# Instants and Instantaneous Velocity

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**Abstract**

## 1 The Problem of Instantaneous Rates of Change

The concept of the instantaneous rate of change of some physical quantity, position most strikingly, poses a significant philosophical dilemma. Notoriously, Zeno argued, in his arrow paradox, that the very concept of an instantaneous rate of change in position is both unavoidable in the explanation of motion and incoherent. Recently worries about instantaneous rates of change have received an additional boost from concerns about the definition and reality of instantaneous velocity in classical physics(cf. Carroll, 2002; Meyer, 2003; Smith, 2003). The clearest recent treatment of the range of possible solutions to the problems posed by instantaneous velocity is by Frank Arntzenius(2000). Although Arntzenius argues that none of the traditional treatments of motion really serve to resolve the tension between Zeno's argument and the unavoidable role of instantaneous velocity in classical physics, Marc Lange(2005) has argued that we can resolve this dilemma with a dispositional analysis of instantaneous velocity.

This paper will argue that the puzzles about instantaneous velocity, and rates of change more generally, are the result of a failure to recognize an ambiguity in the concept of an *instant*, and therefore of an instantaneous state. We will conclude that there are two distinct conceptions of a temporal instant: (i) instants conceived as fundamentally distinct zero-duration temporal atoms and (ii) instants conceived as the boundary of, or between, temporally extended durations. Since the concept of classical instantaneous velocity is well-defined only on the second conception of instants, we will conclude that this distinction allows us to avoid the above dilemma. If instantaneous velocity is well-defined then the states of a system at various instants are not logically distinct and thus we cannot generate Zeno's paradox. However, if we assume that the instants are metaphysically distinct, then instantaneous velocity is not well-defined and thus the second horn of the dilemma about the causal-explanatory role of instantaneous velocity cannot be generated.

The argument proceeds in three basic stages. In the next section, I sketch Frank Arntzenius's objections to 'at-at' and impetus theory resolutions to Zeno's paradox. The following section considers Marc Lange's dispositional account of instantaneous velocity. Section 3 sketches his theory and shows that it is basically, in Arntzenius's terminology, a sophisticated impetus theory. While Lange's theory avoids the most serious of Arntzenius's objections to impetus theories, it does so only at a quite high cost in physical and metaphysical plausibility. The final sections present the positive argument that to the extent that classical physics irreducibly invokes instantaneous velocity, it presupposes a theory of temporal structure in which duration is the primitive concept with instants derived from durations as their boundaries.

## 2 Arntzenius's version of Zeno's argument against change

Zeno's arrow paradox runs basically as follows. If time consists of a collection of durationless instants, then motion is impossible. This follows because if the arrow is moving, then it must make sense to ask at every time how fast the arrow is moving. However, given that instants have no duration, the arrow cannot move at all "during" any particular instant. Therefore, it doesn't have a velocity at that instant. Therefore it cannot be moving at all.

The standard response to this argument invokes the so-called 'at-at' theory of motion. On such a theory of motion speed, rate of change of position, is not a part of the intrinsic instantaneous state of the ball. Rather, it is a purely relational property of the ball's path. Thus, on the "at-at" theory of motion, the ball moves by occupying different locations at different times and the rate of change is determined by the relations between those various locations at a time. Thus, on this theory objects move, but they do so without possessing an instantaneous velocity.

However, as Arntzenius points out this seems to have extremely troubling consequences for our interpretation of classical mechanics. First, it seems to create a severe problem for our ability to formulate any concept of determinism in classical physics. If the instantaneous velocity of a classical particle is not part of its instantaneous state, then nothing about that state serves to determine much of anything about it's future path. More precisely, the motion of a classical particle depends on two kinds of facts about the particle, kinematic and dynamic.[1] Clearly, the future motion of a particle depends on the current forces to which the particle is subject. However, it also depends on the previous motion of the particle, normally encoded in the instantaneous velocity. If instantaneous velocity is not an element of the instantaneous state of the particle, then that state, by itself, cannot determine the particles future motion. Second, Arntzenius rightly points out that classical physics does treat instantaneous velocity as a component of the instantaneous state. If instantaneous velocity is *not* an element of the state, we seem to be required to claim that two particles occupying the same spatial location, but moving in different directions, have the same state.

However, we can, and do normally, define the instantaneous velocity as a limit or neighborhood property of the various instants along the particles trajectory. At first glance this might seem to solve the problems above. However, as both Arntzenius and Lange point out, this is deceptive. First, the relevant limits are defined only over neighborhoods surrounding the instant, not at the instants separately. This seems, minimally, at least in tension with the reasons for adopting the "at-at" theory in the first place. Second, it is not clear that limit velocity actually solves the determinism problem above. The limit property is really a disguised relation between the current state and earlier, or earlier and later, states of the particle. However, we normally look for future states to be determined by the intrinsic state. Or to follow Lange, on the standard understanding of relations, they can only be causally efficacious if at least some of the relata are independently causally efficacious. However, on the limit account of velocity, the relation plays a causal role that it does not inherit from any of the relata.

Given the apparent failure of the "at-at" theory to resolve the problems posed by instantaneous velocity, Arntzenius considers two alternative proposals. The first, impetus theories, propose that the future trajectory is determined by an element of the state which is causally, but not definitionally, connected to the trajectory. The fundamental problem with such theories is that whatever the impetus *is*, it does not seem to be a velocity nor is it properly connected to velocity in general. That is, whatever fulfills the causal role of instantaneous velocity in classical physics must be defined in purely kinematic terms. However, according to the standard impetus theories it is entirely conceptually possible that the impetus could come un-hooked from the relevant properties of the trajectory. In the next section, we will examine a proposal by Marc Lange to avoid this problem, but only at, what I will argue are overly large sacrifices of physical and metaphysical clarity.

The second alternative, the 'no-instants' view, treats time as composed of a collection of finite duration non-atomic regions connected *via* an appropriate algebra. Arntzenius points out two basic problems with so a "no-instants" view. First, it seems to leave the original problem with determinism untouched. It still will be the case that the state of a system at a time fails to determine its future trajectory; if only because the system does not have an instantaneous state. Secondly, our actual ability to do mechanics seems to be entirely parasitic on the more traditional formulations. One formulates and solves problems in the more usual point-set and point function formalism, and then uses the homomorphisms between the point algebra and the Borel algebra to translate into the "no-instants" view.

We will see below that I believe that the "no-instants" view is on the right track, but that there is a more adequate version that does not require us to deprive ourselves of the usual formalism. First, however, we need to consider one more proposal for the nature of instantaneous velocity.

## 3 Instantaneous velocity as a pure disposition

In "How Can Instantaneous Velocity Fulfill Its Causal Role?"(2005) Marc Lange proposed that instantaneous velocity in classical physics be interpreted as the disposition of a classical system to follow a particular future trajectory, as determined by the time derivative from above of its trajectory at each point. In terms of Arntzenius's classification above, Lange proposal is a sophisticated impetus theory of motion. However, it avoids both of Arntzenius's criticisms of such theories since the relevant disposition is defined purely kinematically.

First, Lange's instantaneous velocity is a purely kinematic property of the system. We do not need any knowledge about the dynamics of the system beyond its trajectory to determine its velocity. Second, as long as the trajectory is smooth the instantaneous velocity, the derivative from above, will match the ordinary velocity according to the prior trajectory. Thus, while we require a natural law to link instantaneous velocity to ordinary velocity, it is one that we already possess. This follows from the fact that classical physics implies that all trajectories are continuous and smooth.(cf. Lange, 2005, § 2)

However, Lange's purely subjunctive definition of instantaneous velocity remains unsatisfactory in at least three ways. First, it fails to account for the transformational properties of velocity. Second, these subjunctive properties are not only never actualized in the actual world, it is nomologically impossible that they could be actualized in any world governed by classical physics and containing at least two massive particles. Third, it seems odd that while I can predict the present instantaneous velocity of a particle, I cannot *measure* it. In order to see the problem posed by these three objections, we need to delve a bit more deeply into classical mechanics.

While Lange and Arntzenius do seem to be correct about the unavoidable role that instantaneous velocity plays in classical physics, we should also note how odd a property it is. In particular, although I must assign some value to the instantaneous velocity in order to calculate the future trajectory of a particle, the actual value assigned, barring only the requirement of consistency in the description of other kinematic components of the situation, is almost entirely arbitrary.

In order to see this, consider a single massive particle in classical physics. For simplicity assume that the particle's trajectory is smooth and that the derivative exists to all orders. Then, given Galilean invariance, there is an inertial frame such that the the particle is at the spatial origin of the frame, $\mathbf{r_o} = (x, y, z) = (0, 0, 0)$, the time coordinate, $t_o = 0$, and the particle is at rest in that frame,

$$v_o = \frac{\partial \mathbf{r_o}}{\partial t} = 0 \tag{1}$$

This is the first oddity in about the causal role of velocity. You cannot eliminate velocity in general from classical mechanics, but you can eliminate *any particular velocity*. Instantaneous velocity is causally relevant, but Galilean invariance assures us that the value of velocity is not causally relevant.

It is not of course possible to eliminate velocities in general for systems of more than one particle, because such systems do not generally have a shared rest frame. When we select the instantaneous rest frame of one of the particles, we must, in order to describe the entire system of particles, adjust the values of kinematic variables for all of the other particles according to that frame. But, that brings us to the second oddity about instantaneous velocity, the instantaneous velocity of a particle is defined only relative to other particles within the system and is constrained to vary with them. Notice that this connection between the states of the various particles is not a causal connection but a conceptual or definitional conception. Part of what it is to be a velocity is to transform in this way. However, if one is concerned about the conceptual dependence of velocity on other instantaneous states of the same system, its conceptual dependence on the instantaneous states of physically distinct components should be at least as worrying.

The above issues, related to the transformational structure of kinematic quantities, seem to be quite general problems for any attempt to define velocity as a causally efficacious distinct element of the physical state in classical physics. There are, however, two more particular issues raised by Lange's subjunctive analysis of velocity. First, Lange defines the instantaneous velocity according to the path that the particle *would* follow if no forces *were* present. However, in any possible world containing at least two massive particles and obeying classical physics, all inertial trajectories are *empty*. There is therefore a serious problem with the epistemic status of this disposition. We normally assign dispositional properties to objects because other objects, judged to be relevantly similar to the given object, have actualized that disposition in certain circumstances. However, no particle has ever actualized the disposition to follow the inertial path. That is, we attribute the disposition, *will break if hit by a baseball* to windows, because at least some windows hit by baseballs have broken. However, no classical particle *can* follow the path that, according to Lange, it is disposed to follow.

There is one final puzzle related to the subsistence of instantaneous velocity as a distinct component of the state of a classical system. Given Lange's definition, one can at best calculate or predict the current present instantaneous velocity. Since the present instantaneous velocity is merely causally dependent on the prior trajectory, information about that trajectory provides merely causal information about the current state. We are sometimes practically unable to determine aspects of the current state of the system except by inferring it from other causally relevant information; Lange, however, postulates a distinct component of the *present* state which is *in principle* subject only to prediction rather than measurement.

None of these three issues are fatal for Lange's proposal. However, all of them point to something odd going on with the concept of an instantaneous rate, or even with the concept of an instantaneous state more generally. In the next section, we consider that the underlying problem with instantaneous rates rests not fundamentally with the concept of a rate, but with the concept of an instant. In particular, I will argue that for systems with well-defined instantaneous rates of change in fundamental quantities, the instants, and thus the instantaneous states, are *not* fundamentally distinct.

## 4 Humean instants and non-Humean time

David Hume famously argued that state of the world at distinct times are metaphysically and logically independent of each other. At most these distinct states can be the causal product of earlier states. Hume, equally famously, reduced these causal relations to spatiotemporal contiguity and resemblance. However, the fundamental underlying presumption is precisely that the states of the universe, or any other system, are distinct from each other. We will call this the *Humean presumption* and the instants at which a system possesses each of these distinct states, *Humean instants.*

As an analogy, we can think of the Humean presumption as the belief that we can capture the sequential states of a system in a sequence of photographs of the system, or perhaps, a film strip.[2] We then imagine the "frame-rate" of our camera increasing without limit. The higher the frame rate the more accurately our movie approximates the actual continuous sequence of distinct states. However, even the actual instantaneous state of the system shares with the approximations their fundamentally distinct, although of course not discrete, nature.

Notice two features of this situation. First, this picture of fundamentally distinct states leads naturally to the "at-at" theory of motion. Being in motion is a relation between the various states, the various distinct pictures of the system. Second, it is also the case that if this set theoretic structure is all the structure we have then *instantaneous rates of change are not even definable.* We simply cannot get from a discrete set of distinct states to a truly continuous set of states having a differentiable structure through a sequence of interpolations or approximations. As a matter of mathematics, no matter how many elements we add to a discrete set it remains discrete. That is, if one insists that the evolution of a system is given by a sequence of states at a sequence of Humean instants, then instantaneous rate of change is not defined and one can apply the "at-at" theory of motion.

Instantaneous rates of change are defined only if time, or in relativistic contexts proper time, is isomorphic to the real line. More precisely, the relevant limits are defined only if time can be parametrized by a totally-ordered, Dedekind-complete field. Dedekind completeness is the property that every subset of given ordered set with an upper-bound has a least upper-bound. Equivalently, every Cauchy sequence of the reals converges to a limit in the reals. This is required so that the relevant limits in the definition of the derivative exist. However, notice how this inverts the Humean presupposition. The logically basic element of the situation is not the "instant" or "moment" of time; rather, time construed as a continuous entirety is basic. The instants are the "boundaries" of sub-regions or the limits of sequences of sub-regions of time. Now in many ways these boundaries behave as zero-duration subregions, and thus deserve to be thought of as components of time.

However, they are very different from the Humean instants in that their very nature and existence is bound up with the remainder of time in a way that Humean instants are not. We cannot just "take a snapshot" and lift a continuum instant out of the continuum without fundamentally altering its nature.

## 5 Notes and Morals

Before I draw the fundamental moral of this analysis, two notes are in order. First, I am certainly not the first to notice that the continuity of time should lead us to treat *time, per se* as more fundamental than *times.* Aristotle, in *The Physics*, points out that if time is like a continuous line, then *the nows* are not the atomic constituents of time, but the boundary of regions of time.

> Hence time is not number in the sense in which there is 'number' of the same point because it is beginning and end, but rather as the extremities of a line form a number, and not as the parts of the line do so, . . . because obviously the 'now' is no part of time nor the section any part of the movement, any more than the points are parts of the line-for it is two lines that are parts of one line. [220a15]

In addition, it seems to have been, in part, the need to insure the continuity of time that Immanuel Kant insisted on a pure intuition of time, to parallel that of space. Absent such a pure intuition of time we simply cannot make sense of the a given period or moment as a period or moment *within* time. Thus, in *The Critique of Pure Reason*,

> The infinitude of time signifies nothing more than that every determinate magnitude of time is possible only through limitations of one single time that underlies it. The original representation, *time*, must therefore be given as unlimited.[A32/B47-48][3]

Both discussions emphasize the difficulty of analyzing time as a mere collection of instants.

Second, the claim that a system *is changing*, finishing a change or beginning to change at a particular instant requires that consider a more complicated structure than a mere collection of instants. Consider an attempt to attribute *"is currently changing its position"* to a classical particle as a part of its instantaneous state. To make this claim is precisely to treat the present state of the system as intermediate between the previous state and the later state. Again, if you lift the instantaneous state out of its context and consider it as distinct from the remainder of the states, such a claim makes no sense. As long as one is willing to bite the bullet and do away with instantaneous rates of change, you can get a away with this. But, classical physics seems to require at least instantaneous velocity and thus requires us to recognize that the instantaneous states do not possess the requisite Humean independence.

To recap. We began with a dilemma about instantaneous velocity. Zeno's arrow paradox seems to imply that instantaneous velocity, understood as an instantaneous rate of change, is impossible; classical physics without instantaneous velocity looks at best very little like what we would expect. The "at-at" theory, by itself, makes a hash of classical physics, while the addition of a neighborhood limit property of instantaneous velocity makes a hash of the "at-at" theory. Impetus theories, even the purely kinematic impetus theory of Marc Lange, are extremely hard to make metaphysical sense of and seem not to be theories of *velocity*.

We then considered whether the problem might not be with the concept of a rate, but with the application of the concept of an instant. We saw that the previous attempts to solve the puzzle presupposed that the states of a system at various times are logically and metaphysically distinct, the Humean presupposition. However, if time is continuous then the various instants, and naturally therefore the instantaneous states, are *not* logically or metaphysically distinct. This then allows us to resolve the puzzle in a slightly different way. Instantaneous velocity is precisely what we would expect it to be, the time derivative of the trajectory. However, we cannot generate Zeno's paradox. In a sense, the arrow does move at each instant. More precisely, part of what it means to occupy a particular state at a particular time is to have previously occupied other states at other times and to be going to occupy other states at other times. I simply can't establish the instantaneous state of system in continuous time without establishing its "place" in a larger relational structure.

I wish I had a big metaphysical punch-line for you. But, I don't. Rather, I hope that I have raised serious concerns about the ease with which both philosophers and physicists bandy about the concept of a momentary or instantaneous state of a system. Until we really understand what *time* is, we should be extremely careful thinking we understand its parts.

## Notes

[1]Given that we are restricting attention to classical physics in this essay, I will normally drop the 'classical' modifier.

[2]We can postulate a "metaphysical" lens on our camera if you like, for example one that can take pictures of dispositions as well as occurrent properties.

[3]In this context, one might also note the close connection between the pure intuition of time and the interpretation of calculus in Michael Friedman's interpretation of Kant.(Friedman, 1992)

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