measurements, one finds that noncontextual models cannot even be constructed for a two-dimensional Hilbert space.

But the dichotomy between contextual and noncontextual models is independent of the dichotomy between ψ-ontic and ψ-epistemic models. So, whereas the Bell-Kochen-Specker theorem and variants thereof show the necessity of contextuality, they are silent on the issue of whether one can find an ontological model that is also ψ-epistemic. The ontological models of quantum theory that we do have, such as deBroglie-Bohm, are contextual but ψ-ontic. Bell [30] even provides a very *ad hoc* example of a contextual hidden variable model (an extension of the Bell-Mermin model of Sec. II D 2) to prove that such a model is possible. It too is ψ-ontic (although one must have recourse to the definition appealing to fidelities provided in footnote 10 to properly assess this model) [74].

Many features of the deBroglie-Bohm interpretation of quantum theory have been found to be generalizable to a broad class of ontological models. Nonlocality, contextuality, and

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[23] Hardy was perhaps the first to lay down this challenge explicitly [73].

signalling outside of quantum equilibrium [75] are examples. Inspired by this pattern, Valentini has wondered whether the pilot-wave (and hence ontic) nature of the wave function in the deBroglie-Bohm approach might be unavoidable [76]. On the other hand, it has been suggested by Wiseman that there exists an unconventional reading of the deBroglie-Bohm approach which is not ψ-ontic [77]. A distinction is made between the quantum state of the universe and the conditional quantum state of a subsystem, defined in Ref. [78]. The latter is argued to be epistemic while the former is deemed to be nomic, that is, law-like, following the lines of Ref. [79] (in which case it is presumably a category mistake to try to characterize the universal wave function as ontic or epistemic). We shall not provide a detailed analysis of this claim here, but highlight it as an interesting possibility that is deserving of further scrutiny. Nelson's approach to quantum theory [80] also purports to *not* assume the wave function to be part of the ontology of the theory [81]. However, as pointed out by Wallstrom [82], the theory does not succeed in picking out all and only those solutions of Schrödinger's equation[24]. Consequently, it also fails to provide a ψ-epistemic model of quantum theory.

Recently, Barrett [74] has constructed an ontological model of a quantum system of arbitrary finite dimension that is ψ-epistemic. Although it only works for a countable set of bases of the Hilbert space, it seems likely that this deficiency can be eliminated, in which case it would be the first ψ-epistemic model for a general quantum system. Unfortunately, the model achieves the ψ-epistemic property in a very ad hoc manner, by singling out a pair of non-orthogonal quantum states, and demanding that the epistemic states associated with these have non-zero overlap. Consequently, it does not have the sorts of features, outlined in Refs. [4, 5], that make the ψ-epistemic approach compelling. This suggests that the interesting question is not simply whether a ψ-epistemic model can be constructed, but whether one can be constructed with certain additional properties, such as the property that the classical fidelity between epistemic states associated with a given pair of quantum states is invariant under all unitary transformations of the latter.[25]

Rudolph has devised a ψ-epistemic contextual ontological model that is quantitatively close to the predictions of quantum theory for projective measurements in three-dimensional Hilbert spaces and also has the desired symmetry property [6]. This model does not, however,

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[24] It is assumed that only continuous and single-valued wave functions are valid, a fact that is disputed by Smolin [83].

[25] The Kochen-Specker model discussed in Sec. II D 3 has this feature.

reproduce the quantum predictions exactly.

It is possible that a ψ-epistemic model with the desired symmetry property does not exist. However, a no-go theorem always presumes some theoretical framework. In Sec. II A of the present paper, we have cast ontological models in an operational framework, wherein systems are considered in isolation and the experimental procedures are treated as external interventions. Such a framework may not be able to do justice to all interpretations that have some claim to being judged realist. For instance, in the deBroglie-Bohm interpretation, a system is not separable from the experimental apparatus and consequently it is unclear whether one misrepresents the interpretation by casting it in our current framework (an extension of the formalism used here is, however, developed in Ref. [84]). Ontological models that are fundamentally relational might also fail to be captured by the framework described here. Nonetheless, something would undeniably be learned if one could prove the impossibility of a ψ-epistemic model with the desired symmetry properties within an operational framework of this sort.

VII.   ACKNOWLEDGMENTS

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