A Note on the Logic of (Higher-Order) Vagueness

RICHARD G. HECK JR.

1. The Problem

The vagueness of a predicate '$F\xi$' consists in there being no sharp distinction between the objects which satisfy it and those which do not. Hence, it ought to be possible for there to be objects *a* and *b*, where *a* is F, but *b* is not F, and a (doubly well-ordered) series of objects 'connecting' *a* to *b* such that there is no F-'boundary' between any two adjacent objects in the series. What gives rise to the so-called Sorites Paradox is the thought that, where x' is the next object in such a series after x, the vagueness of '$F\xi$' ought to imply – or even to consist in – the truth of the following:

$$(\alpha) \quad \neg(\exists x)(Fx \,\&\, \neg Fx')$$

That is: There can be no member of the series which is F, the object adjacent to which is not-F. But induction now threatens to produce contradiction: If *a* is F, then the object adjacent to *a*, i.e. a', must be F, lest it be not-F, *contra* (α); and so on. This is the No Sharp Boundaries Paradox.[1]

According to Crispin Wright, and others, the lesson of this paradox is that, 'when dealing with vague expressions, it is essential to have the expressive resources afforded by an operator expressing *definiteness* or *determinacy*' (p. 130).[2] For, employing such an operator, one may deny that (α) is any consequence of the vagueness of '$F\xi$' and so *a fortiori* that the vagueness of '$F\xi$' consists in the truth of (α), insisting instead that what is required is only that no object which is definitely F can be adjacent to one which is definitely not-F:

$$(\alpha') \quad \neg(\exists x)[Def(Fx) \,\&\, Def(\neg Fx')]$$

But it might seem that this only delays the inevitable, for consider the following, the Strengthened No Sharp Boundaries Paradox. If a predicate is *really* vague, one might think, then not only ought the distinction between Fs and not-Fs be fuzzy, the distinction between objects which are definitely F and those which are not definitely F ought itself to be vague (see Dummett [1]). But then, as Wright puts it, 'assent to

$$(\beta) \quad \neg(\exists x)[Def(Fx) \,\&\, \neg Def(Fx')]$$

[1] This term, and this formulation, were first introduced, I believe, in Wright [4].

[2] All page references, in the text, are to Wright [5].

ANALYSIS 53.4, October 1993, pp. 201–208. © Richard G. Heck Jr.

would seem to be compelled even if assent to (α) is not' (p. 130; index altered). And one will then be able to derive, by induction, that, if the first object of our series is definitely F, the last object too must be definitely F.

Now, it is natural to suppose that the operator '*Def*' can also be used to defuse this paradox. If (α'), rather than (α), expresses (or is a consequence of) the vagueness of 'Fξ', then

$$(\beta') \quad \neg(\exists x)[Def(Def(\mathrm{F}x)) \mathbin{\&} Def(\neg Def(\mathrm{F}x'))]$$

rather than (β) should express (or be a consequence of) the vagueness of '*Def*(Fξ)'. A general, and promising, strategy is now obviously available to respond to further strengthenings of the No Sharp Boundaries Paradox, and Wright's explanation of it is not to be improved upon (see pp. 130–31).

The focus of Wright's paper is a formal difficulty which confronts this strategy. Consider the following rule of inference:

$$(\mathrm{DEF}) \quad \frac{Def(\mathrm{A}_1), \ldots, Def(\mathrm{A}_n) \vdash \mathrm{P}}{Def(\mathrm{A}_1), \ldots, Def(\mathrm{A}_n) \vdash Def(\mathrm{P})}$$

The rule DEF reads as follows: If a formula P follows from certain premisses, all of which are 'definitized', then ⌜*Def*(P)⌝ follows from those same premisses. But however reasonable DEF appears, Wright claims that it will enable us to derive

$$(\gamma) \quad Def(\neg Def(\mathrm{F}x')) \rightarrow Def(\neg Def(\mathrm{F}x))$$

from the definitization of (β') – the definitization of (β') being no less plausible than (β') itself. And (γ) is as much of a problem as is (α) or (β) (p. 131).

The majority of Wright's paper consists of criticisms of various attempts to resolve this problem. He concludes that 'the case – which *must*, of course, be flawed! – for thinking that higher-order vagueness is *per se* paradoxical is so far unanswered'. I intend here to answer it.

2. *The Problem with the Problem*

The difficulty, I am going to suggest, is with Wright's derivation of (γ) from the definitization of (β'). The proof uses the rule DEF mentioned above, as well as the following analogue of the modal law T:

$$\text{T-}Def \quad Def(\mathrm{A}) \rightarrow \mathrm{A}$$

The proof, as given semi-formally (in a hopefully self-explanatory system), is then as follows:

[1]	(1) $Def\neg(\exists x)[Def(Def(Fx))\ \&\ Def(\neg Def(Fx'))]$	Premiss
[2]	(2) $Def(\neg Def(Fx'))$	Premiss
[3]	(3) $Def(Fx)$	Premiss
[3]	(4) $Def(Def(Fx))$	3 DEF
[2,3]	(5) $(\exists x)[Def(Def(Fx)\ \&\ Def(\neg Def(Fx'))]$	2,4 $\exists$-intro
[1]	(6) $\neg(\exists x)[Def(Def(Fx))\ \&\ Def(\neg Def(Fx'))]$	1 T-*Def*
[1,2]	(7) $\neg Def(Fx)$	3,5,6 $\neg$-intro
[1,2]	(8) $Def(\neg Def(Fx))$	7 DEF
[1]	(9) $Def(\neg Def(Fx')) \rightarrow Def(\neg Def(Fx))$	2,8 $\rightarrow$-intro

Note carefully the use of *reductio ad absurdum* (line 7) and conditional proof (line 9).

That something odd is happening can best be seen by considering the following question: Why does Wright formulate the rule DEF as he does? The relevant alternative here is the following rule:[3]

$$\text{(DEF*)}\ \frac{A_1, \ldots, A_n \vdash P}{A_1, \ldots, A_n \vdash Def(P)}$$

That is: If P follows from certain premisses, so then $\ulcorner Def(P) \urcorner$ follows from those same premisses. Surely the worry is that, using this rule, one will be able to prove

(δ) $P \rightarrow Def(P)$

by conditional proof:

[1]	(1) P	Premiss
[1]	(2) $Def(P)$	1 DEF
[]	(3) $P \rightarrow Def(P)$	1,2 $\rightarrow$-intro

But (δ), together with T-*Def*, implies that '*Def*' is a redundant operator, which is hardly any good.

Something similar is troubling about DEF. Using DEF, we can prove, by a similar conditional proof:

(ε) $Def(P) \rightarrow Def(Def(P))$

Indeed, a glance at Wright's proof of (γ), sketched above, reveals that the third and fourth steps of that proof constitute just such a proof. More precisely, Wright's use of the rule DEF allows him, at that point in the proof, to pass perfectly freely between '*Def*(F*x*)' and '*Def*(*Def*(F*x*))'. But

[3] It is because Wright does not use DEF* that he can not derive anything troubling from (α') and that the problem seems 'distinctively higher-order' (pp. 131–32). Since, as I shall be arguing, DEF* is actually entirely acceptable, the problem here under discussion does not in fact concern higher-order vagueness in any special way.

surely, anyone who takes higher-order vagueness seriously is going to want to deny (ε) and maintain a distinction between '*Def*(P)' and '*Def*(*Def*(P))'.[4]

3. The Solution to the Problem

Why, then, not just abandon DEF? If DEF leads to (ε), and if we have good reason to reject (ε), then do we not have good reason to reject DEF? Of course. But not just DEF, but the stronger DEF*, is a *valid* rule of inference. DEF* is valid just in case, whenever a sentence P follows from certain premisses, $\ulcorner$*Def*(P)$\urcorner$ also follows from those premisses. Suppose, then, that, whenever certain premisses are true, P also is true. Now, as Wright says, 'there is no apparent way whereby a statement could be true without being definitely so' (p. 130): That is, if P is true, $\ulcorner$*Def*(P)$\urcorner$ must also be true. But then, if the premisses are true, P must be true, whence $\ulcorner$*Def*(P)$\urcorner$ must be true, whence DEF* is valid.

Now, one might conclude from this that (δ) must be valid, that '*Def*' must be redundant, and that vagueness must be illusory. But that would be a mistake. DEF* is formulated as a rule of inference precisely to *distinguish* it from (δ). It is, after all, one thing to say that whenever a sentence P is true, so must $\ulcorner$*Def*(P)$\urcorner$ be true, and entirely another to say that the conditional $\ulcorner$P $\rightarrow$ *Def*(P)$\urcorner$ is valid – even though these often come to the same thing, notably in the case of classical logic.

The problem with the proof of (δ) mentioned above is thus not its reliance on DEF*: The problem lies in the *use of the rule* DEF* *within a subordinate deduction*, here within conditional proof.[5] To allow DEF* to be applied within subordinate deductions is to collapse the distinction between the rule DEF*, which is a valid rule, and the invalid conditional (δ). Similarly, the fallacy in Wright's derivation of (γ), by means of DEF, lies not in its reliance on DEF, but in the application of DEF within subordinate deductions, within conditional proof and *reductio ad absurdum*. Of course, merely to impose this restriction on the application of DEF is not sufficient: We must understand, and explain, the proscription against applying it (and similar rules of inference) within subordinate deductions. There is much to be said about this subject, and I will say none of it here: But it should be clear nevertheless that what we must understand, to

[4] In his [1], Dummett makes this point by suggesting that an appropriate logic for vagueness will have to be weaker than S4, i.e., not contain (ε) as a theorem.

[5] A subordinate deduction, as I am using the term here, is one from a premiss which is later discharged. In many formal systems, main and subordinate deductions are not clearly distinguished, as the distinction is not always important. It is, however, possible to respect the distinction formally.

understand the logic of '*Def*', is *precisely* the distinction between the validity of rules of inference – that is, implication – and the validity of the associated conditionals.[6] *That*, one might say, is the lesson of the Strengthened No Sharp Boundaries Paradox.

Thus is Wright's case that higher-order vagueness is *per se* incoherent answered.

4. *Model-theoretic Details*

It is, of course, one thing to show that Wright's derivation of (γ) from the definitization of (β') is invalid, another to show that no derivation is forthcoming. But that too can be done. For ease of exposition, let us replace Wright's '*Def*' with the more familiar '$\Box$'. Let $\mathcal{L}$ be some normal modal logic, formulated as a natural deduction system, and augmented by the following rule of inference, analogous to DEF, which allows one to infer $\ulcorner \Box A \urcorner$ from A:

$$\text{(V)} \quad \frac{A}{\Box A}$$

Rule (V) is to be applied *only* in main, and never in subordinate, deductions. Call this logic V$\mathcal{L}$; call such logics normal V-logics.

We are familiar enough, thanks to Kripke, with models for normal (quantified) modal logics.[7] To give models for normal V-logics, we need only change the definition of truth *in* a model appropriately. Instead of taking truth in a model to be truth at some 'actual world', we take (absolute) truth to be truth in *all* possible worlds; we define validity, and the like, for V-logics accordingly. It is easy to see that, if M is a model for $\mathcal{L}$, then the model VM (which is just like M, except that the definition of truth is altered as above) is a model for V$\mathcal{L}$. Every axiom of $\mathcal{L}$ will plainly be true in every world in the model and truth in a world (and so absolute truth) will be closed under the inference rules of $\mathcal{L}$. Moreover, (absolute) truth is closed under the rule (V). For whenever A is true in a model VM, A is true in every world in VM; so $\ulcorner \Box A \urcorner$ is true at every world in VM (no matter what the accessibility relation); so $\ulcorner \Box A \urcorner$ will be true in VM.[8]

[6] Wright is absolutely correct that to understand this is to understand the distinction between 'content sense' and 'ingredient sense', in Dummett's terminology (p. 130). Part of the difficulty in understanding this distinction is that it is substantive only when the logic is non-classical.

[7] I shall here ignore the usual complications concerning varying domains. For the present discussion, the domains of the worlds may be taken to be constant. Where we are concerned with vague *objects*, however, we shall want to consider varying domains.

The logic used in Wright's proof is essentially the logic VT, the characteristic axiom of T being what we above called T-*Def*, namely, '$\Box P \rightarrow P$'. In light of the preceding paragraph, and well-known facts about T, standard models of normal modal logics with *reflexive* accessibility relations, (absolute) truth being defined as truth in all worlds, are models of VT. What we must find, then, is such a model in which the following translation of (β')

(ζ) $(\forall x)[\Box\Box Fx \rightarrow \Diamond\Box Fx']$

is true, but in which the following equivalent of a translation of (the universal closure of) (γ)

(η) $(\forall x)[\Diamond\Box Fx \rightarrow \Diamond\Box Fx']$

is not true. That is not hard to do, so long as one is careful about what must be shown.

Consider the following three-world model. Let there be worlds w_0, w_1, and w_2; let w_{i+1} be accessible from w_i, but otherwise only from itself. Let the (constant) domain be {0,1}; let '$F\xi$' be true of both 0 and 1 in w_0, only of 0 in w_1, and of neither in w_2; let $0' = 1' = 1$, in every world:

World	w_0	$\rightarrow$	w_1	$\rightarrow$	w_2
Extension of '$F\xi$'	{0,1}		{0}		{}

It is then clear that (ζ) holds: For neither '$\Box\Box F(0)$' nor '$\Box\Box F(1)$' is true in any world, so (ζ) is true in every world and so is (absolutely) true. But (η) is false in w_0. Consider the following instance of (η), in which 0 has been assigned to 'x':

$$\Diamond\Box F(0) \rightarrow \Diamond\Box F(1)$$

The antecedent, '$\Diamond\Box F(0)$', is then true in w_0, since '$\Box F(0)$' is true in w_0 (w_2 is not accessible from w_0); but the consequent, '$\Diamond\Box F(1)$', is false in w_0, since '$\Box F(1)$' is not true in any world. Hence, (η) is not true in w_0 and so not (absolutely) true.[9]

Note that we have *not* shown that (η) is (absolutely) false in this model, that is, false in every world. It is impossible to find any model in which (ζ) is (absolutely) true and (η) is (absolutely) false, since one can derive a

[8] Indeed, it can be shown that, if $\mathcal{L}$ is complete with respect to some collection of models *and is compact*, then V$\mathcal{L}$ is complete with respect to the corresponding collection of V-models. Indeed, more strongly, the V-models are *strictly characteristic* for V$\mathcal{L}$, in the sense that they correctly characterize deducibility as well as provability. (The assumption of compactness appears to be needed in the case of logics with infinitely many distinct modalities, but I do not know it to be a necessary condition.)

contradiction, in VT, from (ζ) and the negation of (η). But to show that (η) is not derivable from (ζ) in VT, no more is required than to show that it fails to be (absolutely) true in some model of VT in which (ζ) is (absolutely) true.

Appendix: Evans on Vague Objects

As Evans's attempt to prove that there are no vague objects (Evans [2]) has often been discussed in this journal, it is worth noting that similar remarks can be made about it. Evans's argument may be thought of as carried out in VT, his 'indeterminacy' operator, '∇', defined by means of

$$\nabla A \equiv_{df} \Diamond A \;\&\; \Diamond\neg A$$

Evans's 'determinacy' operator may then be defined by means of

$$\triangle A \equiv_{df} \Box A \vee \Box\neg A$$

(All of the facts about these operators which Evans mentions can then be proven in VT.) Evans's argument is then a *valid* derivation of a contradiction from '$\nabla a{=}b$', the statement that it is indeterminate whether *a* is *b*. Evans's proof, as presented, omits two steps, however; the inference from '$\neg a{=}b$' to '$\triangle\neg a{=}b$', by rule V and the definition of '$\triangle$', and then to '$\neg\nabla a{=}b$', by the duality of '∇' and '$\triangle$'. However, because rule V is not applicable in subordinate deductions, '$\neg\nabla a{=}b$', or equivalently, '$\triangle\neg a{=}b$', is not provable, even in a logic so strong as VS4. It is a theorem of VS5, as Evans in effect mentions, but the assumption of the characteristic axiom of S5, '$\Diamond A \rightarrow \Box\Diamond A$', arguably would beg the question against his opponent, and Evans certainly does not intend his argument to rest upon this assumption.

Of course, Evans's derivation of a contradiction from '$\nabla a{=}b$' does show that no instance of it can be (absolutely) true. However, it is unclear that one who believes that there are vague objects must maintain that some

[9] A slightly more complicated model yields a case in which not only (ζ) and the negation of (η) are true, but also '$(\exists x)\Box Fx$' and '$(\exists x)\Box\neg Fx$' are true. Let there be three worlds, w_0, w_1, and w_2, such that w_{i+1} is accessible from w_i, w_0 is accessible from every world, and the accessibily relation is reflexive. Let the (constant) domain consist of {0,1,2}; let the extension of '$F\xi$' be {0,1}, {0,1}, and {0} in w_0, w_1, and w_2, respectively; let $0' = 1$; $1' = 2' = 2$. (ζ) can be verified, and '$\Box F(0)$' and '$\Box\neg F(2)$' are true in every world and so absolutely true (whence '$(\exists x)\Box Fx$' and '$(\exists x)\Box\neg Fx$' are true). However, the instance '$\Diamond\Box F(1) \rightarrow \Diamond\Box F(2)$' of ($\eta$) is false at w_0.

[10] For the reason mentioned above in connection with (δ): Iterations of '$\Box$' would then be redundant. Indeed, it is not difficult to see that '$\triangle\Box A$', '$\triangle\triangle A$', and '$\triangle\nabla A$' are all theorems of S5.

[11] By an instance, I mean one in which the subtituends of the free variables '*a*' and '*b*' are, in the relevant sense, rigid. The importance of this condition has been widely discussed.

instances of '$\nabla a{=}b$' are *true*, rather than only that not all instances are *false*. More precisely, instead of holding that the identity of some objects may actually be indeterminate, she may maintain that '$\triangle a{=}b$' is not *valid*, which is to say that identity is *not* determinate, that not every identity statement is definitely true or definitely false. In my own view, to maintain that there are vague objects, one *need* maintain no more than that not all identity-statements are of determinate truth-value, and so that Evans derived a contradiction from a principle his opponent need not hold: But a discussion of these issues would take us far beyond the purely formal point, which is my focus here.[12]

Harvard University
Cambridge, MA 02138, USA
heck@husc4.harvard.edu

References

[1] Michael Dummett, 'Wang's Paradox', in his *Truth and Other Enigmas* (Cambridge MA: Harvard University Press, 1978), 248–68.
[2] Gareth Evans, 'Can There be Vague Objects?', *Analysis* 38 (1978), 208.
[3] Richard G. Heck, Jnr., 'That There Might Be Vague Objects (So Far as Concerns Logic)', in his *Logic, Semantics, Ontology*, unpublished Ph.D. dissertation, Massachusetts Institute of Technology, 1991.
[4] Crispin Wright, 'Further Reflections on the Sorites Paradox', *Philosophical Topics* 15 (1987), 227–90.
[5] Crispin Wright, 'Is Higher-Order Vagueness Coherent?', *Analysis* 52 (1992), 129–39.

[12] Thanks to George Boolos and to Crispin Wright for helpful comments on an earlier draft of this note.

The appendix summarizes the main results of my [3]. Thanks to George Boolos, and especially to Bob Stalnaker, for discussions of that paper which greatly clarified my thinking about these issues.