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Axiomatic Quantum Mechanics and Completeness

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Abstract The standard axiomatization of quantum mechanics (QM) is not fully explicit about the role of the time-parameter. Especially, the time reference within the probability algorithm (the Born Rule, BR) is unclear. From a probability principle P1 and a second principle P2 affording a most natural way to make BR precise, a logical conflict with the standard expression for the completeness of QM can be derived. Rejecting P1 is implausible. Rejecting P2 leads to unphysical results and to a conflict with a generalization of P2, a principle P3. All three principles are shown to be without alternative. It is thus shown that the standard expression of QM completeness must be revised. An absolutely explicit form of the axioms is provided, including a precise form of the projection postulate. An appropriate expression for QM completeness, reflecting the restrictions of the Gleason and Kochen-Specker theorems is proposed.

Keywords Axioms of quantum mechanics · Completeness · Gleason's theorem · Kochen-Specker theorem · Born rule · Projection postulate · Probabilities as dispositions

1 Introduction

Quantum mechanics (QM), the most elementary of quantum theories, can be shown to be complete in a quite precise sense. *It is impossible to consistently assign values to the observables of a suitable QM system, given two plausible constraints*. Take a physical system S such that its QM representation requires a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) > 2$. (A one-particle spin-1 system is an example.) It is impossible, in this case, to assign values to all QM observables under the constraints that

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(i) there is a one-one correspondence between observables and QM operators representing them (non-contextuality), and (ii) the algebraic relations among such value assignments mirror the algebraic relations among the operators (functional composition) [1]. An *indirect* proof of this fact is immediately obtained from Gleason's Theorem [2]. The theorem entails that, when $\{P\}$ is the set of all projection operators on $\mathcal{H}$, every mapping $\mu: \{P\} \rightarrow [0, 1]$ being interpretable as a probability function must be continuous, while a value assignment obeying (i) and (ii) must induce a mapping $\mu': \{P\} \rightarrow \{0, 1\}$ that is discontinuous [3]. A *direct*, i.e. constructive proof is the Kochen-Specker theorem [4–6],[1] presenting a finite set of operators (called a KS set) for which an assignment obeying (i) and (ii) fails.

But why does the impossibility of assigning values under these constraints tell us anything about QM completeness? After all, the theory's empirical output consists just in probabilities for measurement results and their generalizations: expectation values. In an axiomatic formulation, QM is formally incapable of directly making value assignments, so it cannot generate anything conflicting with any value assignment to S. The natural idea filling this logical gap is the insight that some probability assignments entail value assignments, namely those that predict values with certainty. E.g., a QM prediction to the effect that S, with probability 1, will be found to have a property a_k at time t makes it plausible to conclude that S, at that time, has a_k. Working, from now on, in the Schrödinger picture, writing states as density operators, and taking $\boldsymbol{A}$ as a discrete observable on S with values $\boldsymbol{a}_i$ (with $i = 1, 2, \ldots$) we can express this idea as:

$$\text{If } \mathbf{P}_{a_k}(t), \text{ then } \boldsymbol{a}_k(t).$$

(Here and henceforth boldface '$\boldsymbol{a}_k$' abbreviates 'S has $\boldsymbol{a}_k$' and, accordingly, '$\boldsymbol{a}_k\ (t)$' abbreviates 'S has $\boldsymbol{a}_k$ at t' (where k is some value of i). Likewise, '$\mathbf{P}_{a_k}(t)$' abbreviates 'S is in state $\mathbf{P}_{a_k}(t)$', where $\mathbf{P}_{a_k}(t) = |a(t)\rangle\langle a(t)|$.) Adding such a plausible rule to the QM formalism, we can extract value assignments, but nothing near a set of values big enough to conflict with either the discontinuous assignment used in the corollary of Gleason's Theorem or the assignment to some KS set of operators. This will be the case only if we *limit* ourselves to the value assignments following from the QM state as follows:

$$(\mathbf{EE}) \quad \boldsymbol{a}_k(t) \text{ if and only if } \mathbf{P}_{a_k}(t).$$

This condition establishes a logical link between QM and the two theorems and thus makes precise in which sense they prove QM completeness. Indeed, the condition (often called the eigenstate-eigenvalue link, hence the label 'EE') embodies the classic definition of QM completeness [7, 8]. EE substantiates the generally accepted and most familiar idea that a QM system in a superposition of $\boldsymbol{A}$ eigenstates does not have a value of $\boldsymbol{A}$. This idea plays a special role when interpreters specify how S, being in a superposition of $\boldsymbol{A}$-eigenstates, interacts with an $\boldsymbol{A}$-measurement device. Following a suggestion by von Neumann, it is standardly claimed that S, during the measurement interaction, *takes on* one of the $\boldsymbol{A}$ values, e.g. $\boldsymbol{a}_k$ [9]. If S is found to

[1]While the original argument [4] requires 117 projection operators, the simplest argument now can be given with only 18 projectors [5]. For a proof that this is the smallest possible set, see [6].

have a value of $\boldsymbol{A}$, e.g. $\boldsymbol{a}_k$, at a certain time, then EE dictates that S's state is the pertaining eigenstate, e.g. $\mathbf{P}_{a_k}$, at this time. A slightly less exact form of this consequence would be: If S is found to have value $\boldsymbol{a}_k$, of $\boldsymbol{A}$, then S's state immediately afterwards becomes the pertaining eigenstate e.g. $\mathbf{P}_{a_k}$. This latter requirement is generally called the *projection postulate* [10, 11]. Projection, i.e. S's adopting an $\boldsymbol{A}$-eigenstate during $\boldsymbol{A}$-measurement, is generally thought to be an empirically confirmed fact and with good reason. We can measure copies of S for $\boldsymbol{A}$, filter out the non-$\boldsymbol{a}_k$ results, and then experimentally confirm the remaining state to be $\mathbf{P}_{a_k}$, e.g. via quantum-state tomography.[2] Thus, it will be assumed here that projection is an ineliminable feature of QM, but the discussion will also address interpretations rejecting it.

For future reference, let us explicitly extract a completeness condition (COMP) from EE

$$(\textbf{COMP})\quad \text{If } S \text{ is not in state } \mathbf{P}_{a_k}(t),\ \text{then not } \boldsymbol{a}_k(t).$$

Trivially COMP expresses the backward or 'only if'-direction of EE. Now, the aim of the present paper is to show that COMP is not in harmony with QM, in a standard axiomatization, and to provide a more appropriate expression for completeness. More exactly, I will show COMP to be in conflict with QM as follows. I briefly review a standard axiomatization for QM and point out that two axioms are not fully unambiguous concerning the role of the time parameter. I introduce three reasonable principles, P1–P3, where P3 is a generalized version of P2. These principles explicate basic tenets of interpreting probabilities in a physical theory, in general, and in QM, in particular. For the reader's orientation each of them is given a name besides the numbering. Principle P1 (which may be called the *probability principle*) explicates the idea that probability (in a physical theory) is quantified possibility. The other two principles address the time parameter's role in QM probabilities and remove the ambiguity in the axioms. Principle P2 (called the *simple principle*) provides the most natural disambiguation of the well-known trace formula for QM probabilities. Principle P3 (called the *general principle*) rules that there must be some disambiguation and embraces P2 and the only clear alternative. (All axioms and principles are introduced in Sect. 2.) Now, we can disambiguate the trace formula via the simple principle P2 and then add our proposal for completeness, COMP, to the (now unambiguous) QM axioms. The result may be called *simple complete QM*. But simple complete QM contradicts the probability principle P1—this is the main result (exposed in Sect. 3). The prospects of sacrificing P2 are discussed in Sects. 4 and 5 with a negative result. The resulting truncated version of QM is an unphysical theory as it can neither contain an informative projection postulate nor can respect the general principle, P3. Since the argument is a fundamental attack on COMP it is natural to question the principles despite their plausibility. Accordingly, principles P1–P3 are extensively discussed (in Sect. 6). They are shown to be without reasonable alternative. The remaining culprit thus is COMP, our standard way of expressing QM completeness. This result will initially lead to formally improving the axioms themselves, especially to formulating an acceptable form for the projection postulate (Sect. 7). Finally, a version of completeness will be proposed that both represents the limitations due to the Gleason and Kochen-Specker theorems and does not conflict with QM + P1–P3 (Sect. 8).

[2] See Refs. (11, 12) in [12].

Some preparations for the discussion in Sects. 4 and 5 must be made. Certain interpretations of QM reject the projection postulate and EE; they are now collected under the title of *modal interpretations* [13–16]. This group of interpretations has a weaker expression of completeness at hand:

(**weak COMP**) If S is in a pure state $W(t) \neq \mathbf{P}_{a_k}(t)$, then not $\boldsymbol{a}_k(t)$.

The rationale of weak COMP is that completeness and projection are really independent requirements: While a measurement may leave S in a mixture (obtained by partial tracing of the state of the S-cum-apparatus-supersystem) such that we can say that S has adopted one of $\boldsymbol{A}$'s values without state projection, we still can express the idea that S, having been in a pure state $W(t)$ at interaction onset, did not then have any value of $\boldsymbol{A}$. We may address QM without projection as *weak QM* and weak QM in conjunction with weak COMP as *weak complete QM*. This version (or rather interpretation) of QM is an interesting one, in general, but is not a viable option in the context of rejecting our simple principle P2. The argument for this claim proceeds as follows. Sacrificing P2 must go along with rejecting the projection postulate, thus we are naturally lead to considering weak QM as the remaining form of QM, in view of the completeness results, and weak COMP as the appropriate completeness expression (Sect. 4). However, in conjunction with the negation of P2 there will be no plausible way for this approach to respect the general principle P3 (Sect. 5).

It should be clear that, by nature of the foundational and conceptual questions raised, the reasoning must consist of concept interpretation and logical, not mathematical, argument throughout.

2 Axioms and Principles

Consider the following simple axiomatization of QM,[3] using again the Schrödinger picture and projection operators:

Axiom 1 *Any QM system S is associated with a unique Hilbert space $\mathcal{H}$ and its state is represented by a unique density operator $W(t)$ on $\mathcal{H}$, a function of time.*

Axiom 2 *Any physical quantity* A (*called an observable*) *is represented by a self-adjoint operator $\boldsymbol{A}$ on $\mathcal{H}$ and the possible values of* A (*possible properties of S*) *by the numbers in the spectrum of $\boldsymbol{A}$.*

Axiom 3 *S evolves in time according to $W(t) = U(t)W(t_0)U(t)^{-1}$ where $U(t) = \exp[-i\boldsymbol{H}t]$, a unitary operator, is a function of time and $\boldsymbol{H}$ is an operator representing the total energy of S.*

Axiom 4 *If S is in state $W(t)$ and $\boldsymbol{A}$ is an observable on S, then the expectation value $\langle \boldsymbol{A}\rangle$ is*: $\langle A\rangle = \mathrm{Tr}(W(t)\boldsymbol{A})$.

[3] This axiomatization is standard; see, e.g., [17], Sect. 3.2.

Axiom 5 *If S is found to have value $\boldsymbol{a}_k$ as result of an $\boldsymbol{A}$ measurement, then S's state is $\mathbf{P}_{a_k}$ immediately afterwards.*

Henceforth QM is the theory based on Axioms 1–5 and weak QM the theory based on Axioms 1–4 only. Axiom 2 motivates an identification of physical observables and their mathematical representatives and I will not need to distinguish them. I will also, for simplicity, restrict myself to one discrete observable $\boldsymbol{A}$ throughout. Finally, I will mostly restrict Axiom 4 to probabilities, i.e. expectation values of yes-no observables of type $\mathbf{P}_{a_i}$. Let $\boldsymbol{a}_k$ always be some fixed value of variable $\boldsymbol{a}_i$. Let '$p(\boldsymbol{a}_k)$' mean the probability that S has $\boldsymbol{a}_k$. Then, since $\langle \mathbf{P}_{a_k} \rangle = p(\boldsymbol{a}_k)$, Axiom 4 takes on a simpler, very familiar form, often called the Born Rule (BR):

(**BR**) If S is in state $W(t)$ and $\boldsymbol{A}$ is an observable on S with eigenvalue a_k, then the probability that S has $\boldsymbol{a}_k$ is: $p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$.

Two points should be noted here. First, it should be emphasized that these axioms, though fairly standard, do not constitute a fully satisfactory axiomatization of QM since Axioms 4 and 5, in their present form, leave the role of the time parameter unspecified or vague. The defect in Axiom 4 carries over to BR, in whose equation only the right side, but not the left, carries a time-index. Two of the three principles, to be introduced presently, will have the sole purpose of forcing an unambiguous explication of the time-parameter on the left side of '$p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$' and it should be stressed that the interpretations produced from these principles and considered below exhaust all the reasonable options.

The second point is related. Some interpreters recommend a formulation of BR that mentions measurement and interprets '$p(\boldsymbol{a}_k)$' as *the probability of finding outcome $\boldsymbol{a}_k$ upon measurement.* On reflection, this proposal is more involved and, by invoking our understanding of 'measuring' and 'finding as an outcome', insinuates the mentioned von Neumann picture according to which, in some cases, result $\boldsymbol{a}_k$ is something realized in S some time after measurement onset. This picture deserves explicit discussion, but not insinuation. For that purpose, the present formulation is chosen, which is entirely neutral. Inside '$p(\boldsymbol{a}_k)$' it just mentions $\boldsymbol{a}_k$, the property S eventually has, regardless of whether S has this property *as an outcome* of something or *is found* having it. Moreover no presupposition is made on *when* S has $p(\boldsymbol{a}_k)$. To the contrary, it will be a matter of argument (in Sects. 5 and 6.4) whether we need to answer this question and eventually which answer is compatible with QM. Finally, in this neutral form (with the mentioned ambiguity awaiting clarification) BR *follows* from Axiom 4 and $\langle \mathbf{P}_{a_k} \rangle = p(\boldsymbol{a}_k)$ (see Appendix A)—while the more involved form does not.

Here are three principles. Because of their far-reaching implications for QM, they will be discussed extensively later on (in Sect. 6). The first principle can be motivated from the idea that probability is quantified possibility. More precisely: If a physical theory assigns an event a non-zero probability, then, given the theory's truth, this event is possible. The weakest form of possibility is logical possibility. Thus, yet more precisely:

(**P1**) (*Probability principle*)
If, for a proposition F (describing an event) a theory T yields another proposition $p(F) > 0$, then it is not the case that $T, F \vdash \perp$.

(Here '$T, F \vdash \perp$' means that the set of sentences including F and all sentences of T allows deriving a contradiction in first-order logic.) P1 is beyond reasonable doubt, but it also follows from natural assumptions about probability shared by the main interpretations of that notion (see Appendix B).

The second principle provides the simplest disambiguation of BR, i.e. the simplest way to explicate the time-reference on the left side of '$p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$'. It runs:

(**P2**) (*Simple principle*)
Any expression '$\boldsymbol{a}_k$' such that it names a QM event can be qualified as $\boldsymbol{a}_k(t)'$, where t is a time-parameter.

P2 is motivated by the idea that a fundamental physical theory must explicitly concern space-time events. A theory that builds probability spaces over sets of events must be able to explicitly treat these events as space-time events to qualify as a fundamental physical theory. Hence, all events that are assigned probabilities in QM must explicitly be space-time events. In the present case, these events are just properties (values $\boldsymbol{a}_k$ of $\boldsymbol{A}$) possessed at certain sharp times. We may call events of this special type *sharp space-time events*. We will see later (in Sect. 6) that most theoretical approaches to QM respect P2 and for good reason. P2 mentions all occurrences of '$\boldsymbol{a}_k$' in QM, hence also the ones in Axiom 5. Thus, rejecting P2 will have consequences for Axiom 5 (Sect. 6).

Note, however, that P2 just says that those events denoted by statements of type '$\boldsymbol{a}_k$' within the QM probabilities are sharp space-time events such that the expressions can be explicated as '$\boldsymbol{a}_k(t)$'. The '$\boldsymbol{a}_k$' may not be appropriate expressions of QM events within BR and our simple principle P2 may have no application. The third principle, P3, generalizes P2. It says that whatever the QM events are (and whatever expressions denote them) these events, within BR, can be qualified as sharp space-time events explicitly. Thus:

(**P3**) (*General principle*)
For any expression 'F' such that QM yields an expression '$p(F) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$' there is a parameter t in the formalism qualifying 'F' as '$F(t)$'.

P3 gets its name because it just prescribes a general, not a specific, disambiguation for the Born Rule, BR. The principle is formulated so wide as to appear vague. But it has only two specifications. The first is to place the time-index 'inside the probability' (as commanded by the simple principle, P2), the second 'outside the probability'. Consider the BR expression '$p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$' made precise as '$p(\boldsymbol{a}_k(t)) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$'. This makes QM fulfill P2. But there is an alternative: Read the expression on the left of '$p(a_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$' as '$p(t)(a_k)$' and interpret the latter in the following way: The probability is a disposition of S at time t to display value a_k (make '$\boldsymbol{a}_k$' true). This idea is discussed widely in the foundational literature and is generally explicated as saying that t is the onset time of a measurement interaction on S and $p(t)(\boldsymbol{a}_k)$' *quantifies* S's *strength of disposition at* t *toward displaying* $\boldsymbol{a}_k$ *at some later time*. However, while this notion essentially refers to the idea of probabilities as dispositions it does not need to refer to measurement. We should avoid the impression that anything in our principles or axioms makes essential reference to measurement—as this is in fact unnecessary. (But see Sect. 6 for a discussion of measurement.) We can speak more generally of a region of space containing S and a time t such that '$E(t)$' denotes an event we may call the triggering event. Then, $E(t)$ and state $W(t)$ in conjunction determine S's disposition at t to display $\boldsymbol{a}_k$ at some later time. Accordingly, an alternative to the preceding is to disambiguate '$p(\boldsymbol{a}_k)$' as '$p(t)(\boldsymbol{a}_k)$', which more explicitly reads '$p(\boldsymbol{a}_k$ given $E(t))$'. We thus have an alternative disambiguation of BR that obeys our principle P3. (Note that the argument for only two possibilities is not strict. It would require heavy metalinguistic machinery to show that '$p(\boldsymbol{a}_k(t))$' and '$p(t)(\boldsymbol{a}_k)$' are the *only* ways to specify the time-reference in '$p(\boldsymbol{a}_k)$'.)

3 Simple Complete QM Contradicts the Probability Principle (P1)

P1 and P2 now generate the main argument against COMP. Using P2, we can make BR precise in a most natural way. It can now be rendered more exactly:

(**BR**$'$) If S is in state $W(t)$ and $\boldsymbol{A}$ is an observable on S with eigenvalue $\boldsymbol{a}_k$, then the probability that S has $\boldsymbol{a}_k$ at t is: $p(\boldsymbol{a}_k(t)) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$.

Now suppose that QM is supplemented by the simple principle P2. Assuming that a theory contains all its consequences, QM + P2 will contain BR$'$. Now supplement QM + P2 with COMP. Also, suppose that S is in a state $W(t_1) \neq \mathbf{P}_{a_k}(t_1)$, for some value t_1 of t, such that from BR$'$ it follows that $1 > p(\boldsymbol{a}_k(t_1)) > 0$. Call this assumption N. Finally, let QM + P2, COMP, and N be integrated into one artificial theory, simple complete (scQM). Then, by simple sentential logic:

N	(1)	S is in state $W(t_1)$	(N)
N, BR$'$	(2)	$p(\boldsymbol{a}_k(t_1)) > 0$	(1), (BR$'$)
N	(3)	$\neg\mathbf{P}_{a_k}(t_1)$	(N)
N, COMP	(4)	$\neg\boldsymbol{a}_k(t_1)$	(3), (COMP)

(As usual, the rightmost column indicates the assumptions on which the line in question directly depends and the leftmost column the ones on which the line ultimately depends.) By assumption, BR$'$, COMP, N, are members of scQM which thus entails both line (2), i.e. that a certain proposition is assigned a positive probability, and line (4), i.e. that the negation of that proposition is true. Hence, scQM entails $p(\boldsymbol{a}_k(t_1) > 0$, but also: scQM, $\boldsymbol{a}_k(t_1) \vdash \perp$, in contradiction with P1. Thus given P1, scQM cannot be true.[4]

[4] Alternatively, given P1(a–c) from Appendix B, scQM cannot have a positive probability of being true.

The argument presupposes that scQM, the artificial integration of QM, P2, COMP and assumption N is a *theory*. Is the integration of N an innocuous step? We can, of course, add suitable propositions to QM to create a theory that contradicts virtually any other proposition. But N is a trivially admissible state assignment that QM must be consistent with. So, its integration into scQM is innocuous indeed, but the one of P2 is not. BR′, COMP, N are in conflict with P1, where BR′ is BR, interpreted via the simple principle P2. Given that the probability principle P1 is immune to rejection, QM is in conflict with either P2 or COMP.

4 Rejecting the Simple Principle (P2) Implies Weak QM

Let us anticipate that rejecting the probability principle P1 is not a plausible reaction to the argument (but see below Sect. 6 for a discussion) and that the most plausible defense of COMP consists in rejecting the simple principle P2. To do so implies to give up on the most natural disambiguation of BR. The defender of COMP will just say that within the BR equation '$p(\boldsymbol{a}_k)$' cannot be read as '$p(\boldsymbol{a}_k(t))$', the impression of naturalness notwithstanding. But to reject P2 has consequences for Axiom 5. Recall that this axiom, like Axiom 4 and BR, is vague. We can apply P2 to Axiom 5, yielding:

Axiom 5′ *If S is found to have value $\boldsymbol{a}_k(t)$ as a result of an* $\boldsymbol{A}$ *measurement, then, S's state is* $\mathbf{P}_{a_k}$ *immediately afterwards.*

Note that Axiom 5′ still contains the imprecise 'immediately afterwards' from Axiom 5. But it is an advance over the latter because it contains an exact time to which 'immediately afterwards' can refer. Rejecting P2 would mean that Axiom 5 does *not* contain such a time reference. It would mean, in effect, that expressions like '$\boldsymbol{a}_k$' are *not* explicated as '$\boldsymbol{a}_k(t)$' throughout QM. In this case, Axiom 5 automatically becomes vacuous. In the Schrödinger picture, state evolution cannot start without a precise input state. It is the intention of Axiom 5 to generate such an input—for starting post-measurement state evolution, e.g. when a measurement is a preparation. Axiom 5′ can be seen as a first attempt to make this idea precise. As a necessary condition of precision, Axiom 5′ contains an exact time-index for '$\boldsymbol{a}_k$', but it fails to be sufficiently precise unless the phrase 'immediately after' is made precise. If Axiom 5 is understood as containing an expression '$\boldsymbol{a}_k$' that must not carry a time-index, it is interpreted as not meeting this condition. It is a vacuous statement, not only without any empirical content, but also a formally ineffective addition to the rest of QM. *So, everyone seeking to escape the argument of Sect. 3 by rejecting P2 will have to reject the projection postulate in any substantial form.*

One might object that applying principle P2 to BR is one thing and applying it to Axiom 5 another. But if we reject P2 for the expression '$p(\boldsymbol{a}_k)$' (refuse to read it as '$p(\boldsymbol{a}_k(t))$') we say that these probabilities do not refer to events of type '$\boldsymbol{a}_k(t)$', nor are they tested by observations of events of type '$\boldsymbol{a}_k(t)$'. It is inconsistent under these strictures to allow such an event nevertheless and put it in the antecedent of Axiom 5.

Hence, rejecting P2 implies rejecting Axiom 5 and what I have called weak QM, a theory based on Axioms 1–4 only. The interpretations taking this route are the modal interpretations. Here we must not consider this group of interpretations, in general, but a queer and artificial variant built on negating P2. Note that ¬P2 immediately transforms QM into an unphysical theory. Checking the Axioms, we note that QM (reasonably enough) contains a *unique* time parameter. (The same goes for weak QM.) If the theory supplies a time-index for '$\boldsymbol{a}_k$' in '$p(\boldsymbol{a}_k)$', to obey P2, it must be this one. If, vice versa, '$p(\boldsymbol{a}_k)$' does not inherit the time-index directly from the state, i.e. from the right side of '$p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$', then it does not get any time-reference, at all. QM + ¬P2 does no longer furnish the measurement results, for which it provides probabilities, with exact time-indices.

5 Weak Complete QM Makes the General Principle (P3) Implausible

We have seen that negating the simple principle P2 implies QM without projection, i.e. weak QM. As pointed out in Sect. 2, weak QM has its own version of a completeness expression: weak COMP. Is this weaker version of QM a possible way to maintain the weaker version of COMP? We may call weak QM + weak COMP by the name of weak complete QM. The problem of this approach is that weak COMP allows an argument exactly along the lines of the one against COMP in Sect. 3. (In the sentential logic argument, line (4) also follows from (3) and weak COMP.) So, weak complete QM must reject either the probability principle P1 or the simple principle P2. I anticipated that negating P1 is unreasonable, but negating P2 might be plausible. After all, there seems to be an alternative disambiguation for the Born Rule! The QM events to consider might not be S's possessing values of $\boldsymbol{A}$, but 'takings-on', S's displaying values upon some triggering event. However, it will turn out that this alternative offers no plausible way to respect the general principle P3.

The defender of weak complete QM will have to reject P2 and BR′. Given the assumption, made plausible above, that there is but one alternative way to specify BR, we will now rewrite it as:

(**BR**″) If S is in state $W(t)$ and $\boldsymbol{A}$ is an observable on S with $\boldsymbol{a}_k$, then the probability that S has $\boldsymbol{a}_k$ given $E(t)$ is:
$p(\boldsymbol{a}_k \text{ given } E(t)) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$.

Assuming the triggering event $E(t)$ to be the onset of an $\boldsymbol{A}$-measurement interaction, we recover the idea, found in von Neumann [7] and other classical textbooks [18, 19], that QM probabilities essentially are conditional upon measurement, and the idea that these probabilities are dispositions, possessed by S (or the whole of S and the apparatus) at time t, for S possessing $\boldsymbol{a}_k$ at some later time. However, as has just been pointed out, this later time cannot be referred to in QM because the theory, as axiomatized here, does not have the formal resources to refer to two times. (Similarly, again, for weak QM.)

So, in the expression '$\boldsymbol{a}_k$ given $E(t)$' the '$\boldsymbol{a}_k$', referring to S and the time at which eventually it has $\boldsymbol{a}_k$, cannot bear a time-index. We thus have consciously violated P2, but not necessarily P3, since '$\boldsymbol{a}_k$ given $E(t)$' does contain a time reference, after all. Now, although this possibility exists we are now considering a theory that does not allow time-indices for the measurement results for which it makes predictions. Inevitably there will be difficulties with empirically testing such a theory. So the discussion at this point takes on an unphysical and academic character. But worse is still to come. A theory along these lines cannot plausibly meet the general principle P3, anymore.

Probability expressions of the form '$p(B$ given $A) = z$' (where $z \in [0, 1]$) have been thoroughly investigated in the context of QM [20–22] and three possible analyses have been found: '$p(B \mid A) = z$', '$A \rightarrow p(B) = z$', and '$p(A \rightarrow B) = z$', where '$\rightarrow$' is a conditional connective awaiting further semantic analysis. It should be added that philosophers and logicians have mounted substantial evidence that, in general, $p(B \mid A) \neq p(A \rightarrow B)$, for standard explications of '$\rightarrow$' [23]. So, these two forms of explicating '$p(B$ given $A) = z$' are indeed logically different and we have (at least) three interpretations for the expression. In the present context, we have the special condition that 'B' in '$p(B$ given $A) = z$' must not bear a time-index, i.e. in the relevant BR$''$ expression '$p(\boldsymbol{a}_k$ given $E(t)) = z$' '$\boldsymbol{a}_k$' must not be time-indexed—to escape the contradiction of Sect. 3.

Let's take the three analyses of '$p(\boldsymbol{a}_k$ given $E(t)) = z$', in turn. It is easy to see that '$p(\boldsymbol{a}_k \mid E(t)) = z$' is not a live option. The standard (Kolmogorov) definition of conditional probability is inapplicable, since this would require '$p(\boldsymbol{a}_k \wedge E(t))$' and '$p(E(t))$' to be well-defined, which they are not. Defining them appropriately would mean to import them into QM from elsewhere—something which is clearly inadmissible in a theory dubbed fundamental and, technically, would break the axiomatic closure of the theory. Alternatively, conditional probabilities can be defined as primitive two-place functions from pairs of events into the unit interval [24, 25], but the axioms ruling the interpretation of these functions as probabilities require expressions like '$p(E(t) \mid \boldsymbol{a}_k)$' to be well-defined. Again, no version of BR can supply such probabilities and importing them from elsewhere is out of the question.

Consider second '$E(t) \rightarrow p(\boldsymbol{a}_k) = z$'. This variant contracts two problems. 'z' is a placeholder for '$\mathrm{Tr}(W(t)\mathbf{P}_{a_k})$', in BR$''$. Hence, we have the conditional '$E(t) \rightarrow p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$' containing, as its consequent, an equation '$p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$'. By assumption, this equation is no longer vague, but defined to lack a time-index on the left and carry one on the right. For a mathematical function depending on some parameter, this is an inconsistent requirement. Moreover, exporting the time-reference from the set of events that get assigned probabilities via QM violates our principle P3.

Consider third $p(E(t) \rightarrow \boldsymbol{a}_k) = z$. This possibility respects P3. But the unphysical assumption that its consequent '$\boldsymbol{a}_k$' must not bear a time-index makes it impossible to distinguish a case where '$\boldsymbol{a}_k$' is true at some unspecified time directly after $E(t)$ from a case where '$\boldsymbol{a}_k$' is true at a much later time. This allows constructions of obviously false cases. Suppose that S is a one-particle spin-$\frac{1}{2}$ system in $W(t_1) = \mathbf{P}_{a_m}(t_1)$, where $\boldsymbol{a}_m \neq \boldsymbol{a}_k$ is another eigenvalue of $\boldsymbol{A}$. Suppose that $E(t_1)$ is the onset of a measurement interaction consisting in a series of measurements $\boldsymbol{A} - \boldsymbol{B} - \boldsymbol{A}$ (where $[\boldsymbol{A}, \boldsymbol{B}] \neq 0$). Suppose that, for some copies of S, despite the initial state $\mathbf{P}_{a_m}(t_1)$, the second $\boldsymbol{A}$-measurement yields result '$\boldsymbol{a}_k$'. For these copies '$E(t_1) \rightarrow \boldsymbol{a}_k$' is true and yet $p(E(t_1) \rightarrow \boldsymbol{a}_k) = \mathrm{Tr}\,\mathbf{P}_{a_m}(t_1)\mathbf{P}_{a_k} = 0$, in violation of P1. Of course, we will understand the physics of the experiment and say that the probability of '$\boldsymbol{a}_k$' being true directly after $E(t_1)$ is zero and rises during the course of the whole experiment, but without a time reference we lack the possibility to distinguish different instances of '$\boldsymbol{a}_k$'. The point is not that we cannot come up with an intelligible distinction of instances of '$\boldsymbol{a}_k$', but rather that we cannot do so *within* the present (mutilated) version of QM, where BR is interpreted as BR$''$, which subsequently is read as delivering expressions of type '$p(E(t) \to \boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$'.

So there is no clear possibility at all how, given ¬P2, the general principle P3 can be met, i.e. how the time reference in the Born Rule can be made explicit. Note that this argument for a violation of P3 is non-rigorous because it is built on two unproven meta-assumptions: (i) that the two disambiguations sketched in Sect. 2 and used at the beginning of this section are the only possible ones; (ii) that the three proposed analyses of '$p(\boldsymbol{a}_k$ given $E(t)) = z$' exhaust the possibilities.

6 Are the Principles Reasonable?

Gleason's theorem and Kochen-Specker-type theorems prove that QM, in some sense, is complete. COMP is generally perceived as an appropriate expression of this completeness. COMP implies the von Neumann picture of measurement: a QM system S in a typical case (pure non-eigenstate of $\boldsymbol{A}$ and onset of an $\boldsymbol{A}$-measurement) does not have a certain kind of property (a value of $\boldsymbol{A}$), but 'takes on' a property (an $\boldsymbol{A}$ value) during the measurement interaction. In other words: in QM measurement does not, in general, reveal existing values of observables. The main point of this paper is to attack COMP and the ensuing von Neumann picture. COMP is not in conceptual harmony with the very theory the completeness of which it is meant to express. The disharmony can be transformed into a contradiction by using simple and reasonable principles for physical theories. Obviously, the attack is fundamental and a large amount of our present conceptions of QM is at stake. Hence, it is natural to question the principles on which it is based. Despite their plausibility one or more of them might just be remnants of pre-quantum physics that we are forced to let go. To investigate this intuition, I will consider for every one of the three principles whether it makes sense to abandon it—with a negative result. Moreover, I will briefly discuss whether looking into the process of QM measurement can help to solve the problem—again with a negative result.

6.1 The Probability Principle (P1)

The probability principle, P1, says that no reasonable physical theory should make derivable two statements of the form $\neg F$ and $\mathrm{prob}(F) > 0$. In symbols: If $T \vdash p(F) > 0$, then not $T, F \vdash \perp$. I have emphasized that it is eminently reasonable and follows from natural assumptions shared by the main conceptions of probability.[5] So this principle seems entirely immune to rejection. But is it? Consider the following counterargument. Assume a proposition K to be the conjunction of all value assignments to observables in a KS set (a set such that no consistent value assignment is

[5] See again Appendix B.

possible).[6] Clearly, there are QM states such that every conjunct of K gets a strictly positive probability. So, apparently QM entails $p(K) > 0$, but also QM, $K \vdash \perp$. And so P1 must be false despite appearance to the contrary.

Let's initially suppose, for the sake of argument, that this reasoning were sound. Then we would be caught in a veritable dilemma between QM and scientific reason. Imagine a fantasy theory that, in violation of P1, yields for a certain event F (e.g., heads in a coin toss, z^+ for a spin $\frac{1}{2}$ particle in a suitable state) that $p(F) = \frac{1}{2}$ and also entails that F (heads, z^+) is not the case. (QM is *not* such a theory; we are just fantasizing here!) Imagine how we test this theory. Either the first F (heads or z^+) result will falsify it or the rising number of $\neg F$ (tails or z^-) results (the only possible results, by the theory's lights) in repeated trials with identical copies will inductively prove that $p(F) = 0$, again falsifying the theory. So, the theory is false. Indeed, since F was arbitrary, every theory that entails the negation of P1 is false. Now, QM, by all we know, is not false, but if it *entails* the negation of P1 it must be false (and this is a dilemma).

So, negating P1 is simply not an option and interpreters have followed von Neumann's much more reasonable intuition that no logical conflict arises because for QM events F, F' the propositions $p(F) > 0$ and $\neg F'$ will only be in conflict given $F' = F$, but this will be the case only given another principle, i.e. our simple principle P2 that nails F and F' to the same time and thus identifies them. Von Neumann's idea of S 'taking on' a value during the measurement process is in effect a negation of P2. F and F' are simply events at different times, so $F' \neq F$ and $p(F) > 0$ and $\neg F'$ are not in any conflict. The argument in Sects. 4 and 5 blocks this strategy of negating P2. But that does not bring us back to the dilemma. There is no dilemma because the above reasoning is unsound. It is, of course, true that QM, $K \vdash \perp$ because K in itself is contradictory. (To repeat: K cannot be derived from QM but from the assumption that all observables in a KS set have values that are (i) non-contextual and (ii) respect functional composition!) But it is, of course, false that QM $\vdash$ prob$(K) > 0$. The fact that, for a suitable state, QM for every conjunct of K entails a positive probability does not mean that QM for the conjunction yields a positive probability. As we all know, the formalism itself forbids this because K's members refer to non-commuting observables. And, of course, QM $\vdash p(K) > 0$ would be catastrophic, because K is a contradiction. (The achievement of Kochen and Specker was not to show, for a KS set, value assignments to be contradictory, but just to present a concrete KS set. The contradiction is evident.)

6.2 The Simple Principle (P2)

The trick of blocking (in Sect. 4) the von Neumann response is to ask whether value ascriptions to S (mentioned in BR and Axiom 5) need to be tagged with a value of the time-parameter. If the question is answered in the affirmative it is obvious from the axioms that only one is available: the one that generates the argument against COMP. But it is the simple principle P2 that prescribes that every value ascription *needs* a time-index. So what about P2's plausibility? Let me first show, from examples, that

[6] In this case, K will be a conjunction of no less than 18 propositions. See [5, 6].

this principle is widely accepted in theorizing about QM and then make plausible that it must be so accepted.

First, the reader should recall, from undergraduate days, the Born Rule for the *wave function* according to which "$|\Psi(\mathbf{r},t)|^2 d^3\mathbf{r}$ = the probability of finding the particle at time t in the volume element $d^3\mathbf{r}$" [26]. In *this* equation, the time-parameter appears (most reasonably) on both sides. Moreover, since being in $d^3\mathbf{r}$ is a property that the particle is said to possess at t (i.e. the parameter refers to the event *within* the probability expression) we have a crystal-clear acceptance of P2. General forms of Axiom 3 or BR are not similarly explicit, but of course representations of QM states in projection operator or ket-vector form are easily translated into coordinate representation [27]. The interpretation of the wave-function just quoted will be recovered only if BR, in a more general form, respects P2.

Second, consider representations of QM that recast BR in terms of transition probabilities between S's states [28]. (This approach can be integrated into a full axiomatization of the theory [29]. The purpose here is to highlight the parallelism of Hamiltonian classical mechanics and QM. From the foundational viewpoint such a treatment has the advantage of combining the essential content of Axioms 4 and 5 into one axiom, but the disadvantage of leaving implicit the acceptance of EE.) Adapting the crucial axiom for projection operators, we can specify a QM probability as follows. Let S's state be $W(t) = \mathbf{P}_b$, for some $\mathbf{P}_b$, and let $\mathbf{P}_{a_k}$ be a yes-no observable corresponding to the question 'Does S have value $\boldsymbol{a}_k$ of $\boldsymbol{A}$?', then we write: $p(\mathbf{P}_b, \mathbf{P}_{a_k}) = \mathrm{Tr}(\mathbf{P}_b, \mathbf{P}_{a_k})$. Now, obviously $p(\mathbf{P}_b, \mathbf{P}_{a_k}) = p(\mathbf{P}_{a_k}, \mathbf{P}_b)$, so if $\mathbf{P}_b$ can be time-dependent, so can be $\mathbf{P}_{a_k}$. We have, in effect written out the probability for a transition from $\mathbf{P}_b$ to $\mathbf{P}_{a_k}$ (or vice versa) at some time t. If this transition does take place, then, for a time t' immediately after t (concerning 'immediately after', we here once more encounter the unresolved vagueness from Axiom 5), S will be in $\mathbf{P}_{a_k}(t')$. This state-assignment will be an answer to our yes-no question if and only if EE is adopted, which in turn means that S has a value $\boldsymbol{a}_k$ of $\boldsymbol{A}$ at t'. Again, P2 is respected.

Third, consider fully relativistic versions of QM. These versions explicitly treat space-time events with a time-extension Δt that can be finite. In this more general case, we would require a generalized P2 with a definite Δt for any QM event. In the present, non-relativistic, formulation we have $\Delta t = \delta t$. Here, QM events are just sharp space-time events in the sense introduced. However, the relativistic generalizations always contain the limiting case δt [30]. For that to be possible, they must respect an eventual generalization of P2 and, for this limiting case, P2 itself.

The simple principle P2 is also generally obeyed in the QM description of concrete experiments. A particularly challenging example is a recent correlation experiment [31–36]. The experiment disproves a certain assumption about the objective time-order of events we might be inclined to identify as QM events. If this identification were correct the plausibility of P2 would be cast into doubt. However, a detailed discussion of the experiment (see Appendix C) shows that no identification of the events in question as QM events is necessary. Indeed, it can be shown for the QM events that are addressed in the description (i.e. correlations) that P2 is strictly obeyed. This discussion also further clarifies the status of this principle. P2 is not a claim added gratuitously to QM in order to do interpretational work, but the simplest possibility to meet a consistency requirement. The Born Rule, BR, is vague and P2 just explicates a straightforward way to make it precise. It has turned out that P2 has two properties: first, in conjunction with P1 and trivial assumption N it leads to serious trouble with COMP; second, it is widely accepted in the literature. We have seen its acceptance in theory (interpretation of the wave-function, QM formalism with transition probabilities, relativistic generalization of QM). We can also evidence its acceptance in practice, i.e. in the description of concrete QM experiments. The example chosen is indeed typical for our practice of confronting QM predictions and their experimental tests. Of course, factual ubiquity of an assumption does not replace an argument for its acceptance. It thus remains to argue for P2 systematically.

The simple principle, P2, cannot be proved. However, brief reflection shows that dismissing it would render QM disastrously vague and empirically useless. The correlation experiment just mentioned illustrates how ¬P2 would make QM predictions untestable. This difficulty generalizes. Initially, the main point to keep in mind is this: QM contains a unique time-parameter. QM with BR′ (i.e. BR disambiguated via P2) plus COMP, i.e. simple complete QM (scQM), immediately came to grief because we took a non-eigenstate of some observable $\boldsymbol{A}$ and fixed one value of parameter t. In standard QM we have no second parameter available to tag a statement like '$\mathbf{a}_k$' for time, a statement expressing S's possession of values of an arbitrary $\boldsymbol{A}$. *So, if such a statement cannot have this value it can have none at all*. But this consequence of ¬P2 creates nonsense. Consider once more a concrete state $W(t_1)$ for a fixed value t_1 and a probability generated from it via '$p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t_1)\mathbf{P}_{a_k})$'. Assume that in this equation, because we now reject P2, '$\boldsymbol{a}_k$' is *not* tagged for time. We explicate a time reference via a triggering event ('$p(\boldsymbol{a}_k)$' must be read as '$p(\boldsymbol{a}_k$ given $E(t_1))$'), along the lines of Sect. 5. We thus have a probability, deriving from $W(t_1)$, for S displaying value $\boldsymbol{a}_k$ at some unspecified time. To make any sense of this at all we must add to our axioms that '$\mathbf{a}_k$', though it now must remain unspecified, must nevertheless be specified at least as referring to some unknown $t > t_1$. This in itself is an intellectual challenge. But how do we test the resulting probability? Assume first (reasonably, again) that our measurement results carry time-indices. E.g. assume that, for some times t_2, t_3 with $t_1 < t_2 \leq t_3$, from reading the value of a pointer observable at t_3 we conclude '$\boldsymbol{a}_k(t_2)$'. Does this observation contribute to a test of the probability $p(\boldsymbol{a}_k)$, generated from $W(t_1)$? Is the unspecified time within '$p(\boldsymbol{a}_k)$', after all a time t_x with $t_1 < t_x \leq t_2$? QM simply has no resources to tell us since states are defined for one time-parameter, not two, and using t_1 from $W(t_1)$ is now excluded. Our single measurement result referred to t_2, but a single result is only a test instance, a member of a large set of results for identical copies of S in states $W(t_1)$. Do all results measured in copies of S have to refer to t_2 in order to test a prediction generated from $W(t_1)$ for the unknown t_x? Moreover, in the general case QM probabilities are crucially time-sensitive. Consider a probability generated from a state $W(t_1 + \Delta t)$? Does it also refer to a value $\boldsymbol{a}_k$ that S will 'take on' or 'display' at unknown t_x? Or at $t_x + \Delta t$? Does a result measured at t_2 also count as a test case for probabilities from this latter case? Or does a result measured at $t_2 + \Delta t$?

These questions are so obviously asking for the nonsensical that we should not be misguided to look into QM measurement theory for help. This theory presupposes that QM probabilities make sense, so cannot help us generating such sense. In particular, in QM measurement theory quite often we consider the apparatus M and a pointer observable $\boldsymbol{B}$. The axioms hold for this case, too. But then, given ¬P2, what does it mean that M is in some state $W'(t_3)$ at the end of the measurement interaction? Does it mean that the probabilities produced from $W'(t_3)$ concern pointer values at some unspecified $t > t_3$? If we read the pointer at a specific time, how does this count as testing the probabilities? The misguided questions remain just the same. And the reason for this is that rejecting P2 generates conceptual, not technical problems. *If the QM event for which a prediction is made (the event referred to 'inside' the probability) and a measurement result, which contributes to a test for the prediction, are not of the same form, i.e. carry both a time-reference, we do not have a testable theory before us*.

Assume finally (very, very unreasonably) that our measurement results, like the '$\boldsymbol{a}_k$' in our QM probability given ¬P2, did not carry time-indices. It would, of course, still be unclear whether or not such results test QM predictions. Moreover, without a time-index we could not use any observation result to ascribe a state to S via Axioms 5 or 5′. As emphasized earlier, our measurements could never be state preparations.

6.3 The General Principle (P3)

I have mentioned two options for making precise the time-reference of '$p(\boldsymbol{a}_k)$' in BR: '$p(\boldsymbol{a}_k(t))$' in concordance with P2 and '$p(\boldsymbol{a}_k$ given $E(t))$', the alternative discussed in Sect. 5. I have no argument against further possibilities, but simply see no others. So whether these two cases exhaust the ways to meet the general principle, P3, is an open question. That said P3 is simply a consistency requirement. It can be made plausible with an argument already given. In a mathematical or physical theory using an equation for some parameter (e.g., time), we must be able to specify that parameter on both sides for that theory to make mathematical or physical sense. In concreto: an equation $F(t) = G$, where 'G' cannot be made explicit to specify in which sense F is a function of t, is not an equation specifying a function of t, at all.

6.4 QM Measurement Theory to the Rescue?

A final objection must be discussed. I have just indicated how negating P2 makes the Born Rule unacceptably vague and empirically useless and how this problem can affect the measurement apparatus. However, the axioms and principles in my formulation consciously ignore the notion of measurement. One may form the impression that, because of that ignorance, the problem is either artificially set-up and unrealistic from the start or is just a quirky way to formulate the well-known measurement problem. And if the measurement process were taken into account, could the contradiction perhaps be dissolved?

Initially, it should be emphasized that our problem is not a version of the measurement problem, but a more fundamental conceptual difficulty. Traditionally, the measurement problem is this. Consider that QM is complete in the sense of COMP. Consider that the apparatus also is a QM system possessing a QM state. Due to Axiom 3 (linear state evolution), the system-cum-apparatus supersystem will, in typical cases, evolve into states that cannot, under COMP, be interpreted as the apparatus showing a value of the pointer observable—in contrast with what we really observe. The present argument shows that COMP and QM are in conceptual disharmony, i.e. are mutually inconsistent, given very general and reasonable principles. From this perspective, it is small wonder that ignoring the disharmony and combining COMP and QM anyway we contract the measurement problem as a further, derived difficulty.

Accordingly, QM measurement theory will certainly not resolve our problem. Consider, for an illustration, how a standard approach to QM measurement must either tamper with the axioms or violate principles P2 and P3. Let us again assume an $\boldsymbol{A}$ measurement setting and an S in a state $W(t_1)$, a non-eigenstate of $\boldsymbol{A}$, such that all $\boldsymbol{A}$ eigenvalues receive non-zero probability. Assume further that $t_1 < t_2 \leq t_3$, where measurement interaction starts at t_1, where our approach to QM measurement tries to establish that S has an $\boldsymbol{A}$ value at t_2, and where the pointer is read at t_3 (and it is again immaterial whether $t_2 < t_3$ or $t_2 = t_3$). Reading the pointer at t_3 will give us S's value at t_2, thus enabling us to check predictions of QM for t_2 and blocking the above considerations concerning P2. Namely, this theory is perfectly testable because an individual prediction and its test instance refer to the same t_2. Interpreted this way however, the approach illegitimately modifies BR, i.e. the QM axioms. Suppose first that predictions have been generated from $W(t_1)$, S's state at measurement onset. This state does not have the resources to generate predictions *for a specific time apart from* t_1, so if $t_1 < t_2$ no predictions for the latter can result. If we nonetheless assume that predictions for t_2 can be calculated, via the trace formula, from $W(t_1)$ we must assume reference to two times in this formula. Instead of accepting that standard QM does not offer two time-parameters we have implicitly furnished BR with a second one of them—i.e. tampered with it. If, in order to avoid this inconsistency, we allow that predictions calculated from a QM state do not have to refer to *a specific time*, we violate P2 and produce chaos. Our theory is no longer testable and it is no longer clear how P3 can be met.

Suppose alternatively that predictions do refer to a time t_2 as before, but have been generated from a state $W(t_2)$, S's state after some portion of the interaction, a state where what your favored approach to measurement says about S (S's state has decohered, has collapsed, the world described by it has branched into a plural of worlds, etc.) has happened. Again, the result would be a considerable conceptual mess. First of all, $W(t_2)$ in many approaches differs so crucially from $W(t_1)$ that it does not clearly deliver the probabilities that earlier were naively extracted from $W(t_1)$. In particular, $W(t_2)$ is either an $\boldsymbol{A}$ eigenstate or a mixture of $\boldsymbol{A}$ eigenstates. (Modelling how S moves into such a state is the very point of all approaches to measurement that accept COMP.) An $\boldsymbol{A}$ eigenstate state cannot generate the same non-zero probabilities we calculated from the non-eigenstate $W(t_1)$, so $W(t_2)$ must be a mixture. This mixture can generate the required probabilities, but only given a fixed $\boldsymbol{A}$, not for an arbitrary $\boldsymbol{A}$. Where before the Born Rule had arbitrary $W(t_1)$ and arbitrary $\boldsymbol{A}$ as input, now $W(t_1)$ and $\boldsymbol{A}$ must be chosen, then for the fixed $\boldsymbol{A}$ the $\boldsymbol{A}$-eigenstate mixture $W(t_2)$ must be calculated—not using Axiom 3 (unitary evolution), but using your favored measurement theory. Finally, it is that mixture $W(t_2)$, rather than pure $W(t_1)$, that goes into the trace formula within BR. (For measurement of an S in pure state $W(t_1)$!) As a result, the Born Rule would no longer be neutral vis-à-vis interpretations, but relative to a favored measurement theory. More seriously, it would again have to be substantially revised as now necessarily mentioning *two states of S with different values of t*. We would in effect require that for pure non-eigenstates it is only certain mixtures constructed from them, but not the pure states themselves from which to calculate BR probabilities. In particular, we would be saying that if a system is in pure state $W(t_1)$ and is subjected to a measurement starting at t_1, then not from $W(t_1)$ itself, but only from the evolved $\boldsymbol{A}$-eigenstate mixture $W(t_2)$, probabilities may legitimately be calculated. The Born Rule would be deprived of its general applicability to arbitrary states and, in order to be applicable at all, would have to mention two states explicitly referring to different times. This again constitutes an act of tampering with the axioms. And if again we want to avoid this muddle and *relax* the condition that measurement outcomes need time-indices, the previous arguments concerning P2 and P3 apply once more.

Of course, modifying the QM axioms is not in itself objectionable. A problematic axiom system for a well-working physical theory is never sacrosanct. But changing the axioms underhand is not a legitimate move. There is an unseen conflict of QM, *as axiomatized here*, and COMP, the standard expression of its completeness. To change the axioms would simply mean to evade the problem by switching to some other theory—one perhaps, where a physical system is assigned a state with two independent time-parameters or is assigned two states referring to different times. Such a theory might be in harmony with COMP, but it would just not be standard QM.

Knowledgeable readers may wonder about the so-called two-state formalism of QM here [37–39]. This approach is certainly a non-standard variant of QM and would, if it prescribed that S's state is in general a two-state, substantially change an axiomatization of the theory. So, if the approach did solve the problem raised here for *standard* QM, it would do so by replacing standard QM with another formalism. However, the approach does neither aim to replace the standard formalism, nor does it solve the present problem. The two-state formalism constructs from an initial state $|\Psi(t_i)\rangle$ (representing S's history) and a final state $|\Psi(t_f)\rangle$ (representing S's destiny) a 'two-state' $\rho(t)$ that completely describes S at t, for $t_i < t < t_f$. S thus has a well-defined two-state given fixed initial and final states. For an unspecified $|\Psi(t_f)\rangle$, the original Born Rule is recovered. So neither does the approach assume an S state with two time-parameters that can take different values, nor does it use two S states having a variable time-parameter and both referring to the past of the predicted measurement result.

7 Axioms for QM Consistent with the Principles

Since the three principles are without reasonable alternatives and yield contradictions with our two completeness expressions, the net result is this: *It is impossible to express the completeness of QM by COMP or weak COMP*. But amendments to the axioms, guaranteeing harmony with our principles, are easily made. A set of axioms for QM respecting P1–P3 will consist of Axioms 1 and 2 above plus Axioms 3*, 4*, and 5*, specified as follows:

Axiom 3* *S evolves in time according to $W(t) = U(t)W(t_P)U(t)^{-1}$ where $U(t) = \exp[-\mathrm{i}\boldsymbol{H}t]$, a unitary operator, is a function of time and $\boldsymbol{H}$ is an operator representing the total energy of S, where t_P, some value of t, is called the preparation time, and $W(t_P)$ the prepared state.*

Axiom 4* *If S is in state $W(t) \neq W(t_P)$ and $\boldsymbol{A}$ is an observable on S, then the expectation value $\langle \boldsymbol{A}\rangle(t) = \int(\mathbf{a}\omega, t)p(\boldsymbol{a}\omega, t))\mathrm{d}\omega$ is given by: $\langle \boldsymbol{A}\rangle(t) = \mathrm{Tr}(W(t)\boldsymbol{A})$.*

Axiom 5* *If S has value $\boldsymbol{a}_k(t_1)$ of $\boldsymbol{A}$, then $t_1 = t_P$ and S's state is the state $\mathbf{P}_{a_k}(t_P)$.*

Some remarks should put these revised axioms in perspective. Axiom 4* specifies '$\langle \boldsymbol{A}\rangle(t)$' in accordance with P2. But it does something more. The integral equation in Axiom 4* serves to clearly specify that QM expectations are just ordinary statistical expectations, with the qualification that the basis events are sharp space-time events.[7] Thus, Axiom 4* explicates that the events, the weights of which enter into the QM expectation value, are sharp space-time events. For discrete $\boldsymbol{A}$ the expectation becomes $\langle \boldsymbol{A}\rangle(t) = \sum_i \boldsymbol{a}_i(t)p(\boldsymbol{a}_i(t))$ and we directly see that $p(\boldsymbol{a}_k(t)) = \langle \mathbf{P}_{a_k}\rangle(t)$. Hence, we arrive at a final, fully explicit and fully satisfactory, version of the Born Rule:

(**BR***) If S is in state $W(t) \neq W(t_P)$ and $\boldsymbol{A}$ is an observable on S with eigenvalue a_k, then the probability that S has a_k at t is:
$p(\boldsymbol{a}_k(t)) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$.

BR* is the earlier BR′, with the restriction that $W(t_P)$ is not an admissible input. More explicitly, BR* and Axiom 5* in conjunction rule that if, for some t_1, '$\boldsymbol{a}_k(t_1)$' is true, for some t_1, then no calculation of a number $p(\boldsymbol{a}_k(t_1))$ is allowed.[8] This is not an implausible restriction. It is reasonable indeed to assume that the factual observation of an event at a certain time makes it meaningless to calculate any prediction for that event at that time. Note also that the only axiom making direct reference to the result of a factual observation, i.e. to an actual sharp spacetime event on S, is Axiom 5*. Note finally the three crucial virtues of this axiom system: (1) It is absolutely explicit concerning the time parameter; (2) it does not need to use the notion of measurement in any sense; (3) it allows us to consistently describe measurements as preparations because our findings upon measurement can be used, via Axiom 5*, as a new input for Axiom 3*.

8 An Expression of Completeness

The completeness of QM is embodied in the theorems mentioned: the corollary from Gleason's theorem and versions of the Kochen-Specker theorem. We have seen that

[7] For the relation of QM expectations and statistical expectations, see the discussion in Appendix A.

[8] Cohen-Tannoudji et al. (see [17] Sect. 3.2) have already made a similar proposal when trying, in their version of Axiom 5, to explain the meaning of 'immediately after'. Their version, however, does not clearly exclude an assignment of two different states to the same time, which would be inconsistent with Axiom 1.

COMP is not an admissible way to express the impossibility results incorporated in these theorems. Our principle P1 embodies the most reasonable idea that probability is quantified possibility and P2–P3 represent plausible ways to render precise the imprecise Axioms 1–5. Given these principles, COMP cannot be an expression of the impossibility results, hence of the sense in which QM can be proved to be complete. But what *is* an appropriate expression?

To repeat the first observation of this paper: *It is impossible to consistently assign values to the observables of a suitable QM system, given two plausible constraints*. We will now see that QM, made precise in the sense of Axioms 1–5*, does yield probabilities for existing values. Hence, it cannot be the idea of assigning existing values as such, but the one of doing so under conditions (i) and (ii) which we should interpret as disproved by the completeness theorems. One or both of conditions (i) and (ii) for the assignment of values must be rejected or modified.

It is easy to see that we have produced a general argument for existing values. Consider, once more, S being in a state $W(t_1) \neq \mathbf{P}_{a_k}(t_1)$ such that $p(\boldsymbol{a}_k)$ gets a value other than 1 or 0, where t_1 is the onset time of an $\boldsymbol{A}$-measurement interaction. By BR*, '$p(\boldsymbol{a}_k)$' is explicated as '$p(\boldsymbol{a}_k(t_1))$', the probability that S has $\boldsymbol{a}_k$ at t_1, the onset time. So $W(t_1)$, by our new axioms, collects probabilities for values possessed at the time of measurement onset, t_1. This is nothing but the assumption that S has one of the $\boldsymbol{A}$-values at t_1. The rationale for BR* can be followed back into our principles. If '$p(\boldsymbol{a}_k)$' does not inherit the index t_1 it cannot bear any time-index, at all—in contradiction with P2 and in obvious contrast with reasonable requirements for a fundamental probabilistic theory of space-time events. If we sacrifice P2 nevertheless and take the remaining option for explicating a time-reference in '$p(\boldsymbol{a}_k)$', i.e. '$p(\boldsymbol{a}_k$ given $E(t_1))$', then no established construal of the conditional can both be coherent and respect P3. Respecting both P2 and P3, we end with BR*. Finally, if '$\boldsymbol{a}_k(t_1)$' receives a positive probability, as it does in our case, it must be logically possible to assume it to be true. This is an instance of P1 and says that it must be logically possible to assume S having a value $\boldsymbol{a}_k$ of $\boldsymbol{A}$ at t_1.

As a consequence, it cannot be true that QM is complete in the sense that the QM state $W(t_1)$ provides all properties S has at t_1. Looking only at the axioms (here BR*), $W(t_1)$ does nothing but collect probabilities for S's values at t_1. It is plausible to supplement the axioms with the rule that predictions with certainty entail value ascriptions (i.e. adopting the forward direction of EE: If $\mathbf{P}_{a_k}(t_1)$, then $\boldsymbol{a}_k(t_1)$), but it is implausible to bar all other ascriptions. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ discrete, with values $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots$, and non-degenerate with $[\boldsymbol{A}, \boldsymbol{B}] \neq 0$. Let S be in $W(t_1) = \mathbf{P}_{b_j}(t_1)$. Then, by the rule just adopted, '$b_j(t_1)$' is true and exactly one of '$a_1(t_1)$', '$a_2(t_1)$',... is true. Consider now a set of observables $\{\mathbf{P}\}_{AB}$, that contains the projectors $\mathbf{P}_{b_1}, \mathbf{P}_{b_2}, \ldots, \mathbf{P}_{a_1}, \mathbf{P}_{a_2}, \ldots$ and forms a KS set. What cannot be true, according to the Kochen-Specker theorem, is that value assignments to all members of $\{\mathbf{P}\}_{AB}$ do both of these two: (i) mirror the algebraic relations of the members of $\{\mathbf{P}\}_{AB}$; (ii) are non-contextual, i.e. are unique for every member of $\{\mathbf{P}\}_{AB}$. There are, then, observables $\boldsymbol{A}$ and $\boldsymbol{B}$ such that all of the above assumptions are true, especially '$b_j(t_1)$' is true and exactly one of '$a_1(t_1)$', '$a_2(t_1)$',... is true, and yet it *cannot* be the case that of the $\mathbf{P}_{b_1}, \mathbf{P}_{b_2}, \ldots$ exactly one receives value 1, the others 0, and simultaneously, i.e. noncontextually, one of the $\mathbf{P}_{a_1}, \mathbf{P}_{a_2}, \ldots$ receives value 1, the others 0. In general, we arrive at the following completeness expression for QM: *It is impossible to assign values to the observables of QM systems such that values of submaximal (degenerate) observables mirror the algebraic relations among these observables noncontextually*.[9]

It is an open question what contextual value assignments would look like. As indicated, the context-dependence of pre-assigned values must be one of existing values rather than one depending on measurement influences on S. The prospects for this type of contextuality (sometimes called 'ontological contextuality') have been researched in the past [41–45],[10] but without much resonance. The present argument clearly shows that this possibility merits renewed attention.

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Appendix A: Derivation of the Born Rule (BR) from Axiom 4

Trivially, from $\langle \boldsymbol{A}\rangle = \mathrm{Tr}(W(t)\boldsymbol{A})$ and $\langle \mathbf{P}_{a_k}\rangle = p(\boldsymbol{a}_k)$, we get $p(\boldsymbol{a}_k = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$. But does the expectation $\mathbf{P}_{a_k}$ really equal $p(\boldsymbol{a}_k)$, where the latter is *interpreted as the probability that S has* $\boldsymbol{a}_k$? Yes, it does—given that a QM expectation can be defined as $\langle \boldsymbol{A}\rangle = \int (\boldsymbol{a}(\omega)p(\boldsymbol{a}\omega))\mathrm{d}\omega$ or, for discrete $\boldsymbol{A}$, as $\langle \boldsymbol{A}\rangle = \sum_i \boldsymbol{a}_i\ p(\boldsymbol{a}_i)$ where these expressions are defined as ordinary statistical expectations. Following such a definition, the expression of a summand '$\boldsymbol{a}_i\ p(\boldsymbol{a}_i)$' is understood as a real number representing an event (here S having $\boldsymbol{a}_i$) weighted by the probability of *this very event's* occurrence, i.e. within '$\boldsymbol{a}_i\ p(\boldsymbol{a}_i)$' the two occurrences of '$\boldsymbol{a}_i$' denote the same event $\boldsymbol{a}_i$ (analogously for the 'a' in the integrand '$a\ p(a)$'). Trivially, a QM expectation can be written like an ordinary statistical expectation. Now, if it is also *interpreted* in the same way, then in '$\boldsymbol{a}_i\ p(\boldsymbol{a}_i)$' the first '$\boldsymbol{a}_i$' denotes the event '$S$ has $\boldsymbol{a}_i$' and within '$p(\boldsymbol{a}_i)$' the '$\boldsymbol{a}_i$' denote that same event. In this case, the formulation of the Born Rule as BR follows verbatim from Axiom 4. One can of course object that QM expectations are not ordinary statistical expectations. While we can write $\langle \boldsymbol{A}\rangle = \sum_i \boldsymbol{a}_i\ p(\boldsymbol{a}_i)$ (in the discrete case), the summands are not interpretable in the statistical way just explained. The alternative, suggested by the von Neumann picture, is this. The expression '$p(\boldsymbol{a}_i)$' means the probability for an event that should

[9]The necessity to use sets containing submaximal (degenerate) observables to derive a Kochen-Specker contradiction is well-known since the work reported in [40].

[10]For a discussion of the approaches in note [41–45] see [1], pp. 135–138.

be characterized as S 'taking on' value $\boldsymbol{a}_i$. This probability, within the summand '$\boldsymbol{a}_i\ p(\boldsymbol{a}_i)$' must weight not that taking-on event itself, but the value taken on, hence in '$\boldsymbol{a}_i\ p(\boldsymbol{a}_i)$' the two occurrences of '$\boldsymbol{a}_i$' do not refer to the same event—in contrast with the statistical definition. But this approach is obviously problematic. What happens when $\boldsymbol{A}$ is measured for S in an eigenstate of $\boldsymbol{A}$? There are but two possibilities. Either we interpret '$p(\boldsymbol{a}_i)$' as the probability of S taking on '$\boldsymbol{a}_i$' throughout, but then we have the unwelcome result that even in a state $|\boldsymbol{a}_i\rangle\langle\boldsymbol{a}_i|$ S does not have, but only with certainty take on value $\boldsymbol{a}_i$; or we interpret '$p(\boldsymbol{a}_i)$' differently for eigenstates (probability of having $\boldsymbol{a}_i$) and non-eigenstates (probability of taking on $\boldsymbol{a}_i$) of a chosen $\boldsymbol{A}$. This would mean writing two versions of the Born Rule, for observables and their eigenstates and their non-eigenstates, respectively. (I have heard the proposal that von Neumann's 'takes on' should be replaced by the word 'displays', the suggestion being that 'displays' is neutral between 'has' and 'takes on', but this is implausible. By all reasonable readings, an S can only display (at a specific time) what it has (at the time), so we are back with the simple proposal given in the main text above that, in BR, '$p(\boldsymbol{a}_i)$' is the probability that S has $\boldsymbol{a}_i$.)

Appendix B: Derivation of the Probability Principle (P1) from Four Assumptions

Assume (P1(a)) that contradictions have probability zero; (P1(b)) the conditional probability formula: $p(A \wedge B) = p(A \mid B)p(B)$; (P1(c)) that the probability space for the probabilities delivered by T can be expanded so that $p(T)$ is well-defined; and (P1(d)) that $p(T) > 0$. The non-trivial assumption is P1(c). However, it can be made plausible for all major conceptions of probability. Consider probability being defined as a subjective degree of belief. (For a meaningful integration of this conception of probability into QM see [12].) Then it is rational to define $p(T \mid A)$ for a theory in order to be able to express that T's prediction A, if it comes out true, raises your degree of belief in it: $p(T \mid A) > p(T)$. However, if $p(T \mid A)$ is well-defined, then $p(A)$ and $p(T)$ are well-defined on the same space. Consider, alternatively, probability being defined via conditional probabilities understood as ratios of proportions of logically possible worlds. $p(T)$ then can be defined as $p(T \mid L)$ where L is a logical triviality and $p(A \mid T)$ is defined, on the same space as $p(T)$, as the ratio of logically possible T-worlds where A is true to all logically possible T-worlds. Consider, finally, probability being defined as the limiting relative frequency of possible outcomes in a hypothetical infinite sequence of trials of an experiment. Let a trial of an 'experiment concerning T' be an explicit statement of T with possible 'outcomes' True ($T = 1$) and Not-true ($T = 0$). Then '$T = 1$' is an outcome just like the event reported by A. Thus, $p(T)$ can be defined as $p(T = 1)$ on a superspace of the probability space where $p(A)$ lives. Given P1(a–d), the argument for P1 is very simple. Assume, by P1(c) and P1(d), that $p(T) > 0$. Assume also that $p(A \mid T) > 0$. Then, by P1(b), also $p(A \wedge T) > 0$, whence, by P1(a), $A \wedge T$ is not a contradiction.

Appendix C: Validity of the Simple Principle (P2) in the Description of a QM Experiment (Quantum Correlations vs. Multisimultaneity)

The simple principle P2 is generally obeyed in the QM description of concrete experiments. A recent correlation experiment initially seems to cast doubt on P2, but this impression is spurious. Many experiments evidence correlations of photons from entangled pairs as predicted by QM. The one in question here is a Franson-type experiment where entangled photons are sent into two identical unbalanced Mach-Zehnder interferometers [31]. Those pairs of photons for which path detection within the interferometers is impossible will exhibit a characteristic correlation. Now, one can entertain the classical theory that the photons travel classical paths but one photon's 'choice' of a path behind the photon analyzer (e.g. beam-splitter or polarizer) could be communicated to the second photon influencing its path. In that theory, correlations can find a classical explanation, i.e. they are explained by a causal influence of the photon passing the analyzer in one arm of the apparatus earlier to the one passing the analyzer in the other arm later. In suitable settings such influence would have to travel superluminally, but an objective temporal ordering of the individual photon-analyzer interactions allowing a causal explanation of the correlation would remain possible. The envisaged causal dependence would define a preferred reference frame since the time-ordering between two space-like separated events is not relativistically covariant. The preferred frame could be naturally identified with the inertial frame of the measurement apparatus. This idea has been developed in a theory called Multisimultaneity [32–34]. Here, the preferred frame is identified with the one of the photon analyzer. Accordingly, in a Franson setting with two analyzers there are two preferred frames (hence the name 'Multisimultaneity'). However, a device with analyzers in relative motion can be arranged such that each of the individual photons in an entangled pair meets its analyzer earlier than the other relative to its own reference frame. For this experiment, Multisimultaneity predicts that the correlation vanishes—in contrast with QM. The experiment has been carried out and has confirmed QM [35, 36]. In particular, the possibility of an objective time-ordering of the individual photons passing the analyzers (as a possible source of a causal explanation of the correlations) is thereby disproved. This result seems to cast doubt on our simple principle P2 as follows. It might seem unreasonable to require that every QM event is time-indexed, if such indexing cannot be objective. So, it might not seem entirely implausible to sacrifice P2.

However, a few remarks will help to put this experiment in perspective and show that, within its description, P2 is firmly in place. First, recall that the simple principle P2 makes an assumption only about *QM events*, i.e. those events for which QM generates probabilities. Second, consider that in a correlation experiment the only relevant observable is a correlation. A correlation is a two-valued observable, its values being correlation yes or no, or if correlation is translated into presence at detectors behind the interferometers, detection yes or no. The two individual QM events are 'both photons are/are not correlated' or, when translated into position, 'both photons are/are not detected'. These are indeed the only events in the experiment to which P2 applies. This reflects the fact that the experiment is best interpreted as showing that a two-particle correlation is one inseparable QM event. Of course, we are free to interpret an instance of correlated detector clicks as evidence for the photons both possessing a certain polarization earlier, i.e. when passing the analyzers. But these are not the QM events for the experiment in question. In particular, it is Multisimultaneity that hypothesizes a particular type of event here, i.e. that both photons pass the analyzers *taking particular paths*. For these events, QM by assumption does not yield predictions because which-path information is *complementary* to the correlation measured in this experiment. It is the assumption of Multisimultaneity that these non-QM events, rather than being parts of an indivisible QM event, exist as individual events with an objective time-ordering. But these events are not QM events and QM, in the correlation setting, does not yield probabilities for them. So P2, a principle for disambiguating these probabilities, does not apply to them.

Consider now the QM event proper: the correlation. QM yields predictions for it given by $R(t) = \langle 0|\Psi^{+}\mathbf{r}_a, t)\Psi^{+}(\mathbf{r}_b, t)\Psi(\mathbf{r}_b, t)\Psi(\mathbf{r}_a, t)|0\rangle$ (where $\mathbf{r}_a, \mathbf{r}_b$ are the detector locations, $\Psi(\mathbf{r}_a, t), \Psi(\mathbf{r}_b, t)$ the fields at detectors A and B, and $|0\rangle$ is the vacuum state). $R(t)$ is a probability for correlation (double photon detection) and it can be tested by measuring the coincidence rate (per time-unit) of double photon detections. Let's simplify the real experimental situation and assume that one can directly count double detections for a finite time interval, some multiple of the time unit, and calculate the rate for any time within the interval. Now, fix again a value t_1 of t such that also $\Psi(\mathbf{r}_a, t_1), \Psi(\mathbf{r}_b, t_1)$ are fixed. Interpret $R(t_1)$ as the probability that at t_1 two photons are detected at $\mathbf{r}_a, \mathbf{r}_b$. This prediction is empirically testable: count coincidences in a time interval Δt including t_1 and compute the coincidence rate for all times during Δt, including t_1. The resulting rate then is an instance of a test of the prediction $R(t_1)$. (A full test would of course require a large set of measured rates for identical copies of the entangled photons in the same states $\Psi(\mathbf{r}_a, t_1), \Psi(\mathbf{r}_b, t_1)$.) But from our above considerations of wave-functions we also know that this is an acceptance of the simple principle P2. Suppose for contrast that $R(t_1)$ is the probability, a dispositional property possessed by the whole system at t_1, such that two photons are detected at $\mathbf{r}_a, \mathbf{r}_b$ at some unspecified later time. This is a rejection of P2, since the predicted outcome itself is no longer time-dependent. For testing this prediction we would need additional theory, e.g. a prescription that the unspecified later time can be specified at least insofar as the predicted coincidence happens within an interval very much shorter than Δt. Without such a prescription the prediction would not be testable. But there is no such prescription in the QM description, yet $R(t)$ is considered testable because it has actually been tested. Hence, the description of these experiments implies an acceptance of the simple principle P2.

An additional remark is in order here. If $R(t_1)$ is interpreted as the probability that at t_1 two photons are detected at r_a, r_b this may be interpreted as a probability of *simultaneous* detection of the two photons. Given the background of an experimentum crucis for QM vs. Multisimultaneity we may then ask: In which reference frame are these two events simultaneous? While this question is generally legitimate it is not one that can be answered by QM. (This is pointed out by Suarez and Scarani (see [30]). They write that a correlation measurement "produces events which are simultaneously strictly correlated in space-like separated regions. But in which inertial frame are these correlated events simultaneous? Quantum mechanics does not answer this question.") It would not even be adequate to say that QM treats both photon detections as simultaneous, but without explicating the reference frame. If QM yields $R(t_1)$ this is just the probability for the single event of double photon detection at t_1, not the probability for two correlated events of single photon detection both at t_1. This was captured above by distinguishing the correlation as one indivisible QM event from the single photon detections which are non-QM events because a QM event is one for which there is a BR probability and $R(t_1)$ is a probability just for the correlation. Of course, from the state $\Psi(r_b, t)\Psi(r_a, t)$ of the whole (entangled) two-photon system we can also produce probabilities $p(r_a(t))$, $p(r_b(t))$ for the detection of a single photon at r_a or r_b. Given these probabilities, we can refer to individual detection events as QM events, namely by choosing values of t for $r_a(t)$ and $r_b(t)$. It is, of course, a matter of choice which values to choose and, if we choose the same for both, to fix a reference frame against which it makes sense to call them simultaneous. But this is nothing that QM itself can do. Moreover, if $R(t_1)$ *could* be interpreted as a probability for two QM events $r_a(t_1)$ and $r_b(t_1)$ (instead of one complex event as done here) the situation would not be changed substantially. The state would then endow both events with the same value t_1, thus treat them as simultaneous, but the question of the appropriate frame would still be open. Consider finally two events $r_a(t_1)$ and $r_b(t_2)$ which are time-like separated, i.e. $t_1 \neq t_2$ in all inertial frames. We can regard them as QM events individually because there is a state $\Psi(r_b, t_1)\ \Psi(r_a, t_1)$ such that $r_a(t_1)$ gets a probability $p(r_a(t_1))$, and similarly there is a state for $p(r_b(t_2))$. It is, however, impossible to calculate a correlation for these two events for the simple reason that a correlation is calculated from one state with one time-reference and no single complex QM event can be named that could be identified with the two QM events $r_a(t_1)$ and $r_b(t_2)$. And if, again $R(t_1)$ could be interpreted as a probability for two QM events (instead of one), it could not treat the two events $r_a(t_1)$ and $r_b(t_2)$ because we have one parameter t available taking value t_1, so unable to have any implications for an event at t_2 (and vice versa for $R(t_2)$).

These remarks, it should be emphasized, are not intended as steps toward an interpretation of the experiment in question. P2 is not a 'metaphysical' claim added to QM in order to do any interpretational work, hence its addition is not on a par with attempts to explain correlations of any kind. P2 is not an interpretational addition to QM, but one of two possible ways to make a time-index explicit for the left side of '$p(\boldsymbol{a}_k) = \mathrm{Tr}(W(t)\mathbf{P}_{a_k})$'. The second possibility is excluded in Sect. 5, and having no time-index on the left is not a consistent option (see Sect. 6.3). So, ultimately P2 is itself a consistency requirement, not an interpretive feature. Now, P2 implies (as shown in Sect. 8) that QM measurements must reveal existing values. There are proposals in the literature for explaining nonlocal correlations using (a) an objective time-order of certain events and (b) measurements that reveal existing values in QM systems. (Multisimultaneity is but one such proposal.) How is P2 related to this group of proposals? Concerning feature (a) P2 has absolutely nothing to say. If the events in question are non-QM events (e.g. both photons passing the analyzers *taking particular paths* in the correlation case) QM will not give them probabilities and P2 does not apply. If the events are detection events $r_a(t)$ and $r_b(t)$ such that, by BR, they individually get probabilities $p(r_a(t))$ and $p(r_b(t))$, P2 applies to them (rules that any of these events is individually time-indexed). Choosing the same value t_1 for both types of QM events is possible (given a state referring to t_1), thus fixing reference to the individual events $r_a(t_1)$ and $r_b(t_1)$. Likewise it is possible to fix reference to events $r_a(t_1)$ and $r_b(t_2)$ given two *different* states (referring to t_1 and t_2, respectively). It is possible to fix reference frames such that it makes sense to speak of simultaneity and it is possible to choose t_1 and t_2 such that $t_2 > t_1$ in all inertial frames. All this is a matter of choice and the result a matter of interpretation P2 is entirely tacit about. By contrast, P2 is not tacit concerning feature (b). As emphasized, the text presents an argument for the necessity to understand BR via P2 and, in addition, P2 implies that measurements reveal existing values. The main text thus is an argument for (b), not just as a feature of explanations of nonlocal correlations, but any interpretation of QM. But P2, all the other principles and the whole argument are, again, tacit about what such an interpretation should look like. To repeat: neither giving an interpretation of QM nor explaining QM effects is the subject of this paper. P2 has the sole purpose of disambiguating the Born Rule (with respect to the time-index) and it is the simplest conceivable disambiguation (hence its name 'simple principle').

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