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A representation of time discounting

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Abstract

The concept of time discounting introduces weights on future goods to make these less valuable. Famously, both the specific functional form of time discounting and its normative status are contested. To address these problems, this paper provides a measurement-theoretic framework of representation for time discounting. The general representation theorem characterises time discounting factors as ratio-scale representations of differences in temporally extended prospects. This framework of representation is used to reconsider interpretations of time discounting factors such as time preferences, uncertainty and preference change. It also allows to compare these interpretations with regards to their normative appeal, the kinds of reductionism regarding time and preference they imply and the specific functional form of time discounting they suggest. Farther, multiple-self interpretations of decision-makers become available, such that a time discounting factor represents the degree of connectedness between temporal selves in a person.

1 Introduction

Intertemporal decisions require evaluations of prospects that are spread out over time. Such intertemporal decisions are of particular significance: for individuals, decisions about education, migration and pensions come to mind. For collectives, such decisions concern investments in infrastructure and dealing with environmental problems such as climate change. Notoriously, how to deal with the intertemporal element in such decisions is answered differently by expected utility theory on the one hand and economics on the other hand. Standard expected utility theory does not evaluate the temporal element in prospects separately from other features. Rather, the overall subjective evaluation of a prospect is supposed to include any possible influence of the temporal dimension, since prospects are taken to be complete descriptions of world histories. In contrast, it is standard practice in economics to introduce weightings to evaluate the temporal dimension of a prospect separately. Such weightings are performed by time discounting factors that make goods in the far future less valuable than those in the near future.

For an illustration of the different evaluations of intertemporal decisions in expected utility theory and economics, take the example of deciding between the two prospects of instantaneous consumption and long-term saving. Expected utility theory requires of the decision-maker to form preferences over the two prospects, taking them as full descriptions of the world histories they induce. Rational preferences will satisfy consistency conditions and represent the decision-makers state of mind with regards to the prospects. In economics, such intertemporal prospects are evaluated differently: decision-makers are assumed to form preferences over consumption and saving and then combine these with a weighting for the time difference that is inherent in their description. This implies that in addition to standard preferences, there is also the domain of time difference for which a separate evaluation is formed that is summarised by a time discounting factor.

While most models in economics assume standard expected utility theory, so-called time discounting factors are nevertheless introduced as an additional feature when modelling intertemporal decisions. The practice of introducing discounting factors to analyse intertemporal decisions is hence a significant departure from the way in which expected utility theory is usually integrated in economic theory. There is no underlying theory for such a separate treatment of time. Foundations for time discounting have largely been restricted to the development of axioms for time discounting that capture the idea of the impatience of agents, introducing technical conditions that are solely motivated by the desire to preserve expected utility theory (Samuelson 1937, Koopmans 1960, Lancaster 1963). This fact makes the use of time discounting problematic from a foundational perspective.

Time discounting is also a heavily contested concept, both normatively and descriptively (Loewenstein/Elster 1992). Normatively, it is often questioned whether time discounting is justified at all. Despite its common usage in economics, philosophers tend to deny the justifiability of time discounting (Sidgwick 1907, Rawls 1971). Descriptively, there is no consensus on the correct functional form of discount factors (Frederick et al 2002). In its most basic use, the discounting rate remains the same regardless of how far prospects extend through time. This most commonly used form of discounting originated with Samuelson (1937) and is called exponential discounting, due to the shape of the value function it induces. More recently, hyperbolic discounting, initially proposed by Ainslie (1975), postulates a declining discount rate, based on empirical evidence. Loewenstein/Read (2003) survey the rich literature that has emerged on hyperbolic discounting, dealing with the questions of how to derive hyperbolic discounting functions and whether they can be used instead of exponential discounting functions.

This paper embarks from the supposition that the underlying problem of these normative and descriptive debates is the aforementioned breakdown of foundations between expected utility theory and economics with regards to intertemporal decisions. Since there is no general theory or representation of time discounting, it is difficult to assess the overall justifiability of time discounting as a concept as well as to argue about the specific functional form it should take. More generally, as it is unclear what discounting factors are supposed to measure and represent, a framework is lacking that guides discussions about time discounting. This paper provides such a framework. It gives a representation for discounting factors in absolute-difference structures. In this framework, a discounting factor is a ratio-scale representation of differences that can be related to the temporal parts in a temporally extended prospect. Formally, this provides a structure to assess the temporal element of prospects in intertemporal decisions. In extension to standard expected utility theory, the representation for time discounting allows one to consider cases in which some temporal parts are given different weights.

The general representation for time discounting can accommodate different interpretations. Fundamentally, those interpretations determine the exact nature of the temporal differences that a discounting factor represents. A number of possible interpretations will be discussed in the framework of the representation theorem. Firstly, the standard interpretation of time discounting as representing pure time preferences is considered. Secondly, time discounting as a representation of uncertainty about the future is discussed. Thirdly, time discounting is developed as a representation of differences in the decision-makers preferences over time. Fourthly, generalising the possible interpretations, time discounting is seen as capturing a number of time-related factors.

The interpretations of the representation are used to revisit the normative and descriptive debates about time discounting. Normatively, plausible interpretations of time discounting that are compatible with expected utility theory can be used to endorse discounting factors in some contexts. Descriptively, specific interpretations of the representation are employed to defend both exponential and hyperbolic discounting, albeit by committing to different underlying interpretations of their respective representation. The framework also makes available multiple-self interpretations of the decision-maker. In this interpretation, we understand the temporal parts in a prospect as having a bearing on different temporal parts of the decision-maker. A model of a multiple-self is introduced in which differences between selves can be interpreted and formalised according to connectedness criteria drawn from the literature of personal identity over time, such as psychological, sympathy or memory connectedness. These criteria, in turn, can provide subjectivist interpretations of the differences that a time discounting factor represents.

The paper proceeds as follows. Section two gives the general measurement-theoretic representation for time discounting in terms of time differences. The third section introduces four special representations of time discounting, interpreting difference by time preferences, uncertainty, preference change as well as by a multidimensional account. The fourth section discusses the normative justification of time discounting, a multiple-self interpretation of the four special representations, the debate between exponential and hyperbolic discounting and problems of reductionism with regards to time and preference. The fifth section concludes.

2 A General Representation of Time Discounting

2.1 Time and Utility

Discounting, in its most general meaning, is the lowering of the value of an object, good or prospect for a specific and separate reason. For instance, shops may lower the price of goods if a consumer purchases a large number or a specific bundle of them, policy-makers may disregard the opinion of someone with a vested interest and individuals are often less affected by the suffering of people that they do not know personally, and so on. Time discounting refers to the practice of weighing the goodness of a prospect with the time of its occurrence. More specifically, after evaluating the goodness of a temporally extended prospect, it is then weighted by the temporal distance between the present and when its temporal parts occur. For instance, after evaluating the goodness of the prospect of having dinner, it is then asked when the dinner will be taking place. Depending on whether it is in the near or far future, the goodness of the dinner will be diminished to a smaller or larger extent.

Applying time discounting to expected utility theory, the utility of a prospect is multiplied by a discounting factor (usually assumed to take a value between 0 and 1), thereby diminishing its value. In the dinner example, the *discounted* utility of having dinner tomorrow, $D(0,1) \times U(\text{Dinner}_{\text{tomorrow}})$ will be less than the initial evaluation of the dinner $U(\text{Dinner})$. One important effect of time discounting is that it makes time differences comparable over the utility dimension. After assigning a value to the utility of a prospect, time discounting then assesses the effect of the temporal location of the prospect. The goodness of having dinner today, $U(\text{Dinner}_{\text{today}})$, will be larger than having dinner at later times, for instance $U(\text{Dinner}_{\text{today}}) \geq D(0,1) \times U(\text{Dinner}_{\text{tomorrow}})$, as the utility of the prospect that is further away in time is diminished.

Weighing goodness with a time discounting factor implies that utility and time are viewed as separate. If this were not the case, the temporal dimension could already be fully included in the evaluation of the goodness of the prospect. Indeed, this is what standard expected utility theory recommends: as utilities are derived from preferences over a complete set of prospects whose elements contain full world histories, there is no place in the theory for a separate evaluation of specific features of prospects. Crucially, Savage's (1954) subjective expected utility theory also requires that the ordering of world histories inherent in prospects is not supposed to matter over and above a subjective evaluation of the goodness of outcomes. Indifference is required between two prospects that only differ in the temporal ordering of otherwise identical outcomes. Other expected utility theory frameworks (notably Jeffrey 1983) avoid the indifference requirement. However, even those frameworks do not advocate an explicit and separate treatment of the temporal dimension as it would undermine the idea that complete preferences over a single domain are sufficient for the evaluation of goodness. Time discounting thus represents a departure from expected utility theory.

There are some reasons for subscribing to this departure. Firstly, time plays an important role in most decisions. Indeed, it is the most significant feature of many decision problems. In the dinner example mentioned above, time is the only dimension on which the two prospects differ. Analysing the time dimension separately can help to identify how exactly it influences the goodness of prospects. Secondly, time can also be overwhelmingly important, for instance in long-run decisions. In collective decisions, such long-run intertemporalities occur in public investment programmes or environmental problems, for example climate change. For individual decisions, such long-run intertemporal decisions occur in education, insurances and housing. Analysing the temporal dimension of such problems separately allows a more specific discussion of the role of time and the attitudes that agents ought to take to it. Thirdly, when analysing the time dimension separately, a number of time-specific features can be used for its analysis. For instance, the concept of tenses becomes available and states of the world, utilities and actions can be analysed relative to them occurring in the past, present or future. Likewise, a succession of integers can be used to identify positions in time and such time series can be used to characterise consequences. Using such theories of time that deal with tenses or time series only becomes possible when separating time from other features of a prospect.[1] Fourthly, the two dimensions (i) evaluations of goodness, and (ii) temporal occurrence of goodness, are often at odds; for instance when individuals fail to follow prudent plans. Treating the two dimensions separately allows one to better analyse such trade-offs both empirically and normatively. Fifthly, there is an important methodological consideration in favour of exploring the available options of such a strategy. Despite the lack of time discounting foundations in decision theory, it is the received practice in economics to analyse the temporal dimension separately in intertemporal decisions by introducing discounting factors.

Discounting factors are given by discounting functions. A discounting function is a mapping from a set of time points to the real numbers, $D : T \to \mathbb{R}$. T can be continuous or discrete, and in most applications it is assumed to be a set of positive integers $T = \{0, 1, 2, \ldots\}$ which represent points in time, with 0 being the present and all other points representing points at future times. The number in $\mathbb{R}$ that is assigned to a time point is then used as a discounting factor for utility that occurs at that point in time. For instance, if x_3 is a consequence indexed by the point in time it occurs at, then its discounted utility $U(x_3)$ is obtained by weighting its utility $U(x)$ with the discounting factor $D(3) \in \mathbb{R}$ such that $U(x_3) = D(3)U(x_0)$.

In most of its applications, time discounting results in valuing future utility less than the same amount of these objects in the present or without discounting. For instance, in the aforementioned example, discounting usually leads to $U(x_0), \ldots, \geq U(x_3)$. Furthermore, discounting factors are usually lower for points in time that are further away. Hence, most discounting functions are decreasing, such that $D(t) \geq D(t + 1)$. The range of the discounting function is usually restricted to a real interval such as $(0, 1)$. A discounting factor of 0 would completely devalue the utility of future consequences, while negative discounting factors would result in

[1] This argument could apply to many other dimensions as well. For instance, there are many theories and measurement procedures available for dimensions like space, colour and sound. However, time appears to be more fundamental with respect to the evaluation of goodness. This does not preclude to also isolate the treatment of other dimensions for a specific class of decision problems.

negative utilities and discounting factors larger than 1 would increase the utility assigned to future consequences. While such values for discounting factors are not logically impossible, most discounting functions do not include such values. A discounting factor of 1 would result in no discounting at all. Sometimes, the range is only restricted to $(0, 1]$ to include the possibility to assign weight 1 to the present. As an illustration, take a discounting function such that the discounting factors are $D(0) = 1$, $D(1) = .9$, $D(2) = .8$, $D(3) = .7$ and so on. Then the utility of consequences at times $0, 1, \ldots$ is multiplied by the respective discounting factor. For instance, discounted utility at $t = 2$ will be $.8U(x_0)$ and so on.

The general characterisation of a discounting function is summarised more formally in the next definition. Most discounting functions in the literature are variants of this formulation, offering more specific restrictions on what values discounting factors can take.

Definition 1 (Time discounting function) *A discounting function D is a decreasing mapping from a set of time points T to a real interval, $D : T \rightarrow (0, 1]$ such that*

$$D(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 < D(t) < 1 & \text{if } t > 0, \end{cases}$$

where for all $s < t$, $D(s) \geq D(t)$.

This general definition of a discounting function shows that values between 0 and 1 are assigned to time points such that the later the point in time, the lower the value that is assigned. This reflects the idea that time discounting values the distant future less than the nearer future and that the present is not discounted.

There is a large literature on how to further specify discounting functions which will be reviewed more closely in section 4. Some discounting functions derive discounting factors from discount rates, other functions employ concepts like delays and some functions combine those elements in different ways. These different methods result in different shapes of the discounting function, the most common being exponential and hyperbolic discounting functions.

In exponential discounting functions, a constant discount rate $r \in [0, 1]$ is introduced that assures that each succeeding point in time is discounted with the same discounting factor, multiplied by the number of time points that lie between it and the present.

Definition 2 (Exponential discounting) *A discounting function is exponential iff the discounting factor is determined by a constant discount rate $r \in [0, 1]$, such that:*

$$D(t) = \delta^t,$$

where $\delta = \left(\frac{1}{1+r}\right)$ and $D(t)$ is time-invariant.

Exponential discounting was initially proposed by Samuelson (1937) and is the most commonly applied time discounting method in economics. More recently, hyperbolic discounting has emerged from empirical study of how real-world agents discount for temporal distance. Such hyperbolic discounting functions decrease more drastically for small distances in time and less so for larger ones. This results in more time discounting for short horizons than in exponential discounting. In the literature, many variants of functions that fulfil this requirement have been proposed (for an overview see the review by Frederick 2002). Since hyperbolic discounting is informed by empirical research, there are a number of proposals that each captures a variety of data.

All the following functions determine a discounting factor by delays, discount rates and/or constants such that discounting factors are smaller than in exponential discounting for small $t \in T$:

- Discounting for delay: $D(t) = 1/t$, where t equals the length of delay (Ainslie 1975, 1992, 2001),
- Discounting for delay and discount rate: $D(t) = 1/(1 + rt)$, where r is the discount rate and t the delay (Herrnstein (1981) and Mazur (1987),
- Generalised discounting: $D(t) = 1(1 + \alpha t)^{\gamma/\alpha}$, where $\alpha > 0$ measures how much the function departs from constant rate discounting and $\gamma > 0$ is a parameter related to time preferences (Loewenstein/Prelec 1992, Laibson et al 1998),
- Quasi-hyperbolic discounting:

$$D(t) = \begin{cases} 1 & \text{if } t = 0, \\ \beta\delta^t & \text{if } t > 0 \ , \end{cases}$$

  where $\beta < 1$ can be constant or decline as t increases and δ^t is the exponential discounting function $(1/(1 + r))^t$ (Pollack/Phelps 1986, Laibson 1997, 1998, Barro 1999).

The question which of the aforementioned time discounting functions is the correct one has generated a lot of interest and debate (for an overview see Loewenstein/Read 2003). Indeed, the question of the correct functional form for time discounting is one of the two most widely debated questions concerning time discounting. An equally important question concerning time discounting is a normative one: why is it justifiable at all to introduce time discounting factors as diminishing weights on utilities? If utility is an all-encompassing evaluation of goodness, how can it be changed due to temporal location?

These two debates should be discussed in a representational framework. Indeed, the absence of a general theory of discounting or general representational framework is a methodological concern in itself. A framework of representation will answer the question what exactly time discounting factors measure and represent. For many other contentious concepts, similar questions to those that have been raised about time discounting can be answered or at least explored systematically by going back to underlying theories of measurement and representation. Consider as an example expected utility theory itself, where normative and empirical problems have been debated with regards to the structure of the representation of expected utility (for instance, whether certain axioms are normatively justifiable, formally avoidable and empirically verifiable). This suggests that a framework for representation of time discounting is needed in order to more carefully examine the problems associated with it. The next section provides such a general framework. The representation offered represents time discounting as a decreasing function of time differences. In the section thereafter, specific interpretations of time difference are discussed which is followed by discussing both the functional form and normative significance of time discounting.

2.2 A General Measurement-Theoretic Representation of Time Discounting

In this section, a general representational framework for time discounting that avoids precommitment to a particular interpretation is given. In the next section, it is linked to specific variants that have already been proposed in the literature.

Discounting factors are weights that are included in calculations of utility, expected utility and expected value of intertemporal prospects. As shown above, discounting factors $D(t) \in (0, 1]$ can lower the value of intertemporal prospects, depending upon the temporal location of their parts, determined by a time index t. A representation theorem for discounting factors establishes what exactly they measure or represent. The question for the representation is thus twofold:

firstly, how exactly can values for $D(t) \in (0, 1]$ be established and secondly, what do these values represent? In this section, we give a general, measurement-theoretic answer to the first question. It will be shown that time discounting factors can be represented as decreasing functions of time differences. The general representation will then be used in the next section to develop representations that not only have the desired mathematical properties of time discounting factors, but can also provide substantial interpretations as to what kind of time differences time discounting represents and whether and in what way it is permissible to diminish utility with such a discounting factor.

In order to establish a representation of time discounting, this section presents an axiomatic treatment of time differences in absolute-difference structures. The representation theorem for discounting developed here rests on a formal framework of measurement and representation developed in Krantz et al (1971a, 1971b, 1971c). In Krantz et al (1971a:136), difference measurement is introduced as a way to measure intervals between elements of a set. A specific way of measuring differences can be given in absolute-difference structures. In these structures, absolute differences along a single dimension can be represented up to an interval scale (Krantz et al 1971b:170ff.).

Definition 3 (Absolute-difference structure. Krantz et al (1971a):172) *Suppose Q is a set with at least two elements and $\succeq$ is a binary relation on $Q \times Q$. The pair $\langle Q \times Q, \succeq \rangle$ is an absolute-difference structure iff, for all $q, r, s, t, q', r', t' \in Q$, and all sequences $q_1, q_2, \ldots, q_i, \ldots \in Q$ the following axioms hold:*

1. *Weak ordering. $\langle Q \times Q, \succeq \rangle$ is a weak order.*
2. *Symmetry. If $q \neq r$, then $qr \sim rq \succ qq \sim rr$.*
3. *Well-Behavedness.*
   - *(i) If $r \neq s$, $qs \succeq qr, rs$ and $rt \succeq rs, st$, then $qt \succeq qs, rt$.*
   - *(ii) If $qs \succeq qr, rs$ and $qt \succeq qs, st$, then $qt \succeq rt$.*
4. *Weak Monotonicity. Suppose that $qs \succeq qr, rs$. If $qr \succeq q'r'$ and $rs \succeq r's'$, then $qs \succeq q's'$; moreover if either $qr \succ q'r'$ or $rs \succ r's'$, then $qs \succ q's'$.*
5. *Solvability. If $qr \succeq st$, then there exists $t' \in Q$, such that $qr \succeq t'r$ and $qt' \sim st$.*
6. *Archimedean property. If $q_1, q_2, \ldots, q_i, \ldots$ is a strictly bounded standard sequence (i.e., there exist $t', t'' \in Q$, such that for all $i = 1, 2, \ldots,$ $t', t'' \succ q_{i+1}q_1 \succeq q_i q_1$ and $q_{i+1}q_i \sim q_2 q_1 \succ q_1 q_1$), then the sequence is finite.*

Theorem 4 (Interval representation. Krantz et al (1971a):173) *If $\langle Q \times Q, \succeq \rangle$ is an absolute-difference structure, then there exists a function $\varphi : Q \to \mathbb{R}$ such that for all $q, r, s, t \in Q$,*

$$(qr) \succeq (st) \text{ iff } \mid \varphi(q) - \varphi(r) \mid \geq \mid \varphi(s) - \varphi(t) \mid$$

If φ' is another function with the same property, then $\varphi' = \alpha\varphi + \beta$, where $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$.

Krantz et al (1971a:173) show that upon $\langle Q \times Q, \succeq \rangle$ satisfying all conditions of an absolute-difference structure, it is possible to numerically represent the differences between elements in Q. Here, we interpret Q as a set of indexed consequences and any $P \subseteq Q$ as a prospect. Such a prospect P could consist in a collection of consequences $q, r, s, t, \ldots \in Q$, for example starter, main and dessert in the prospect of a three-course menu. Usually, not much attention is devoted to different parts of a prospect, as the latter is just taken as a full description of the world and evaluated as such. However, when different parts of prospects are contingent on different states of the world, then some form of separability between parts of a prospect is assumed.

In the context of difference measurement, we assume prospects are a collection of different parts $q, r, s, t, \ldots \in P \subseteq Q$ and single out a dimension on which these parts can be compared. Such dimensions could be colour, sound, sweetness, spatial location or temporal location and so on. For instance, P could be the prospect of summer in the Northern hemisphere and its parts $q, r, s, t, \ldots \in P$ could describe all the locations in the Northern hemisphere. These locations can be compared with respect to their spatial distance from a point of view the observer chooses and the above definition lists six conditions that differences between pairs of locations would need to fulfil in order for $\langle Q \times Q, \succeq \rangle$ to be an absolute-difference structure. Another possible interpretation is to view a prospect P as a temporally extended prospect such that $q, r, s, t, \ldots \in P$ are its indexed temporal parts. For instance, take the temporally extended prospect of consumption which consists of the temporal parts {buying, enjoying, recycling}. To give a different example, take the temporally extended prospect of receiving an apple on Tuesday, a banana on Wednesday, an orange on Thursday and so on.

The above definition can be linked to the temporal structure of prospects quite intuitively. Firstly, a temporally extended prospect $P \subseteq Q$ can be written as a set with at least two temporal parts which are labelled $q, r, s, t, \ldots \in P$. Secondly, the binary relation $\succeq$ on $Q \times Q$ indicates comparisons of the length of time intervals between pairs of temporal parts. For instance, take any two pairs $qt, rs \in P$ and compare their time differences such that $qt \succ rs$ iff the time interval between q and t is strictly greater than the time interval between r and s. The six axioms on the binary relation $\succeq$ would follow indeed immediately from the idea that time can be represented by a succession of integers and that in the complete description of a temporally extended prospect, each temporal part corresponds to exactly one of those integers. Even if this were not the case, then the axioms could be established by plausibility of temporal succession, for instance the elements in a temporally extended prospect of consumption that consists of the three temporal parts {buying, enjoying, recycling} could then be ordered by comparing the intervals between the temporal parts, i.e. buying, recycling $\succeq$ buying, enjoying and so on.

Theorem 4 asserts that absolute differences can be represented on an interval scale. That is, in the context of time differences of temporal parts, a number φ can be assigned to any temporal part q, r, s, t of a temporally extended prospect $P \subseteq Q$ such that for any two temporal parts, the absolute difference of their numbers adequately reflects their time difference when compared to any other pair in the prospect. Moreover, the assigned numbers are unique up to a positive affine transformation.

It could be objected at this point that the framework and the axioms are much weaker indeed than what is available for intertemporal decisions in virtue of the way we usually understand and measure time as a so called B-series (i.e. a succession of integers or clock-time). The purpose of the definition used here is however not so much to capture measurement of time as a B-series, but to develop a general framework that helps to compare competing theories about the role time differences should play in the evaluation of goodness. Hence, the measurement framework introduced here can carry different interpretations of how to measure time differences. Such interpretations can include various aspects of individual time perception. Time differences thus understood could indeed be harder to measure than a B-series. Here, we first develop time discounting as a representation of the ratio-scale measurement of time differences before turning to turn interpretations in the next section.

In order to obtain a representation suitable for discounting factors, a ratio scale is needed which can be obtained by normalising an interval scale. Generally, such a normalisation is permissible if there is an absolute (or true) zero point. For the purpose of representing differences on a ratio scale, it is natural that there is a viewpoint which is assigned zero difference. For instance, when comparing spatial differences, then there will be a point in space to which the observer has zero distance to. Likewise, when comparing the temporal differences between temporal parts, a point of view of an agent will be taken from which these differences are assessed. Indeed, for most cases, the present – as the natural viewpoint from which prospects and courses of actions are assessed – can be normalised as the true zero. Hence, an interval scale representation of time differences of temporal parts can normalised to the present via a positive affine transformation.

Corollary 5 (Normalisation) *Suppose an absolute-difference structure $\langle Q \times Q, \succeq \rangle$ and its representation φ. Then φ^* is the normalisation of φ to p by $\varphi^* = \alpha\varphi + \beta$ such that $\varphi^*(p) = 0$ and φ^* is ratio. If φ'^* is another function with the same property, then $\varphi'^* = \alpha\varphi^*$, where $\alpha \in \mathbb{R}, \alpha > 0$.*

Proof sketch. The permissibility of this transformation follows immediately from the properties of the interval representation, i.e. that φ is unique up to a positive affine transformation.

This condition asserts that by a transformation permissible by the interval representation, the interval scale can be normalised to an absolute zero which gives a ratio scale on which only multiplicative transformations are allowed. Regarding the interpretation of the intervals as time intervals, normalising the present as an absolute zero is also plausible. Indeed, the normalisation also captures the idea of tenses in the formalism, since all temporal parts in the past take a value smaller than zero (i.e. $\varphi^* \in \mathbb{R}^-$ for all $l, m, n, o, \ldots \in Q < p$) and all temporal parts in the future take a value larger than zero (i.e. $\varphi^* \in \mathbb{R}^+$ for all $q, r, s, t, \ldots \in Q > p$). This gives a ratio-scale representation.

It is possible to formulate a discounting function which transforms the normalised differences into discounting factors, which can be used as weights assigned to parts of prospects that corresponds to their differences.

Definition 6 (Difference Discounting) *Suppose an absolute difference structure $\langle Q \times Q, \succeq \rangle$ and its representation φ. Upon φ satisfying normalisation, a difference discounting function Disc is a map from differences to a real interval, $Disc : \varphi^* \to (0, 1]$, such that*

$$Disc\,(\varphi^*(q)) = \begin{cases} 1 & \text{if } \varphi^*(q) = 0 \ , \\ 0 < Disc(q) < 1 & \text{if } \varphi^*(q) \neq 0 \ , \end{cases}$$

where for all $\varphi^*(r) > \varphi^*(q) > \varphi^*(p)$, $Disc(r) \leq Disc(q)$ and for all $\varphi^*(n) < \varphi^*(o) < \varphi^*(p)$, $Disc(n) \geq Disc(o)$.

In this definition, the representation of differences in a prospect is used to formulate weights $Disc(0, 1]$. The weighting assigns value 1 to the element to which the difference representation is normalised to. Hence, if this weight is used to discount value for difference, it results in no discounting at all. Farther, the weightings assigns a number in the real interval $(0, 1)$ to all other differences such that the larger the difference, the lower the weight. This makes it possible to use the function *Disc* as a weighting for differences of parts of prospects.

From the above representation and definitions, one additional assumptions is needed to obtain a definition of time discounting in terms of differences. Firstly, the prospect Q has to be understood as a temporally extended prospect, such that there are different points in time for each of which a prospect gives a complete description (in consumer theory, such temporally extended prospects are referred to as consumption bundles $c_0, c_1, \ldots, c_t$). That is to say, it is now assumed that $q, r, s, t \in Q$ are temporal parts of such a temporally extended prospect.

From this follows that φ is a interval representation of time differences and that the ratio-scale normalisation φ^* is due to understanding p as the temporal part with zero time difference, i.e. the present. As alluded to above, this does not mean that the time differences necessarily corresponds to time as commonly understood as clock-time. It could be the case that the time differences obtained by the representation are different from clock-time. For instance, the above measurement procedure could be understood as an elicitation of an agent's subjective perception of time differences which does not necessarily correspond perfectly to clock-time. Indeed, in the next section we will discuss four plausible interpretations of time differences in the above representation.

In order to obtain a representation of time discounting in terms of time differences, however, there has to be a correspondence between some externally given time index and the time differences in the above representation.

Axiom 7 (Correspondence between difference and time) *Suppose an absolute difference structure $\langle Q \times Q, \succeq \rangle$ and its normalised representation φ^*. Then, a mapping C is a correspondence between a set of time points $T = \{0, 1, \ldots\}$ and normalised differences φ^* iff $C : T \rightarrow \varphi^*$ such that:*

$$C(t) = \begin{cases} 0 & \text{if } \varphi^* = 0, \\ 0 < C(t) < 1 & \text{if } \varphi^* > 0, \end{cases}$$

where for all $t < u$, $C(t) \leq C(u)$.

This condition asserts that each point in time is associated with a temporal part such that the representation of its time difference φ^* is increasing.

If it is the case that there is such a correspondence between a set of time points and the normalised representation of time difference φ^*, then time discounting according to time differences as stated in the above definition is possible. There are several ways in which specific time discounting functions can be formulated.

For bounded prospects or bounded horizons, a maximally distant $t \in Q$ can be selected to obtain discounting factors in the following way:

$$D(q) = 1 - \left(\frac{\epsilon\varphi^*(q)}{\epsilon\varphi^*(max)} \right),$$

where a small $\epsilon > 0$ transforms the values of φ^*. The disadvantage with the approach is that it implies a maximally distant point max for which the discounting factor is $D(max) = 0$.

In a more general approach, a transformation of all differences for all $q \in Q$ with a small $\epsilon > 0$ yields $\epsilon\varphi^*(q) < 1$, for all $q \in Q$ of future temporal parts, i.e. for which also holds that $\varphi^*(q) \geq \varphi^*(p)$. Then, discounting factors can be obtained by

$$D(q) = 1 - \epsilon\varphi^*(q).$$

As mentioned in the beginning of the section, it is a matter of convention that time discounting factors take values in the interval $(0, 1)$. The discussion of how to obtain such values from time differences and the way the time differences are scaled in order to do so highlights again that the range between zero and 1 is but a convention.

For exponential discounting, so-called per-period discount rates $r \in [0, 1]$ are introduced which are on absolute scales. Discount rates ascribe weights from one period to the other, placing a value of 1 on the current (or preceding) period and expressing how much values in the next period are devalued relative to this period. In exponential discounting, r is supposed to be the same between all periods or points in time. With such an absolute valuation in place, exponential discounting factors $D(t) = \delta^t$ are then obtained by $\delta = 1/1 + r$.

To obtain exponential discounting in absolute difference structures, we introduce the requirement that pairs of temporal parts which are immediate successors are equally spaced. However, again, the values obtained will not be on absolute scales.

Axiom 8 (Equal Spacedness) *Suppose an absolute-difference structure $\langle Q \times Q, \succeq \rangle$. Let all elements $q, r, s, t, \ldots \in Q$ have an immediate successor, such that for all $r \in Q$, there are exactly two $q, s \in Q$ such that $r \neq s \neq q$ and $qs \succ qr \sim rs$ and there is no $t \neq r \in Q$ such that $qt, st \prec qr, rs$. Immediate successors are equally spaced iff for all immediate successors $qr, st \in Q \times Q$, $qr \sim st$ such that*

$$\mid \varphi(q) - \varphi(r) \mid = \mid \varphi(s) - \varphi(t) \mid .$$

This condition asserts that all temporal parts have an immediate successor and that the absolute difference between immediate successors is the same for all temporal parts.

Corollary 9 (Exponential discounting with difference representation) *Suppose an absolute-difference structure $\langle Q \times Q, \succeq \rangle$ and its normalised representation φ^* that satisfies correspondence C. If $\langle Q \times Q, \succeq \rangle$ is equally spaced, then time discounting with difference representation is exponential discounting where $\mid \varphi^*(q) - \varphi^*(r) \mid$ of immediate successors $qr \in Q \times Q$ determines the constant discount factor δ such that:*

$$D(t) = \mid \epsilon\varphi^*(q) - \epsilon\varphi^*(r) \mid^t,$$

where $\epsilon > 0$ yields $\epsilon\varphi^*(q) < 1$, for all $q \in Q$ of future temporal parts, i.e. for which also holds that $\varphi^*(q) \geq \varphi^*(p)$ and $D(t)$ is time-invariant.

Proof sketch. We show that the exponential discounting function is a special case of the general time discounting function with difference representation. Then we show that equal spacedness implies a constant difference between immediate successors.

Upon temporal parts satisfying equal spacedness, the time discounting function has a constant discounting factor that corresponds to time differences.

Accordingly, similar conditions on $\langle Q \times Q, \succeq \rangle$ can lead to hyperbolic discounting, for instance, when postulating that intervals between temporal parts that are close to p are generally larger than those further away. Ascribing larger differences to those intervals leads to steeper discounting functions for the near future than the far future.

We have developed a general representational structure for time discounting functions in absolute-difference structures. The discounting factor has been established by considering time differences in temporally extended prospects. Taking any pair of temporal parts of a prospect and comparing their time difference with others, each future temporal part can be assigned a number $\varphi^* \in [0, 1)$ that indicates their time difference on a ratio scale with the present being assigned the value zero. φ^* is then used to obtain a weight which behaves like a discounting factor.

The representational structure developed in this section gives an answer to the question what discounting factors measure and represent. Formally, discount factors are weights that diminish the utilities of temporally extended prospects. The above representation gives a general measurement-theoretic foundation of the functional form of such weights, establishing discounting factors as representing time differences on a ratio scale. However, in order to discuss the justifiability of using such discounting factors to weigh utility, a motivation and interpretation has to be provided alongside with the formal representation. Otherwise, utility would be weighted by a discounting factor due to temporal distance *simpliciter*. While this is a possibility and indeed the most general way to understand time discounting, linking the representational framework to interpretations will provide specific ways to discuss the problems associated with time discounting, namely the correct functional form and indeed the normative justification for discounting for temporal distance. In the literature, a multitude of interpretations and justifications is available. The framework presented here allows their unified comparison.

3 Special Representations for Time Discounting

3.1 Interpreting Time Difference

The next sections will introduce four approaches that provide both time discounting factors with the desired mathematical properties and a substantial interpretation that links them with utility theory. Accordingly, the four special representations can be compared by the way they modify the above general representation.

As mentioned in the previous section, the general representation shows that there is a discounting function that corresponds to a difference evaluation. Upon assuming that the difference evaluation is concerned with time and assuming that it corresponds to an externally given time series, there is a time discounting function. The distinction between some concept of time difference and the externally given time series allows to give many interpretations to what we take time difference to be.

Time difference as used in the representation could as well be subject to the individual perception of an agent. That is to say, introducing a distinction between an externally given time series (clock-time, for instance) and subjective time difference, leaves open the possibility that individual agents have different time perceptions. The subjective time difference between one period and the next might be different for a person that enjoys herself than for a person that is in pain, even though they both experience the same passage of time according to clock-time.

Interpreting the absolute-difference structure as a framework to measure such subjective evaluations of time difference makes it possible to compare how well such interpretations fare when their numerical representation is used to derive a time discounting factor. The four approaches interpret the binary relation $\succeq$ and its representation φ^* as follows:

Time preferences. In this interpretation, agents are assumed to exhibit pure and positive time preference, i.e. it is put forward the idea that agents take the present to be best and utility at earlier points in time is always preferred to later points in time. In this interpretation, given a few further assumptions, φ^* is the representation of time preferences on a set of time points.

Uncertainty. Here, distance in time is assumed to be linked to an increase in fundamental uncertainty which cannot be captured by a standard representation of uncertainty in a probability function. Rather, time intervals can be compared with regards to the fundamental uncertainty they produce and φ^* is the representation of the uncertainty associated with a set of time points.

Preference Change. In this view, contrary to expected utility theory which depicts agents as possessing a diachronically stable set of preferences, agents can also change their preferences. The evaluation of each temporal part is associated with a temporally indexed set of preferences. The binary relation $\succeq$ captures the difference in evaluation of different sets of preferences and accordingly φ^* is the representation of how different or similar preferences are at different points in time.

Multitude of factors. In a very general interpretation, the passage of time can be associated with a multitude of effects. This interpretation is compatible with any motivation for time discounting that endorses it as a residual term that captures if prospects are temporally extended. For instance, φ^* could be a positive function of any mixture of the aforementioned interpretations, expressing the extent to which temporal parts are different on a number of dimensions for each point in time.

After developing the representations in the next sections, their interpretations will be compared on several dimensions and used to discuss descriptive and normative problems associated with the method of time discounting.

3.2 Time Preference

The canonical interpretation of time discounting in economics interprets the discount rate r in the exponential discounting function as a representation of time preference. At the heart of the time preference interpretation of time discounting lies the idea that time impatience plays a major role in intertemporal decisions. That is to say, it is assumed that agents have a preference for the present, and a preference over earlier rewards rather than later ones. Such preferences are supposed to be captured by the discount rate r which is then used to obtain discount factors (Definition 2).

It is worth noting that assuming time preference is a significant departure from and addition to standard expected utility theory. Agents as commonly modelled in decision theory and microeconomics have complete and transitive preferences over a set of prospects from which an additive utility function is derived. However, in addition to those preferences, the concept of *pure and positive time preference* is also commonly used in economic models when dealing with intertemporal decisions in which prospects are temporally extended and/or differ with respect to when they occur. That is to say, the concept of time preference introduces a new and additional type of preference. It can be further described as a structural preference as it is supposed to hold over the temporal dimension of all prospects. In addition, the concept of pure and positive time preference also implies that an agent has a preference for utility at earlier points in time over later points in time, in addition to his preferences over prospects. More specifically, *pure* time preference refers to the fact that the betterness which is expressed by time preference arises from distance in time. Farther, *positive* time preference refers to the fact that time preferences capture the idea that 'earlier' is better than 'later', with the present being the earliest point in time considered such that positive outcomes at it are preferred to all later points in time.

Frederick et al (2002) point our that for many of the precursors of time preference theories of time discounting, like Boehm-Bawerk, Fisher, Jevon and Pigou, the concept of time impatience was widely taken to be psychologically plausible and central in developing the idea of time discounting due to time preference. Pure and positive time preferences are also one of the key assumptions in Samuelson's (1937) model of discounted utility, which has served as the standard derivation of time discounting factors in economics. Several axiomatisations of discounted utility that follow Samuelson's original derivation have been proposed, the most influential being the ones by Koopmans (1960) and Lancaster (1963). The structure shared by these axiomatisations is to postulate that agents have pure and positive time preferences and then develop conditions on those preferences that jointly capture time impatience and preserve the additive utility function from expected utility theory. The conditions used in these derivation suggest that the rate of time preference r is constant for all periods.

Rubinstein/Fishburn (1982) make transparent the correspondence between conditions commonly introduced in utility theory and conditions on time preference. They provide an axiomatisation of pure and positive time preference in difference structures that gives exponential discounting, like the standard representations by Koopmans (1960) and Lancaster (1963). Their formal framework is based on a difference structure $\langle T \times X, \succeq, \rangle$, where T is a set of time points, X is a set of options that can occur at those times and $\succeq$ is a preference relation on such pairs of time points and consequences. Rubinstein/Fishburn (1982) use the standard conditions on rational preferences as a starting point and add further conditions to prove that the preference relation $\succeq$ can be represented as discounted utility.

The Rubinstein/Fishburn (1982) representation and, indeed, the aforementioned standard representations of time discounting, have the technical advantage of linking pure and positive time preferences with utility theory which makes available some motivations for the constraints on time preferences, e.g. transitivity and completeness. However, the additional conditions that lead to the representation of discounted utility with an exponential discounting function are introduced without explicit further motivation (i.e. motivations that go beyond the preservation of the utility function). This does not mean to deny that the preservation of the utility function cannot be motivated. An argument for introducing time discounting based on time preferences that lead to exponential discounting could point out the following: while preferences over prospects and the utility derived from such preferences are the standard for evaluating goodness, other evaluations of goodness based on specific domains such as time (or pairs of times and consequences) can be introduced when dealing with decisions for which such a dimension becomes relevant (for instance intertemporal decisions) and should be integrated into utility theory in a way that can amend it but preserves its basic properties.

Such an interpretation of the time preference approach raises the question as to what exactly should be the domain of time preference. Most approaches deal with pairs of times and consequences, while Ok/Masatlioglu (2007) provide an axiomatisation of time preference that also uses axioms on pairs of points in time. However, they use the latter only for deriving hyperbolic discounting functions while at the same time assuming standard time preferences on pairs of times and dates. In this approach, there are hence three domains of preference: consequences (for utility), pairs of consequences and times (for time preferences that give constant r and hence exponential discounting) and pairs of points in time (to derive hyperbolic discounting). Here, the question remains open as to why a multitude of domains for preferences can be introduced and according to what criteria such a step can be justified.

In the general framework presented in the previous section, the idea of time preference can be accommodated by seeing it as a preference for small time intervals between temporal parts and the present, such that $pp \in Q \times Q$ is preferred to all other intervals. In absolute-difference structures, pure and positive time preference can therefore be described as a binary relation on a set of points in time which are ordered according to closeness (or remoteness) to the present. The closer a point in time is to the present, the more points it will be preferred to. This idea of a preference for the present and a preference for closeness to the present has to be combined with the assumption that temporal parts do all carry positive utility.

Interpreting time preference in the representational framework of absolute-difference structures allows to distinguish between conditions on preference on the one hand and time preference on the other hand. While it is possible to postulate preferences that give well-behaved representations of time preferences in absolute-difference structures and indeed the standard exponential discounting function, the conditions will have to be motivated separately. This makes transparent the requirement to provide justifications for why time preferences have to satisfy, for instance, weak ordering, well-behavedness and solvability. Crucially, it makes transparent the fact that to obtain the exponential function representation of time discounting, there has to be an assumption that time preferences satisfy the condition of equal spacedness and that they correspond to some externally given time series.

More generally, upon describing the time preference interpretation of time discounting in terms of the general representational structure offered in the previous section, it becomes transparent that it is based on the idea of time impatience and preservation of the utility function.

3.3 Risk, Uncertainty and Ignorance

It has been argued that the passage of time is associated with an increase in fundamental uncertainty and that in general, evaluations of and information about the future are imprecise. Accordingly, time discounting could be justified with an appeal to uncertainty and the ambiguity over risk assessments. This amounts to claim that time requires additional structure in a model, even though there is already a probability function which represents an agent's beliefs which in turn is supposed to capture his ignorance about states of the world, facts, causal relationships and so on. On a general level, it is possible to motivate this claim by an appeal to the idea that such probability functions cannot capture all ignorance and uncertainty. The latter idea goes back to Knight (1921).

While not explicitly drawing on the idea that there is unmeasurable uncertainty, some authors have proposed to derive discounting functions from increasing uncertainty over time. For instance, Wu (1999), Weitzman (2001), Gollier (2001, 2002), Noussair/Wu (2006) and Epper et al (2009) motivate discounting functions by an increasing uncertainty over time, using time-indexed probability functions and risk evaluations to derive the correct functional form of time discounting functions. Conceptually, this amounts to a claim that probability evaluations become less credible the further away in time the temporal part of a prospect.

The uncertainty interpretation of time discounting has not led to specific representations of time discounting. Rather, variants of it have been used primarily for establishing specific functional forms of time discounting in empirical work. In finance, resource economics, environmental economics and public economics it is a standard practice to use time-index evaluations of risk to determine discount rates and discounting functions. While the literature is too diverse to review, it is important that the uncertainty approach to time discounting also highlights the problem of domain of preference and more generally the problem of "structural" preferences. Indeed, the papers cited above draw also on research about "risk preferences" which are understood as structural preferences alongside time preferences. Following the uncertainty interpretation of time discounting hence raises broader methodological concerns about the domain of preferences.

In absolute-difference structures, the influence on uncertainty can be integrated by providing an interpretation of $\langle Q \times Q, \succeq \rangle$. In the ignorance and uncertainty interpretation, the binary relation $\succeq$ and its representation φ^* can be viewed as measuring the increase of uncertainty and ignorance over time. The subjective time difference between temporal parts is hence interpreted as a difference of time-induced uncertainty. It is then possible to postulate constraints on time differences thus understood, resulting in specific functional forms that are due to time-induced uncertainty.

3.4 Preference Change

Associating time with fundamental changes, it can be argued to have an impact on the preferences of an agent. Indeed, it is possible to interpret time discounting as summarizing possible changes in propositional attitudes such as preferences and beliefs (e.g. Frederick et al. 2002:389). However, in expected utility theory and in economics, it is usually assumed that the preferences of an agent do not change over time. That is to say, a decision-maker is assumed to be completely stable with regards to his preferences. Relaxing that assumption can be used to account for time difference in terms of differences of preferences. In this interpretation, time difference is associated with changing preferences of a decision-maker. More formally, a decision-maker can be depicted as having a set of preferences at each point in time and preference change can be measured by comparing the sets. A measure of time difference thus obtained can then be used to represent time discounting as dependent on the extent of such changes.

In order to interpret the representational framework introduced in the previous section, its elements need to be restated. In analogy to temporal parts $p, q, r, s, t, \ldots$ in a prospect P, we can characterise different sets of preferences $s_1, s_2, s_3, s_4, s_5, \ldots$ in a decision-maker S. It is then possible to measure the differences between the different sets of preferences. More specifically, this interpretation makes available distance measures between sets to characterise the extent of preference change in a decision-maker. The usual assumption of a perfectly stable decision-maker over time with the same set of preferences is characterised by zero distance between all sets of preferences. In a general sense, time discounting is then determined by the extent to which preferences change.

Different distance measures can be introduced to characterise distances between sets of preferences more specifically. For instance, the Hamming distance measures the distance between two sets of binary relations by determining how many binary relations are the same between the two sets. As a simple example, let a domain of preference consist of the following options $\Omega = \{a, b, c, x, y, z\}$. Considering transitivity and completeness, there are 15 binary relations which are implied in an ordering of these elements. Let one set of preferences in S be $s_1 = \{a > b > c > x > y > z\}$ and another one $s_4 = \{a > b > c > x \sim y \sim z\}$. Then, the Hamming distance between these two sets is 3, since when comparing all binary relations there are 3 which are different between these two sets.[2] Time discounting factors can then be determined by considering all Hamming distances between all sets of preferences.

In the time discounting representation of absolute differences, the distances between sets of preferences can thus be used to determine the length of intervals between temporal parts. For this, it has to be assumed that sets of preferences coincide with temporal parts, in addition to all assumptions for the general time difference representation given in the previous section. In the preference change interpretation of time discounting, time differences can thus not only be suitably interpreted as preference change, but there is also a quantification of the preference change possible via distance measures. It is then also possible to determine the specific functional form of time discounting by discussing plausible behaviour of such distance measures. For instance, exponential discounting would imply that an agent changes her preferences with the same rate while hyperbolic discounting would imply that there are more changes in preference in the short run than in the long run.

3.5 Multi-dimensional Representation

The interpretations of time difference put forward so far have attempted to explain time difference on a single dimension. Interpreting time difference with impatience, uncertainty or rate of preference change, these representations of time discounting are therefore reductionist accounts of temporal distance. However, it is also possible to consider combinations of the aforementioned accounts. Farther, it is possible to simply postulate that there are a number of effects that arise with temporal distance and that an interpretation of time difference should therefore be multidimensional.

In this multi-dimensional view, time difference is determined by a number of factors. The subjective evaluation of intervals between elements in $Q \times Q$ can therefore be seen as a positive function of the aforementioned factors, and possible other factors as well. Hence, in addition

[2] The binary relations between xy, yz, xz are changed and the ones between $ab, bc, cx, ac, bx, cy, ax, by, cz, ay, bz, az$ are unchanged.

to satisfying the definition of an absolute-difference structure, and applying the representations that follow from it and the normalisation and transformation, is also holds that

$$\varphi^* = f^+(\text{impatience}, \text{uncertainty}, \text{rate of preference change}, \ldots).$$

A theory of time discounting that is based on such a multidimensional view of time difference would need to make specific claims about how exactly those factors influence time difference. The view is included here to show that time difference as used in the representation does not preclude such non-reductionist accounts of time difference. Indeed, it is possible to interpret the above function in a loose and informal way, taking it as showing that subjective perception of time difference is necessarily too complex to be captured more precisely. Conversely, it is also possible to use distance functions as showed in the previous section to provide measurements for the influence of specific factors.

More generally, such a multi-dimensional account can be viewed as the correspondence of what can be called "residual-factor" interpretations of time discounting in the framework of representation presented in the previous section. In such interpretations, time discounting factors are introduced to include the effects of time without further analysing these. In absolute-difference structures, this means that there are a number of determinants for the intervals between temporal parts that cannot be further specified.

4 Interpretation

4.1 Representing Time Discounting

In the previous sections, we have provided a general measurement-theoretic representation for time discounting as well as discussed four more specific interpretations of time difference. In this section, these interpretations are compared in four respects.

Firstly, the interpretations are discussed with regards to (i) the reductionism or non-reductionism they imply towards the fundamental concepts of time and preference and whether they are applicable under the assumption of pure intertemporal decision and (ii) the temporal structure of the agent (multiple-selves). Both of these comparisons yield further insights into the scope of the representations and their respective interpretations. In a general sense, note that the general framework of representation introduced here is capable of accommodating a diverse and large range of specific representations that differ widely with regards to the reductionism concerning time and utility they imply and the possibilities for their subjectivist interpretations in multiple-self model of decision-makers.

Secondly, the interpretations are discussed with regards to (iii) the permissibility of weighing utility with a discounting factor so derived and (iv) the functional form of time discounting (exponential v hyperbolic) they can endorse. Both of these comparisons yield insights into the way in which the representations and their respective interpretations deal with the widely debated descriptive and normative problems of time discounting. In a general sense, note that the framework of representation adopted here allows to distinguish between the derivation of the measurement-theoretic properties of time discounting, their interpretation and the various repercussions both have for the functional form of discounting and its overall justification.

4.2 Reductionism and Pure Intertemporal Decision

Integrating the idea of discounting utility for temporal distance into economic theory can be achieved by adhering to any of the interpretations introduced in the previous section. In this context, the question arises how the representations introduced here are compatible with decision theory. Ideally, a representation of time discounting would rest on assumptions that are identical with decision theory while at the same time capturing what is relevant about time for decision-making. However, it is impossible to jointly fulfil these two requirements: if we were to exclusively adopt the assumptions of decision theory, then the idea of temporal distance cannot be adequately expressed. In the Savage (1954) framework, indifference is required with regards to the temporal structure of prospects. In other frameworks, state-dependent utility can be used to do so but it remains open as to why another domain of preference can be introduced. Conversely, if we adopt a concept of time that includes intuitions about time differences, a preference for the present or some notion of temporal distance, more than the standard assumptions of decision theory are used to achieve the desired representation. Hence, any representation of time discounting is necessarily in complement to standard frameworks of decision theory. This suggests to investigate how exactly the representations of time discounting complement decision theory. The representations introduced in this paper can be read as formulating different ways to achieve a trade-off between the two requirements of staying close to the orthodoxy of decision theory and to adequately capture the relevance of time for decision-making.

Concerning time, it can be asked which one of the representations does best capture the relevance of time for decision-making. Time, after all, has a variety of complex and interesting features which have a profound impact on decision-makers. By its very nature, discounting for *temporal distance* reduces time's relevance for decision-making quite considerably. Accordingly, all approaches are limited by the way the general representation has been set up as it exclusively provides a representation of *time difference* or temporal distance. This already narrows down the scope of time discounting quite considerably. However, by providing a substantial interpretation of time distance, the representations introduced in this paper enrich this notion of time difference considerably, for instances when introducing multiple sets of preferences or a multitude of factors in the generalised interpretation. In contrast, the time preference and the uncertainty approach provide a very narrow understanding of time difference.

A further way to describe the nature of the trade-offs achieved is by characterising in how far a representation succeeds in staying close to the framework of expected utility theory. Which of the representations can reduce time difference to preferences, thereby staying close to the framework of standard decision theory? Apart from the preference change representation, all other representations introduce new concepts in order to interpret time differences. In the time preferences representation, preferences on a new domain are introduced to deal with time, namely preferences on time points (or pairs of utilities and time points). In the uncertainty representation, a measure of fundamental uncertainty that increases with time is introduced. In the generalised interpretation, any of those additional concepts can be used. In contrast, the preference change interpretation does not introduce an entirely new concept but defines the notion of distances between preference sets. While introducing more than one set of preferences and constructing their distance is a significant addition to the standard decision-theoretic framework, this approach can still be characterised as reducing time to a notion that is related to preference. However, staying close to the notion of preference will not alone settle the question as to which approach stays closest to frameworks of decision theory. It can be argued that in introducing time discounting as a representation of time preferences or fundamental uncertainty, the two approaches stay closer to the decision-theoretic framework than the preference change approach. Firstly, time preferences and fundamental uncertainty do share a great deal of the formal properties of preferences and beliefs. Secondly, it could be argued that assuming more than one set of preference represents a much more significant departure from the received picture of rational agency than introducing additional structure to capture specific types of preferences or beliefs.

Having compared how the four representations fare with regards to reducing time to preference and how they characterise time, many other interrelations between time and preference could be discussed. For instance, the phenomena of dynamic inconsistency, weakness of will, forming and carrying out plans and making self-promises have been widely discussed in the literature on time and decision. Such problems have been discussed in conjunction with changing preferences and the role that risk and uncertainty play for decision-making. It is convenient to assume away that there can be a change in preferences or uncertainty over time. The combined assumptions of stability of propositional attitudes and of certainty (or fully compensated risk), I call the assumption of pure intertemporal decision.

Definition 10 (Pure intertemporal decision) *A pure intertemporal decision is an intertemporal decision in which the agent has stable preferences and in which any uncertainty is fully represented by a probability function and compensated in the expected utility calculation.*

In such pure intertemporal decisions, the problem of the effect of temporal distance on the evaluation of goodness of a prospect is laid bare. None of the standard ingredients can be evoked to change in order to account for the influence of the time dimension. That means that both the time preference and the uncertainty interpretation of time discounting are unavailable und the assumption of pure intertemporal decision.

The assumption of pure intertemporal decision is introduced to be able to distinguish between two rather complicated and intertwined features, namely "temporal distance" and "temporal dynamics". In many discussions, intertemporality in decisions is treated in an all-encompassing way, treating problems of distance in time, changes in the propositional attitudes of the agent and factors in the decision environment all at once. Frequently, one phenomenon is "explained" by the other. For instance, many motivations for hyperbolic discounting draw on preference change (e.g. Ainslie 1992). Conversely, exponential discounting is motivated by stable preferences (e.g. Koopmans 1960, Lancaster 1963). In dynamic decision theory, appeals to discounting are used to motivate solutions to puzzles in intertemporal settings (e.g. Strotz 1956). Such problems can broadly be classified as arising from the dynamics of time, or constituting the dynamics of time in the realm of decision-making. In order to analyse the problem of time discounting and the way it is interpreted and represented with regards to time and preference, it is convenient to be able to assume away such temporal decision dynamics. This is not to deny that decision dynamics and the intertemporality of decisions are closely related and that in any real-world decision problem they will both play an important role. Indeed, much of the challenge of analysing intertemporal decisions lies precisely in capturing the interplay between those two factors. However, the concept of pure intertemporal decisions allows to set aside these complexities, isolating the intertemporal problem of *temporal distance*. In the representations of time discounting developed in this paper, only the time preference and the generalised interpretation are applicable under the assumption of pure intertemporal decision.

More generally, the representational framework introduced here makes explicit the fact that discounting for temporal distance needs a motivation in terms of a general measurable concept of time difference. Once an interpretation of temporal distance has settled, further more challenging problems of decision dynamics can be analysed. In such dynamic problems, for instance the problem to adhere to a prudent plan, a decision-maker has to evaluate various conditional propositions. Such conditionals may well have time-indexed antecedents and consequents as well as different evaluations, depending on the time of evaluations. However, these are already much more complex problems than pure intertemporal decisions. Making the assumption of pure intertemporal decision allows to explicitly rule out problems that can exist in addition to the problem of temporal distance. More generally, it highlights that in analysing intertemporal decisions, respecting the distinction of temporal distance and temporal dynamics is important.

4.3 Multiple-Selves

In order to understand the specific challenge of intertemporal elements in individual decisions, this section discusses intertemporality in decisions with regards to the internal temporal structure of the deciding person. If a decision-maker faces an intertemporal choice, her deliberations involve considerations about her beliefs and desires at different points in time. Hence an account of the temporal nature of the deciding subject's agency plays a key role in understanding this relationship. This is important to discuss the temporal occurrence of value and its relation to the decision-making subject which, in turn, is key for the subjectivist framework of decision theory.

Here, we interpret the significance of distance in time for agency with a "multiple-self" account of personal identity over time which treats different temporal parts of a person as distinct. This offers a unifying interpretative framework for representations of discounting, namely that the deciding person exhibits a looser connection to her more distant temporal parts (i.e. her later selves). If furthermore it is assumed that rational agency of a person can be based on such imperfect intrapersonal connectedness, then this representation also offers a justification of time discounting. This section introduces the underlying theories and uses them for an interpretation of the time discounting representations.

The notion of the multiple-self is studied in the context of theories of personal identity over time. Such theories, overviews of which can be found in Noonan (2003) or Raymond and Barresi (2003), investigate how a person both persists and changes over time. The seemingly contradictory nature of sameness and change in persons generates two main concerns in theories of personal identity over time. Firstly, emphasis is either given to sameness or to change of persons over time. At the extreme ends of the spectrum, theories focus either exclusively on sameness, such as the idea that persons are constituted by the soul which is assumed to be stable over time, or on difference over time, such as Hume's (1739) idea that there is great variation between different time slices within persons. Secondly, various criteria have been adopted to describe the substantive nature of personal identity over time. Examples for such criteria include different psychological features, the body, the brain, memory, emotions and consciousness.

Multiple-self theories, such as those of Parfit (1984), Elster (1986), and Ainslie (1992), understand persons as distinct yet interconnected selves. In these accounts, selves are capable of reasoning and acting, and are interconnected with each other to form a multiple-self. A multiple-self model of personal identity over time consists of three elements: a set of selves, a notion of intrapersonal connectedness between these selves and an interpretation of connectedness through a criterion of personal identity over time. Such models can be formulated in the absolute difference structures used in the representations of time discounting. Accordingly, they facilitate a substantive interpretation of such representations.

Formally, a multiple-self can be understood as a difference structure $\langle S \times S, \succeq, \rangle$ where S is a set of selves and $\succeq$ and its representations φ and φ^* characterise the connectedness between the selves. More specifically, $\succeq$ is a binary relation amongst pairs of selves that measures their difference and φ and φ^* their representations. The latter can be interpreted as connectedness functions that measure the strength of intrapersonal connectedness in the multiple-self, with intrapersonal connectedness being inversely related to the difference or distance measured. That is, for any two pairs of selves, the difference and distance measures give a value that characterises their connectedness. An interpretation of absolute difference structures in this vein gives both a formal model of a multiple-self and the possibility to link it to representations of discounting. The isomorphism between temporal parts of prospects and temporal selves of a decision-maker is immediate as the latter is the foundation for an evaluation of the former. Indeed, it can be argued that this is a more plausible interpretation of the absolute-difference structure in the realm of decision theory. As decision theory formalises the evaluation of goodness in a subjectivist way, taking into account the temporal structure of the deciding subject rather than the prospect under evaluation is closer to its spirit. This offers the possibility to also link the interpretations of time discounting with an interpretation of intrapersonal connectedness.

As mentioned above, in addition to a set of selves and a formal understanding of their connectedness, a substantive interpretation is needed as to what connectedness between selves amounts to. It is on this level that the parallels to the interpretations of the specific time discounting representations can be drawn. Adopting such a multiple-self model of personal identity over time, three substantive interpretations of underlying connectedness are now considered and linked to the interpretations of time discounting given in the previous section.

Sympathetic connectedness, such as proposed by Schechtmann (2001), measures the degree to which temporal selves can sympathize with each other. Such a sympathetic access expresses the strength of emotional bond between selves, supervening on physical and psychological features. This interpretation appears to mirror the time impatience interpretation that lies at the heart of time preferences. In the multiple-self view, a diminishing sympathy with later selves is the subjectivist motivation for the time impatience of an agent.

Memory connectedness, as originally proposed by Locke (1694) and further developed by Shoemaker and Swinburne (1984), measures the degree to which a self remembers having had an experience at an earlier time and thus expresses the extent of access to experiences of earlier selves. Viewing the connectedness between selves as constituted by memory has some parallels in the interpretation of time discounting as uncertainty in the role that information about specific points in time (or temporal selves) plays for both those accounts.

Psychological connectedness, mainly due to Parfit (1984), measures the degree of similarity between psychological traits of different selves, such as preferences. Accordingly, the temporal self is depicted as an agent acting on the basis of his preferences. This interpretation of connectedness in a multiple-self reflects the preference change interpretation of time discounting. Upon granting that a set of preferences is a sufficient description of the psychological make-up of a temporal self, the distance between those sets can both quantify the degree of connectedness as well as to determine the time discounting function.

Note also that by adopting and combining further criteria of personal identity over time, as reviewed by Noonan (2003) and Raymond and Barresi (2003), various multi-dimensional, more realistic accounts of connectedness are possible in the multiple-self model. The parallel option in the time discounting consists in combining any of the specific interpretations into a multidimensional representation.

This discussion shows that the general representation of time discounting can be interpreted directly with a multiple-self model which, in turn, makes accessible accounts in the literature of personal identity over time to discuss time discounting.

4.4 Justifying Discounting

Is time discounting not justifiable, permissible or even required of rational individuals? More specifically, can a future good be discounted for occurring far away in time or should it be weighted equally against less distant goods? In philosophy, it is mostly denied that time discounting of future goods is justifiable (Sidgwick 1907, Rawls 1971, Broome 1991). Yet in economics and finance, discounting factors based on Samuelson's (1937) DU-model are widely employed and in everyday life, interest rates and other incentives offer a premium to those who discount future goods less than others.

There is the question as to what kind of goods are discounted when employing time discounting. It is easier to justify time discounting when applied to money, capital or other goods or objects that deteriorate over time. In such contexts, time discounting can be justified and even required in terms of *preserving* the evaluation of goodness rather than conceptually changing or adding to it. However, can goodness understood more fundamentally, for instance as wellbeing or utility, also be discounted for temporal distance? This is a much harder question. Broome (1991:44) goes so far as to say that there is "more disagreement than misunderstanding" between philosophers and economists, asserting that typically, economists do not employ time discounting for wellbeing and utility whereas philosophers focus on the justifiability of the latter. While it is true that many applications of time discounting are not concerned with wellbeing and utility, the foundations of time discounting in economics are clearly concerned with discounting utility, as reviewed in the previous section. However, it is true that time discounting that there are more available routes in justifying time discounting were we to ignore the problem of discounting utility.

Discounting goodness or utility for distance in time has commonly been rejected in philosophy. An early instance of this can be found in Plato's *Protagoras*:

> "... if any one says: "Yes, Socrates, but immediate pleasure differs widely from future pleasure and pain" – to that I should reply: And do they differ in anything but pleasure and pain? There can be no other measure of them."

More recently, Rawls (1971:259) follows Sidgwick (1907) in saying that discounting future value as such is not permissible:

> "The mere difference of location in time, of something's being earlier or later, is not a rational ground for having more or less regard for it."

These assertions question that the value-making features of a state of the world could depend on its temporal occurrence. Moreover, they also establish that evaluating goodness is separate and should be prioritised. Indeed, when evaluating goodness, the standard that is chosen (for instance, pleasure, wellbeing or utility), should not be amended with other considerations.

This is in stark contrast to fact that the discounted utility-model is widely applied in economics. The somewhat unresolved situation is summarised in the following quote by Ramsey (1928):

> "It is assumed that we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible . . . we shall, however, . . . include such a rate of discount in some of our investigations."

These views establish that a separate reason is needed in order to justify time discounting. Farther, they suggest that time discounting will be in conflict with the assumed generality of the standard of evaluation that ethical theories usually evoke. That is to say, upon committing to utilitarianism or expected utility, little if any room is left for influencing evaluations beyond those deemed to determine goodness in the given theory. This raise the issue of whether the aforementioned representations and interpretations of time discounting can also serve as a normative justification for discounting.

Little is offered by the time discounting interpretations themselves; i.e. time preferences, uncertainty and preference change do not carry with them a set of arguments for their normative appeal. However, it has been shown that the interpretations do come close to expected utility theory in different ways: the time preference interpretation simply extends by its very nature the familiar concept of preferences, the uncertainty account also draws on the familiar concept of risk preferences or indeed probability functions and the preference change interpretation duplicates the preference set that is used in standard accounts. While this discussion makes explicit what kinds of assumptions are made by the accounts, it does not suggest how to make these also normatively plausible.

Consider again the multiple-self interpretations of personal identity over time. Accepting the multiple-self as a basis for rational agency would make it more plausible to accept the additions to decision theory that the time discounting interpretations carry with them. However, Sidgwick's and Rawls' normative rejection of time discounting already contains a statement about this problem, namely by Rawls (1971:259), who said:

> "Rationality requires an impartial concern for all parts of our life."

In contrast, consider Parfit (1984):

> "My concern for my future may correspond to the degree of connectedness between me now and myself in the future ... since connectedness is nearly always weaker over long periods, I can rationally care less about my further future."

This is countered by Williams (1970) who maintains that imperfect intrapersonal connectedness does not disburden from concern for future selves and Elster (1986) who claims that agency unifies temporal selves despite imperfect connectedness.

Due we assume that rational agency does not require to make (or indeed consist in making) such sweeping assumptions about the overall diachronic coherence and consistency of a decision-maker as Willams and Elster suggest, then it is possible to make the time discounting interpretations accessible via the multiple-self model. One argument to not make such general requirements of rational agency in expected utility theory is that the latter is not concerned with an as broad picture of rational agency as other ethical theories. This does, however, limit the scope of expected utility theory. In the same vein, there is an instrumental methodological argument in favour of taking the "disuniting metaphysics of personhood" Broome (2004) as basis for rational agency: just as decision theory implies separability of outcomes to draw a decision matrix and apply the consistency requirements of rational choice (Broome 1991), it can be argued that it also implies separability of selves across time (Broome 2004). The latter is not a mere formal requirement – it also ensures to make sense of rational agency at a time.

Hence, the general representation of time discounting offered in this paper makes explicit the crucial normative steps that would make time discounting normatively justifiable. Namely, it requires to make broader assumptions about rational agency and the role of time than decision theory usually does: in addition to the usual assumption of rational preferences, there is also an assumption that permits one of the time discounting interpretations to be applied. At the same time, this does not mean that a full picture of rational agency is offered, which on many accounts would preclude the permissibility of time discounting.

4.5 Exponential v Hyperbolic Discounting

The debate in economics about which method of time discounting to employ has been gaining notoriety in the last years and is also held in the context of the rise of behavioural economics. In this paper, no attempt to review the debate is made (an overview can be found in Angeletos et al (2001) and Frederick et al (2002)). How to prescribe the correct discounting function? Here, we briefly discuss the two methods of exponential and hyperbolic discounting, as defined in section 2, in the context of the interpretations offered in the previous section. The question, in terms of the general representation, is what kinds of further structural conditions on the absolute-difference structure can be motivated? The condition of equal spacedness between temporal parts (Axiom 8) that leads to exponential discounting (Corollary 9) has already been introduced. Conditions for hyperbolic discounting would postulate larger time differences in the near future and smaller time difference in the far future.

Concerning time preference, the approach has mainly been used in exponential discounting. However, this is mainly due to the close relations between the standard time preference representations and utility theory. When considering a time preference interpretation of discounting in the representational framework used in this paper, then it becomes clear that time impatience is at the core of the account. But even considered independently, time impatience can be motivated to satisfy equal spacedness. Take the example of a time impatient person which does not satisfy equal spacedness. Then she would be more impatient to wait for something one day but might consider herself equally impatient to wait 900 or 901 days. However, on the 900th day, she would then be very impatient and furthermore experience regret over not having had equally spaced regret. This suggests that the time preference account can justify exponential discounting.

Concerning uncertainty, this account has been mainly used to motivate hyperbolic discounting functions. In the representational framework presented in the previous section, this is also plausible. Postulating that uncertainty satisfies equal spacedness would rule out that we can perceive of differences in uncertainty in the near future as much more relevant than those in the far future. Indeed, in most real-life decision situations, the understanding of contingencies in a given time horizon is much better and thus makes more of a difference than in the long-run where such perceptions cannot be formed. This suggests the the uncertainty interpretation can justify hyperbolic discounting.

Concerning preference change, the account can be considered to be more plausible for hyperbolic discounting than for exponential discounting. Interpreting time difference with the rate of preference change, it is plausible that preference change does not behave uniformly over time. Indeed, it is not plausible that when considering distances between preferences, that these distances are constant which would be needed to support the assumption of equal spacedness. Moreover, when considering that the interpretation of preference change does not refer to an external observer but to a perceived or projected preference change on part of the decision-maker herself, then the assumption of a time horizon is much more plausible. Such time horizons could be given by rites of passage in one'e life, such that up until a specific event such as retirement, changes in preference are perceived to be rather drastic whereas from a given point of view in one's working life one assumes the rate of preference change to be fairly stable thereafter. Note how an inconsistency as alluded to in the time impatience case would not present an argument against such a hyperbolic horizon here, as it would imply that agents experience regret over not having had better access to the extent of their future preference change. This suggests that in the preference change interpretation, hyperbolic discounting is easier to justify.

The discussion suggests that exponential discounting is supported by a time impatience interpretation (corresponding to sympathy connectedness in the multiple-self), whereas hyperbolic discounting is better supported by a preference change interpretation (corresponding to psychological connectedness in the multiple-self) and an uncertainty interpretation (corresponding to memory connectedness in the multiple-self).

5 Conclusions

The paper has provided a representational framework for time discounting, giving a representation for discounting factors in absolute-difference structures. In this framework, a discounting factor is a ratio-scale representation of differences that can be related to the temporal parts in a temporally extended prospect. Formally, this provides a structure to assess the temporal element of prospects in intertemporal decisions. Conceptually, it can be related to a number of interpretations of time differences, including time preferences, uncertainty and preference change.

The framework has been employed to revisit the normative and descriptive debates about time discounting. The time difference interpretations have been compared with regards to their implied reductionism about time and preference, the kinds of multiple-self accounts they can accommodate, the functional forms of time discounting they best support and the kinds of normative justifications they make permissible.

The paper thus clarifies a number of problems with regards to time discounting, distinguishing the problem of temporal distance from time-induced decision dynamics, making explicit assumptions about agency, domain of preference and interpretation of time difference that underlie the current descriptive and normative debates of time discounting and making accessible multiple-self interpretations of rational agents.

References

Ainslie, G. 1975. Specious Reward: A Behavioral Theory of Impulsiveness and Impulse Control. *Psychological Bulletin* **82(4)**:463–96.

Ainslie, G., 1992. *Picoeconomics.* Cambridge University Press.

Ainslie, G., 2001. *Breakdown of Will.* Cambridge University Press.

Angeletos, G. et al., 2001. The hyperbolic consumption model: Calibration, Simulation, and empirical evaluation. *Journal of Economic Perspectives* **15**:47–68.

Barro, R. et al. 1986. Ramsey Meets Laibson in the Neoclassical Growth Model. *Quarterly Journal of Economics* **114(4)**:1125–52.

Bleichrodt, H. et al., 2008. Koopmans' constant discounting for intertemporal choice: A simplification and generalization. *Journal of Mathematical Psychology* **52**:341–7.

Broome, J., 1991. *Weighing Goods.* Basil Blackwell.

Broome, J., 1999. *Ethics out of Economics.* Cambridge University Press.

Broome, J., 2004. *Weighing Lives.* Oxford University Press.

Elster, J., 1986. *The Multiple Self.* Cambridge University Press.

Epper, T. et al, 2009. Uncertainty Breeds Decreasing Impatience: The Role of Risk Preferences in Time Discounting. Institute for Empirical Research in Economics University of Zurich, Working Paper Series, 412.

Fishburn, P.C., Rubinstein, A., 1982. Time Preference. *International Economic Review* **23(3)**:677–94.

Frederick, S., 1999. *Discounting, Time Preference and Identity.* PhD Thesis, Carnegie Mellon University.

Frederick, S. et al., 2002. Time Discounting and Time Preference: A Critical Review. *Journal of Economic Literature* 40(2):351–401.

Gollier, C., 2001. *The Economics of Risk and Time.* Cambridge,. MA: The MIT Press.

Gollier, C., 2002. Discounting an uncertain future. *Journal of Public Economics* **85**:149–66.

Herrnstein, R. 1981. Self-Control as Response Strength. In C.M. Bradshaw et al (eds) Quantification of Steady-State Operant Behavior, North Holland: Elsevier, 3-20.

Hume, D., 1739. *A Treatise of Human Nature.* Clarendon, Oxford.

Koopmans, T., 1960. Stationary Ordinal Utility and Impatience. *Econometrica* **28(2)**:287–309.

Knight, F., 1921. *Risk, Uncertainty, and Profit.* Boston, MA: Hart, Schaffner & Marx.

Krantz, D. et al., 1971a. *Foundations of Measurement.* Vol. I. Dover Publications.

Krantz, D. et al., 1971b. *Foundations of Measurement.* Vol. II. Dover Publications.

Krantz, D. et al., 1971c. *Foundations of Measurement.* Vol. III. Dover Publications.

Laibson, D. 1986. Golden Eggs and Hyperbolic Discounting. *Quarterly Journal of Economics* **112(2)**:443–77.

Laibson, D. et al. 1986. Self-Control and Saving for Retirement. *Brookings Papers on Economic Activity* **1**:91–196.

Lancaster, K., 1963. An Axiomatic Theory of Consumer Time Preference. *International Economic Review* **4(2)**:221–31.

Locke, J., 1694. *An Essay concerning Human Understanding.* Book II, chapter XXVII. Clarendon, Oxford.

Loewenstein, G., Elster, J., 1992. *Choice over Time.* Russell Sage.

Loewenstein, G., Prelec, D. 1992. Anomalies in Intertemporal Choice: Evidence and an Interpretation. *Quarterly Journal of Economics* **107(2)**:573–97.

Loewenstein, G., Read, D., et al., 2003. *Time and Decision: Economic and Psychological Perspectives on Intertemporal Choice.* Russell Sage.

Manzini, P., Mariotti, M., 2007. Choice over Time. *IZA Discussion Paper Series*, No. 2993.

Mazur, J. 1987. An Adjustment Procedure for Studying Delayed Reinforcement. In: Commons et al (eds) The Effect of Delay and Intervening Events on Reinforcement Value. Hillsdale: Erlbaum, 55-73.

Noonan, H. W., 2003. *Personal Identity.* Routledge, London.

Noussair, C., Wu, P., 2006. Risk Tolerance in the Present and the Future: An Experimental Study. *Managerial and Decision Economics* **27**:401–12.

Ok, E., Masatlioglu, Y., 2007. A Theory of (Relative) Discounting. *Journal of Economic Theory* **137**:214–45.

Parfit, D., 1984. *Reasons and Persons.* Clarendon, Oxford.

Phelps, E., Pollak, R. 1986. On Second-Best National Saving and Game-Equilibrium Growth. *Review of Economic Studies* **35(2)**:185–99.

Rawls, J., 1971. *A Theory of Justice.* Harvard University Press.

Raymond, M., Barresi, J. (Eds.), 2003. *Personal Identity.* Blackwell, Oxford.

Ramsey, F.P., 1928. A Mathematical Theory of Saving. *Economic Journal* **38(152)**:543–59.

Samuelson, P. 1937. A Note on Measurement of Utility. *Review of Economic Studies* **4**:155–61.

Samuelson, P. 1939. The Rate of Interest Under Ideal Conditions. *The Quarterly Journal of Economics* **53(2)**:286–97.

Schechtmann, M., 2001. Empathic Access. *Philosophical Explorations*, **4**, 95–111.

Shoemaker, S., Swinburne, R.G., 1984. *Personal Identity.* Blackwell, Oxford.

Sidgwick, H., 1962(1907). *Methods of Ethics.* 7th Ed. Macmillan.

Strotz, R., 1956. Myopia and Inconsistency in Dynamic Utility Maximization. *Review of Economic Studies* **13**:165-80.

Tversky, A., Krantz, D., 1970. The Dimensional Representation and the Metric Structure of Similarity Data. *Journal of Mathematical Psychology* **7**:572–96.

Weitzman, M.L., 2001. Gamma Discounting. *The American Economic Review* **91(1)**:260–71.

Williams, B., 1970. The Self and the Future. *Philosophical Review* **79**:161–80.

Wu, P., 1999. Anxiety and Decision Making with Delayed Resolution of Uncertainty. *Theory and Decision* **46**:159–98.