Truthmaker Semantics for Epistemic Logic*

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Abstract

We explore some possibilities for developing epistemic logic using truthmaker semantics. We identify three possible targets of analysis for the epistemic logician. We then list some candidate epistemic principles and review the arguments that render some controversial. We then present the classic Hintikkan approach to epistemic logic and note - as per the 'problem of logical omniscience' - that it validates all of the aforementioned principles, controversial or otherwise. We then lay out a truthmaker framework in the style of Kit Fine and present six different ways of extending this semantics with a conditional knowledge operator, drawing on notions of implication and content that are prominent in Fine's work. We demonstrate that different logics are thereby generated, bearing on the aforementioned epistemic principles. Finally, we offer preliminary observations about the prospects for each logic.

Keywords: [truthmaker semantics; epistemic logic; epistemic paradox]

1 Introduction

Can truthmaker semantics improve on Hintikka-style semantics for epistemic logic?[1] That depends on the answers to various questions:

- What is one's purpose in designing a system for epistemic logic? For instance, is it to capture ordinary knowledge ascription; in-principle knowability relative to a given body of empirical evidence; or the transmission of strong justification?
- What are the limitations, if any, of the Hintikkan approach relative to the selected goal? In particular, do cogent philosophical arguments bear against the validities that fall out of this approach?
- What are the unequivocal successes of the Hintikkan approach, and can a truthmaker approach emulate them without incurring large costs in complexity?

Our purpose is to briefly explore the scope of a truthmaker approach to epistemic logic and the extent to which it can best the Hintikkan approach. §2 identifies three possible

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[1] Hintikka (1962) kick-started the Hintikkan tradition. See (Fagin et al., 1995), (van Ditmarsch et al., 2008), (van Benthem, 2011) and (Humberstone, 2016) for comprehensive introductions and overviews.

targets of analysis for the epistemic logician (as per the first item above). Then, we offer a list of candidate epistemic principles and review the arguments that render some controversial. §3 presents the Hintikkan approach and notes - as per its well-known susceptibility to the 'problem of logical omniscience' - that it validates all of the aforementioned principles, controversial or otherwise. §4 lays out a truthmaker framework in the style of Fine (2016, 2017a, forthcoming). §5 presents six different ways of extending this semantics with a (conditional) knowledge operator, drawing on notions of implication and content that are prominent in Fine's work. We demonstrate that different logics are thereby generated, bearing on the principles from §2. §6 offers preliminary observations about the prospects for each logic, relative to (i) a target of analysis for epistemic logic and (ii) philosophical commitments that bear on the candidate principles in §2. Proofs are presented in a technical appendix.

The Hintikkan approach - in its relative elegance, simplicity, utility and familiarity - is a useful point of initial comparison when stress-testing novel systems for epistemic logic. We don't pretend that our evaluation is comprehensive. The issue of logical omniscience has inspired a universe of variations on and competitors to Hintikkan epistemic logic. See (Fagin et al., 1995, Ch.9), Humberstone (2016) and (Berto and Jago, 2019, Chs. 5, 9, 10) for a sense of this. We make no claims about the connections or relative (dis)advantages of our truthmaker-based approach to these alternatives. Further, a truthmaker-based approach is flexible, and admits a variety of systems worth studying under the heading of 'epistemic logic'. No doubt, the systems we identify offer a mere (hopefully instructive) sample.

2 Principles of Interest

We will work with epistemic language $\mathscr{L}_e$, defined by the grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_\varphi \psi \mid A\varphi \mid \varphi \Rightarrow \psi,$$

where $p \in \mathsf{Prop} = \{p, q, \dots\}$, a countable set of atoms. We employ the usual abbreviation for disjunction $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, material conditional $\varphi \supset \psi := \neg(\varphi \wedge \neg\psi)$, and bi-conditional $\varphi \equiv \psi := (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Read $K_\varphi \psi$ as 'knowing φ is sufficient for knowing ψ'; $A\varphi$ as 'φ is knowable apriori'; $\varphi \Rightarrow \psi$ as 'ψ is an apriori implication of φ' i.e. 'it is knowable apriori that φ implies ψ'.[2]

This language has non-standard features. For one, standard formal languages for epistemic logic aim to express *unconditional* knowledge or knowability claims (via a unary K operator).[3] For another, it is unusual to include sentences for expressing claims of apriority.[4] However, as will become evident momentarily, $\mathscr{L}_e$ is well-suited for capturing the epistemic principles that occupy us in this paper: as we see it, standard languages

[2] Another approach would include an appropriate conditional $>$ in the language so as to render $\varphi \Rightarrow \psi$ definable as $A(\varphi > \psi)$. The nature of the underlying conditional $>$ is peripheral to our current interests, however - including the statement of our principles of interest.

[3] Though see (Berto and Hawke, forthcoming).

[4] Though see (Anderson, 1993).

can express some but not all of these principles.[5] What's more, as we'll see in §3, $\mathscr{L}_e$ receives a natural Hintikkan interpretation, laying pertinent features of the Hintikkan approach especially bare.

Since *apriori knowability* is a relatively standard philosophical notion, we offer little clarification of the intended reading for $A\varphi$ and $\varphi \Rightarrow \psi$, besides the standard gloss ('knowable on the basis of pure reasoning alone') and, respectively, standard examples: on one hand, mathematical truths, such as that one and one is two; on the other hand, claims of logical or analytic entailment (if such there be), such as that Jon is a bachelor only if he is unmarried, or that two is both even and prime only if it is even.

What, however, is it for knowledge of ϕ to be *sufficient* for knowledge of ψ? We focus on three elaborations, relative to three natural goals in designing an epistemic logic.

First, one may aim to capture the logic of ordinary knowledge ascription, without restriction. In this case, read $K_\varphi \psi$ as: 'knowing φ entails knowing ψ'.

Second, one may aim for a logic of in-principle knowability, relative to a fixed body of empirical evidence. That is: the logic of ordinary knowledge ascription but restricted to cognitively ideal agents, where a cognitively ideal agent is unbounded with respect to computational and conceptual resources. In this case, read $K_\varphi \psi$ as: 'for cognitively ideal agents, knowing φ entails knowing ψ'. We sometimes gloss this as: 'knowing φ entails that ψ is knowable in-principle'.

Third, one may aim to capture knowledge-level warrant transmission. The term 'epistemic warrant' labels subtly distinct notions in the literature. Pollock (1983) targets ideally rational belief; Plantinga (1993) targets whatever knowledge adds to true belief, Gettier examples in mind; Pryor (2000) targets beliefs that are epistemically appropriate for an agent; Moretti and Piazza (2018) take 'warrant' as interchangeable with 'justification'. We use it as follows: an agent has *warrant* for φ exactly when she has propositional justification in support of φ. She has *strong warrant* for φ exactly when she is warranted to the degree necessary for knowing φ. By *propositional justification*, we mean that if the agent *were* to believe φ on the basis of an appropriate part of her total epistemic state, then her belief *would* be justified. We thus evoke the traditional contrast with *doxastic justification*, where having doxastic justification for φ implies that the agent actually believes φ. We read $K_\varphi \psi$ as: 'knowing φ provides strong warrant for ψ', or more concisely 'knowing φ strongly warrants ψ'.

To appreciate the intuitive contrast between our three readings of 'sufficient', consider:

(1) Knowing that Jane is a lawyer is sufficient for knowing that Jane is a fisherman.

(2) Knowing that Jane is an expert lawyer is sufficient for knowing that Jane is a lawyer.

[5] Standard languages have their own expressive advantages. For example, they are well-suited for capturing various principles of introspection e.g. the principle that knowledge entails knowing that one knows. Of course, language $\mathscr{L}_e$ has its own advantages on this front: for example, it can express the principle that $K_p q$ requires $K_p(K_p q)$, i.e., that knowing p is sufficient for knowing q only if knowing p is sufficient for knowing that knowing p is sufficient for knowing q. At any rate, nested modalities won't play a role in the present paper.

(3) Knowing that Jane is a lawyer is sufficient for knowing Cantor's theorem.

(4) Knowing the conjunction of the ZF axioms (and basic logical principles) is sufficient for knowing Cantor's theorem.

Claim (1) seems false on any reading. An agent that knows Jane is a lawyer needn't know that Jane is a fisherman. Nor is she thereby *positioned* to know Jane is a fisherman without accrual of further empirical evidence. Nor does this knowledge provide warrant (strong or otherwise) for believing that Jane is a fisherman.

Only (2) seems true when (1)-(4) are paraphrased in terms of ordinary knowledge ascriptions. An agent that knows that Jane is an expert lawyer also knows, presumably, that Jane is a lawyer. To know the latter is intuitively *part* of knowing the former.

In contrast, (2), (3), (4) seem true when interpreted as claims about in-principle knowability. A cognitively ideal agent that knows that Jane is a lawyer also knows Cantor's theorem. For the latter is presumably *apriori*: under *no* circumstance is further empirical evidence necessary for establishing it, putting aside agents' contingent psychological foibles.

In contrast, only (4) seems unequivocally true when (1)-(4) are interpreted in terms of knowledge-level warrant transmission. If one knows the conjunction of the ZF axioms then, presumably, one *thereby receives* strong propositional justification for believing Cantor's theorem. On the other hand (regarding (3)), knowing that Jane is a lawyer clearly provides no justification for believing Cantor's theorem. Claim (2), meanwhile, raises subtle issues. It isn't possible to know that Jane is an expert lawyer without knowing that Jane is a lawyer. Hence, it isn't clear that from knowing the former one can *thereby receive* justification for believing the latter. The partial converse is more tempting: by knowing that Jane is a lawyer one thereby receives partial warrant for believing that she is an expert lawyer (establishing *expertise* requires, of course, a further source). Compare: one hesitates to say that from knowing that two is both even and prime one thereby receives justification that two is even, as warrant for the latter is a prerequisite for knowing the former. Compare: one hesitates to say that from knowing that two is even one thereby receives justification that two is even, as warrant for the latter is (obviously) a prerequisite for knowing the former. Compare: one *doesn't* hesitate to say that from knowing the ZF axioms one thereby receives (ultimate) justification for Cantor's Theorem, as warrant for the latter is (obviously) *not* a prerequisite for knowing the former.

Here's another way to see the intuitive point: an argument for 'Jane is a lawyer' that relies on the premise 'Jane is an expert lawyer' seems viciously circular. Likewise, an argument for 'Two is even' from the premise 'Two is both even and prime' seems viciously circular. Likewise, an argument for 'Two is even' from the premise 'Two is even' is viciously circular.

We propose to deflate the issue: we identify vicious circularity with *degenerate warrant transmission*, where warrant transmission from φ to ψ is degenerate, intuitively, if warrant for ψ is a prerequisite for warranted belief in φ (so any chain of justifications for ψ that appeals to φ contains redundancy, since ψ must appear earlier in the chain).

Non-degenerate warrant transmission is, we submit, an irreflexive and anti-symmetric relation. Its logic is an especially delicate matter that we can afford to put aside. In contrast, (partial) warrant transmission in general, subsuming degenerate warrant transmission, is *not* irreflexive and anti-symmetric.

We can now turn to candidate logical principles, organized in three groups. In what follows, read φ as $\Vdash \varphi$ (i.e. φ is valid) and $\varphi_1, \ldots, \varphi_n \Vdash \psi$ as logical consequence.

Uncontroversial:

> **Simplification:** $K_{p\wedge q}p$, $K_{p\wedge q}q$
>
> **Reflexivity:** $K_p p$
>
> **Cautious Agglomeration:** $K_p q \Vdash K_p(p \wedge q)$
>
> **Cautious Transitivity:** $K_p q, K_{p\wedge q} r \Vdash K_p r$
>
> **Cautious Monotonicity:** $K_p q, K_p r \Vdash K_{p\wedge q} r$

Relatively uncontroversial:

> **Double Negation:** $K_p(\neg\neg p)$
>
> **Weak Simplification:** $K_{p\wedge q}(p \vee q)$
>
> **Weak Omniscience:** $K_p(p \vee \neg p)$
>
> **Apriority:** $Ap \Vdash K_\varphi p$
>
> **Transitivity:** $K_p q, K_q r \Vdash K_p r$
>
> **Monotonicity:** $K_p r \Vdash K_{p\wedge q} r$

Controversial:

> **Negative Addition:** $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
>
> **Agglomeration:** $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
>
> **Single-Premise Closure:** $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
>
> **Disjunctive Syllogism:** $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$

In what follows, we often abbreviate the names of our candidate principles: **Simp**, **Refl**, **C-Trans**, **C-Mon**, **DN**, **W-Simp**, **W-Omni**, **Apriority**, **Trans**, **Mon**, **Neg Add**, **Agg**, **SPC**, **DS**.

We proceed on the assumption that any epistemic logician should accept the principles we label 'uncontroversial'. This isn't beyond dispute. Nevertheless, there seems a strong *prima facie* case for each principle, for each of our three interpretations: the principles seem easily recovered from judgment about mundane cases, via generalization, and we aren't aware of pressing counter-examples.[6]

[6] See Smith (2018) for the intuitive case for **Cautious Transitivity** and **Cautious Monontonicity** for a logic of conditional justification.

To illustrate, consider **Cautious Agglomeration** for ordinary knowledge ascription. This principle finds support from mundane judgment. Again, consider (2): if Sam knows (p) that Jane is an expert lawyer, it follows that Sam knows (q) that Jane is a lawyer. Now compare 'Sam knows that Jane is an expert lawyer' and 'Sam knows that Jane is both an expert lawyer and a lawyer'. The latter merely seems an awkward rephrasing of the former: it sounds odd to assert one but deny the other. Standard theoretical considerations bolster this. Suppose that knowing p entails knowing q. One explanation (Yablo, 2014, Ch. 7): the content of q is part of the content of p. In this case, $p \wedge q$ seems likewise part of the content of p: the former's content partitions into that of p and q, each of which is part of the content of the latter. An alternative explanation (Cherniak, 1986): propositional attitude ascriptions apply only, some philosophers think, to minimally rational agents, and minimal rationality implies the agent's knowledge is closed under sufficiently simple inferences - like that from p to q. But the inference from p and q to $p \wedge q$ is then likewise simple. Hence, if knowing p entails knowing q, then minimal rationality plausibly demands that knowing p entails knowing $p \wedge q$.

Now consider the case for **Cautious Transitivity** for ordinary knowledge ascription. Momentarily, we'll defend **Transitivity** for ordinary knowledge ascription. Given that arbitrary propositional formulae may replace atoms without loss of validity, **Transitivity** yields $K_p(p \wedge q), K_{p\wedge q}r \vDash K_p r$, and so **Transitivity** and **Cautious Agglomeration** entail **Cautious Transitivity**.

In contrast, our 'relatively uncontroversial' principles are only controversial relative to a certain interpretation of the logic. They thereby help to discriminate logical systems that are best suited for only particular interpretations.

The first three are only questionable, we take it, for a logic of ordinary knowledge ascription - and here our intuitions are murky. Suppose that an ordinary agent knows that Jane is a lawyer. Does it follow that she knows that either Jane is a lawyer or not a lawyer? On one hand, our agent might not be familiar with simple logical principles, and even if she *is* familiar with, say, disjunction introduction, perhaps knowing p needn't include her knowing $p \vee \neg p$: perhaps the latter knowledge is only available via an (admittedly simple) inference that an agent might fail to draw. On the other hand, it is difficult to say what coming to know that Jane is either a lawyer or not a lawyer *adds* to knowing that Jane is a lawyer. The former presents as *vacuous* knowledge about Jane and her profession. Can an agent lack vacuous knowledge? Does agency require minimal rationality, and does minimal rationality entail that knowledge ascription is closed under the simple inference from p to $p \vee \neg p$? Similar remarks apply to **Double Negation**. Suppose one knows that Jane is friendly. Does it follow that one knows that Jane is not unfriendly? Can one really learn something *new* about Jane if one infers that Jane is not unfriendly?

Apriority, meanwhile, is uncontroversial for a logic of in-principle knowability; uncontroversially wrong for a logic of ordinary knowledge ascriptions; and (at least) controversial for a logic of warrant transmission. Examples (3) and (4) illustrate this.

If logical entailment is transitive, then **Transitivity** is uncontroversial for the logic of ordinary knowledge ascription and in-principle knowability. If knowing p entails knowing q, and knowing q entails knowing r, then knowing p entails knowing r. *Mutatis mutandis* when we restrict attention to cognitively ideal agents.

In this case, **Monotonicity** should be taken as uncontroversial for the logic of ordinary knowledge ascription and in-principle knowability. For **Transitivity** entails: $K_{p\wedge q}p, K_p r \Vdash K_{p\wedge q}r$. Thus, **Transitivity** and **Simplification** together entail **Monotonicity**.

However, both **Transitivity** and **Monotonicity** are highly controversial for the logic of strong warrant transmission, in light of the purported *defeasibility* of (even strong) justification. To see this for **Transitivity**, we adapt a counter-example from Smith (2018), itself adapted from Cohen (2002). Knowing that a certain wall is white but bathed in red light strongly warrants that the wall *appears* to be red. Plausibly, knowing that the wall *appears* to be red strongly warrants that the wall *is* red (lest basic perceptual knowledge be called into doubt). However, knowing that the wall is white but bathed in red light doesn't warrant a belief that the wall is red. Likewise for **Monotonicity**: plausibly, knowing that the wall appears to be red strongly warrants that the wall is red; but knowing that the wall both appears to be red *and* is white but bathed in red light doesn't warrant a belief that the wall is red.

As for the 'controversial' principles: though the debates are unresolved, arguments exist for rejecting these for all *three* interpretations of the logic. In brief, the line is: various (alleged) philosophical paradoxes are best understood, on reflection, as identifying counter-examples to the offered principles.

- **Preface counter-example to Agg:**[7] A historian can know every claim $p_1, \ldots, p_n$ in her new book, but rightly acknowledge that books of this length frequently have at least one error. Hence, she isn't positioned to know, or anyway thereby warranted to believe, $p_1 \wedge \ldots \wedge p_n$.
- **Cartesian counter-example to Neg Add and SPC:**[8] Agent A knows that she has hands. Not being a (handless) brain-in-vat is an apriori implication of having hands. Yet our agent is not positioned to know, or anyway thereby warranted to believe, that she is not a (handless) brain-in-vat.
- **Dogmatism counter-example to Neg Add and SPC:**[9] Suppose agent A knows p but doesn't know e, where e counts as non-deductive evidence against p.[10]

[7] The preface paradox was introduced by Makinson (1965). It is usually framed and debated as concerning rational belief, rather than knowledge per se. See Douven (2003) and Leitgeb (2013) for recent discussion.

[8] Dretske (1970) and Nozick (1981) pioneered the rejection of 'epistemic closure' as a solution to the skeptical conundrum. See Hawke (2016) and Berto and Hawke (forthcoming) for more recent developments of this approach. See Stine (1976), Lewis (1996) and Hawthorne (2004, 2005) for spirited defenses of epistemic closure. Wright (2004) rejects the necessary transmission of warrant across implication; see Pryor (2004) for push-back.

[9] The dogmatism paradox is due to Kripke, in a work that eventually appeared as (Kripke, 2011). The paradox first appeared in the literature in a discussion by Harman (1973). It is explicitly waged against 'epistemic closure' by Sharon and Spectre (2010, 2017). Recent discussions include Sorensen (1988), Lasonen-Aarnio (2014), Beddor (forthcoming) and Berto and Hawke (forthcoming).

[10] For instance, e might be the potential testimony of an expert on the broad subject matter of p who is -

Hence (as it happens), e misleads on the question of p: a rational agent ought to lower her credence in p upon coming to know e, despite p being true. However, knowing p needn't position A to *know* (or be warranted in believing) that e is misleading. That is: she may not be positioned to know, without further ado, that $\neg(e \wedge \neg p)$. For, if A *were* so positioned, it would presumably be reasonable for her to ignore e's usual evidential import for p, were she to come to know e. At least, she would reasonably *resist* coming to know e, since she knows such inquiry threatens to undermine her knowledge. Upon generalization it follows, counter-intuitively, that agents are right to adopt an extreme dogmatism, ignoring or actively avoiding counter-evidence to any claims they take themselves to know.

- **Criterion counter-example to Neg Add and SPC:**[11] Agent A knows p on the non-deductive basis of knowing her total empirical evidence e. However, knowing p doesn't position her to know, or anyway provide warrant in believing, that $\neg(e \wedge \neg p)$ i.e. that e isn't misleading on the question of p. After all, $e \wedge \neg p$ is perfectly consistent with the total empirical evidence and, anyway, beliefs based on e about the accuracy of e's predictions would be viciously circular.

- **Criterion counter-example to DS:** Agent A knows $\neg p$ - that she is not disembodied - on the basis of her empirical evidence. She also knows $p \vee a$: either she is disembodied (p) or an accurate verdict on the question of p is supported by her empirical evidence (a). But she is not thereby positioned to know, or thereby warranted in believing, that her empirical evidence supports an accurate verdict on the question of p. If she were, she would presumably be so positioned on the basis of her empirical evidence, since a is contingent and empirical. However, it is objectionably circular to claim that an agent's total empirical evidence supports knowledge or warranted belief about the accuracy of verdicts drawn from that agent's total empirical evidence.

- **Surprise exam counter-example to DS:**[12] A teacher announces that there will be a surprise exam the following week. The students thereby know $p \vee q$: either the exam is on Friday (p) or earlier in the week (q). They also rightly conclude $\neg p$, on the basis that if the exam arrived on Friday, the element of surprise would be eliminated by Thursday night. But the students cannot pool this knowledge to come to know (or be warranted in believing) q: if they could, they would be able to iterate their reasoning and (absurdly) come to know that a surprise exam is impossible, and so they didn't know $p \vee q$ in the first place.

We do not claim that the above (alleged) counter-examples are universally accepted by philosophers, nor that rejection of the controversial principles is a popular or cost-free resolution of the associated epistemic paradoxes (indeed, an obvious cost is that the controversial principles have pre-theoretic appeal, at least for the logic of knowability

unusually - misguided on the particular question of p.

[11] For a classic discussion of the problem of the criterion, see Chisholm (1973). The puzzle has re-emerged in recent discussions on the issue of 'easy knowledge': see, for instance, Cohen (2002) and Sosa (2009).

[12] See Sorensen (2018, Sect. 1) for an overview of the paradox and some responses in the literature. Kripke (2011) offers another recent discussion.

or warrant transmission). Rather, our point is that treating the paradoxes as yielding such counter-examples deserves serious discussion. The associated paradoxes aren't amenable to a cost-free resolution, and it is striking that giving up the controversial principles provides one option. A formal system that invalidates these principles potentially serves as a neutral tool for framing the philosophical debate or, at least, a tool for epistemologists that accept the force of the alleged counter-examples. A formal system that validates one of these principles stakes a strong position in a perplexing debate.

3 The Classical Approach to Epistemic Logic

It is well-known[13] that the Hintikkan approach to epistemic logic - taking it as a normal modal logic - validates every (relatively) controversial principle in the above list. This rules it out as capturing the logic of ordinary knowledge ascription or warrant transmission. As a logic of knowability, it is controversially strong.

We spell out the Hintikkan approach - both its semantics and syntax - somewhat unusually: as a system for conditional knowledge. We respect its core ideas, however: a body of knowledge κ at world w is modeled as a set of possible worlds, and knowledge of φ is rightly ascribed to the knower just in case (the proposition expressed by) φ is entailed by κ. A key underlying idea is that content - i.e. a proposition - is well modeled as a set of possible worlds and entailment, therefore, as set containment. Besides offering continuity with our overall discussion, working with conditional knowledge operators allows us to put aside the vexed issue of *fragmentation* - the question as to whether an agent's total knowledge state is best modeled as, invariably, a single unified body of knowledge, or, more flexibly, as a collection of distinct bodies of knowledge. (See, e.g., Lewis (1982), Stalnaker (1984, Ch.5), Fagin et al. (1995, Ch.9), Spectre (2019).)

Definition 1 (Hintikka Model)**.** *A* Hintikka model *is a pair $\mathscr{M} = \langle W, \mathrm{v} \rangle$ where W is a non-empty set of possible worlds and $\mathrm{v} : \mathsf{Prop} \to 2^W$ is a valuation function that assigns a subset of W (a proposition) to each atom.*

Definition 2 (Hintikka Semantics)**.** *Given a Hintikka model $\mathscr{M} = \langle W, \mathrm{v} \rangle$ and a possible world $w \in W$, the $\vDash$-semantics for $\mathscr{L}_e$ is recursively defined as:*

$$
\begin{array}{lll}
w \vDash p & \textit{iff} & w \in v(p) \\
w \vDash \neg\varphi & \textit{iff} & \textit{not } w \vDash \varphi \\
w \vDash \varphi \wedge \psi & \textit{iff} & w \vDash \varphi \textit{ and } w \vDash \psi \\
w \vDash A\varphi & \textit{iff} & u \vDash \varphi \textit{ for all } u \in W \\
w \vDash \varphi \Rightarrow \psi & \textit{iff} & \textit{for all } u \in W (\textit{if } u \vDash \varphi \textit{ then } u \vDash \psi) \\
w \vDash K_\varphi \psi & \textit{iff} & \textit{for all } u \in W (\textit{if } u \vDash \varphi \textit{ then } u \vDash \psi)
\end{array}
$$

Truth in a model, logical consequence and *validity* are defined in standard ways: with $\varphi_1, \ldots, \varphi_n, \psi \in \mathscr{L}_e$:

- ψ is true in $\mathscr{M}$, denoted by $\mathscr{M} \vDash \psi$, iff $w \vDash \psi$ for every $w \in W$,

[13]Issues of logical omniscience have been obvious from the start: for this reason, Hintikka (1962, Sect. 2) is careful to specify the exact interpretation of his logic.

- ψ is a logical consequence of $\{\varphi_1, \ldots, \varphi_n\}$, denoted by $\varphi_1, \ldots, \varphi_n \vDash \psi$, iff $w \vDash \varphi_1 \wedge \ldots \wedge \varphi_n$ materially implies $w \vDash \psi$ for every w in every $\mathscr{M}$, and
- ψ is logically valid, denoted by $\vDash \psi$, iff $\mathscr{M} \vDash \psi$ for all $\mathscr{M}$.

The above modal components exhibit various redundancies: $\varphi \Rightarrow \psi$ and $K_\varphi \psi$ have exactly the same interpretation; and $A\psi$ and $K_\varphi \psi$ (and, thus, $\varphi \Rightarrow \psi$) are interdefinable via the equivalences $A\psi \equiv K_\top \psi$ and $K_\varphi \psi \equiv A(\varphi \supset \psi)$, where $\top$ is a propositional tautology. However, the truthmaker semantics examined in §4 can discern these elements of the language, and we prefer to keep the object language fixed throughout.

Thus, the Hintikkan approach formalizes both conditional knowledge and apriori implication as strict implication. Hence:

Proposition 1. *The classical approach validates every principle: uncontroversial, relatively uncontroversial, and controversial.*

Thus, someone who both adopts certain goals for epistemic logic and is sympathetic to certain philosophical arguments has cause to seek a refinement of the Hintikkan approach. In some cases, this is obvious: for instance, a logic for strong warrant transmission should not validate **Apriority**.

4 An Exact Truthmaker Semantics

We now introduce the so-called "inclusive" truthmaker semantics of Fine (2016, Sect. 3) for the language of propositional logic. We again utilize $\mathscr{L}_e$ and various fragments thereof. The fragment without the conditional knowledge modality $K_\varphi \psi$ is denoted by $\mathscr{L}$. The fragment that, in addition, lacks $\varphi \Rightarrow \psi$ and $A\varphi$ is denoted by $\mathscr{L}_{pl}$ (*pl* for propositional logic).

Definition 3 (State Space)**.** *A* state space *is a tuple $\langle S, \leq \rangle$ where S is a non-empty set of states and $\leq$ is a partial order on S. In other words, a state space is a non-empty poset.*

Relation $\leq$ is the *parthood* relation on S: $s \leq t$ reports that state s is a part of ('contained in') state t. Given a state space $\langle S, \leq \rangle$ and a subset $T \subseteq S$, we say $s \in S$ is an *upper bound* of T iff $t \leq s$ for all $t \in T$. We call $s \in S$ *the least upper bound of T* if s is an upper bound of T and for any upper bound $s' \in S$ of T, $s \leq s'$. We call a state space $\langle S, \leq \rangle$ *complete* if every non-empty subset $T \subseteq S$ has a least upper bound. For any non-empty $T \subseteq S$ we denote the least upper bound of T by $\bigsqcup T$ and call it the *fusion* of T. In particular, the least upper bound of a two-element set $\{s, t\} \subseteq S$ is denoted by $s \sqcup t$ and called the fusion of s and t. Finally, we call a subset $T \subseteq S$ *downward closed* iff for all $s, t \in S$, $s \in T$ and $t \leq s$ implies $t \in T$.

Definition 4 (Modalized State Space)**.** *A* modalized state space *is a tuple $\langle S, P, \leq \rangle$ where $\langle S, \leq \rangle$ is a complete state space and P is a non-empty, downward closed subset of S.*

Set P is the subspace of *possible* states. States s, t are *compatible* when $s \sqcup t \in P$.

Intuitively, the picture is as follows. States are to be thought of as *situations*: roughly, a collection of particular objects, standing in particular relations.[14] Situations have a mereological structure, captured by $\leq$. For example, consider any situation where, say, both Socrates is wearing a toga and Aristotle is pontificating. This includes a smaller situation that concerns *only* Socrates and his attire. A 'world' w, on this picture, is taken to be a 'maximal' possible state (i.e. situation), in the following sense: for all $s \in S$, if s is compatible with w then $s \leq w$. Further, we here (primarily) understand 'possibility' in an epistemic sense.[15] In particular, P is understood to contain *basic empirical possibilities*: each requires (non-degenerate) empirical information to be ruled out. States in $S - P$ are thus thought of as ruled out apriori i.e. without any requirement of empirical information. Thus, apriori claims can be identified with those that are true at every world in P (an 'epistemic necessity'), and so can be known independently of the available empirical information.

A (unilateral) *proposition* is a subset of S that is closed under fusions and non-empty. A *bilateral proposition* is a pair of propositions. Propositions A, B are incompatible when $s \sqcup t \notin P$ for every $s \in A$ and $t \in B$. Fine (2017a, Sect. 2) considers three natural conditions on unilateral propositions in the state-based setting: closure under fusions; non-emptiness; and convexity. A set of states A is convex if $s \leq t \leq u$ and $s \in A$ and $u \in A$ implies $t \in A$. Fine (2016) shows that the class of propositions admit a natural account of content parthood only if propositions are assumed to be closed, non-empty and convex. For simplicity, we operate in the more general setting where convexity isn't imposed, since, as far as we can see, this has no bearing on our results.

Definition 5 (Model)**.** *A model is a tuple* $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ *where* $\langle S, P, \leq \rangle$ *is a modalized state space and* $\mathtt{v} : \mathsf{Prop} \to (2^S \times 2^S)$ *assigns a bilateral proposition* $\langle p^+, p^- \rangle$ *to each atom* $p \in \mathsf{Prop}$*, with* p^+ *and* p^- *incompatible.*

Definition 6 (Exact verification & falsification)**.** *Given a model* $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ *and a state* $s \in S$*, exact verification* $\vdash$ *and exact falsification* $\dashv$ *for* $\mathscr{L}$ *is recursively defined as:*

- $s \vdash p$ *iff* $s \in p^+$
- $s \dashv p$ *iff* $s \in p^-$
- $s \vdash \neg\varphi$ *iff* $s \dashv \varphi$
- $s \dashv \neg\varphi$ *iff* $s \vdash \varphi$
- $s \vdash \varphi \wedge \psi$ *iff there exists* $t, u \in S$ *such that* $s = t \sqcup u$ *and* $t \vdash \varphi$ *and* $u \vdash \psi$
- $s \dashv \varphi \wedge \psi$ *iff* $s \dashv \varphi$ *or* $s \dashv \psi$ *or there exists* $t, u \in S$ *such that* $s = t \sqcup u$ *and* $t \dashv \varphi$ *and* $u \dashv \psi$
- $s \vdash A\varphi$ *iff for all* $t \in P$ *there is* $t' \in P$ *such that* $t' \sqcup t \in P$ *and* $t' \vdash \varphi$
- $s \dashv A\varphi$ *iff there is* $t \in P$ *such that for all* $u \in P$ *either* $t \sqcup u \notin P$ *or* $u \dashv \varphi$

[14] Compare Barwise and Perry (1983).

[15] See Chalmers (2010) for a detailed discussion of the nature of an 'epistemic' space of possibilities, in contrast to, say, a space of (merely) logical, metaphysical, or nomological possibilities.

- *$s \vdash \varphi \Rightarrow \psi$ iff for all $t \in P$ (if there is $t' \in P$ such that $t' \leq t$ and $t' \vdash \varphi$, then there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash \psi$)*
- *$s \dashv \varphi \Rightarrow \psi$ iff there is $t \in P$ such that there exists $t' \in P$ with $t' \leq t$ and $t' \vdash \varphi$ and there is no $u \in P$ such that $t \sqcup u \in P$ and $u \vdash \psi$*

As we define $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, we obtain the following exact verification and falsification clauses for disjunction:

- $s \vdash \varphi \vee \psi$ iff $s \vdash \varphi$ or $s \vdash \psi$ or $s \vdash \varphi \wedge \psi$
- $s \dashv \varphi \vee \psi$ iff there exists $t, u \in S$ such that $s = t \sqcup u$ and $t \dashv \varphi$ and $u \dashv \psi$

We say that *s exactly verifies φ* (or *makes exactly φ true*) when $s \vdash \varphi$; that *s exactly falsifies φ* (or *makes exactly φ false*) when $s \dashv \varphi$; that *s verifies φ* (or *inexactly* verifies φ, for emphasis) when there exists $t \leq s$ such that $t \vdash \varphi$; and that *s falsifies φ* (or *inexactly* falsifies φ, for emphasis) when there exists $t \leq s$ such that $t \dashv \varphi$.

The $\vdash$-clause for $A\varphi$ is intended to echo the definition of a necessary state in (Fine, forthcoming, Sect. 5) i.e. a state is necessary just in case it is compatible with every possible state. For us, $A\varphi$ is made true just in case every possible state is compatible with a state that makes exactly φ true. In particular, every possible world will be compatible with a φ verifier, and so will inexactly verify φ. The $\vdash$-clause for $\varphi \Rightarrow \psi$, meanwhile, relativizes the clause for $A\varphi$ to the possible φ verifiers. Claim $\varphi \Rightarrow \psi$ is made true just in case: every possible state that (inexactly) verifies φ can be extended to a possible state that (inexactly) verifies ψ. This is intended to echo the definition of *loose verification* (and so *loose/classical consequence*) in the appendix of (Fine, 2017a): a state s loosely verifies φ just in case any state compatible with s is compatible with a state that verifies φ.

Verification by a model, logical consequence and *validity* are defined as follows: with $\varphi_1, \ldots, \varphi_n, \psi \in \mathscr{L}$:

- ψ is verified by $\mathtt{M}$, denoted by $\mathtt{M} \Vdash \psi$, iff $s \vdash \psi$ for every $s \in S$,
- ψ is a logical consequence of $\{\varphi_1, \ldots, \varphi_n\}$, denoted by $\varphi_1, \ldots, \varphi_n \Vdash \psi$, iff $s \vdash \varphi_1, \ldots, s \vdash \varphi_n$ materially implies $s \vdash \psi$ for every s in every $\mathtt{M}$, and
- ψ is logically valid, denoted by $\Vdash \psi$, iff $\mathtt{M} \Vdash \psi$ for all $\mathtt{M}$.[16]

The *exact unilateral content* of φ, relative to a given model, is the proposition:

$$|\varphi| = \{s \in S : s \text{ exactly verifies } \varphi\}$$

The *inexact unilateral content* of φ, relative to a given model, is the proposition:

$$||\varphi|| = \{s \in S : s \text{ inexactly verifies } \varphi\}$$

[16]These definitions extend to the whole language $\mathscr{L}_e$ in the same way.

Then, the exact and inexact bilateral contents for φ are, respectively, the bilateral propositions $\langle|\varphi|,|\neg\varphi|\rangle$ and $\langle||\varphi||,||\neg\varphi||\rangle$. We here ape the introduction of unilateral and bilateral content in (Fine, 2017a).

Sentences $A\varphi$ and $\varphi \Rightarrow \psi$ express *global* properties of a model: they are either verified by every state, or by none (see Lemma 3 in Appendix). Our proposed clauses for $K_\varphi \psi$ will follow suit. Admittedly, this abuses the notion of 'exact verification': intuitively, these sentences are made true by the overall structure of the model, not by any particular state.[17] They are aberrant from a Finean perspective for a second reason: if a sentence has no verifiers, its content cannot be identified with a proposition. However, our aim is not to offer a compelling account (formal or otherwise) of what makes a knowledge claim true, or apriori. Rather, we want to explore the opportunities provided by a truthmaking model for capturing a body of knowledge, ideal or otherwise.[18] 'Verification by a model' may be thought of as our chief tool in this regard, with 'verification by a state' a mere stepping stone.

5 Possible Definitions of Conditional Knowledge

5.1 Hintikka-Style Conditional Knowledge

We now introduce four accounts of conditional knowledge. They are naturally grouped into pairs. Accounts (1) and (3) are 'ruling in' accounts: roughly, $K_\varphi \psi$ is (made) true because restriction to the possible φ states amounts to a restriction to the possible ψ states. Accounts (2) and (4) are the corresponding 'ruling out' accounts: roughly, $K_\varphi \psi$ is (made) true because elimination of the possible $\neg\varphi$ states amounts to elimination of the possible $\neg\psi$ states.

We label this selection 'Hintikka-style' since, as we see it, each translates one of two (equivalent) conceptions of the classic Hintikkan account of $K_\varphi \psi$ into the truthmaker setting: every φ world is a ψ world; and every $\neg\psi$ world is a $\neg\varphi$ world. In contrast, as we shall see, their analogues can generate different logics in the truthmaker setting.

The accounts are as follows:

(1) $\mathtt{M} \Vdash K_\varphi \psi$ iff every possible state that makes φ true can be extended to a possible state that also makes ψ true.

To achieve this effect, we adopt the following definitions for the exact verification and falsification clauses:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if there is $t' \in P$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash \psi$)

[17] One natural way to render their verification as robustly state dependent is to render the set of possible states as itself state dependent, with each state related to the set of states that are, intuitively, *made possible* by that state.

[18] Our accounts of $K_\varphi \psi$, and pertinent discussion, could be executed entirely in the meta-language.

- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that there exists $t' \in P$ with $t' \leq t$ and $t' \vdash \varphi$ and there is no $u \in P$ such that $t \sqcup u \in P$ and $u \vdash \psi$

(2) $\mathtt{M} \Vdash K_\varphi \psi$ iff every possible state that makes ψ false can be extended to a possible state that also makes φ false.

To achieve this effect, we adopt the following definitions for $\vdash$ and $\dashv$:

- $s \vdash K_\varphi \psi$ iff if for all $t \in P$ (if there is $t' \in P$ such that $t' \leq t$ and $t' \dashv \psi$, then there is $u \in P$ such that $t \sqcup u \in P$ and $u \dashv \varphi$)
- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that there exists $t' \in P$ with $t' \leq t$ and $t' \dashv \psi$ and there is no $u \in P$ such that $t \sqcup u \in P$ and $u \dashv \varphi$

(3) $\mathtt{M} \Vdash K_\varphi \psi$ iff every exact truthmaker for φ verifies ψ.

We use the following definitions for $\vdash$ and $\dashv$:

- $s \vdash K_\varphi \psi$ iff for all $u \in S$ (if $u \vdash \varphi$ then there is $u' \in S$ such that $u' \leq u$ and $u' \vdash \psi$)
- $s \dashv K_\varphi \psi$ iff there is $u \in S$ such that $u \vdash \varphi$ and there is no $u' \in S$ with $u' \leq u$ and $u' \vdash \psi$.

(4) $\mathtt{M} \Vdash K_\varphi \psi$ iff every exact falsemaker for ψ falsifies φ.

We use the following definitions for $\vdash$ and $\dashv$:

- $s \vdash K_\varphi \psi$ iff for all $u \in S$ (if $u \dashv \psi$ then there is $u' \in S$ such that $u' \leq u$ and $u' \dashv \varphi$)
- $s \dashv K_\varphi \psi$ iff there is $u \in S$ such that $u \dashv \psi$ and there is no $u' \in S$ with $u' \leq u$ and $u' \dashv \varphi$.

The first account is especially strongly analagous to the Hintikkan approach: it takes $K_\varphi \psi$ and $\varphi \Rightarrow \psi$ as semantically equivalent, i.e., as two notations for making the same claim. Evidently, it is only viable as a candidate for capturing in-principle knowability.

Proposition 2.

(1) Definition (1) validates every principle in §2: uncontroversial, relatively uncontroversial and controversial.

*(2) Definition (2) validates every principle in §2 except **SPC** and **DS**.*

*(3) Definition (3) and (4) validate every principle in §2 except **Apriority**, **SPC**, and **DS**.*

5.2 Immanent Conditional Knowledge

We introduce two further accounts of $K_\varphi \psi$, both under the heading of *immanent* accounts of conditional knowledge. Yablo (2014, Sect. 7.3) observes that, for some φ and ψ, knowing ψ is part of knowing φ. In this case, we have an instance of *immanent closure*: knowing φ entails knowing ψ because knowledge is closed under parts. It is natural to elaborate as follows: knowing ψ is part of knowing φ just in case content ψ is part of content φ. To know Jane is a lawyer and a fisherman is to know she is a lawyer, since the proposition that Jane is a lawyer is part of the proposition that Jane is a lawyer and a fisherman. To know Jane is an expert lawyer is to know she is a lawyer, since the proposition that Jane is a lawyer is part of the proposition that Jane is an expert lawyer.

Fine (2016, 2017a) explicates this sentiment in terms of partial verification. Let A and B be unilateral propositions on Fine's account: each a set of states, thought of as exact verifiers. Then, by Fine's lights, B is part of A just in case (i) every exact verifier for A has an exact verifier for B as a part and (ii) every exact verifier for B is contained in some exact verifier for A. By (i), if A is (made) true then B is (made) true; by (ii), if B is (made) true, then A is partly (made) true. The connection to subject matter is drawn by way of the account in Fine (2017b) of the subject matter of a unilateral proposition: A's subject matter is the fusion of the verifiers for A. It follows that if B is part of A then B's subject matter is contained in A's subject matter.

Fine (2017a, Sect. 5) extends his definition of content parthood to the bilateral case: $\mathbf{B} = \langle B, \overline{B} \rangle$ is part of $\mathbf{A} = \langle A, \overline{A} \rangle$ just in case (i) B is part of A and (ii) $\overline{B} \subseteq \overline{A}$ i.e. every falsifier of $\mathbf{B}$ is a falsifer of $\mathbf{A}$.

All this suggests an account of $K_\varphi \psi$: $K_\varphi \psi$ holds just in case the content (expressed by) ψ is part of the content (expressed by) φ, by Finean lights.

(5) $\mathtt{M} \Vdash K_\varphi \psi$ iff, for all s, (i) if s is an exact verifier for φ then s verifies ψ, (ii) if s is an exact falsifier for ψ then it is an exact falsifier for φ, (iii) if s is an exact verifier for ψ then s is contained in an exact verifier for φ.

Corresponding exact verification and falsification clauses are obtained by strengthening and weakening, respectively, those for the third truthmaker semantics for $K_\varphi \psi$ (given as the first items above) as follows[19]:

$$
\begin{aligned}
s \vdash K_\varphi \psi \text{ iff } & (1) \text{ for all } u \in S \text{ (if } u \vdash \varphi \text{ then there is } u' \in S \text{ such that } u' \leq u \text{ and } u' \vdash \psi\text{), and} \\
& (2) \text{ for all } u \in S \text{ (if } u \dashv \psi \text{ then } u \dashv \varphi\text{), and} \\
& (3) \text{ for all } u \in S \text{ (if } u \vdash \psi \text{ then there is } u' \in S \text{ such that } u \leq u' \text{ and } u' \vdash \varphi\text{).}
\end{aligned}
$$

$$
\begin{aligned}
s \dashv K_\varphi \psi \text{ iff } & (1) \text{ there is } u \in S \text{ such that } u \vdash \varphi \text{ and there is no } u' \in S \text{ with } u' \leq u \text{ and } u' \vdash \psi \text{, or} \\
& (2) \text{ there is } u \in S \text{ (} u \dashv \psi \text{ and } u \not\dashv \varphi\text{), or} \\
& (3) \text{ there is } u \in S \text{ (} u \vdash \psi \text{ and there is no } u' \in S \text{ such that } u \leq u' \text{ and } u' \vdash \varphi\text{).}
\end{aligned}
$$

A variation of the fifth account is nearby. Fine (2017b) defines the subject matter of a *bilateral* content $\mathbf{A} = \langle A, \overline{A} \rangle$ as the fusion of the states in $A \cup \overline{A}$ i.e. the fusion of all of $\mathbf{A}$'s exact verifiers and falsifiers. This is naturally explicated further as: the subject matter of $\mathbf{B}$ is contained in that of $\mathbf{A}$ exactly when every exact verifier of $\mathbf{B}$ is contained in either an exact verifier or exact falsifier of $\mathbf{A}$; ditto for $\mathbf{B}$'s exact falsifiers. Then we may develop an 'immanent' account of $K_\varphi \psi$ as: $K_\varphi \psi$ holds exactly when φ implies ψ and the subject matter of ψ is contained in that of φ, by the lights of the forgoing

[19]We use $s \not\vdash \varphi$ ($s \not\dashv \varphi$) as an abbreviation for "it is not the case that $s \vdash \varphi$ ($s \dashv \varphi$)".

account.[20] This serves, roughly, as our sixth account of $K_\varphi\psi$. Notably, the logic thereby generated is significantly different to that generated by the fifth account.

(6) $\mathtt{M} \Vdash K_\varphi\psi$ iff (i) for every possible truthmaker for φ there is a compatible state that exactly verifies ψ and (ii) the subject matter of ψ is contained in the overall subject matter of φ.

Corresponding exact verification and falsification clauses are obtained by strengthening and weakening, respectively, the first truthmaker semantics for $K_\varphi\psi$ (given as the first items above) as follows:

$s \vdash K_\varphi\psi$ iff (1) for all $t \in P$ (if there is $t' \in P$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash \psi$), and
(2) for all $u \in S$ (if $u \vdash \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$), and
(3) for all $u \in S$ (if $u \dashv \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$)

$s \dashv K_\varphi\psi$ iff (1) there is $t \in P$ such that there exists $t' \in S$ with $t' \leq t$ and $t' \vdash \varphi$, and there is no $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$, or
(2) there is $u \in S$ ($u \vdash \psi$ and there is no $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$), or
(3) there is $u \in S$ ($u \dashv \psi$ and there is no $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$)

Proposition 3.

*(1) Definition (5) validates every principle in §2 except **W-Omni**, **Apriority**, **Neg Add**, **SPS**, and **DS**,*

*(2) Definition (6) validates every principle in §2 except **Apriority**, **Neg Add**, **SPS**.*

Table 1 summarizes Propositions 2 and 3.

6 Discussion

With Table 1 at hand, we offer preliminary remarks on the capacity of truthmaker semantics to help epistemic logic escape Hintikkan confines.

The flexibility and power of truthmaker semantics is in full evidence: we already have six distinct accounts of $K_\varphi\psi$ on the table that deserve serious attention. In particular, the corresponding logics all deliver the uncontroversial validities of §2, but are otherwise distinctive (with, notably, the possible exception of the third and fourth accounts). Those suspicious of **Neg Add**, **SPC** and **DS** will mark progress: various natural set-ups in the truthmaker setting invalidate these principles. What's more, we have located a number of set-ups that reject **Apriority**. As its validity distinguishes the logic of knowability-in-principle from those of knowledge ascription or strong warrant transmission, the truthmaker setting opens the door to serious investigation of the latter two.

[20] Compare Hawke et al. (forthcoming).

	1	2	3	4	5	6
Simplification	✓	✓	✓	✓	✓	✓
Reflexivity	✓	✓	✓	✓	✓	✓
Cautious Agglomeration	✓	✓	✓	✓	✓	✓
Cautious Transitivity	✓	✓	✓	✓	✓	✓
Cautious Monotonicity	✓	✓	✓	✓	✓	✓
Double Negation	✓	✓	✓	✓	✓	✓
Weak Simplification	✓	✓	✓	✓	✓	✓
Weak Omniscience	✓	✓	✓	✓	X	✓
Apriority	✓	✓	X	X	X	X
Transitivity	✓	✓	✓	✓	✓	✓
Monotonicity	✓	✓	✓	✓	✓	✓
Negative Addition	✓	✓	✓	✓	X	X
Agglomeration	✓	✓	✓	✓	✓	✓
Single-Premise Closure	✓	X	X	X	X	X
Disjunctive Syllogism	✓	X	X	X	X	✓

Table 1: Validities (✓) and invalidities (X). Numbers in the top row refer to the proposed truthmaker accounts of $K_\varphi \psi$.

Nevertheless, the ultimate status of our candidate logics is enmeshed in various philosophical debates. To validate a disputed principle is, after all, to take up a strong philosophical stance, and each candidate validates many (relatively) controversial principles. (In contrast, *invalidating* a principle may be regarded as a position of neutrality, raising the question as to which restricted classes of models enforce a certain validity.)

For instance, none of our candidate systems invalidate **Agg**, **Trans**, or **Mon**. Those sympathetic to alleged counter-examples will see little progress here. In particular, sympathy for the defeasibility of strong warrant transmission is not catered for.

Further, the *pattern* of validities generated by some of our systems will give some pause. The second, third and fourth invalidate **SPC** without invalidating **Neg Add**.[21] However, as developed in §2, the alleged counter-examples to **SPC** and **Neg Add** go hand-in-hand. If so, this puts pressure on any claim to having usefully departed from a classic Hintikkan explication of knowability-in-principle. Only the first and second accounts observe **Apriority**, but the former logic is apparently indistinguishable from the classic Hintikkan approach, while the latter drops **SPC** while validating **Neg Add**. However, the issues are far from clear cut: see (Roush, 2010) for an intriguing case for accepting **Neg Add** while rejecting **SPC**. Whether Roush's arguments survive scrutiny bears directly on the evaluation of our candidate logics.

Turn to the fifth and sixth systems. Both (perhaps pleasingly) invalidate **SPC** and **Neg Add** in unison. Since they also invalidate **Apriority**, they stand out (one might think) as particularly promising accounts of the logic of ordinary knowledge ascription or strong warrant transmission. Further subtleties are in play, however. The fifth account

[21] Hawke (2016) uses this same point to critique a number of closure-denying theories.

invalidates **W-Omni**, a potentially strange outcome for an account of strong warrant transmission: isn't $p \vee \neg p$ strongly warranted by knowing p? Further, there is an interesting debate to be had over the relative superiority of the fifth and sixth systems as accounts of the logic of ordinary knowledge. If this logic indeed reflects facts about minimal rationality (as might motivate the validation of, say, **DN**), then it is tempting to champion both **W-Omni** and **DS** (it is hard to imagine simpler or more intuitively immediate inference patterns). If so, we have a *prima facie* argument for our sixth account over our fifth account. On the other hand, if **DS** is indeed undermined by, say, 'surprise exam' considerations, then the fifth account has the edge.

7 Conclusion

The considerations of the last section are obviously not decisive. The firmest conclusions we can draw are as follows. First, truthmaker semantics allows for the development of various logical systems that are worth taking seriously as candidate epistemic logics. Second, the sample we consider in this paper establishes that such logics can depart from a normal modal logic (and each other), yielding patterns of validities and invalidities that interact with longstanding epistemic paradoxes.

Much work remains. We have barely scratched the surface in assessing the relative merits of our candidate logics. Further, we have said nothing on the subject of metalogical results. Finally, subtle variations of our candidate systems can no doubt be produced by tweaking various technical parameters. The costs and gains of such tweaks remain to be seen.

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Proofs

A Auxiliary Lemmas

The following lemmas help us prove Propositions 2 and 3.

Lemma 1. *Given a model $\langle S, P, \leq \mathtt{v}\rangle$ and $s,t \in S$: if $s \notin P$ then $s \sqcup t \notin P$.*

Proof. Let $s,t \in S$ such that $s \notin P$ and suppose $s \sqcup t \in P$. This implies, since P is downward closed and $s \leq s \sqcup t$, that $s \in P$, contradicting the assumption. $\square$

Lemma 2. *Given a model $\mathtt{M} = \langle S, P, \leq, \mathtt{v}\rangle$ and $\varphi \in \mathscr{L}_e$: $|\neg\varphi| = \{s \in S : s \dashv \varphi\}$.*

Proof. Follows immediately by the exact verification and falsification clauses for $\neg$ and the definition of $|\varphi|$. $\square$

Lemma 3. *Given a model $\mathtt{M} = \langle S, P, \leq, \mathtt{v}\rangle$ and $\varphi, \psi \in \mathscr{L}_e$, the following hold for all six interpretations of $K_\varphi \psi$:*

(1) if $|A\varphi| \neq \emptyset$ then $|A\varphi| = S$,

(2) if $|\neg A\varphi| \neq \emptyset$ then $|\neg A\varphi| = S$,

(3) if $|\varphi \Rightarrow \psi| \neq \emptyset$ then $|\varphi \Rightarrow \psi| = S$,

(4) if $|\neg(\varphi \Rightarrow \psi)| \neq \emptyset$ then $|\neg(\varphi \Rightarrow \psi)| = S$,

(5) if $|K_\varphi \psi| \neq \emptyset$ then $|K_\varphi \psi| = S$, and

(6) if $|\neg K_\varphi \psi| \neq \emptyset$ then $|\neg K_\varphi \psi| = S$.

Proof. Easy consequence of the corresponding exact truthmaker semantics since none of the exact verification and falsification clauses for $A\varphi, \varphi \Rightarrow \psi$, and $K_\varphi \psi$ is state dependent. $\square$

Lemma 4. *Given a model $\mathtt{M} = \langle S, P, \leq, \mathtt{v}\rangle$ and $\varphi \in \mathscr{L}_e$: $|\varphi|$ and $|\neg\varphi|$ are closed under non-empty fusions for all six interpretations of $K_\varphi \psi$.*

Proof. The proof follows by induction on the structure of φ.

Case for $p \in \mathsf{Prop}$: holds by the definition of a model (Definition 5).

Now suppose inductively that the statement holds for $\psi, \chi \in \mathscr{L}_e$.

Case for $\neg\psi$: By the induction hypothesis (IH), we already have that $|\neg\psi|$ is closed under fusion. Moreover, observe that $|\neg\neg\psi| = |\psi|$, by the exact verification and falsification clauses for $\neg$. Therefore, again by IH, $|\neg\neg\psi|$ is a closed under fusion.

Case for $\psi \wedge \chi$: Let $\emptyset \neq T \subseteq |\psi \wedge \chi|$. This means, by the exact verification clause for $\wedge$, that for all $t \in T$, there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \vdash \psi$, and $u' \vdash \chi$. Denote $T_\psi = \{u \in S : u \vdash \psi$ and there is $u' \in S$ such that $u \sqcup u' \in T$ and $u' \vdash \chi\}$, and similarly, $T_\chi = \{u' \in S : u' \vdash \chi$ and there is $u \in S$ such that $u \sqcup u' \in T$ and $u \vdash \psi\}$. Since $T \neq \emptyset$,

we have $T_\psi \neq \emptyset$ and $T_\chi \neq \emptyset$. Moreover, by IH, $\bigsqcup T_\psi \in |\psi|$ and $\bigsqcup T_\chi \in |\chi|$. Finally, since $T \subseteq |\psi \wedge \chi|$, we also have that $\bigsqcup T = \bigsqcup T_\psi \sqcup \bigsqcup T_\chi$. Therefore, by the exact verification clauses for $\wedge$, we obtain that $\bigsqcup T \in |\psi \wedge \chi|$. For $|\neg(\psi \wedge \chi)|$: let $\emptyset \neq T \subseteq |\neg(\psi \wedge \chi)|$. This means, by Lemma 2 and the exact falsification clause for $\wedge$, that for all $t \in T$, either $t \dashv \psi$ or $t \dashv \chi$, or there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \dashv \psi$, and $u' \dashv \chi$. Denote $T_{\neg\psi} = \{u \in S : u \dashv \psi$ and either $u \in T$ or there is $u' \in S$ such that $u \sqcup u' \in T$ and $u' \dashv \chi\}$, and, similarly,
$T_{\neg\chi} = \{u' \in S : u' \dashv \chi$ and either $u' \in T$ or there is $u \in S$ such that $u \sqcup u' \in T$ and $u \dashv \psi\}$. Since $T \neq \emptyset$, $T_{\neg\psi} \cup T_{\neg\chi} \neq \emptyset$. The rest follows similarly to the case for $|\psi \wedge \chi|$.

Case for $A\psi$, $\psi \Rightarrow \chi$, and $K_\psi \chi$: Follows from Lemma 3 and the fact that $\langle S, \leq \rangle$ is a complete state space. □ □

In what follows, counter-models are given in figures immediately below the corresponding explanations. It is easy to see that the given counter-models are of type described in Definition 5. In figures of models, white diamonds represent impossible states and black dots represent possible states. Exact verification and falsification are given by labelling nodes together with symbols $\vdash$ and $\dashv$, respectively. We sometimes write "$\vdash \neg p$" instead of "$\dashv p$" in order to keep figures clean.

B Proof of Proposition 2

Let $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ be a model and $s \in S$ be a state.

(1) Recall the exact verification clause:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if there is $t' \in P$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash \psi$)

Simplification: $K_{p\wedge q} p, K_{p\wedge q} q$:
We prove only the former, the latter follows similarly: let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p \wedge q$. Thus, by the exact verification clause for $\wedge$, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash p$, and $u' \vdash q$. Since $u \leq t' \leq t$, we have $t \sqcup u = t \in P$ and $u \in P$ (since P is downward closed and $t \in P$). As $u \vdash p$ as well, we obtain that $s \vdash K_{p\wedge q} p$.

Reflexivity: $K_p p$
Let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p$. Since $t \sqcup t' = t \in P$ as well, we obtain that $s \vdash K_p p$.

Cautious Aggloremation: $K_p q \Vdash K_p(p \wedge q)$
Suppose that $s \vdash K_p q$ and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p$. Since $s \vdash K_p q$, the latter implies that there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash q$. Observe that $t' \sqcup u \vdash p \wedge q$ (by the exact verification clause for $\wedge$) and $t \sqcup (t' \sqcup u) = (t \sqcup t') \sqcup u = t \sqcup u$ (since $\sqcup$ is associative and $t' \leq t$). Therefore, $t \sqcup (t' \sqcup u) \in P$ and $t' \sqcup u \in P$, thus, $s \vdash K_p(p \wedge q)$.

Cautious Transitivity: $K_p q, K_{p\wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p\wedge q} r$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p$. Then, by (a), there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash q$. Thus, by the exact verification clause for $\wedge$, we obtain $t' \sqcup u \vdash p \wedge q$. Then, as $t' \sqcup u \leq t \sqcup u \in P$ and $t' \sqcup u \in P$, by (b), we have that there is $u' \in P$ such that $(t \sqcup u) \sqcup u' \in P$ and $u' \vdash r$. Now consider $t \sqcup u'$. Since $t \sqcup u' \leq (t \sqcup u) \sqcup u' \in P$, we have $t \sqcup u' \in P$ (recall that P is downward closed). Since $u' \vdash r$ as well, we conclude that $s \vdash K_p r$.

Cautious Monotonicity: $K_p q, K_p r \Vdash K_{p\wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_p r$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p \wedge q$. Hence, by the exact verification clause for $\wedge$, there are $u, u' \in S$ with $u \sqcup u' = t'$, $u \vdash p$, and $u' \vdash q$. Then, since $u \leq t' \leq t$, we have $u \in P$. Since $u \vdash p$, by (b), we obtain that there is $s' \in P$ such that $s' \sqcup t \in P$ and $s' \vdash r$. Therefore, $s \vdash K_{p\wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|p| = |\neg\neg p|$.

Weak Simplification: $K_{p\wedge q}(p \vee q)$
Let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p \wedge q$. This implies, by the exact verification clause for $\vee$, that $t' \vdash p \vee q$. Since $t' \sqcup t = t \in P$, we obtain that $s \vdash K_{p\wedge q}(p \vee q)$.

Weak Omniscience: $K_p(p \vee \neg p)$
Let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p$. This implies, by the exact verification clause for $\vee$, that $t' \vdash p \vee \neg p$. Since $t' \sqcup t = t \in P$, we obtain that $s \vdash K_p(p \vee \neg p)$.

Aprioriity: $Ap \Vdash K_\varphi p$.
Suppose $s \vdash Ap$. This means that for all $t \in P$ there is $t' \in P$ such that $t' \sqcup t \in P$ and $t' \vdash p$. Therefore, $s \vdash K_\varphi p$.

Transitivity: $K_p q, K_q r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_q r$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash p$. Then, by (a), there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash q$. Then, since $t \sqcup u \in P$, $u \leq t \sqcup u$, and $u \vdash q$, by (b), we obtain that there is $u' \in P$ such that $(t \sqcup u) \sqcup u' \in P$ and $u' \vdash r$. As $(t \sqcup u') \leq (t \sqcup u) \sqcup u'$ and P is downward closed, we have $t \sqcup u' \in P$. Therefore, we have established that $s \vdash K_p r$.

Monotonicity: $K_p r \Vdash K_{p\wedge q} r$
Same as the proof for Cautious Monotonicity.

Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Suppose $s \vdash K_\varphi p$ and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash \varphi$. Then, by the assumption that $s \vdash K_\varphi p$, there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash p$. This means that $u \dashv \neg p$, and that $u \dashv \neg p \wedge q$. Therefore, $u \vdash \neg(\neg p \wedge q)$. As $t \sqcup u \in P$ as well, we conclude that $s \vdash K_\varphi \neg(\neg p \wedge q)$.

Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$

Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash \varphi$. Then, by (a), there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash p$. Since $t' \leq t \leq t \sqcup u$, we obtain by (b) that there is $u' \in P$ such that $(t \sqcup u) \sqcup u' \in P$ and $u' \vdash q$. Then, $u \sqcup u' \vdash p \wedge q$. As $t \sqcup (u \sqcup u') = (t \sqcup u) \sqcup u' \in P$ and $u \sqcup u' \in P$ (since P is downward closed), we obtain that $s \vdash K_\varphi(p \wedge q)$.

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash p \Rightarrow q$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash \varphi$. Then, by (a), there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash p$. Since $u \leq t \sqcup u \in P$, we obtain by (b) that there is $u' \in P$ such that $(t \sqcup u) \sqcup u' \in P$ and $u' \vdash q$. Since $u' \leq t \sqcup u' \leq (t \sqcup u) \sqcup u' \in P$, we have $t \sqcup u' \in P$ and $u' \in P$. As $u' \vdash q$ as well, we conclude that $s \vdash K_\varphi q$.

Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Suppose (a) $s \vdash K_\varphi \neg p$ and (b) $s \vdash K_\varphi(p \vee q)$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \vdash \varphi$. Then, by (b), there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash p \vee q$. Since $t' \leq t \sqcup u \in P$, we have by (a) that there is $u' \in P$ such that $(t \sqcup u) \sqcup u' \in P$ and $u' \vdash \neg p$. Therefore, there is no $s' \leq u$ such that $s' \vdash p$: otherwise $(t \sqcup u) \sqcup u' \notin P$, by Lemma 1. Then, since $u \vdash p \vee q$, we have that $u \vdash q$. Since $t \sqcup u \in P$ as well, we conclude that $s \vdash K_\varphi q$.

(2) Recall the exact verification clause:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if there is $t' \in P$ such that $t' \leq t$ and $t' \dashv \psi$, then there is $u \in P$ such that $t \sqcup u \in P$ and $u \dashv \varphi$)

Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
We prove only the former, the latter follows similarly: let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv p$. Thus, by the exact falsification clause for $\wedge$, we have $t' \dashv p \wedge q$. Since $t \sqcup t' = t \in P$ as well, we conclude that $s \vdash K_{p \wedge q} p$.

Reflexivity: $K_p p$
Let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv p$. Since $t \sqcup t' = t \in P$ as well, we have that $s \vdash K_p p$.

Cautious Aggloremation: $K_p q \Vdash K_p(p \wedge q)$
Suppose that $s \vdash K_p q$ and $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv p \wedge q$. This means, by the exact falsification clause for $\wedge$, that either (1) $t' \dashv p$ or (2) $t' \dashv q$ or (3) there are $u, u' \in S$ such that $t' = u \sqcup u'$, $u \dashv p$ and $u' \dashv q$. If (1) is the case, since $t \sqcup t' = t \in P$, we obtain that $s \vdash K_p(p \wedge q)$. If (2) is the case, then, by the assumption that $s \vdash K_p q$, there is $u'' \in P$ such that $t \sqcup u'' \in P$ and $u'' \dashv p$. Therefore, $s \vdash K_p(p \wedge q)$. And, finally, assume that (3) is the case, that is, $t' = u \sqcup u'$ for some $u, u' \in S$ such that $u \dashv p$ and $u' \dashv q$. Then, as $u \leq t' \leq t$, we have $t \sqcup u = t \in P$ and $u \in P$ (since P is downward closed). As $u \dashv p$, we have that $s \vdash K_p(p \wedge q)$.

Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p \wedge q} r$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv r$. Then, by (b), there is $u \in P$ such that $t \sqcup u \in P$ such that $u \dashv p \wedge q$. This means that, either (1) $u \dashv p$, or (2) $u \dashv q$, or (3) there are $u', s' \in S$ such that

$u = u' \sqcup s'$, $u' \dashv p$ and $s' \dashv q$. If (1) is the case, we are done. If (2) is the case: since $u \leq u \sqcup t$ we obtain by (a) that there is a $t'' \in P$ such that $(u \sqcup t) \sqcup t'' \in P$ and $t'' \dashv p$. Since $t \sqcup t'' \leq (u \sqcup t) \sqcup t'' \in P$, we have $t \sqcup t'' \in P$. If (3) is the case: since $u' \leq u$, we have $u' \leq t \sqcup u' \leq t \sqcup u \in P$. We therefore have $t \sqcup u' \in P$, $u' \in P$, and $u' \dashv p$. Hence, we can conclude that $s \vdash K_p r$.

Cautious Monotonicity: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_p r$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv r$. Then, by (b), there is $u \in P$ such that $t \sqcup u \in P$ and $u \dashv p$. Then, by the exact falsification clause of $\wedge$, we have $u \dashv p \wedge q$. This implies that $s \vdash K_{p \wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|\neg p| = |\neg\neg\neg p|$.

Weak Simplification: $K_{p \wedge q}(p \vee q)$
Let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv p \vee q$. Thus, by the exact falsification clause for $\vee$, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \dashv p$, and $u' \dashv q$. This implies that $t' \dashv p \wedge q$. Since $t \sqcup t' = t \in P$, we have $s \vdash K_{p \wedge q}(p \vee q)$.

Weak Omniscience: $K_p(p \vee \neg p)$
Since no possible state exactly falsifies $p \vee \neg p$, weak omniscience is vacuously valid.

Apriority: $Ap \Vdash K_\varphi p$
Suppose $s \vdash Ap$. This means that for all $t \in P$ there is $t' \in P$ such that $t' \sqcup t \in P$ and $t' \vdash p$. This implies that there is no $t \in P$ such that $t \dashv p$, thus, $s \vdash K_\varphi p$ is vacuously the case.

Transitivity: $K_p q, K_q r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_q r$, and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv r$. Then, by (b), there is $u \in P$ such that $t \sqcup u \in P$ such that $u \dashv q$. Then, as $u \leq t \sqcup u \in P$, by (a), we have that there is $u' \in P$ such that $(t \sqcup u) \sqcup u' \in P$ such that $u' \dashv p$. Since P is downward closed and $t \sqcup u' \leq (t \sqcup u) \sqcup u' \in P$, we have $t \sqcup u' \in P$. Therefore, $s \vdash K_p r$.

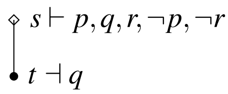
Monotonicity: $K_p r \Vdash K_{p \wedge q} r$
Same as the proof for Cautious Monotonicity.

Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Suppose $s \vdash K_\varphi p$ and let $t \in P$ such that there is $t' \in P$ with $t' \leq t$ and $t' \dashv \neg(\neg p \wedge q)$. The latter means, by the exact falsification of $\neg$, that $t' \vdash \neg p \wedge q$. Thus, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash \neg p$, and $u' \vdash q$. Since $u \leq t' \leq t$ and P is downward closed, we have $u \in P$. Moreover, as $u \dashv p$, we obtain by the first assumption that there is $s' \in P$ such that $t \sqcup s' \in P$ and $s' \dashv \varphi$. We then conclude that $s \vdash K_\varphi \neg(\neg p \wedge q)$.

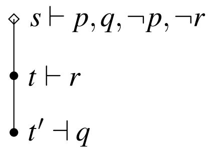
Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$, and let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \dashv p \wedge q$. Therefore, either $t' \dashv p$, or $t' \dashv q$, or there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \dashv p$, and $u' \dashv q$. If $t' \dashv p$, then by (a) there is $s' \in P$ such that

$t \sqcup s' \in P$ and $s' \dashv \varphi$. If $t' \dashv q$, then by (b) there is $s' \in P$ such that $t \sqcup s' \in P$ and $s' \dashv \varphi$. If there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \dashv p$, and $u' \dashv q$, we obtain the same results by (a) and (b) since $u, u' \leq t' \leq t$ and, thus, $u, u' \in P$. Therefore, $s \vdash K_\varphi(p \wedge q)$.

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Counter-example: Since none of the possible states exactly falsifies or verifies p, we have $K_r p$ and $p \Rightarrow q$ exactly verified at every state. However, $t \in P$, $t \dashv q$ and $t \leq t$ but there is no state $t' \in P$ such that $t \sqcup t' \in P$ and $t' \dashv r$. Therefore, no state exactly verifies $K_r q$.



Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Counter-example: Since none of the possible states exactly falsifies $\neg p$ or $p \vee q$, we have $K_r \neg p$ and $K_r(p \vee q)$ exactly verified at every state. Moreover, $t' \in P$, $t' \dashv q$ and $t' \leq t'$ but there is no state $s' \in P$ such that $s' \sqcup t' \in P$ and $s' \dashv r$. Therefore, no state exactly verifies $K_r q$.



(3) Recall the exact verification clause:

- $s \vdash K_\varphi \psi$ iff for all $u \in S$ (if $u \vdash \varphi$ then there is $u' \in S$ such that $u' \leq u$ and $u' \vdash \psi$)

Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
Let $u \in S$ such that $u \vdash p \wedge q$. This means that there are $t, t' \in S$ with $t \sqcup t' = u$, $t \vdash p$, and $t' \vdash q$. Since $t, t' \leq u$, we conclude that $s \vdash K_{p \wedge q} p$ and $s \vdash K_{p \wedge q} q$.

Reflexivity: $K_p p$
Follows from the fact that for all $u \in S$, $u \leq u$.

Cautious Aggloremation: $K_p q \Vdash K_p(p \wedge q)$
Suppose that $s \vdash K_p q$ and let $u \in S$ such that $u \vdash p$. Then, by the assumption, there is $u' \in S$ such that $u' \leq u$ and $u' \vdash q$. Observe also that $u \sqcup u' = u$. Then, by the exact verification clause of $\wedge$, we obtain that $u \vdash p \wedge q$. Since $u \leq u$, we can conclude that $s \vdash K_p(p \wedge q)$.

Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p \wedge q} r$, and let $u \in S$ such that $u \vdash p$. Then, by (a), there is $u' \in S$ such that $u' \leq u$ and $u' \vdash q$. This implies, by the exact verification clause of $\wedge$, that $u \vdash p \wedge q$. Then, by (b), we conclude that there is $s' \in S$ such that $s' \leq u$ and $s' \vdash r$. Therefore, $s \vdash K_p r$.

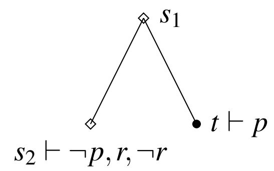
Cautious Monotonicity: $K_p q, K_p r \Vdash K_{p\wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $K_p r$, and let $u \in S$ such that $u \vdash p \wedge q$. This means that there are $t, t' \in S$ such that $u = t \sqcup t'$, $t \vdash p$, and $t' \vdash q$. Then, by (b), there is $u' \leq t$ such that $u' \vdash r$. As $u' \leq t \leq u$, we conclude that $s \vdash K_{p\wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|p| = |\neg\neg p|$.

Weak Simplification: $K_{p\wedge q}(p \vee q)$
Let $u \in S$ such that $u \vdash p \wedge q$. This implies, by the exact verification clause of $\vee$, that $u \vdash p \vee q$. Since $u \leq u$, we obtain that $s \vdash K_{p\wedge q}(p \vee q)$.

Weak Omniscience: $K_p(p \vee \neg p)$
Let $u \in S$ such that $u \vdash p$. This means, by the exact verification clause of $\vee$, that $u \vdash p \vee \neg p$. Since $u \leq u$, we obtain that $s \vdash K_p(p \vee \neg p)$.

Apriority: $Ap \Vdash K_\varphi p$
Counter-example: Ap is exactly verified everywhere in the model since the only possible state is t and $t \vdash p$. However, although $s_2 \vdash r$ there is no $u \leq s_2$ such that $u \vdash p$. Therefore, no state exactly verifies $K_r p$.



Transitivity: $K_p q, K_q r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_q r$, and let $u \in S$ such that $u \vdash p$. Then, by (a), there is $u' \in S$ such that $u' \leq u$ and $u' \vdash q$. This implies, by (b), that there is $t \in S$ such that $t \leq u'$ and $t \vdash r$. Since $t \leq u' \leq u$, we conclude that $s \vdash K_p r$.

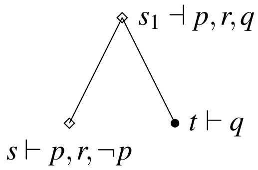
Monotonicity: $K_p r \Vdash K_{p\wedge q} r$
Same as the proof for Cautious Monotonicity.

Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Suppose $s \vdash K_\varphi p$ and let $u \in S$ such that $u \vdash \varphi$. Then, by the assumption, there is $u' \in S$ such that $u' \leq u$ and $u' \vdash p$. Then, following the exact verification and falsification clauses for $\neg$ and $\wedge$, we obtain that $u' \vdash \neg(\neg p \wedge q)$. Therefore, we obtain that $s \vdash K_\varphi \neg(\neg p \wedge q)$.

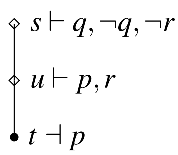
Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$, and let $u \in S$ such that $u \vdash \varphi$. Then, by (a), there is $t \in S$ such that $t \leq u$ and $t \vdash p$. And, by (b), there is $t' \in S$ such that $t' \leq u$ and $t' \vdash q$. Therefore, $t \sqcup t' \vdash p \wedge q$. Moreover, $t \sqcup t' \leq u$ since both $t \leq u$ and $t' \leq u$. Therefore, $s \vdash K_\varphi(p \wedge q)$.

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Counter-example: There is no possible state that verifies p, thus, $p \Rightarrow q$ is vacuously exactly verified by every state. Moreover, s is the only state that exactly

verifies r, $s \vdash p$ and $s \leq s$. Therefore, $K_r p$ is also exactly verified by every state. However, there is no state u such that $u \leq s$ and $u \vdash q$. Hence, $K_r q$ is not exactly verified.



Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Counter-example: u is the only state that exactly verifies r. We then have that $t \leq u$ and $t \vdash \neg p$, and $u \leq u$ and $u \vdash p \vee q$ (since $u \vdash p$). Therefore, both $K_r \neg p$ and $K_r(p \vee q)$ are exactly verified by every state in the model. However, there is no state u' such that $u' \leq u$ and $u' \vdash q$. Therefore, $K_r q$ is not exactly verified in this model.



(4) Recall the exact verification clause:

- $s \vdash K_\varphi \psi$ iff for all $u \in S$ (if $u \dashv \psi$ then there is $u' \in S$ such that $u' \leq u$ and $u' \dashv \varphi$)

Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
We prove only the former, the latter follows similarly: let $u \in S$ such that $u \dashv p$. This implies, by the exact falsification clause of $\wedge$, that $u \dashv p \wedge q$. Since $u \leq u$, we conclude that $s \vdash K_{p \wedge q} p$.

Reflexivity: $K_p p$
Follows from the fact that for all $u \in S$, $u \leq u$.

Cautious Agglorемation: $K_p q \Vdash K_p(p \wedge q)$
Suppose that $s \vdash K_p q$ and let $u \in S$ such that $u \dashv p \wedge q$. Then, by the exact falsification clause of $\wedge$, we have that either (a) $u \dashv p$, or $u \dashv q$, or (c) there is $t, t' \in S$ such that $u = t \sqcup t'$, $t \dashv p$, and $t' \dashv q$. If (a) is the case, we obtain the result since $u \leq u$. If (b) is the case, then by the first assumption, there is $u' \in S$ such that $u' \leq u$ and $u' \dashv p$. If (c) is the case, we again obtain the result since $t \leq u$.

Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p \wedge q} r$, and let $u \in S$ such that $u \dashv r$. Then, by (b), there is $u' \in S$ such that $u' \leq u$ and $u' \dashv p \wedge q$. This means that either (1) $u' \dashv p$, or (2) $u' \dashv q$, or (3) there are $t, t' \in S$ such that $t \sqcup t' = u'$, $t \dashv p$ and $t' \dashv q$. If (1) is the case, since $u' \leq u$, we have the desired result. If (2) is the case, by (a), there is $s' \in S$ such that $s' \leq u'$ and $s' \dashv p$. As $s' \leq u' \leq u$ and $\leq$ is transitive, we have

that $s' \leq u'$. If (3) is the case: since $t \leq u' \leq u$, we have $t \leq u$. Then, as $t \dashv p$, we obtain that the desired conclusion. Therefore, $s \vdash K_p r$.

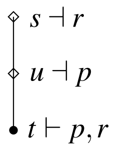
Cautious Monotonicity: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $K_p r$, and let $u \in S$ such that $u \dashv r$. Then, by (b), there is $u' \in S$ such that $u' \leq u$ and $u' \dashv p$. Then, by the exact falsification clause for $\wedge$, we have $u' \dashv p \wedge q$. Therefore, $s \vdash K_{p \wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|\neg p| = |\neg\neg\neg p|$.

Weak Simplification: $K_{p \wedge q}(p \vee q)$
Let $u \in S$ such that $u \dashv p \vee q$. This means that there are $t, t' \in S$ such that $t \sqcup t' = u$, $t \dashv p$, and $t' \dashv q$. This implies that $t \dashv p \wedge q$. Since $t \leq u$, we obtain that $s \vdash K_{p \wedge q}(p \vee q)$.

Weak Omniscience: $K_p(p \vee \neg p)$
Let $u \in S$ such that $u \dashv p \vee \neg p$. This means that there are $t, t' \in S$ such that $t \sqcup t' = u$, $t \dashv p$, and $t' \dashv \neg p$. Since $t \leq u$, we obtain that $s \vdash K_p(p \vee \neg p)$.

Apriority: $Ap \Vdash K_\varphi p$
Counter-example: Ap is exactly verified everywhere in the model since the only possible state is t and $t \vdash p$. However, although $u \dashv p$ there is no $u' \leq u$ such that $u' \dashv r$. Therefore, no state exactly verifies $K_r p$.



Transitivity: $K_p q, K_q r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_q r$, and let $u \in S$ such that $u \dashv r$. Then, by (b), there is $u' \in S$ such that $u' \leq u$ and $u' \dashv q$. This implies, by (a), that there is $t \in S$ such that $t \leq u'$ and $t \dashv p$. Since $t \leq u' \leq u$, we conclude that $s \vdash K_p r$.

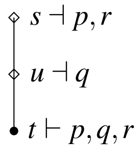
Monotonicity: $K_p r \Vdash K_{p \wedge q} r$
Same as the proof for Cautious Monotonicity.

Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Suppose $s \vdash K_\varphi p$ and let $u \in S$ such that $u \dashv \neg(\neg p \wedge q)$. The latter means that $u \vdash \neg p \wedge q$. Thus, there are $t, t' \in S$ such that $t \sqcup t' = u$, $t \vdash \neg p$, and $t' \vdash q$. Therefore, $t \dashv p$. Then, by the first assumption, there is $t'' \leq t$ such that $t'' \dashv \varphi$. Since $t'' \leq t \leq u$, we conclude that $s \vdash K_\varphi \neg(\neg p \wedge q)$.

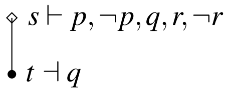
Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$, and let $u \in S$ such that $u \dashv p \wedge q$. Therefore, either (1) $u \dashv p$, or (2) $u \dashv q$, or (3) there are $t, t' \in S$ such that $t \sqcup t' = u$, $t \dashv p$, and $t' \dashv q$. If (1) is the case, then, by (a), there is $u' \in S$ such that $u' \leq u$ and $u' \dashv \varphi$. If (2) is the case, then by (b), there is $u' \in S$ such that $u' \leq u$ and $u' \dashv \varphi$. If (3)

is the case, we obtain the same results by (a) and (b) since $t, t' \leq u$. Therefore, $s \vdash K_\varphi(p \wedge q)$.

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Counter-example: Since the only possible state is t and $t \vdash p$ and $t \vdash q$, we have that $p \Rightarrow q$ is exactly verified by every state in the model. The only state that exactly falsifies p is s, and we moreover have that $s \leq s$ and $s \dashv r$. Therefore, $K_r p$ is also exactly verified by every state of the model. However, $u \dashv q$ but there is no u' such that $u' \leq u$ and $u' \dashv r$. Therefore, $K_r q$ is not exactly verified.



Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Counter-example: The only state that exactly falsifies $\neg p$ and $p \vee q$ is s. It moreover exactly falsifies r and $s \leq s$. Therefore, both $K_r \neg p$ and $K_r(p \vee q)$ are exactly verified by every state in the model. However, t is the only state that exactly falsifies q and there is no u' such that $u' \leq t$ with $u' \dashv r$. Therefore, $K_r q$ is not exactly verified.



$\square$

C Proof of Proposition 3

Let $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ be a model and $s \in S$ be a state.

(5) Recall the exact verification clause:

$s \vdash K_\varphi \psi$ iff (1) for all $u \in S$ (if $u \vdash \varphi$ then there is $u' \in S$ such that $u' \leq u$ and $u' \vdash \psi$), and
(2) for all $u \in S$ (if $u \dashv \psi$ then $u \dashv \varphi$), and
(3) for all $u \in S$ (if $u \vdash \psi$ then there is $u' \in S$ such that $u \leq u'$ and $u' \vdash \varphi$).

Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
We prove only the former, the latter follows similarly. Item (1) is proved for the third definition of knowledge. For (2), let $u \in S$ such that $u \dashv p$. This implies, by the falsification clause for $\wedge$, that $u \dashv p \wedge q$. For (3) let $u \in S$ such that $u \vdash p$. Since $q^+ \neq \emptyset$, there is $u' \in S$ such that $u' \vdash q$. Therefore, $u \sqcup u' \vdash p \wedge q$. Moreover, $u \sqcup u' \geq u$. Hence, we have (3).

Reflexivity: $K_p p$
Item (1) is proven for the third definition of knowledge. Item (2) is vacuously true. Item (3) follows from the fact that $u \leq u$ for all $u \in S$.

Cautious Aggloremation: $K_p q \Vdash K_p(p \wedge q)$
Suppose that $s \vdash K_p q$ and show $s \vdash K_p(p \wedge q)$. Item (1) is proven for the third definition of knowledge. For (2): let $u \in S$ such that $u \dashv p \wedge q$. This implies that either (a) $u \dashv p$, or (b) $u \dashv q$, or (c) there are $t, t' \in S$ such that $u = t \sqcup t'$, $t \dashv p$, and $t' \dashv q$. If (a) is the case, we are done. If (b) is the case, then $u \dashv p$ since $s \vdash K_p q$. If (c) is the case, then $t' \dashv p$ again by $s \vdash K_p q$. Then, since p^- is closed under fusion, $t \dashv p$, and $t \sqcup t' = u$, we have that $u \in p^-$. I.e., $u \dashv p$. For (3): let $u \in S$ such that $u \vdash p \wedge q$. This means, by the exact verification clause of $\wedge$, that there are $t, t' \in S$ such that $u = t \sqcup t'$, $t \vdash p$, and $t' \vdash q$. Since $s \vdash K_p q$, the fact that $t' \vdash q$ implies that there is $u' \in S$ such that $t' \leq u'$ and $u' \vdash p$. Since p^+ is closed under fusion, we have that $u' \sqcup t \in p^+$, i.e., $u' \sqcup t \vdash p$. Since $u = t' \sqcup t$ and $t' \leq u'$, we also have that $u \leq u' \sqcup t$. We therefore obtain the result.

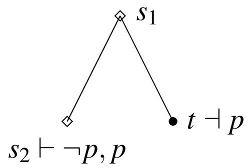
Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$:
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p \wedge q} r$. Item (1) is proven for the third definition of knowledge. For (2): let $u \in S$ such that $u \dashv r$. Then, by (b), we have that $u \dashv p \wedge q$. Then, either (a*) $u \dashv p$, or (b*) $u \dashv q$, or (c*) there are u_1, u_2 such that $u = u_1 \sqcup u_2$, $u_1 \dashv p$, and $u_2 \dashv q$. If (a*) is the case, we are done. If (b*) is the case, by (a), we have that $u \dashv p$. If (c*) is the case, by (a), we have that $u_2 \dashv p$. Then, since p^- is closed under fusion, we obtain that $u = u_1 \sqcup u_2 \dashv p$. For (3): let $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \geq t$ such that $t' \vdash p \wedge q$. This means that there are $u, u' \in S$ such that $t' = u \sqcup u'$, $u \vdash p$, and $u' \vdash q$. Then, by (a), there is $s' \in S$ such that $s' \geq u'$ and $s' \vdash p$. Observe that, since $t' = u \sqcup u'$ and $s' \geq u'$, we have that $u \sqcup s' \geq t' \geq t$. And, since p^+ is closed under fusion, $u \sqcup s' \vdash p$. Therefore, $s \vdash K_p r$.

Cautious Monotonicity: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $K_p r$. Item (1) is proven for the third definition of knowledge. To prove (2) let $u \in S$ such that $u \dashv r$. Then, by (b), $u \dashv p$. This immediately implies that $u \dashv p \wedge q$. To prove (3), suppose $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \in S$ such that $t' \geq t$ and $t' \vdash p$. We know that $q^+ \neq \emptyset$ so there is $s' \in S$ such that $s' \vdash q$. This means that $t' \sqcup s' \vdash p \wedge q$. Since $t' \geq t$, we have $t' \sqcup s' \geq t$, therefore, we obtain (3).

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity.

Weak Simplification: $K_{p \wedge q}(p \vee q)$
Item (1) is proved for the third definition of knowledge. To prove (2) let $u \in S$ such that $u \dashv p \vee q$. This implies, by the exact falsification clauses of $\vee$ and $\wedge$, that $u \dashv p \wedge q$. To prove (3), suppose $t \in S$ such that $t \vdash p \vee q$. Then, either (a) $t \vdash p$, or (b) $t \vdash q$, or (c) there are $u, u' \in S$ such that $t = u \sqcup u'$, $u \vdash p$, and $u' \vdash q$. If (a) is the case: we know that $q^+ \neq \emptyset$ so there is $s' \in S$ such that $s' \vdash q$. This implies that $t \sqcup s' \vdash p \wedge q$. Since $t \sqcup s' \geq t$, we obtain the desired result. If (b) is the case: we know that $p^+ \neq \emptyset$ so there is $s' \in S$ such that $s' \vdash p$. This means that $t \sqcup s' \vdash p \wedge q$. Since $t \sqcup s' \geq t$, we obtain the desired result. If (c) is the case, we have that $t \vdash p \wedge q$. As $t \geq t$, we obtain (3).

Weak Omniscience: $K_p(p \vee \neg p)$
Counter-example: For $t \vdash \neg p$, we have $t \vdash p \vee \neg p$. However, there is no $t' \in S$ such that $t' \geq t$ and $t' \vdash p$. This violates item (3) in the above exact verification clause. Therefore, no state in the model exactly verifies $K_p(p \vee \neg p)$.

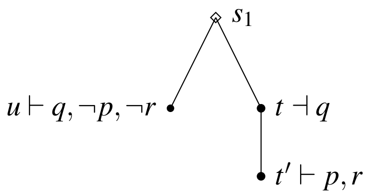


Apriority: $Ap \Vdash K_\varphi p$:
See the counterexample in the proof of Proposition 2 for the third definition of knowledge. The same counterexample violates item (1).

Transitivity: $K_p q, K_q r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_q r$. Item (1) is proved for the third definition of knowledge. To prove (2) let $u \in S$ such that $u \dashv r$. Then, by (b), we have that $u \dashv q$. Then, similarly by (a), we have that $u \dashv p$. To prove (3), let $u \in S$ such that $u \vdash r$. Then, by (b), there is $u' \in S$ such that $u \leq u'$ and $u' \vdash q$. Then, similarly by (a), there is $u'' \in S$ such that $u' \leq u''$ and $u'' \vdash p$. Since $\leq$ is transitive, we also have $u \leq u''$. We therefore conclude $s \vdash K_p r$.

Monotonicity: $K_p r \Vdash K_{p \wedge q} r$
Same as the proof for Cautious Monotonicity.

Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Counter-example: $K_r p$ is exactly verified at every state of the model. However, for example, $t \vdash \neg(\neg p \wedge q)$ but there is no $u \geq t$ such that $u \vdash r$. This violates item (3) in the above exact verification clause. Thus, no state of the model exactly verifies $K_\varphi \neg(\neg p \wedge q)$.



Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$. Item (1) is proven for the third definition of knowledge. To prove item (2), let $t \in S$ such that $t \dashv p \wedge q$. Then, either (a*) $t \dashv p$, or (b*) $t \dashv q$, or (c*) there are u_1, u_2 such that $t = u_1 \sqcup u_2$, $u_1 \dashv p$, and $u_2 \dashv q$. If (a*) is the case, by (a), we have $t \dashv \varphi$. If (b*) is the case, by (b), we have that $t \dashv \varphi$. If (c*) is the case, by (a) and (b), we have that $u_1 \dashv \varphi$ and $u_2 \dashv \varphi$. Then, by Lemma 4, we obtain that $t = u_1 \sqcup u_2 \dashv \varphi$. To prove (3) suppose $t \in S$ such that $t \vdash p \wedge q$. Then, there are $u, u' \in S$ such that $t = u \sqcup u'$, $u \vdash p$, and $u' \vdash q$. Then, by (a) and (b), there are $s', t' \in S$ such that $u \leq s'$ and $s' \vdash \varphi$, and $u' \leq t'$ such that

$t' \vdash \varphi$. Then, by Lemma 4, $s' \sqcup t' \vdash \varphi$. As $t = u \sqcup u'$, $u \leq s'$, and $u' \leq t'$, we also have $t \leq s' \sqcup t'$, which proves item (3).

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
See the counterexample in the proof of Proposition 2 for the third definition of knowledge. The same counterexample violates item (1).

Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi (p \vee q) \Vdash K_\varphi q$
See the counterexample in the proof of Proposition 2 for the third definition of knowledge. The same counterexample violates item (1).

(6) Recall the exact verification clause:

$s \vdash K_\varphi \psi$ iff (1) for all $t \in P$ (if there is $t' \in P$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in P$ such that $t \sqcup u \in P$ and $u \vdash \psi$), and
(2) for all $u \in S$ (if $u \vdash \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$), and
(3) for all $u \in S$ (if $u \dashv \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$)

Simplification: $K_{p\wedge q} p, K_{p\wedge q} q$
Item (1) is proven for the first definition of knowledge. For (2) let $t \in S$ such that $t \vdash p$. Since $q^+ \neq \emptyset$, there is $s' \in S$ such that $s' \vdash q$. Then, $t \sqcup s' \vdash p \wedge q$. This implies that $t \sqcup s' \vdash (p \wedge q) \vee \neg(p \wedge q)$. As $t \sqcup s' \geq t$, we obtain (2). Item (3) follows easily since $t \dashv p$ implies that $t \dashv p \wedge q$, i.e., $t \vdash \neg(p \wedge q)$. Therefore, $t \vdash (p \wedge q) \vee \neg(p \wedge q)$. The result then follows since $t \leq t$.

Reflexivity: $K_p p$
Item (1) is proven for the first definition of knowledge. (2) and (3) follows from the facts that for all $t \in S$: $t \leq t$, and if $t \vdash p$ or $t \vdash \neg p$, we have $t \vdash p \vee \neg p$.

Cautious Aggloremation: $K_p q \Vdash K_p (p \wedge q)$
Suppose that $s \vdash K_p q$. Item (1) is proven for the first definition of knowledge. To prove (2), let $u \in S$ such that $u \vdash p \wedge q$. This means that there are $t, t' \in S$ such that $u = t \sqcup t'$, $t \vdash p$ and $t' \vdash q$. Therefore, $t \vdash p \vee \neg p$. Moreover, $t' \vdash q$ implies by the first assumption that there is $u' \in S$ such that $t' \leq u'$ and $u' \vdash p \vee \neg p$. As $u = t \sqcup t'$ and $t' \leq u'$, we have that $u \leq t \sqcup u'$. Moreover, by Lemma 4, we have that $t \sqcup u' \vdash p \vee \neg p$. Thus, we proved (2). To prove (3), let $u \in S$ such that $u \dashv p \wedge q$. We then have the following cases:

If $u \dashv p$, we have that $u \vdash \neg p$ and thus $u \vdash p \vee \neg p$. Since $u \leq u$, we obtain the result.

If $u \dashv q$, then, by the first assumption, there is $u' \in S$ such that $u \leq u'$ and $u' \vdash p \vee \neg p$.

If there are $t, t' \in S$ such that $u = t \sqcup t'$, $t \dashv p$ and $t' \dashv q$, we obtain that $t \vdash p \vee \neg p$. Moreover, by $t' \dashv q$ and the first assumption, there is $u' \in S$ such that $t' \leq u'$ and $u' \vdash p \vee \neg p$. As $u = t \sqcup t'$ and $t' \leq u'$, we have that $u \leq t \sqcup u'$. Moreover, by Lemma 4, we have that $t \sqcup u' \vdash p \vee \neg p$. Thus, we proved (3).

Cautious Transitivity: $K_p q, K_{p\wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p\wedge q} r$. Item (1) is proven for the first definition

of knowledge. For (2): let $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \in S$ such that $t' \geq t$ and $t' \vdash (p \wedge q) \vee \neg(p \wedge q)$. We then have three cases:

Case $t' \vdash p \wedge q$: This means that there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash p$, and $u' \vdash q$. The former implies that $u \vdash p \vee \neg p$. Moreover, $u' \vdash q$ implies by (a) there is $s' \in S$ such that $s' \geq u'$ and $s' \vdash p \vee \neg p$. Since $s' \geq u'$, we obtain that $s' \sqcup u \geq u \sqcup u' = t' \geq t$. Since $|p \vee \neg p|$ is closed under fusion (by Lemma 4), we also have that $s' \sqcup u \vdash p \vee \neg p$.

Case $t' \vdash \neg(p \wedge q)$: Then, we have three subcases.
If $t' \vdash \neg p$, then $t' \vdash p \vee \neg p$. Since $t' \geq t$, we obtain the desired result.
If $t' \vdash \neg q$, then by (a), there is $s' \geq t'$ such that $s' \vdash p \vee \neg p$. Since $s' \geq t' \geq t$, we obtain the desired result.
If there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash \neg p$, and $u' \vdash \neg q$, it follows similar to the first case.

Case $t' \vdash (p \wedge q) \wedge \neg(p \wedge q)$: Then, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash (p \wedge q)$, and $u' \vdash \neg(p \wedge q)$. By a similar argument as in the first case, there is $s' \in S$ such that $s' \geq u$ and $s' \vdash p \vee \neg p$. Again, by a similar argument as in the second case, there is $s'' \in S$ such that $s'' \geq u'$ and $s'' \vdash p \vee \neg p$. Since $s' \geq u$ and $s'' \geq u'$, we have $s'' \sqcup s' \geq u \sqcup u' = t' \geq t$. Moreover, since $|p \vee \neg p|$ is closed under fusion, we also have that $s'' \sqcup s' \vdash p \vee \neg p$. Therefore, we have proven (2).

Item (3) follows similarly.

Cautious Monotonicity: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_p r$. Item (1) is proven for the first definition of knowledge. For (2): let $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \geq t$ such that $t' \vdash p \vee \neg p$. We then have three cases:

Case $t' \vdash p$: Then, since $q^+ \neq \emptyset$, there is $s' \in S$ such that $s' \vdash q$. Therefore, $t' \sqcup s' \vdash p \wedge q$. This implies that $t' \sqcup s' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t' \sqcup s' \geq t' \geq t$, we obtain the desired result.

Case $t' \vdash \neg p$: This implies that $t' \dashv p \wedge q$, i.e., that $t' \vdash \neg(p \wedge q)$. Therefore, $t' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t' \geq t$, we obtain the desired result.

Case $t' \vdash p \wedge \neg p$: this means that there are $u, u' \in S$ such that $t' = u \sqcup u'$, $u \vdash p$, and $u' \vdash \neg p$. Similar to the first case, there is $s'' \in S$ such that $s'' \geq u$ and $s'' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Moreover, similar to the second case, $u' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $s'' \geq u$, we have that $s'' \sqcup u' \geq u \sqcup u' = t' \geq t$. Moreover, since $|(p \wedge q) \vee \neg(p \wedge q)|$ is closed under fusion (by Lemma 4), we have $s'' \sqcup u' \vdash (p \wedge q) \vee \neg(p \wedge q)$. We can then conclude that (2) is the case. Item (3) follows in a similar way.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity.

Weak Simplification: $K_{p \wedge q}(p \vee q)$
Item (1) is proven for the first definition of knowledge. For (2), let $t \in S$ such that $t \vdash p \vee q$. We then have three cases:

Case $t \vdash p$: Then, since $q^+ \neq \emptyset$, there is $s' \in S$ such that $s' \vdash q$. Therefore, $t \sqcup s' \vdash p \wedge q$. This implies that $t \sqcup s' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t \sqcup s' \geq t$, we obtain the desired result.

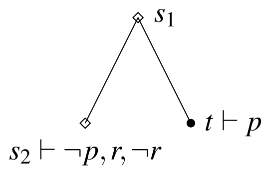
Case $t \vdash q$: Similar to the above case, use $p^+ \neq \emptyset$.

Case $t \vdash p \wedge q$: this implies that $t \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t \geq t$, we conclude that (2) is the case.

For (3), let $t \in S$ such that $t \dashv p \vee q$. This means that there are $u, u' \in S$ such that $t = u \sqcup u'$, $u \dashv p$ and $u' \dashv q$. Therefore $t \dashv p \wedge q$, i.e., $t \vdash \neg(p \wedge q)$. This implies that $t \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t \geq t$, we conclude that (3) is the case.

Weak Omniscience: $K_p(p \vee \neg p)$
Item (1) is proven for the first definition of knowledge. For (2), let $t \in S$ such that $t \vdash p \vee \neg p$. We then obtain (2) since $t \geq t$. For (3), let $t \in S$ such that $t \dashv p \vee \neg p$. This implies that $t \vdash p \wedge \neg p$. Then, by the exact verification clause for $\vee$, we obtain that $t \vdash p \vee \neg p$. Since $t \geq t$, we have (3).

Apriority: $Ap \Vdash K_\varphi p$
Counter-example: Ap is exactly verified everywhere in the model since the only possible state is t and $t \vdash p$. However, $t \vdash p$ but there is no u such that $u \geq t$ and $u \vdash r \vee \neg r$. This violates item (2). Therefore, no state in the model exactly verifies $K_r p$.



Transitivity: $K_p q, K_q r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_q r$. Item (1) is proved for the first definition of knowledge. To prove (2) let $u \in S$ such that $u \vdash r$. Then, by (b), we have that there is $u' \in S$ such that $u \leq u'$ and $u' \vdash q \vee q'$. We then have three cases:

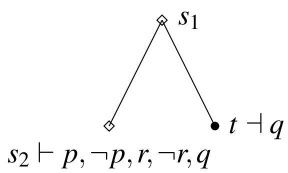
If $u' \vdash q$, then by (a), we obtain that there is $u'' \in S$ such that $u' \leq u''$ and $u'' \vdash p \vee \neg p$. Since $u \leq u' \leq u''$ and $\leq$ is transitive, we obtain the result.

If $u' \vdash \neg q$, then $u' \dashv q$. Then, again by (a), we obtain that there is $u'' \in S$ such that $u' \leq u''$ and $u'' \vdash p \vee \neg p$. Since $u \leq u' \leq u''$ and $\leq$ is transitive, we obtain the result.

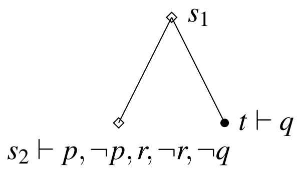
If there are $t, t' \in S$ such that $u' = t \sqcup t'$, $t \vdash q$ and $t' \vdash \neg q$, we have by (a) that there are $u_1, u_2 \in S$ such that $t \leq u_1$ and $u_1 \vdash p \vee \neg p$ and $t' \leq u_2$ and $u_2 \vdash p \vee \neg p$. By Lemma 4, we have that $u_1 \sqcup u_2 \vdash p \vee \neg p$. It is also easy to see that $u \leq u' = t \sqcup t' \leq u_1 \sqcup u_2$, so we obtain the desired result. We can then conclude that (2) is the case. Item (3) follows in a similar way.

Monotonicity: $K_p r \Vdash K_{p \wedge q} r$
Same as the proof for Cautious Monotonicity.

Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Counter-example: It is easy to check that $K_r p$ is exactly verified at every state. However, $t \vdash \neg(\neg p \wedge q)$ but there is no u such that $u \geq t$ and $u \vdash r \vee \neg r$. This violates item (2). Therefore, no state in the model exactly verifies $K_r \neg(\neg p \wedge q)$.



Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Counterexample: It is easy to check that $K_r p$ and $p \Rightarrow q$ are exactly verified at every state. However, $t \vdash q$ but there is no u such that $u \geq t$ and $u \vdash r \vee \neg r$. This violates item (2). Therefore, no state in the model exactly verifies $K_r q$.



Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Follows similarly to the proof of Agglomeration for definition (5).

Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Suppose (a) $s \vdash K_\varphi \neg p$ and (b) $s \vdash K_\varphi(p \vee q)$. Item (1) is proven for the first definition of knowledge. For (2): let $t \in S$ such that $t \vdash q$. This implies that $t \vdash p \vee q$. Then, by (b), we conclude that there is $t' \geq t$ such that $t' \vdash \varphi \vee \neg\varphi$. For (3): let $t \in S$ such that $t \dashv q$. Since $p^- \neq \emptyset$, there is $s' \in S$ such that $s' \dashv p$. This means that $t \sqcup s' \dashv p \vee q$. Then, by (b), there is $u' \geq t \sqcup s'$ such that $u' \vdash \varphi \vee \neg\varphi$. Since $t \leq t \sqcup s' \leq u'$, we conclude that (3) is the case.

□