There may be infinitely many near coherence classes under $\mathfrak{u} < \mathfrak{d}$

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Outline

Mappings between filters

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A filter is a non-principal proper filter on ω.

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Let $f : \omega \to \omega$ be finite-to-one. We set $f(\mathscr{F}) = \{X : f^{-1}X \in \mathscr{F}\}$.

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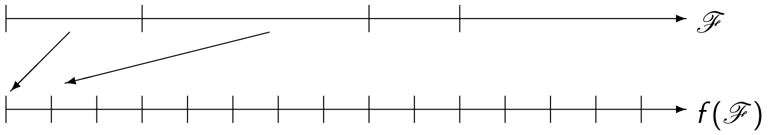

Near coherence of filters

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Two filters $\mathscr{F}$ and $\mathscr{G}$ on ω are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a proper filter.

If $\mathscr{U}$ is an ultrafilter, then also $f(\mathscr{U})$ is an ultrafilter.

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If $f(\mathscr{U}) = f(\mathscr{V})$ and $g(\mathscr{V}) = g(\mathscr{W})$, then there is a slower growing finite-to-one function h such that $h(\mathscr{U}) = h(\mathscr{W})$.

Fact

The near-coherence relation is an equivalence relation on the ultrafilters on ω.

Its classes are called near-coherence classes of ultrafilters.

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Possible numbers of near-coherence classes

Theorem. Booth, Galvin, Mary-Ellen Rudin, Blass

Under CH, there are 2^{2^ω} near-coherence classes of ultrafilters.

Theorem. Blass, Shelah, 1987

It is consistent relative to ZFC that there is just one near-coherence class of ultrafilters.

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Theorem. Banakh, Blass, 2005

If there are infinitely many near-coherence classes of ultrafilters then there are 2^{2^ω} classes.

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$\mathfrak{d} \leq \mathfrak{u}$ implies that there are infinitely many near-coherence classes of ultrafilters.

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The role of $\mathfrak{u}$ and $\mathfrak{d}$

$\mathfrak{u}$ comes in as the minimal number of steps in constructing one representative of one class.

Proposition. Blass, 1987

There is a set D, a so-called test set, of size $\mathfrak{d}$ such that any two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent, if there is some $f \in D$ with $f(\mathscr{U}) = f(\mathscr{V})$.

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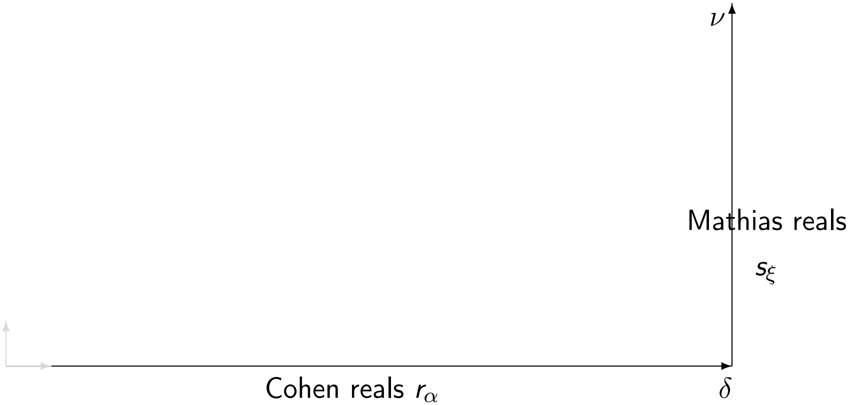

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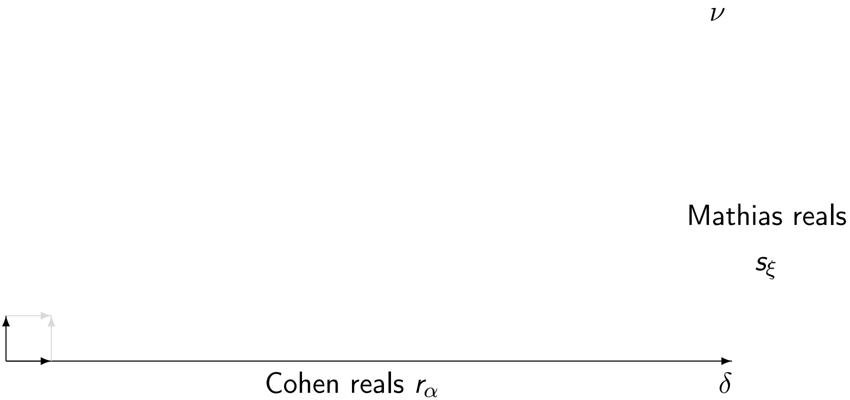

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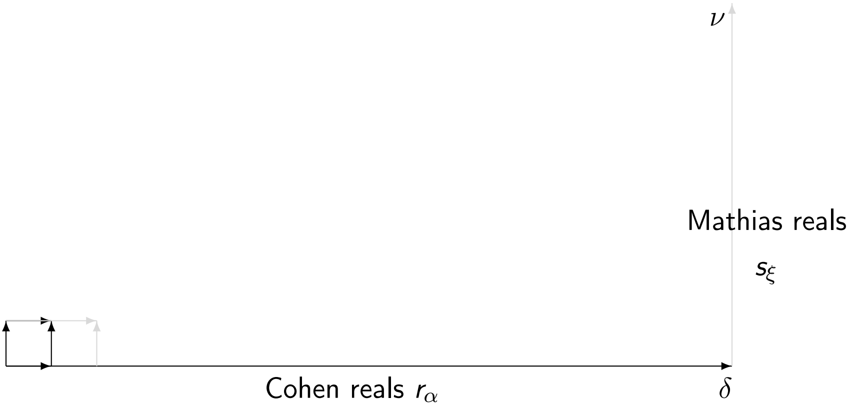

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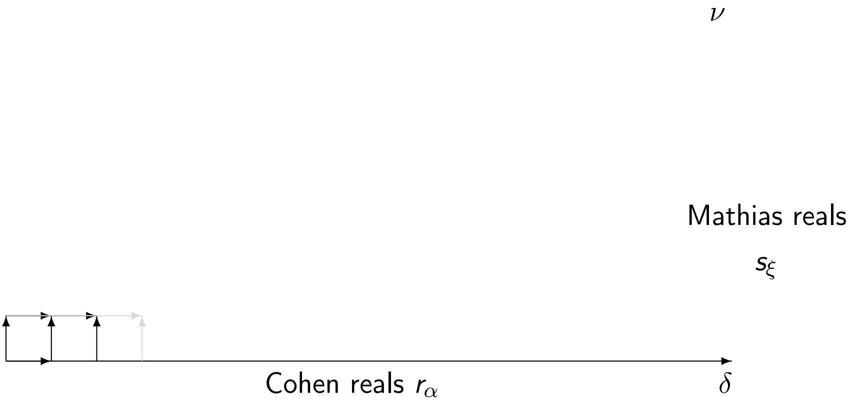

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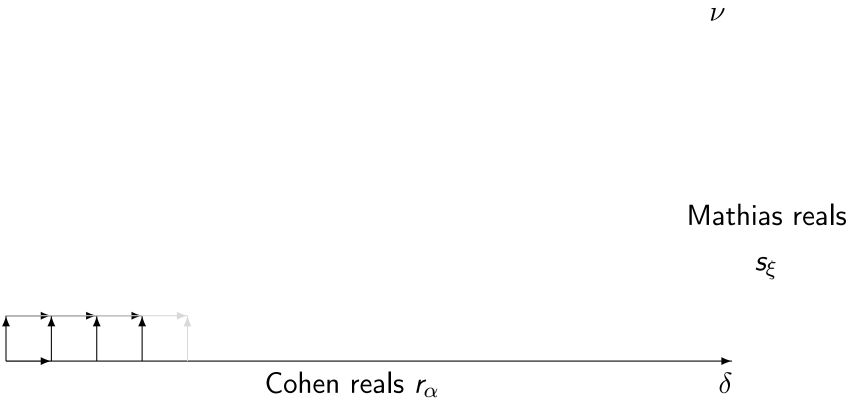

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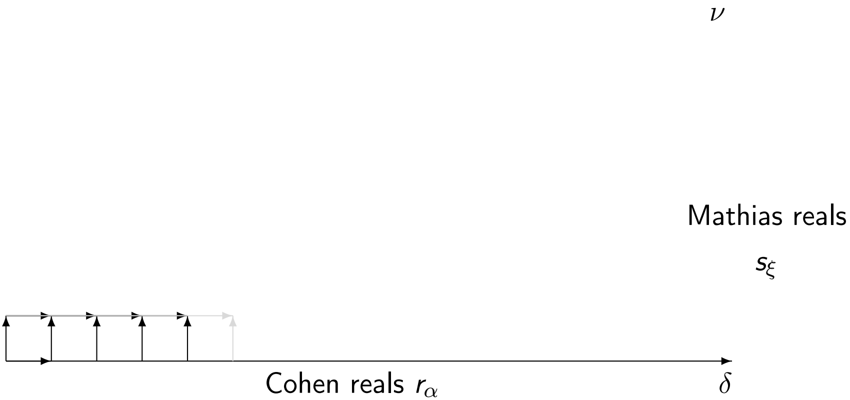

An iteration with a rectangular structure

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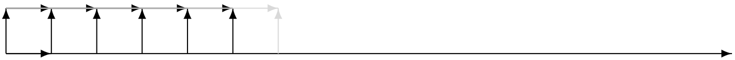
Mathias reals

s_ξ



Cohen reals r_α δ

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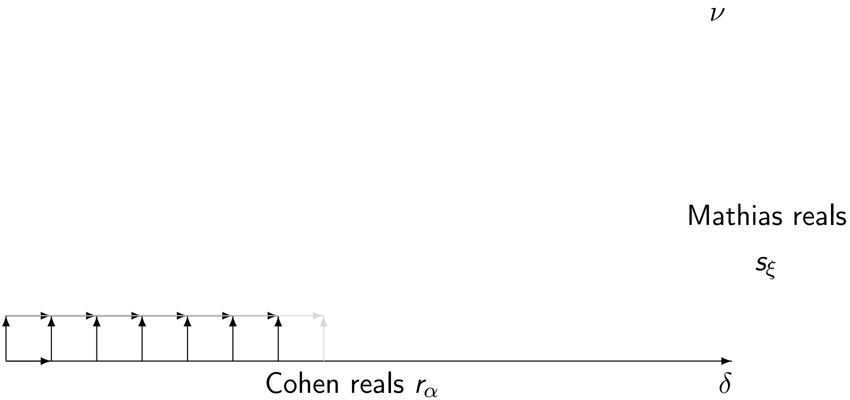

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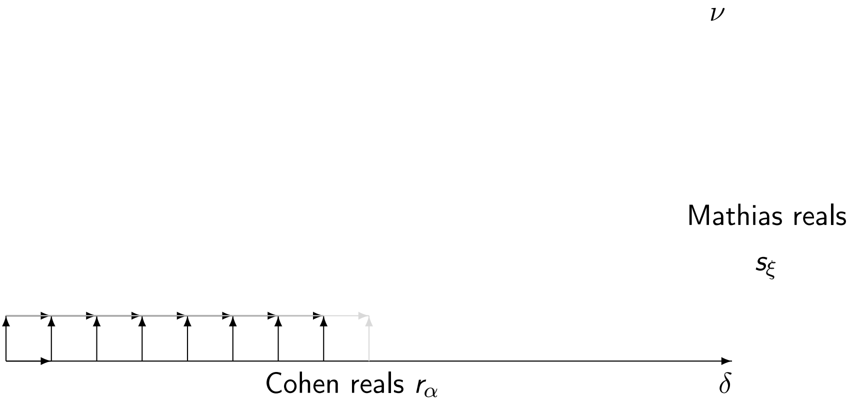

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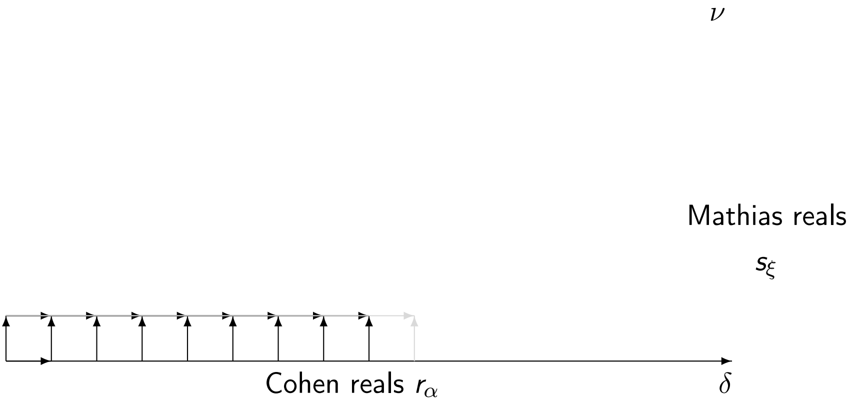

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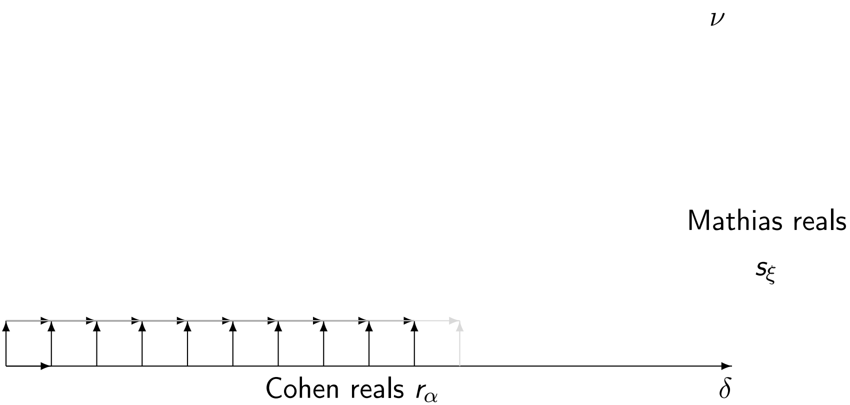

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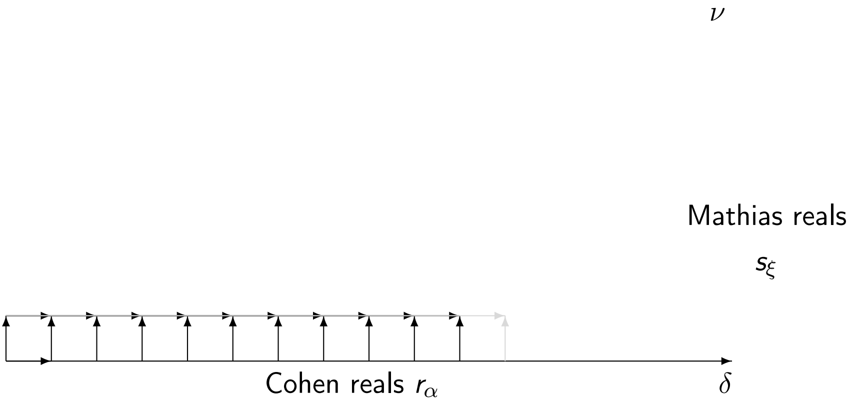

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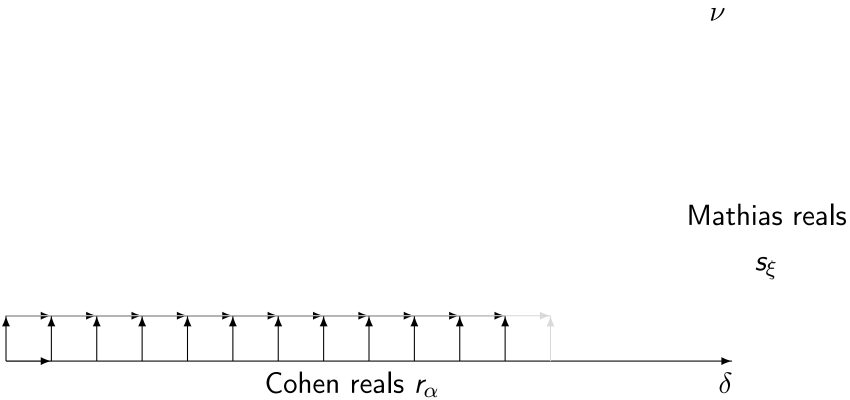

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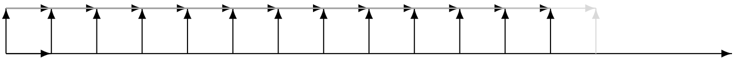

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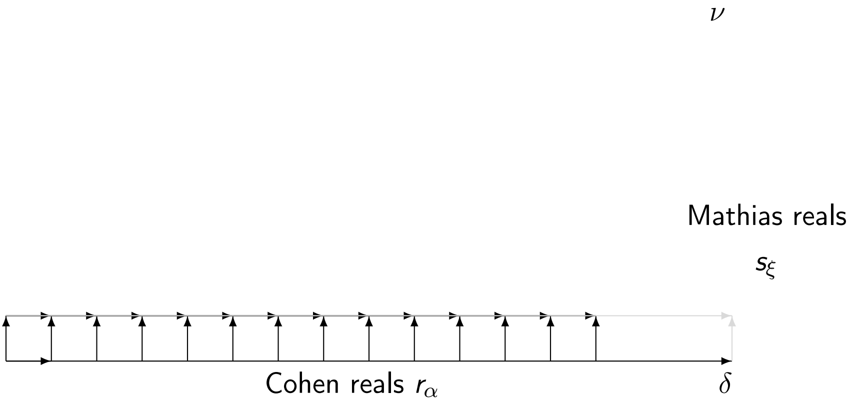

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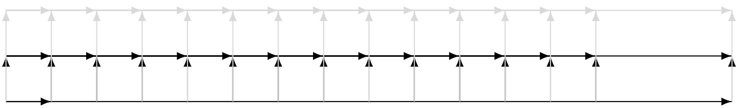

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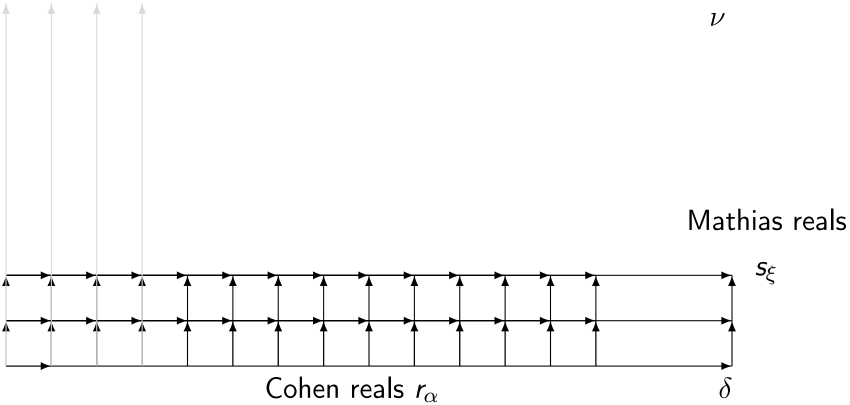

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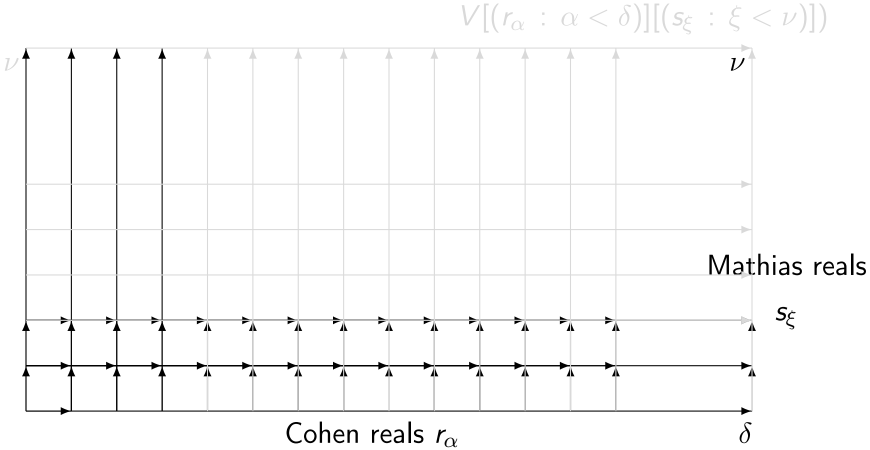

Figure 1: A sketch of $V[(r_\alpha : \alpha < \delta)][(s_\xi : \xi < \nu)]$

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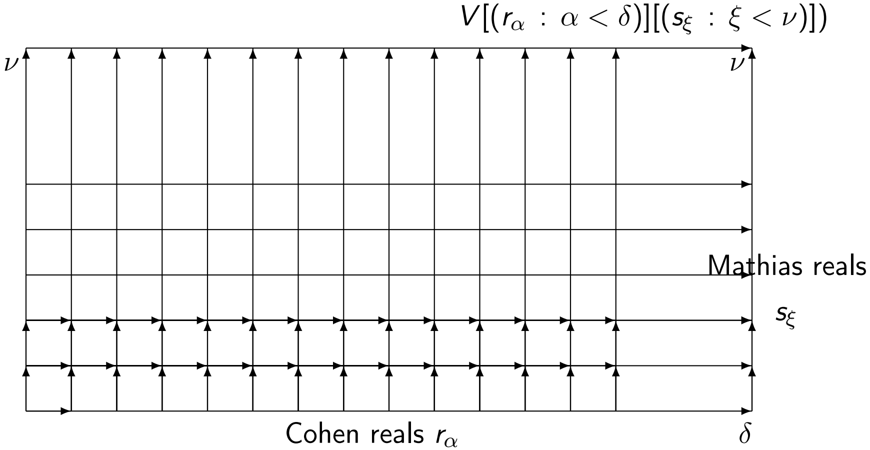

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The forcing construction

Let V be a ground model of CH. Let ν and δ be regular cardinals such that $\aleph_1 \leq \nu < \delta$.

First δ Cohen reals are added (or something else) in a finite support iteration, call them r_α, $\alpha < \delta$. Thereafter ν Mathias reals are added by Mathias forcings $Q(\mathscr{U}_\xi)$, $\xi < \nu$, in a finite support support iteration. We call the whole forcing $\mathbb{P}$.

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A variant of Mathias forcing

A forcing condition in $Q(\mathscr{U}_\xi)$ is a pair (a, A), such that a is a finite set of natural numbers and $A \in \mathscr{U}_\xi$ and $\max(a) < \min(A)$.

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We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

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If a filter $\mathscr{F}$ and a ultrafilter $\mathscr{U}$ are not nearly coherent, then $\mathfrak{d}(\mathscr{F}) \leq \chi(\mathscr{U})$.

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