The Succinctness of First-order Logic over Modal Logic via a Formula Size Game

Lauri Hella Miikka Vilander

School of Information Sciences
University of Tampere

Abstract

We propose a new version of formula size game for modal logic. The game characterizes the equivalence of pointed Kripke-models up to formulas of given numbers of modal operators and binary connectives. Our game is similar to the well-known Adler-Immerman game. However, due to a crucial difference in the definition of positions of the game, its winning condition is simpler, and the second player (duplicator) does not have a trivial optimal strategy. Thus, unlike the Adler-Immerman game, our game is a genuine two-person game. We illustrate the use of the game by proving a nonelementary succinctness gap between bisimulation invariant first-order logic FO and (basic) modal logic ML.

Keywords: Succinctness, formula size game, bisimulation invariant first-order logic, n-bisimulation.

1 Introduction

Succinctness is an important research topic that has been quite active in modal logic for the last couple of decades; see, e.g., [3,13,11,14,1,12,5] for earlier work on this topic and [6,20,7,10,19,21] for recent research. If two logics $\mathcal{L}$ and $\mathcal{L}'$ have equal expressive power, it is natural to ask, whether there are properties that can be expressed in $\mathcal{L}$ by a substantially shorter formula than in $\mathcal{L}'$ (or vice versa). For example, $\mathcal{L}$ is *exponentially more succinct* than $\mathcal{L}'$, if for every integer n there is an $\mathcal{L}$-formula φ_n of length $\mathcal{O}(n)$ such that any equivalent $\mathcal{L}'$-formula ψ_n is of length at least 2^n.

Often such a gap in succinctness comes together with a similar gap in the complexity of the logics. For example, Etessami, Vardi and Wilke [3] proved that, over ω-words, the two-variable fragment FO^2 of first-order logic has the same expressive power as unary-TL (a weak version of temporal logic), but FO^2 is exponentially more succinct than unary-TL, and furthermore, the complexity of satisfiability for FO^2 is NEXPTIME-complete, while the complexity of unary-TL is in NP [17]. However, succinctness does not always lead to a penalty in terms of complexity: an example is public announcement logic PAL which is exponentially more succinct than epistemic logic EL, but both have the same complexity, as proved by Lutz in [12].

In order to prove succinctness results we need a method for proving lower bounds for the length of formulas expressing given properties. The two most common methods used in the recent literature are the *formula size game* introduced by Adler and Immerman [1], and *extended syntax trees* due to Grohe and Schweikardt [8]. The latter was inspired by the former, and in fact, an extended syntax tree is essentially a witness for the existence of a winning strategy in the Adler-Immerman game. Thus, these two methods are equivalent, and the choice between them is often a matter of convenience.

Originally, Adler and Immerman [1] formulated their game for the branching-time temporal logic CTL. They used it for proving an $n!$ lower bound on the size of CTL-formulas for expressing that there is a path on which each of the propositions $p_1, \ldots, p_n$ is true. As it is straightforward to express this property by a formula of CTL$^+$ of size linear in n, their result established that CTL$^+$ is $n!$ times more succinct than CTL, thus improving an earlier exponential succinctness result of Wilke [22].

After its introduction in [1], the Adler-Immerman game, as well as the method of extended syntax trees, has been adapted to a host of modal languages. These include epistemic logic [6], multimodal logics with union and intersection operators on modalities [20] and modal logic with contingency operator [21], among others.

The Adler-Immerman game can be seen as a variation of the Ehrenfeucht-Fraïssé game, or, in the case of modal logics, the bisimulation game. In the Adler-Immerman game, quantifier rank (or modal depth) is replaced by a parameter, usually called formula size, that is closely related to the length of the formula. Moreover, in order to use the game for proving that a property is not definable by a formula of a given size, it is necessary to play the game on a pair $(\mathbb{A}, \mathbb{B})$ of sets of structures instead of just a pair of single structures.

The basic idea of the Adler-Immerman game is that one of the players, S (spoiler), tries to show that the sets $\mathbb{A}$ and $\mathbb{B}$ can be separated by a formula of size n, while the other player, D (duplicator), aims to show that no formula of size at most n suffices for this. The moves that S makes in the game reflect directly the logical operators in a formula that is supposed to separate the sets $\mathbb{A}$ and $\mathbb{B}$. Any pair (σ, δ) of strategies for the players S and D produces a finite game tree $T_{\sigma,\delta}$, and S wins this play if the size of $T_{\sigma,\delta}$ is at most n. The strategy σ is a winning strategy for S if using it, S wins every play of the game. If this is the case, then there is a formula of size at most n that separates the sets, and this formula can actually be read from the strategy σ.

A peculiar feature of the Adler-Immerman game is that the second player, duplicator, can be completely eliminated from it. This is because D has an optimal strategy $\delta_{\max}$, which is to always choose the maximal allowed answer; this strategy guarantees that the size of the tree $T_{\sigma,\delta}$ is as large as possible. Thus, in this sense the Adler-Immerman game is not a genuine two-person game, but rather a one-person game.

In the present paper, we propose another type of formula size game for modal logic. Our game is a natural adaptation of the game introduced by

Hella and Väänänen [9] for propositional logic and first-order logic. The basic setting in our game is the same as in the Adler-Immerman game: there are two players, S and D, and two sets of structures that S claims can be separated by a formula of some given size. The crucial difference is that in our game we define positions to be tuples $(m, k, \mathbb{A}, \mathbb{B})$ instead of just pairs $(\mathbb{A}, \mathbb{B})$ of sets of structures, where m and k are parameters referring to the number of modal operators and binary connectives in a formula. In each move S has to decrease at least one of the parameters m or k. The game ends when the players reach a position $(m^*, k^*, \mathbb{A}^*, \mathbb{B}^*)$ such that either there is a literal separating $\mathbb{A}^*$ and $\mathbb{B}^*$, or S cannot make any moves, usually because $m^* = k^* = 0$. In the former case, S wins the play; otherwise D wins.

Thus, in contrast to the Adler-Immerman game, to determine the winner in our game it suffices to consider a single "leaf-node" $(m^*, k^*, \mathbb{A}^*, \mathbb{B}^*)$ of the game tree. This also means that our game is a real two-person game: the final position $(m^*, k^*, \mathbb{A}^*, \mathbb{B}^*)$ of a play depends on the moves of D, and there is no simple optimal strategy for D that could be used for eliminating the role of D in the game.

We believe that our game is more intuitive and thus, in some cases it may be easier to use than the Adler-Immerman game. On the other hand, it should be remarked that the two games are essentially equivalent: The moves corresponding to connectives and modal operators are the same in both games (when restricting to the sets $\mathbb{A}$ and $\mathbb{B}$ in a position $(m, k, \mathbb{A}, \mathbb{B})$). Hence, in principle, it is possible to translate a winning strategy in one of the games to a corresponding winning strategy in the other.

We illustrate the use of our game by proving a nonelementary succinctness gap between first-order logic FO and (basic) modal logic ML. More precisely, we define a bisimulation invariant property of pointed Kripke-models by a first-order formula of size $\mathcal{O}(2^n)$, and show that this property cannot be defined by any ML-formula of size less than the exponential tower of height $n-1$. Furthermore, we show that the same property of pointed Kripke-models is already definable by a formula of size $\mathcal{O}(2^n)$ in ML^2, which is a version of 2-dimensional modal logic defined by Otto in [16]. Hence the same nonelementary succinctness result holds for ML^2 over ML.

A similar gap between FO and temporal logic follows from a construction in the PhD thesis [18] of Stockmeyer. He proved that the satisfiability problem of FO over words is of nonelementary complexity. Etessami and Wilke [4] observed that from Stockmeyer's proof it is possible to extract FO-formulas of size $\mathcal{O}(n)$ whose smallest models are words of length nonelementary in n. On the other hand, it is well known that any satisfiable formula of temporal logic has a model of size $\mathcal{O}(2^n)$, where n is the size of the formula.

2 Preliminaries

In this section we fix notation, define the syntax and semantics of basic modal logic and define our notions of formula size. For more on the notions used in the paper, we refer to the textbook [2] of Blackburn, de Rijke and Venema.

Basic modal logic and first-order logic

Let Φ be a set of proposition symbols, and let $\mathcal{M} = (W, R, V)$, where W is a set, $R \subseteq W \times W$ and $V : \Phi \to \mathcal{P}(W)$, and let $w \in W$. The structure $(\mathcal{M}, w)$ is called a *pointed Kripke-model for* Φ.

Let $(\mathcal{M}, w)$ be a pointed Kripke-model. We use the notation

$$\Box(\mathcal{M}, w) := \{(\mathcal{M}, v) \mid v \in W, wR^{\mathcal{M}}v\}.$$

If $\mathbb{A}$ is a set of pointed Kripke-models, we use the notation

$$\Box\mathbb{A} := \bigcup_{(\mathcal{M},w)\in\mathbb{A}} \Box(\mathcal{M}, w).$$

Furthermore, if f is a function $f : \mathbb{A} \to \Box\mathbb{A}$ such that $f(\mathcal{M}, w) \in \Box(\mathcal{M}, w)$ for every $(\mathcal{M}, w) \in \mathbb{A}$, then we use the notation

$$\Diamond_f\mathbb{A} := f(\mathbb{A}).$$

Now we define the syntax and semantics of basic modal logic for pointed models.

Definition 2.1 Let Φ be a set of proposition symbols. The set of formulas of $\mathrm{ML}(\Phi)$ is generated by the following grammar

$$\varphi := p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \Phi$.

As is apparent from the definition of the syntax, we assume that all ML-formulas are in negation normal form. This is useful for the formula size game that we introduce in the next section.

Definition 2.2 The satisfaction relation $(\mathcal{M}, w) \vDash \varphi$ between pointed Kripke-models $(\mathcal{M}, w)$ and $\mathrm{ML}(\Phi)$-formulas φ is defined as follows:

(1) $(\mathcal{M}, w) \vDash p \Leftrightarrow w \in V(p)$,

(2) $(\mathcal{M}, w) \vDash \neg p \Leftrightarrow w \notin V(p)$,

(3) $(\mathcal{M}, w) \vDash (\varphi \wedge \psi) \Leftrightarrow (\mathcal{M}, w) \vDash \varphi$ and $(\mathcal{M}, w) \vDash \psi$,

(4) $(\mathcal{M}, w) \vDash (\varphi \vee \psi) \Leftrightarrow (\mathcal{M}, w) \vDash \varphi$ or $(\mathcal{M}, w) \vDash \psi$,

(5) $(\mathcal{M}, w) \vDash \Diamond\varphi \Leftrightarrow$ there is $(\mathcal{M}, v) \in \Box(\mathcal{M}, w)$ such that $(\mathcal{M}, v) \vDash \varphi$,

(6) $(\mathcal{M}, w) \vDash \Box\varphi \Leftrightarrow$ for every $(\mathcal{M}, v) \in \Box(\mathcal{M}, w)$ it holds that $(\mathcal{M}, v) \vDash \varphi$.

Furthermore, if $\mathbb{A}$ is a class of pointed Kripke-models, then

$$\mathbb{A} \vDash \varphi \Leftrightarrow (\mathcal{A}, w) \vDash \varphi \text{ for every } (\mathcal{A}, w) \in \mathbb{A}.$$

For the sake of convenience we also use the notation

$$\mathbb{A} \vDash \neg\varphi \Leftrightarrow (\mathcal{A}, w) \nvDash \varphi \text{ for every } (\mathcal{A}, w) \in \mathbb{A}.$$

In Section 4, we also consider the case $\Phi = \emptyset$. For this purpose, we add the atomic constants $\top$ and $\bot$ to ML, where $(\mathcal{M}, w) \vDash \top$ and $(\mathcal{M}, w) \nvDash \bot$ for all pointed Kripke-models $(\mathcal{M}, w)$.

The syntax and semantics for first-order logic are defined in the standard way. Each ML-formula φ defines a class $\mathrm{Mod}(\varphi)$ of pointed Kripke-models:

$$\mathrm{Mod}(\varphi) := \{(\mathcal{M}, w) \mid (\mathcal{M}, w) \vDash \varphi\}.$$

In the same way, any FO-formula $\psi(x)$ in the vocabulary consisting of the accessibility relation symbol R and unary relation symbols U_p for $p \in \Phi$ defines a class $\mathrm{Mod}(\psi)$ of pointed Kripke-models:

$$\mathrm{Mod}(\psi) := \{(\mathcal{M}, w) \mid \mathcal{M} \vDash \psi[w/x]\}.$$

The formulas $\varphi \in \mathrm{ML}$ and $\psi(x) \in \mathrm{FO}$ are *equivalent* if $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\psi)$.

The well-known link between ML and FO is the following theorem.

Theorem 2.3 (van Benthem Characterization Theorem) *A first-order formula $\psi(x)$ is equivalent to some formula in* ML *if and only if* $\mathrm{Mod}(\psi)$ *is bisimulation invariant.*

If a property of pointed Kripke-models is n-bisimulation invariant for some $n \in \mathbb{N}$, then it is also bisimulation invariant. Thus, FO-definability and n-bisimulation invariance imply ML-definability for any property of pointed Kripke-models. We will use this version of van Benthem's characterization in Section 4.1 for showing that certain property is ML-definable. For the sake of easier reading, we give here the definition of n-bisimulation.

Definition 2.4 Let $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ be pointed Φ-models. We say that $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are *n-bisimilar*, $(\mathcal{M}, w) \leftrightarrows_n (\mathcal{M}', w')$, if there are binary relations $Z_n \subseteq \cdots \subseteq Z_0$ such that for every $0 \leq i \leq n-1$ we have

(1) $(\mathcal{M}, w) Z_n (\mathcal{M}', w')$,

(2) if $(\mathcal{M}, v) Z_0 (\mathcal{M}', v')$, then $(\mathcal{M}, v) \vDash p \Leftrightarrow (\mathcal{M}', v') \vDash p$ for each $p \in \Phi$,

(3) if $(\mathcal{M}, v) Z_{i+1} (\mathcal{M}', v')$ and $(\mathcal{M}, u) \in \Box(\mathcal{M}, v)$ then there is $(\mathcal{M}', u') \in \Box(\mathcal{M}', v')$ such that $(\mathcal{M}, u) Z_i (\mathcal{M}', u')$,

(4) if $(\mathcal{M}, v) Z_{i+1} (\mathcal{M}', v')$ and $(\mathcal{M}', u') \in \Box(\mathcal{M}', v')$ then there is $(\mathcal{M}, u) \in \Box(\mathcal{M}, v)$ such that $(\mathcal{M}, u) Z_i (\mathcal{M}', u')$.

It is well known that if Φ is finite, two pointed Φ-models are n-bisimilar if and only if they are equivalent with respect to ML(Φ)-formulas of modal depth at most n.

Formula size

We define notions of formula size for ML and FO. These notions are related to the length of the formula as a string rather than the DAG-size[1] of it. For ML

[1] The DAG-size of a formula φ is the number of edges of the syntactic structure of φ in the form of a DAG. Thus since the fan-out in the DAG is at most two, the DAG-size is at most two times the number of subformulas of φ.

we define separately the number of modal operators and the number of binary connectives in the formula.

Definition 2.5 The *modal size* of a formula $\varphi \in \mathrm{ML}$, denoted $\mathrm{ms}(\varphi)$, is defined recursively as follows:

(1) If φ is a literal, then $\mathrm{ms}(\varphi) = 0$.

(2) If $\varphi = \psi \vee \vartheta$ or $\varphi = \psi \wedge \vartheta$, then $\mathrm{ms}(\varphi) = \mathrm{ms}(\psi) + \mathrm{ms}(\vartheta)$.

(3) If $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$, then $\mathrm{ms}(\varphi) = \mathrm{ms}(\psi) + 1$.

Definition 2.6 The *binary connective size* of a formula $\varphi \in \mathrm{ML}$, denoted by $\mathrm{cs}(\varphi)$, is defined recursively as follows:

(1) If φ is a literal, then $\mathrm{cs}(\varphi) = 0$.

(2) If $\varphi = \psi \vee \vartheta$ or $\varphi = \psi \wedge \vartheta$, then $\mathrm{cs}(\varphi) = \mathrm{cs}(\psi) + \mathrm{cs}(\vartheta) + 1$.

(3) If $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$, then $\mathrm{cs}(\varphi) = \mathrm{cs}(\psi)$.

The size of an ML formula is defined as the sum of modal size and connective size. We do not count literals or parentheses since their number can be derived from the number of binary connectives.

Definition 2.7 The *size* of a formula $\varphi \in \mathrm{ML}$ is $\mathrm{s}(\varphi) = \mathrm{ms}(\varphi) + \mathrm{cs}(\varphi)$.

Similarly we define formula size for FO to be the number of binary connectives and quantifiers in the formula. In general this could lead to an arbitrarily large difference between formula size and actual string length. For an example if f is a unary function symbol, then atomic formulas of the form $f(x) = x$, $f(f(x)) = x$ and so on, all have size 0. In this paper however, we only consider formulas with one binary relation so this is not an issue.

Definition 2.8 The *size* of a formula $\varphi \in \mathrm{FO}$, denoted by $\mathrm{s}(\varphi)$, is defined recursively as follows:

(1) If φ is a literal, then $\mathrm{s}(\varphi) = 0$.

(2) If $\varphi = \neg\psi$, then $\mathrm{s}(\varphi) = \mathrm{s}(\psi)$.

(3) If $\varphi = \psi \vee \vartheta$ or $\varphi = \psi \wedge \vartheta$, then $\mathrm{s}(\varphi) = \mathrm{s}(\psi) + \mathrm{s}(\vartheta) + 1$.

(4) If $\varphi = \exists x\psi$ or $\varphi = \forall x\psi$, then $\mathrm{s}(\varphi) = \mathrm{s}(\psi) + 1$.

To refer to some rather large formula sizes we need the exponential tower function.

Definition 2.9 We define the function $\mathrm{twr} : \mathbb{N} \to \mathbb{N}$ recursively as follows:

$$\mathrm{twr}(0) = 1$$
$$\mathrm{twr}(n+1) = 2^{\mathrm{twr}(n)}.$$

We will also use in the sequel the binary logarithm function, denoted by log.

Separating classes by formulas

The definition of the formula size game in the next section is based on the notion of separating classes of pointed Kripke-models by formulas.

Definition 2.10 Let $\mathbb{A}$ and $\mathbb{B}$ be classes of pointed Kripke-models.
(a) We say that a formula $\varphi \in \mathrm{ML}$ *separates the classes* $\mathbb{A}$ *and* $\mathbb{B}$ if $\mathbb{A} \vDash \varphi$ and $\mathbb{B} \vDash \neg\varphi$.
(b) Similarly, a formula $\psi(x) \in \mathrm{FO}$ separates the classes $\mathbb{A}$ and $\mathbb{B}$ if for all $(\mathcal{M}, w) \in \mathbb{A}$, $\mathcal{M} \vDash \psi[w/x]$ and for all $(\mathcal{M}, w) \in \mathbb{B}$, $\mathcal{M} \vDash \neg\psi[w/x]$.

In other words, a formula $\varphi \in \mathrm{ML}$ separates the classes $\mathbb{A}$ and $\mathbb{B}$ if $\mathbb{A} \subseteq \mathrm{Mod}(\varphi)$ and $\mathbb{B} \subseteq \overline{\mathrm{Mod}(\varphi)}$, where $\overline{\mathrm{Mod}(\varphi)}$ is the complement of $\mathrm{Mod}(\varphi)$.

3 The formula size game

As in the Adler-Immerman game, the basic idea in our formula size game is that there are two players, S (spoiler) and D (duplicator), who play on a pair $(\mathbb{A}, \mathbb{B})$ of two sets of pointed Kripke-models. The aim of S is to show that $\mathbb{A}$ and $\mathbb{B}$ can be separated by a formula with modal size at most m and connective size at most k, while D tries to refute this. The moves of S reflect the connectives and modal operators of a formula that is supposed to separate the sets. The parameters m and k decrease with every move and act as resources indicating how many connectives and modal operators S has left to spend.

The crucial difference between our game and the Adler-Immerman game is that we define positions in the game to be tuples $(m, k, \mathbb{A}, \mathbb{B})$ instead of just pairs $(\mathbb{A}, \mathbb{B})$. This means that in the connective moves, D has a genuine choice to make. Furthermore, the winning condition of the game is based on a natural property of single positions instead of the size of the entire game tree.

We give now the precise definition of our game.

Definition 3.1 Let $\mathbb{A}_0$ and $\mathbb{B}_0$ be sets of pointed Φ-Kripke-models and let $m_0, k_0 \in \mathbb{N}$. *The formula size game between the sets* $\mathbb{A}_0$ *and* $\mathbb{B}_0$, denoted $\mathrm{FS}_{m_0,k_0}(\mathbb{A}_0, \mathbb{B}_0)$, has two players, S and D. The number m_0 is the *modal parameter* and k_0 is the *connective parameter* of the game. The starting position of the game is $(m_0, k_0, \mathbb{A}_0, \mathbb{B}_0)$. Let the position after n moves be $(m, k, \mathbb{A}, \mathbb{B})$. To continue the game, S has the following four moves to choose from:

- *Left splitting move*: First, S chooses natural numbers m_1, m_2, k_1 and k_2 and sets $\mathbb{A}_1$ and $\mathbb{A}_2$ such that $m_1 + m_2 = m$, $k_1 + k_2 + 1 = k$ and $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$. Then D decides whether the game continues from the position $(m_1, k_1, \mathbb{A}_1, \mathbb{B})$ or the position $(m_2, k_2, \mathbb{A}_2, \mathbb{B})$.
- *Right splitting move*: First, S chooses natural numbers m_1, m_2, k_1 and k_2 and sets $\mathbb{B}_1$ and $\mathbb{B}_2$ such that $m_1 + m_2 = m$, $k_1 + k_2 + 1 = k$ and $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}$. Then D decides whether the game continues from the position $(m_1, k_1, \mathbb{A}, \mathbb{B}_1)$ or the position $(m_2, k_2, \mathbb{A}, \mathbb{B}_2)$.
- *Left successor move*: S chooses a function $f : \mathbb{A} \to \Box\mathbb{A}$ such that $f(\mathcal{A}, w) \in \Box(\mathcal{A}, w)$ for all $(\mathcal{A}, w) \in \mathbb{A}$ and the game continues from the position $(m - 1, k, \Diamond_f \mathbb{A}, \Box\mathbb{B})$.
- *Right successor move*: S chooses a function $g : \mathbb{B} \to \Box\mathbb{B}$ such that $g(\mathcal{B}, w) \in \Box(\mathcal{B}, w)$ for all $(\mathcal{B}, w) \in \mathbb{B}$ and the game continues from the position $(m - 1, k, \Box\mathbb{A}, \Diamond_g \mathbb{B})$.

The game ends and S wins in a position $(m, k, \mathbb{A}, \mathbb{B})$ if there is a Φ-literal φ which separates the sets $\mathbb{A}$ and $\mathbb{B}$. The game ends and D wins in a position $(m, k, \mathbb{A}, \mathbb{B})$ if S cannot move and S does not win in this position.

The modal and connective parameters m and k can be thought of as resources for S, since in a position $(m, k, \mathbb{A}, \mathbb{B})$ S cannot make a successor move if $m = 0$ or a splitting move if $k = 0$. Note also that if $\Box(\mathcal{M}, w) = \emptyset$ for some $(\mathcal{M}, w) \in \mathbb{A}$ $(\in \mathbb{B})$ then S cannot make a left (right) successor move.

We prove now that the formula size game indeed characterizes the separation of two sets of pointed Kripke-models by a formula of a given size.

Theorem 3.2 *Let $\mathbb{A}$ and $\mathbb{B}$ be sets of pointed Φ-models and let m and k be natural numbers. Then the following conditions are equivalent:*

$(\text{win})_{m,k}$ *S has a winning strategy in the game* $\text{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.

$(\text{sep})_{m,k}$ *There is a formula $\varphi \in \text{ML}(\Phi)$ such that $\text{ms}(\varphi) \leq m$, $\text{cs}(\varphi) \leq k$ and the formula φ separates the sets $\mathbb{A}$ and $\mathbb{B}$.*

Proof. The proof proceeds by induction on the number $m+k$. If $m+k = 0$, no moves can be made. Thus if S wins, then there is a literal φ that separates the sets $\mathbb{A}$ and $\mathbb{B}$. In this case $\text{s}(\varphi) = 0$ so $(\text{win})_{0,0} \Rightarrow (\text{sep})_{0,0}$. On the other hand, if there is a formula φ such that $\text{s}(\varphi) \leq 0$ and φ separates the sets $\mathbb{A}$ and $\mathbb{B}$, then φ is a literal. Thus S wins the game, and we see that $(\text{sep})_{0,0} \Rightarrow (\text{win})_{0,0}$.

Suppose then that $m + k > 0$ and $(\text{win})_{n,l} \Leftrightarrow (\text{sep})_{n,l}$ for all $n, l \in \mathbb{Z}_+$ such that $n + l < m + k$. Assume first that $(\text{win})_{m,k}$ holds. Consider the following cases according to the first move in the winning strategy of S.

(a) Assume that the first move of the winning strategy of S is a left splitting move choosing numbers $m_1, m_2, k_1, k_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$ and $k_1 + k_2 + 1 = k$, and sets $\mathbb{A}_1, \mathbb{A}_2 \subseteq \mathbb{A}$ such that $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$. Since this move is given by a winning strategy, S has a winning strategy for both possible continuations of the game, $(m_1, k_1, \mathbb{A}_1, \mathbb{B})$ and $(m_2, k_2, \mathbb{A}_2, \mathbb{B})$. Since $m_i + k_i < m_i + k_i + 1 \leq m + k$ for $i \in \{1, 2\}$, by induction hypothesis there is a formula ψ such that $\text{ms}(\psi) \leq m_1$, $\text{cs}(\psi) \leq k_1$ and ψ separates the sets $\mathbb{A}_1$ and $\mathbb{B}$ and a formula ϑ such that $\text{ms}(\vartheta) \leq m_2$, $\text{cs}(\vartheta) \leq k_2$ and ϑ separates the sets $\mathbb{A}_2$ and $\mathbb{B}$. Thus $\mathbb{A}_1 \vDash \psi$ and $\mathbb{A}_2 \vDash \vartheta$ so $\mathbb{A} \vDash \psi \vee \vartheta$. On the other hand $\mathbb{B} \vDash \neg\psi$ and $\mathbb{B} \vDash \neg\vartheta$ so $\mathbb{B} \vDash \neg(\psi \vee \vartheta)$. Therefore the formula $\psi \vee \vartheta$ separates the sets $\mathbb{A}$ and $\mathbb{B}$. In addition $\text{ms}(\psi \vee \vartheta) = \text{ms}(\psi) + \text{ms}(\vartheta) \leq m_1 + m_2 = m$ and $\text{cs}(\psi \vee \vartheta) = \text{cs}(\psi) + \text{cs}(\vartheta) + 1 \leq k_1 + k_2 + 1 = k$ so $(\text{sep})_{m,k}$ holds.

(b) Assume that the first move of the winning strategy of S is a right splitting move choosing numbers $m_1, m_2, k_1, k_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$ and $k_1 + k_2 + 1 = k$, and sets $\mathbb{B}_1, \mathbb{B}_2 \subseteq \mathbb{B}$ such that $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}$. Since this move is given by a winning strategy, player I has a winning strategy for both possible continuations of the game, $(m_1, k_1, \mathbb{A}, \mathbb{B}_1)$ and $(m_2, k_2, \mathbb{A}, \mathbb{B}_2)$. By induction hypothesis there is a formula ψ such that $\text{ms}(\psi) \leq m_1$, $\text{cs}(\psi) \leq k_1$ and ψ separates the sets $\mathbb{A}$ and $\mathbb{B}_1$ and a formula ϑ such that $\text{ms}(\vartheta) \leq m_2$, $\text{cs}(\vartheta) \leq k_2$ and ϑ separates the sets $\mathbb{A}$ and $\mathbb{B}_2$. Thus $\mathbb{A} \vDash \psi$ and

$\mathbb{A} \vDash \vartheta$ so $\mathbb{A} \vDash \psi \wedge \vartheta$. On the other hand $\mathbb{B}_1 \vDash \neg\psi$ and $\mathbb{B}_2 \vDash \neg\vartheta$ so $\mathbb{B} \vDash \neg(\psi \wedge \vartheta)$. Therefore the formula $\psi \wedge \vartheta$ separates the sets $\mathbb{A}$ and $\mathbb{B}$. In addition $\mathrm{ms}(\psi \wedge \vartheta) = \mathrm{ms}(\psi) + \mathrm{ms}(\vartheta) \leq m_1 + m_2 = m$ and $\mathrm{cs}(\psi \wedge \vartheta) = \mathrm{cs}(\psi) + \mathrm{cs}(\vartheta) + 1 \leq k_1 + k_2 + 1 = k$ so $(\mathrm{sep})_{m,k}$ holds.

(c) Assume that the first move of the winning strategy of S is a left successor move choosing a function $f : \mathbb{A} \to \Box\mathbb{A}$ such that $f(\mathcal{A}, w) \in \Box(\mathcal{A}, w)$ for all $(\mathcal{A}, w) \in \mathbb{A}$. The game continues from the position $(m-1, k, \Diamond_f\mathbb{A}, \Box\mathbb{B})$ and S has a winning strategy from this position. By induction hypothesis there is a formula ψ such that $\mathrm{ms}(\psi) \leq m-1$, $\mathrm{cs}(\psi) \leq k$ and ψ separates the sets $\Diamond_f\mathbb{A}$ and $\Box\mathbb{B}$. Now for every $(\mathcal{A}, w) \in \mathbb{A}$ we have $f(\mathcal{A}, w) \in \Box(\mathcal{A}, w)$ and $f(\mathcal{A}, w) \vDash \psi$. Therefore $\mathbb{A} \vDash \Diamond\psi$. On the other hand $\Box\mathbb{B} \vDash \neg\psi$ so for every $(\mathcal{B}, w) \in \mathbb{B}$ and every $(\mathcal{B}, v) \in \Box(\mathcal{B}, w)$ we have $(\mathcal{B}, v) \nvDash \psi$. Thus $\mathbb{B} \vDash \neg\Diamond\psi$. So the formula $\Diamond\psi$ separates the sets $\mathbb{A}$ and $\mathbb{B}$ and since $\mathrm{ms}(\Diamond\psi) = \mathrm{ms}(\psi) + 1 \leq m$ and $\mathrm{cs}(\Diamond\psi) = \mathrm{cs}(\psi) \leq k$, $(\mathrm{sep})_{m,k}$ holds.

(d) Assume that the first move of the winning strategy of player I is a right successor move choosing a function $g : \mathbb{B} \to \Box\mathbb{B}$ such that $g(\mathcal{B}, w) \in \Box(\mathcal{B}, w)$ for every $(\mathcal{B}, w) \in \mathbb{B}$. The game continues from the position $(m-1, k, \Box\mathbb{A}, \Diamond_g\mathbb{B})$ and player I has a winning strategy from this position. By induction hypothesis there is a formula ψ such that $\mathrm{ms}(\psi) \leq m-1$, $\mathrm{cs}(\psi) \leq k$ and ψ separates the sets $\Box\mathbb{A}$ and $\Diamond_g\mathbb{B}$. Thus $\Box\mathbb{A} \vDash \psi$ so for every $(\mathcal{A}, w) \in \mathbb{A}$ and every $(\mathcal{A}, v) \in \Box(\mathcal{A}, w)$ we have $(\mathcal{A}, v) \vDash \psi$ so $\mathbb{A} \vDash \Box\psi$. On the other hand $\Diamond_g\mathbb{B} \vDash \neg\psi$ so for every $(\mathcal{B}, w) \in \mathbb{B}$ we have $g(\mathcal{B}, w) \in \Box(\mathcal{B}, w)$ and $g(\mathcal{B}, w) \nvDash \psi$. Thus $\mathbb{B} \vDash \neg\Box\psi$. Therefore the formula $\Box\psi$ separates the sets $\mathbb{A}$ and $\mathbb{B}$ and since $\mathrm{ms}(\Box\psi) = \mathrm{ms}(\psi) + 1 \leq m$ and $\mathrm{cs}(\Box\psi) = \mathrm{cs}(\psi) \leq k$, $(\mathrm{sep})_{m,k}$ holds.

Now assume $(\mathrm{sep})_{m,k}$ holds, and φ is the formula separating $\mathbb{A}$ and $\mathbb{B}$. We obtain a winning strategy of S for the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$ using φ as follows:

(a) If φ is a literal, S wins the game with no moves.

(b) Assume that $\varphi = \psi \vee \vartheta$. Let $\mathbb{A}_1 := \{(\mathcal{A}, w) \in \mathbb{A} \mid (\mathcal{A}, w) \vDash \psi\}$ and $\mathbb{A}_2 := \{(\mathcal{A}, w) \in \mathbb{A} \mid (\mathcal{A}, w) \vDash \vartheta\}$. Since $\mathbb{A} \vDash \varphi$ we have $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$. In addition, since $\mathbb{B} \vDash \neg\varphi$, we have $\mathbb{B} \vDash \neg\psi$ and $\mathbb{B} \vDash \neg\vartheta$. Thus ψ separates the sets $\mathbb{A}_1$ and $\mathbb{B}$ and ϑ separates the sets $\mathbb{A}_2$ and $\mathbb{B}$. Since $\mathrm{ms}(\psi) + \mathrm{ms}(\vartheta) = \mathrm{ms}(\varphi) \leq m$, there are $m_1, m_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$, $\mathrm{ms}(\psi) \leq m_1$ and $\mathrm{ms}(\vartheta) \leq m_2$. Similarly since $\mathrm{cs}(\psi) + \mathrm{cs}(\vartheta) + 1 = \mathrm{cs}(\varphi) \leq k$, there are $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 + 1 = k$, $\mathrm{cs}(\psi) \leq k_1$ and $\mathrm{cs}(\vartheta) \leq k_2$. By induction hypothesis S has winning strategies for the games $\mathrm{FS}_{m_1,k_1}(\mathbb{A}_1, \mathbb{B})$ and $\mathrm{FS}_{m_2,k_2}(\mathbb{A}_2, \mathbb{B})$. Since $k \geq \mathrm{cs}(\varphi) \geq 1$, S can start the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$ with a left splitting move choosing the numbers m_1, m_2, k_1 and k_2 and the sets $\mathbb{A}_1$ and $\mathbb{A}_2$. Then S wins the game by following the winning strategy for whichever position D chooses.

(c) Assume that $\varphi = \psi \wedge \vartheta$. Let $\mathbb{B}_1 := \{(\mathcal{B}, w) \in \mathbb{B} \mid (\mathcal{B}, w) \nvDash \psi\}$ and $\mathbb{B}_2 := \{(\mathcal{B}, w) \in \mathbb{B} \mid (\mathcal{B}, w) \nvDash \vartheta\}$. Since $\mathbb{B} \vDash \neg\varphi$, we have $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}$. In addition, since $\mathbb{A} \vDash \varphi$, we have $\mathbb{A} \vDash \psi$ and $\mathbb{A} \vDash \vartheta$. Thus ψ separates the sets $\mathbb{A}$ and $\mathbb{B}_1$ while ϑ separates the sets $\mathbb{A}$ and $\mathbb{B}_2$. As in the previous case, there are $m_1, m_2, k_1, k_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$, $\mathrm{ms}(\psi) \leq m_1$, $\mathrm{ms}(\vartheta) \leq m_2$, $k_1 + k_2 = k$, $\mathrm{cs}(\psi) \leq k_1$ and $\mathrm{cs}(\vartheta) \leq k_2$. By induction hypothesis player I has a winning strategy for the games $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B}_1)$ and $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B}_2)$. Player I wins the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$ by starting with a right splitting move choosing the numbers m_1, m_2, k_1, and k_2 and the sets $\mathbb{B}_1$ and $\mathbb{B}_2$ and proceeding according to the winning strategies for the games $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B}_1)$ and $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B}_2)$.

(d) Assume that $\varphi = \Diamond\psi$. Since $\mathbb{A} \models \varphi$, for every $(\mathcal{A}, w) \in \mathbb{A}$ there is $(\mathcal{A}, v_w) \in \Box(\mathcal{A}, w)$ such that $(\mathcal{A}, v_w) \models \psi$. We define the function $f : \mathbb{A} \to \Box\mathbb{A}$ by $f(\mathcal{A}, w) = (\mathcal{A}, v_w)$. Clearly $\Diamond_f\mathbb{A} \models \psi$. On the other hand $\mathbb{B} \models \neg\varphi$ so for each $(\mathcal{B}, w) \in \mathbb{B}$ and each $(\mathcal{B}, v) \in \Box(\mathcal{B}, w)$ we have $(\mathcal{B}, v) \not\models \psi$. Therefore $\Box\mathbb{B} \models \neg\psi$ and the formula ψ separates the sets $\Diamond_f\mathbb{A}$ and $\Box\mathbb{B}$. Moreover, $\mathrm{ms}(\psi) = \mathrm{ms}(\varphi) - 1 \leq m - 1$ and $\mathrm{cs}(\psi) = \mathrm{cs}(\varphi) \leq k$ so by induction hypothesis S has a winning strategy for the game $\mathrm{FS}_{m-1,k}(\Diamond_f\mathbb{A}, \Box\mathbb{B})$. Since $m \geq \mathrm{ms}(\varphi) \geq 1$, S can start the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$ with a left successor move choosing the function f. Then S wins the game by following the winning strategy for the game $\mathrm{FS}_{m-1,k}(\Diamond_f\mathbb{A}, \Box\mathbb{B})$.

(e) Assume finally that $\varphi = \Box\psi$. Since $\mathbb{A} \models \varphi$, as in the previous case we obtain $\Box\mathbb{A} \models \psi$. On the other hand, since $\mathbb{B} \models \neg\varphi$, for every $(\mathcal{B}, w) \in \mathbb{B}$ there is $(\mathcal{B}, v_w) \in \Box(\mathcal{B}, w)$ such that $(\mathcal{B}, v_w) \not\models \psi$. We define the function $g : \mathbb{B} \to \Box\mathbb{B}$ by $g(\mathcal{B}, w) = (\mathcal{B}, v_w)$. Clearly $\Diamond_g\mathbb{B} \models \neg\psi$ so the formula ψ separates the sets $\Box\mathbb{A}$ and $\Diamond_g\mathbb{B}$. By induction hypothesis player I has a winning strategy for the game $\mathrm{FS}_{m-1,k}(\Box\mathbb{A}, \Diamond_g\mathbb{B})$. Player wins the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$ by starting with a right successor move choosing the function g and proceeding according to the winning strategy of the game $\mathrm{FS}_{m-1,k}(\Box\mathbb{A}, \Diamond_g\mathbb{B})$.

□

Note that in Theorem 3.2 we allow the set of proposition symbols Φ to be infinite. This is in contrast with other similar games, such as the bisimulation game and the n-bisimulation game. For an example let $\Phi = \{p_i \mid i \in \mathbb{N}\}$ and $W = \{w\} \cup \{w_i \mid i \in \mathbb{N}\}$. Furthermore let $(\mathcal{A}, w)$ be a pointed model, where $\mathrm{dom}(\mathcal{A}) = W$, $R^{\mathcal{A}} = \{(w, w_i) \mid i \in \mathbb{N}\}$ and $V^{\mathcal{A}}(p_i) = \{w_j \mid j \geq i\}$ for each $i \in \mathbb{N}$. Let $(\mathcal{B}, w)$ be the same model with the addition of a point $w_{\mathbb{N}}$ in which all propositions are true. In other words $\mathrm{dom}(\mathcal{B}) = W \cup \{w_{\mathbb{N}}\}$, $R^{\mathcal{B}} = R^{\mathcal{A}} \cup \{(b, w_{\mathbb{N}})\}$ and $V^{\mathcal{B}}(p_i) = V^{\mathcal{A}}(p_i) \cup \{w_{\mathbb{N}}\}$ for each $i \in \mathbb{N}$.

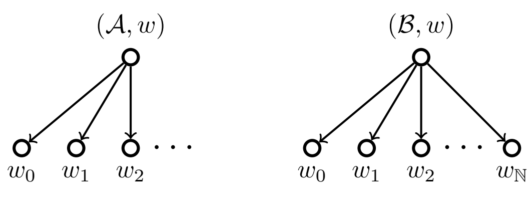

Fig. 1. The pointed models $(\mathcal{A}, w)$ and $(\mathcal{B}, w)$.

We see that by moving to $w_{\mathbb{N}}$, S wins the (n-)bisimulation game between the models $(\mathcal{A}, w)$ and $(\mathcal{B}, w)$, even though the models satisfy exactly the same ML-formulas.

We prove next that m-bisimilarity implies that D has winning strategy in the formula size game with modal parameter m. This simple observation is used in the next section, when we apply the game $\mathrm{FS}_{m,k}$ for proving a succinctness result for FO over ML.

Theorem 3.3 *Let $\mathbb{A}$ and $\mathbb{B}$ be sets of pointed models and let $m, k \in \mathbb{N}$. If there are m-bisimilar pointed models $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$, then D has a winning strategy for the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.*

Proof. The proof proceeds by induction on the number $m+k \in \mathbb{N}$. If $m+k = 0$ and $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$ are m-bisimilar, then they are 0-bisimilar and thus satisfy the same literals. Thus there is no literal $\varphi \in \mathrm{ML}$ that separates the sets $\mathbb{A}$ and $\mathbb{B}$. Since S cannot make any moves and S does not win the game in this position, D wins the game $\mathrm{FS}_{0,0}(\mathbb{A}, \mathbb{B})$.

Assume that $m + k > 0$ and $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$ are m-bisimilar. As in the basic step, S does not win the game in this position. We consider the cases of the first move of S in the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.

If S starts with a left splitting move choosing the numbers m_1, m_2, k_1 and k_2 and the sets $\mathbb{A}_1$ and $\mathbb{A}_2$, then since $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$, D can choose the next position $(m_i, k_i, \mathbb{A}_i, \mathbb{B})$, $i \in \{1, 2\}$ in such a way that $(\mathcal{A}, w) \in \mathbb{A}_i$. Then we have $m_i \leq m$ and $m_i + k_i < m + k$ so by induction hypothesis D has a winning strategy for the game $\mathrm{FS}_{m_i,k_i}(\mathbb{A}_i, \mathbb{B})$. The case of a right splitting move is similar.

If S starts with a left successor move choosing a function $f : \mathbb{A} \to \Box\mathbb{A}$, then since $(\mathcal{A}, w)$ and $(\mathcal{B}, v)$ are m-bisimilar, there is a pointed model $(\mathcal{B}, v') \in \Box(\mathcal{B}, v)$ that is $m-1$-bisimilar with the pointed model $f(\mathcal{A}, w)$. Since $m-1+k < m+k$, by induction hypothesis D has a winning strategy in $\mathrm{FS}_{m-1,k}(\Diamond_f \mathbb{A}, \Box\mathbb{B})$. The case of a right successor move is similar. □

4 Succinctness of FO over ML

In this section, we illustrate the use of the formula size game $\mathrm{FS}_{m,k}$ by proving a nonelementary succinctness gap between bisimulation invariant first-order logic and modal logic. We also show that this gap is already present between the 2-dimensional modal logic ML^2 introduced in [16] and basic modal logic.

4.1 A property of pointed frames

For the remainder of this paper we consider only the case where the set Φ of propositional symbols is empty. This makes all points in Kripke-models propositionally equivalent so we call pointed models in this section pointed frames. The only formulas available for the win condition of S in the game $\mathrm{FS}_{m,k}$ are $\bot$ and $\top$. Thus S only wins the game from the position $(m, k, \mathbb{A}, \mathbb{B})$ if either $\mathbb{A} = \emptyset$ and $\mathbb{B} \neq \emptyset$, or $\mathbb{A} \neq \emptyset$ and $\mathbb{B} = \emptyset$.

We will use the following two classes in our application of the formula size game $\mathrm{FS}_{m,k}$:

- $\mathbb{A}_n$ is the class of all pointed frames $(\mathcal{A}, w)$ such that for all $(\mathcal{A}, u), (\mathcal{A}, v) \in \Box(\mathcal{A}, w)$, the frames $(\mathcal{A}, u)$ and $(\mathcal{A}, v)$ are n-bisimilar.
- $\mathbb{B}_n$ is the complement of $\mathbb{A}_n$.

Lemma 4.1 *For each $n \in \mathbb{N}$ there is a formula $\varphi_n(x) \in \mathrm{FO}$ that separates the classes $\mathbb{A}_n$ and $\mathbb{B}_n$ such that the size of $\varphi_n(x)$ is exponential with respect to n, i.e.,* $\mathrm{s}(\varphi_n) = \mathcal{O}(2^n)$.

Proof. We first define formulas $\psi_n(x, y) \in \mathrm{FO}$ such that $(\mathcal{M}, u) \leftrightarrows_n (\mathcal{M}, v)$ if and only if $\mathcal{M} \vDash \psi_n[u/x, v/y]$. The formulas $\psi_n(x, y)$ are defined recursively as follows:

$$\begin{aligned}\psi_1(x, y) :=& \exists s R(x, s) \leftrightarrow \exists t R(y, t)\\ \psi_{n+1}(x, y) :=& \forall s (R(x, s) \to \exists t (R(y, t) \wedge \psi_n(s, t))\\ & \wedge \forall t (R(y, t) \to \exists s (R(x, s) \wedge \psi_n(s, t)).\end{aligned}$$

Clearly these formulas express n-bisimilarity as intended. When we interpret the equivalences and implications as shorthand in the standard way, we get the sizes $\mathrm{s}(\psi_1) = 11$ and $\mathrm{s}(\psi_{n+1}) = 2 \cdot \mathrm{s}(\psi_n) + 13$. Thus $\mathrm{s}(\psi_n) = 3 \cdot 2^{n+2} - 13$.

Now we can define the formulas φ_n:

$$\varphi_n(x) := \forall y \forall z (R(x, y) \wedge R(x, z) \to \psi_n(y, z)).$$

Clearly for every $(\mathcal{A}, w) \in \mathbb{A}_n$ we have $\mathcal{A} \vDash \varphi_n[w/x]$ and for every $(\mathcal{B}, v) \in \mathbb{B}_n$ we have $\mathcal{B} \vDash \neg\varphi_n[w/x]$ so the formula φ_n separates the classes $\mathbb{A}_n$ and $\mathbb{B}_n$. Furthermore, $\mathrm{s}(\varphi_n) = \mathrm{s}(\psi_n) + 6 = 3 \cdot 2^{n+2} - 7$ so the size of φ_n is exponential with respect to n. □

Lemma 4.2 *For each $n \in \mathbb{N}$, the formula φ_n is $n + 1$-bisimulation invariant.*

Proof. Let $(\mathcal{A}, w)$ and $(\mathcal{B}, v)$ be $n + 1$-bisimilar pointed models. Assume that $\mathcal{A} \vDash \varphi_n[w/x]$. If $(\mathcal{B}, v_1), (\mathcal{B}, v_2) \in \Box(\mathcal{B}, v)$, by $n + 1$-bisimilarity there are $(\mathcal{A}, w_1), (\mathcal{A}, w_2) \in \Box(\mathcal{A}, w)$ such that $(\mathcal{A}, w_1) \leftrightarrows_n (\mathcal{B}, v_1)$ and $(\mathcal{A}, w_2) \leftrightarrows_n (\mathcal{B}, v_2)$. Since $\mathcal{A} \vDash \varphi_n[w/x]$, we have $(\mathcal{B}, v_1) \leftrightarrows_n (\mathcal{A}, w_1) \leftrightarrows_n (\mathcal{A}, w_2) \leftrightarrows_n (\mathcal{B}, v_2)$ so $\mathcal{B} \vDash \psi_n[v_1/x, v_2/y]$. Thus, we see that $\mathcal{B} \vDash \varphi_n[v/x]$. □

It follows now from van Benthem's characterization theorem that each φ_n is equivalent to some ML-formula. Thus, we get the following corollary.

Corollary 4.3 *For each $n \in \mathbb{N}$, there is a formula $\vartheta_n \in \mathrm{ML}$ that separates the classes $\mathbb{A}_n$ and $\mathbb{B}_n$.*

4.2 Set theoretic construction of pointed frames

We have shown that the classes $\mathbb{A}_n$ and $\mathbb{B}_n$ can be separated both in ML and in FO. Furthermore the size of the FO-formula is exponential with respect to n. It only remains to ask: what is the size of the smallest ML-formula that separates the classes $\mathbb{A}_n$ and $\mathbb{B}_n$? To answer this we will need suitable subsets of $\mathbb{A}_n$ and $\mathbb{B}_n$ to play the formula size game on.

Definition 4.4 Let $n \in \mathbb{N}$. *The finite levels of the cumulative hierarchy* are defined recursively as follows:

$$\mathsf{V}_0 = \emptyset$$
$$\mathsf{V}_{n+1} = \mathcal{P}(\mathsf{V}_n)$$

For every $n \in \mathbb{N}$, V_n is a *transitive set*, i.e., for every $a \in \mathsf{V}_n$ and every $b \in a$ it holds that $b \in \mathsf{V}_n$. Thus it is reasonable to define a frame $\mathcal{F}_n = (\mathsf{V}_n, R_n)$, where for all $a, b \in \mathsf{V}_n$ it holds that $(a, b) \in R_n \Leftrightarrow b \in a$.

For every point $a \in \mathsf{V}_n$ we denote by $(\mathcal{M}_a, a)$ the pointed frame, where $\mathcal{M}_a$ is the subframe of $\mathcal{F}_n$ generated by the point a.

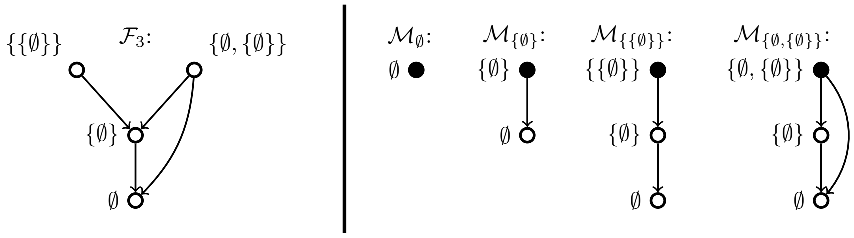

Fig. 2. The frame $\mathcal{F}_3$ and its generated subframes

Lemma 4.5 *Let $n \in \mathbb{N}$ and $a, b \in \mathsf{V}_{n+1}$. If $a \neq b$, then $(\mathcal{M}_a, a) \not\Leftrightarrow_n (\mathcal{M}_b, b)$.*

Proof. We prove the claim by induction on n. The basic step $n = 0$ is trivial since V_1 only has one element. For the induction step, assume that $a, b \in \mathsf{V}_{n+1}$ and $a \neq b$. Assume further for contradiction that $(\mathcal{M}_a, a) \Leftrightarrow_n (\mathcal{M}_b, b)$. Since $a \neq b$, by symmetry we can assume that there is $x \in a$ such that $x \notin b$. By n-bisimilarity there is $y \in b$ such that $(\mathcal{M}_x, x)$ and $(\mathcal{M}_y, y)$ are $n-1$-bisimilar. Since $x \in a \in \mathsf{V}_{n+1}$ and $y \in b \in \mathsf{V}_{n+1}$, we have $x, y \in \mathsf{V}_n$. By induction hypothesis we obtain $x = y$. This is a contradiction, since $x \notin b$ and $y \in b$. $\square$

If $\mathbb{A}$ is a set of pointed frames we use the notation $\bigtriangleup\mathbb{A}$ for the pointed frame which is formed by taking all the pointed frames of $\mathbb{A}$ and connecting a new root to their distinguished points as illustrated in Figure 3. To make sure that $(\bigtriangleup\mathbb{A}, v)$ is bisimilar with $(\mathcal{A}, v)$ for any $(\mathcal{A}, v) \in \bigtriangleup\mathbb{A}$, we require that the frames in $\mathbb{A}$ are compatible in possible intersections. The precise definition is the following.

Let $\mathbb{A}$ be a set of pointed frames such that for all $(\mathcal{A}, v), (\mathcal{A}', v') \in \mathbb{A}$ it holds that $R^{\mathcal{A}} \upharpoonright (\mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{A}')) = R^{\mathcal{A}'} \upharpoonright (\mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{A}'))$ and let $w \notin \mathrm{dom}(\mathcal{A})$ for all $(\mathcal{A}, v) \in \mathbb{A}$. We use the notation $\bigtriangleup\mathbb{A} := (\mathcal{M}, w)$, where

$$\mathrm{dom}(\mathcal{M}) = \{w\} \cup \bigcup\{\mathrm{dom}(\mathcal{A}) \mid (\mathcal{A}, v) \in \mathbb{A}\}, \text{ and}$$
$$R^{\mathcal{M}} = \{(w, v) \mid (\mathcal{A}, v) \in \mathbb{A}\} \cup \bigcup\{R^{\mathcal{A}} \mid (\mathcal{A}, v) \in \mathbb{A}\}.$$

For each $n \in \mathbb{N}$ we define the following sets of pointed frames:

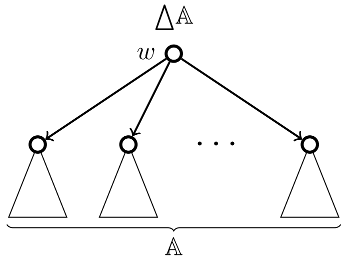

Fig. 3. The pointed frame $\triangle\mathbb{A}$

$$\mathbb{C}_n := \{\triangle\{(\mathcal{M}_a, a)\} \mid a \in \mathsf{V}_{n+1}\}$$
$$\mathbb{D}_n := \{\triangle\{(\mathcal{M}_a, a), (\mathcal{M}_b, b)\} \mid a, b \in \mathsf{V}_{n+1}, a \neq b\}.$$

In other words the pointed frames in $\mathbb{C}_n$ have a single successor from level $n+1$ of the cumulative hierarchy, whereas the pointed frames in $\mathbb{D}_n$ have two different successors from the same set. Therefore clearly $\mathbb{C}_n \subseteq \mathbb{A}_n$ and by Lemma 4.5 also $\mathbb{D}_n \subseteq \mathbb{B}_n$. In the next subsection we will use these sets in the formula size game.

It is well known that the cardinality of V_n is the exponential tower of $n-1$. Thus, the cardinality of $\mathbb{C}_n$ is $\mathrm{twr}(n)$.

Lemma 4.6 *If $n \in \mathbb{N}$, we have $|\mathbb{C}_n| = |\mathsf{V}_{n+1}| = \mathrm{twr}(n)$.* □

4.3 Graph colorings and winning strategies in $\mathrm{FS}_{m,k}$

Our aim is to prove that any ML-formula ϑ_n separating the sets $\mathbb{C}_n$ and $\mathbb{D}_n$ is of size at least $\mathrm{twr}(n-1)$. To do this, we make use of a surprising connection between the chromatic numbers of certain graphs related to pairs of the form $(\mathbb{V}, \mathbb{E})$, where $\mathbb{V} \subseteq \mathbb{C}_n$ and $\mathbb{E} \subseteq \mathbb{D}_n$, and existence of a winning strategy for D in the game $\mathrm{FS}_{m,k}(\mathbb{V}, \mathbb{E})$.

Let $n \in \mathbb{N}$, $\emptyset \neq \mathbb{V} \subseteq \mathbb{C}_n$ and $\mathbb{E} \subseteq \mathbb{D}_n$. Then $\mathcal{G}(\mathbb{V}, \mathbb{E})$ denotes the graph (V, E), where

$$V = \{(\mathcal{M}, w) \mid \triangle\{(\mathcal{M}, w)\} \in \mathbb{V}\}, \text{ and}$$
$$E = \{((\mathcal{M}, w), (\mathcal{M}', w')) \in V \times V \mid \triangle\{(\mathcal{M}, w), (\mathcal{M}', w')\} \in \mathbb{E}\}.$$

Definition 4.7 Let $\mathcal{G} = (V, E)$ be a graph and let C be a set. A function $\chi : V \to C$ is a *coloring* of the graph $\mathcal{G}$ if for all $u, v \in V$ it holds that if $(u, v) \in E$, then $\chi(u) \neq \chi(v)$. If the set C has k elements, then χ is called a *k-coloring* of $\mathcal{G}$.

The *chromatic number* of $\mathcal{G}$, denoted by $\chi(\mathcal{G})$, is the smallest number $k \in \mathbb{N}$ for which there is a k-coloring of $\mathcal{G}$.

When playing the formula size game $\mathrm{FS}_{m,k}(\mathbb{V}, \mathbb{E})$, splitting moves correspond with dividing either the vertex set or the edge set of the graph $\mathcal{G}(\mathbb{V}, \mathbb{E})$

into two parts, forming two new graphs. In the next lemma we get simple arithmetic estimates for the behaviour of chromatic numbers in such divisions.

Lemma 4.8 *Let $\mathcal{G} = (V, E)$ be a graph.*

(i) *Let $V_1, V_2 \subseteq V$ be nonempty such that $V_1 \cup V_2 = V$ and let $\mathcal{G}_1 = (V_1, E \restriction V_1)$ and $\mathcal{G}_2 = (V_2, E \restriction V_2)$. Then we have $\chi(\mathcal{G}) \leq \chi(\mathcal{G}_1) + \chi(\mathcal{G}_2)$.*

(ii) *Let $E_1, E_2 \subseteq E$ such that $E_1 \cup E_2 = E$ and let $\mathcal{G}_1 = (V, E_1)$ and $\mathcal{G}_2 = (V, E_2)$. Then $\chi(\mathcal{G}) \leq \chi(\mathcal{G}_1)\chi(\mathcal{G}_2)$.*

Proof.

(i) Let V_1, V_2, $\mathcal{G}_1$ and $\mathcal{G}_2$ be as in the claim and let $k_1 = \chi(\mathcal{G}_1)$ and $k_2 = \chi(\mathcal{G}_2)$. Let $\chi_1 : V_1 \to \{1, \ldots, k_1\}$ be a k_1-coloring of the graph $\mathcal{G}_1$ and let $\chi_2 : V_2 \to \{k_1+1, \ldots, k_1+k_2\}$ be a k_2-coloring of the graph $\mathcal{G}_2$. Then it is straightforward to show that $\chi = \chi_1 \cup (\chi_2 \restriction (V_2 \setminus V_1))$ is a $k_1 + k_2$-coloring of the graph $\mathcal{G}$, whence $\chi(\mathcal{G}) \leq k_1 + k_2 = \chi(\mathcal{G}_1) + \chi(\mathcal{G}_2)$.

(ii) Let $\chi_1 : V \to \{1, \ldots, k_1\}$ and $\chi_2 : V \to \{1, \ldots, k_2\}$ be colorings of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then it is easy to verify that the map $\chi : V \to \{1, \ldots, k_1\} \times \{1, \ldots, k_2\}$ defined by $\chi(v) = (\chi_1(v), \chi_2(v))$ is a coloring of $\mathcal{G}$. Thus we obtain $\chi(\mathcal{G}) \leq |\{1, \ldots, k_1\} \times \{1, \ldots, k_2\}| = \chi(\mathcal{G}_1)\chi(\mathcal{G}_2)$.

□

Lemma 4.9 *Assume $\emptyset \neq \mathbb{V} \subseteq \mathbb{C}_n$ and $\mathbb{E} \subseteq \mathbb{D}_n$ for some $n \in \mathbb{N}$ and let $m, k \in \mathbb{N}$. If $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \geq 2$ and $k < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})))$, then D has a winning strategy in the game $\mathrm{FS}_{m,k}(\mathbb{V}, \mathbb{E})$.*

Proof. Let $n, m, k \in \mathbb{N}$ and assume that $\emptyset \neq \mathbb{V} \subseteq \mathbb{C}_n$, $\mathbb{E} \subseteq \mathbb{D}_n$, $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \geq 2$ and $k < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})))$. We prove the claim by induction on k.

If $k = 0$, S can only make successor moves. Since $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \geq 2$, there are $(\mathcal{M}, w), (\mathcal{M}', w') \in V$ such that $((\mathcal{M}, w), (\mathcal{M}', w')) \in E$. Thus $\triangle\{(\mathcal{M}, w)\}$, $\triangle\{(\mathcal{M}', w')\} \in \mathbb{V}$ and $\triangle\{(\mathcal{M}, w), (\mathcal{M}', w')\} \in \mathbb{E}$. If S makes a left or right successor move, then in the resulting position $(m-1, 0, \mathbb{V}', \mathbb{E}')$ it holds that $(\mathcal{M}, w) \in \mathbb{V}' \cap \mathbb{E}'$ or $(\mathcal{M}', w') \in \mathbb{V}' \cap \mathbb{E}'$. Thus the same pointed model is present on both sides of the game and by Theorem 3.3, D has a winning strategy for the game $\mathrm{FS}_{m,k}(\mathbb{V}', \mathbb{E}')$.

Assume then that $k > 0$. If S starts the game with a successor move, then D wins as described above.

Assume that S begins the game with a left splitting move choosing the numbers $m_1, m_2, k_1, k_2 \in \mathbb{N}$ and the sets $\mathbb{V}_1, \mathbb{V}_2 \subseteq \mathbb{V}$. Consider the graphs $\mathcal{G}(\mathbb{V}, \mathbb{E}) = (V, E)$, $\mathcal{G}(\mathbb{V}_1, \mathbb{E}) = (V_1, E_1)$ and $\mathcal{G}(\mathbb{V}_2, \mathbb{E}) = (V_2, E_2)$. Since $\mathbb{V}_1 \cup \mathbb{V}_2 = \mathbb{V}$, we have $V_1 \cup V_2 = V$. In addition, by the definition of the graphs $\mathcal{G}(\mathbb{V}, \mathbb{E})$, $\mathcal{G}(\mathbb{V}_1, \mathbb{E})$ and $\mathcal{G}(\mathbb{V}_2, \mathbb{E})$ we see that $E_1 = E \restriction V_1$ and $E_2 = E \restriction V_2$. Thus by Lemma 4.8, we obtain $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \leq \chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E})) + \chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E}))$. It must hold that $k_1 < \log(\chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E})))$ or $k_2 < \log(\chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E})))$, since otherwise we would have

$$\begin{aligned} k &< \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}))) \leq \log(\chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E})) + \chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E}))) \\ &\leq \log(\chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E}))) + \log(\chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E}))) + 1 \leq k_1 + k_2 + 1 = k. \end{aligned}$$

Thus D can choose the next position of the game, $(m_i, k_i, \mathbb{V}_i, \mathbb{E})$, in such a way that $k_i < \log(\chi(\mathcal{G}(\mathbb{V}_i, \mathbb{E})))$. By induction hypothesis D has a winning strategy in the game $\mathrm{FS}_{m_i,k_i}(\mathbb{V}_i, \mathbb{E})$.

Assume then that S begins the game with a right splitting move choosing the numbers $m_1, m_2, k_1, k_2 \in \mathbb{N}$ and the sets $\mathbb{E}_1, \mathbb{E}_2 \subseteq \mathbb{E}$. Consider now the graphs $\mathcal{G}(\mathbb{V}, \mathbb{E}) = (V, E)$, $\mathcal{G}(\mathbb{V}, \mathbb{E}_1) = (V_1, E_1)$ and $\mathcal{G}(\mathbb{V}, \mathbb{E}_2) = (V_2, E_2)$. Clearly $V_1 = V_2 = V$ and since $\mathbb{E}_1 \cup \mathbb{E}_2 = \mathbb{E}$, we have $E_1 \cup E_2 = E$. Thus by Lemma 4.8, we obtain $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \leq \chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_1))\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_2))$. It must hold that $k_1 < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_1)))$ or $k_2 < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_2)))$, since otherwise we would have

$$\begin{aligned} k &< \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}))) \leq \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_1))\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_2))) \\ &= \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_1))) + \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_2))) \leq k_1 + k_2 + 1 = k. \end{aligned}$$

Thus D can again choose the next position of the game, $(m_i, k_i, \mathbb{V}, \mathbb{E}_i)$, in such a way that $k_i < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_i)))$. By induction hypothesis D has a winning strategy in the game $\mathrm{FS}_{m_i,k_i}(\mathbb{V}, \mathbb{E}_i)$. □

Lemma 4.10 *If* $k < \mathrm{twr}(n-1)$ *and* $m \in \mathbb{N}$, *then D has a winning strategy in the game* $\mathrm{FS}_{m,k}(\mathbb{C}_n, \mathbb{D}_n)$.

Proof. By Lemma 4.6, we have $|\mathbb{C}_n| = \mathrm{twr}(n)$ and the set $\mathbb{D}_n$ consists of all the pointed frames $\triangle\{(\mathcal{M}, w), (\mathcal{M}', w')\}$, where $(\mathcal{M}, w), (\mathcal{M}', w') \in \mathbb{C}_n$, $(\mathcal{M}, w) \neq (\mathcal{M}', w')$. Thus the graph $\mathcal{G}(\mathbb{C}_n, \mathbb{D}_n)$ is isomorphic with the complete graph $K_{\mathrm{twr}(n)}$. Therefore we obtain

$$\chi(\mathcal{G}(\mathbb{C}_n, \mathbb{D}_n)) = \chi(K_{\mathrm{twr}(n)}) = \mathrm{twr}(n).$$

By the assumption, $k < \mathrm{twr}(n-1) = \log(\mathrm{twr}(n)) = \log(\chi(\mathcal{G}(\mathbb{C}_n, \mathbb{D}_n)))$, so by Lemma 4.9, D has a winning strategy in the game $\mathrm{FS}_{m,k}(\mathbb{C}_n, \mathbb{D}_n)$. □

Theorem 4.11 *Let* $n \in \mathbb{N}$. *If a formula* $\vartheta_n \in \mathrm{ML}$ *separates the classes* $\mathbb{A}_n$ *and* $\mathbb{B}_n$, *then* $\mathrm{s}(\vartheta_n) \geq \mathrm{twr}(n-1)$.

Proof. Assume that a formula $\vartheta_n \in \mathrm{ML}$ separates the classes $\mathbb{A}_n$ and $\mathbb{B}_n$. As observed in the end of Subsection 4.2, it holds that $\mathbb{C}_n \subseteq \mathbb{A}_n$ and $\mathbb{D}_n \subseteq \mathbb{B}_n$. Therefore ϑ_n also separates the sets $\mathbb{C}_n$ and $\mathbb{D}_n$.

Assume for contradiction that $\mathrm{s}(\vartheta_n) < \mathrm{twr}(n-1)$. By Theorem 3.2, S has a winning strategy in the game $\mathrm{FS}_{m,k}(\mathbb{C}_n, \mathbb{D}_n)$ for $m = \mathrm{ms}(\vartheta_n)$ and $k = \mathrm{cs}(\vartheta_n)$. On the other hand, $k < \mathrm{twr}(n-1)$, whence by Lemma 4.10, D has a winning strategy in the same game. □

We now have everything we need for proving the nonelementary succinctness of FO over ML. By Lemma 4.1, for each $n \in \mathbb{N}$ there is a formula $\varphi_n(x) \in \mathrm{FO}$ such that φ_n separates the classes $\mathbb{A}_n$ and $\mathbb{B}_n$ with $s(\varphi) = \mathcal{O}(2^n)$. On the other hand by Corollary 4.3, there is an equivalent formula $\vartheta_n \in \mathrm{ML}$, but by Theorem 4.11 the size of ϑ_n must be at least $\mathrm{twr}(n-1)$. So the property of a pointed models all successors being n-bisimilar with each other can be expressed in FO with a formula of exponential size, but in ML expressing it requires a formula of non-elementary size.

Corollary 4.12 *Bisimulation invariant* FO *is nonelementarily more succinct than* ML*.*

Remark 4.13 It is well known that the DAG-size of any formula φ is greater than or equal to the logarithm of the size of φ. Thus if ϑ_n is a formula as in Theorem 4.11, the DAG-size of ϑ_n must be at least $\mathrm{twr}(n-2)$. Consequently the result of Corollary 4.12 also holds for DAG-size.

4.4 Succinctness of 2-dimensional modal logic

Our proof for the nonelementary succinctness gap between bisimulation invariant FO and ML is based on the fact that n-bisimilarity of two points $u, v \in W$ of a Kripke-frame $\mathcal{M} = (W, R)$ is definable by an FO-formula $\psi_n(x, y)$ (see the proof of Lemma 4.1). However, it is not difficult to see that the property $(\mathcal{M}, u) \rightleftharpoons_n (\mathcal{M}, v)$ is already expressible in 2*-dimensional modal logic*.

The idea in 2-dimensional modal logic is that the truth of formulas is evaluated on pairs (u, v) of elements of Kripke-models instead of single points. We refer to the book [15] of Marx and Venema for a detailed exposition on 2-dimensional and multi-dimensional modal logics. For our purposes it suffices to consider the modal fragment ML^2 of the 2-dimensional modal μ-calculus L^2_μ, introduced by Otto [16].

A Kripke-model $\mathcal{T}$ for ML^2 consists of a set W of points, a binary accessibility relation R, and a valuation V. Note that proposition symbols are interpreted as sets of pairs, whence V is a function $\Phi \to \mathcal{P}(W^2)$. Since accessibility is defined separately for the two components of pairs $(u, v) \in W^2$, there are two modal operators $\Diamond_1$ and $\Diamond_2$ in ML^2. The semantics of these operators and their duals are defined as follows:

- $(\mathcal{T}, (u, v)) \vDash \Diamond_1\varphi \Leftrightarrow$ there is $u' \in W$ such that uRu' and $(\mathcal{T}, (u', v)) \vDash \varphi$,
- $(\mathcal{T}, (u, v)) \vDash \Diamond_2\varphi \Leftrightarrow$ there is $v' \in W$ such that vRv' and $(\mathcal{T}, (u, v')) \vDash \varphi$,
- $(\mathcal{T}, (u, v)) \vDash \Box_1\varphi \Leftrightarrow$ for all $u' \in W$, if uRu', then $(\mathcal{T}, (u', v)) \vDash \varphi$,
- $(\mathcal{T}, (u, v)) \vDash \Box_2\varphi \Leftrightarrow$ for all $v' \in W$, if vRv', then $(\mathcal{T}, (u, v')) \vDash \varphi$.

In addition to proposition symbols, connectives and modal operators, the logic ML^2 has variable substitution operators (see [16], p. 242–43), but we will not need them here.

Any pointed Kripke-model $(\mathcal{M}, w) = ((W, R, V), w)$ can be interpreted as the 2-dimensional pointed model $(\mathcal{M}_2, (w, w))$, where $\mathcal{M}_2 = (W, R, V_2)$ and $V_2(p) = \{(w, w) \mid w \in V(p)\}$ for each $p \in \Phi$. This gives us a meaningful way of defining properties of pointed models $(\mathcal{M}, w)$ by formulas of ML^2. In particular, we say that a formula $\varphi \in \mathrm{ML}^2$ separates two classes $\mathbb{A}$ and $\mathbb{B}$ of pointed models if for all $(\mathcal{M}, w) \in \mathbb{A}$, $(\mathcal{M}_2, (w, w)) \vDash \varphi$ and for all $(\mathcal{M}, w) \in \mathbb{B}$, $(\mathcal{M}_2, (w, w)) \nvDash \varphi$.

The *size* $\mathrm{s}(\varphi)$ of a formula $\varphi \in \mathrm{ML}^2$ is defined in the same way as for formulas of ML; see Definitions 2.5, 2.6 and 2.7. In other words, $\mathrm{s}(\varphi)$ is the total number of modal operators and binary connectives occurring in φ.

Observe now that two pointed frames $(\mathcal{M}, u)$ and $(\mathcal{M}, v)$ are 1-bisimilar if and only if $(\mathcal{M}_2, (u, v)) \models \rho_1$, where $\rho_1 := \Diamond_1 \top \leftrightarrow \Diamond_2 \top$. Furthermore if $\rho_n \in \mathrm{ML}^2$ defines the class of all 2-dimensional pointed frames $(\mathcal{M}_2, (u, v))$ such that $(\mathcal{M}, u) \leftrightarrows_n (\mathcal{M}, v)$, then $\rho_{n+1} := \Box_1 \Diamond_2 \rho_n \wedge \Box_2 \Diamond_1 \rho_n$ defines the class of all $(\mathcal{M}_2, (u, v))$ such that $(\mathcal{M}, u) \leftrightarrows_{n+1} (\mathcal{M}, v)$.

Lemma 4.14 *For each $n \in \mathbb{N}$ there is a formula $\zeta_n \in \mathrm{ML}^2$ that separates the classes $\mathbb{A}_n$ and $\mathbb{B}_n$ such that the size of ζ_n is exponential with respect to n, i.e., $\mathrm{s}(\zeta_n) = \mathcal{O}(2^n)$.*

Proof. Let ζ_n be the formula $\Box_1 \Box_2 \rho_n$. Then $(\mathcal{M}_2, (w, w)) \models \zeta_n$ if and only if $(\mathcal{M}, u)$ and $(\mathcal{M}, v)$ are n-bisimilar for all $(\mathcal{M}, u), (\mathcal{M}, v) \in \Box(\mathcal{M}, w)$, whence ζ_n separates $\mathbb{A}_n$ from its complement $\mathbb{B}_n$. An easy calculation shows that the size of ζ_n is $3 \cdot 2^{n+1} - 5$. □

By Theorem 4.3, for each $n \in \mathbb{N}$ there is a formula $\vartheta_n \in \mathrm{ML}$ that is equivalent with ζ_n. On the other hand, by Theorem 4.11 the size of ϑ_n is at least $\mathrm{twr}(n-1)$. Thus, we obtain the nonelementary succinctness gap already between ML^2 and ML.

Corollary 4.15 *The 2-dimensional modal logic ML^2 is nonelementarily more succinct than ML.*

Acknowledgement. We are grateful to an anonymous referee for pointing out the possibility of using 2-dimensional modal logic for separating the classes $\mathbb{A}_n$ and $\mathbb{B}_n$.

References

[1] Adler, M. and N. Immerman, *An* n! *lower bound on formula size*, ACM Trans. Comput. Log. **4** (2003), pp. 296–314.
URL http://doi.acm.org/10.1145/772062.772064

[2] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge University Press, New York, NY, USA, 2001.

[3] Etessami, K., M. Y. Vardi and T. Wilke, *First-order logic with two variables and unary temporal logic*, Inf. Comput. **179** (2002), pp. 279–295.
URL http://dx.doi.org/10.1006/inco.2001.2953

[4] Etessami, K. and T. Wilke, *An until hierarchy and other applications of an ehrenfeucht-fraïssé game for temporal logic*, Inf. Comput. **160** (2000), pp. 88–108.
URL http://dx.doi.org/10.1006/inco.1999.2846

[5] Figueira, S. and D. Gorín, *On the size of shortest modal descriptions*, in: *Advances in Modal Logic 8, papers from the eighth conference on "Advances in Modal Logic," held in Moscow, Russia, 24-27 August 2010*, 2010, pp. 120–139.
URL http://www.aiml.net/volumes/volume8/Figueira-Gorin.pdf

[6] French, T., W. van der Hoek, P. Iliev and B. P. Kooi, *Succinctness of epistemic languages*, in: *IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence, Barcelona, Catalonia, Spain, July 16-22, 2011*, 2011, pp. 881–886.
URL http://ijcai.org/papers11/Papers/IJCAI11-153.pdf

[7] French, T., W. van der Hoek, P. Iliev and B. P. Kooi, *On the succinctness of some modal logics*, Artif. Intell. **197** (2013), pp. 56–85.
URL http://dx.doi.org/10.1016/j.artint.2013.02.003

[8] Grohe, M. and N. Schweikardt, *The succinctness of first-order logic on linear orders*, Logical Methods in Computer Science **1** (2005).
URL http://dx.doi.org/10.2168/LMCS-1(1:6)2005

[9] Hella, L. and J. Väänänen, *The size of a formula as a measure of complexity*, in: *Logic Without Borders - Essays on Set Theory, Model Theory, Philosophical Logic and Philosophy of Mathematics*, 2015 pp. 193–214.
URL http://dx.doi.org/10.1515/9781614516873.193

[10] Iliev, P., "On the Relative Succinctness of Some Modal Logics," Ph.D. thesis, University of Liverpool (2013).

[11] Laroussinie, F., N. Markey and P. Schnoebelen, *Temporal logic with forgettable past*, in: *17th IEEE Symposium on Logic in Computer Science (LICS 2002), 22-25 July 2002, Copenhagen, Denmark, Proceedings*, 2002, pp. 383–392.
URL http://dx.doi.org/10.1109/LICS.2002.1029846

[12] Lutz, C., *Complexity and succinctness of public announcement logic*, in: *5th International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2006), Hakodate, Japan, May 8-12, 2006*, 2006, pp. 137–143.
URL http://doi.acm.org/10.1145/1160633.1160657

[13] Lutz, C., U. Sattler and F. Wolter, *Modal logic and the two-variable fragment*, in: *Computer Science Logic, 15th International Workshop, CSL 2001. 10th Annual Conference of the EACSL, Paris, France, September 10-13, 2001, Proceedings*, 2001, pp. 247–261.
URL http://dx.doi.org/10.1007/3-540-44802-0_18

[14] Markey, N., *Temporal logic with past is exponentially more succinct, concurrency column*, Bulletin of the EATCS **79** (2003), pp. 122–128.

[15] Marx, M. and Y. Venema, "Multi-dimensional modal logic," Applied Logic Series **4**, Kluwer Academic Publishers, Dordrecht, 1997, xiv+239 pp.
URL http://dx.doi.org/10.1007/978-94-011-5694-3

[16] Otto, M., *Bisimulation-invariant PTIME and higher-dimensional μ-calculus*, Theor. Comput. Sci. **224** (1999), pp. 237–265.
URL http://dx.doi.org/10.1016/S0304-3975(98)00314-4

[17] Sistla, A. P. and E. M. Clarke, *The complexity of propositional linear temporal logics*, J. ACM **32** (1985), pp. 733–749.
URL http://doi.acm.org/10.1145/3828.3837

[18] Stockmeyer, L. J., "The Complexity of Decision Problems in Automata Theory and Logic," Ph.D. thesis, Massachusetts Institute of Technology (1974).

[19] van der Hoek, W. and P. Iliev, *On the relative succinctness of modal logics with union, intersection and quantification*, in: *International conference on Autonomous Agents and Multi-Agent Systems, AAMAS '14, Paris, France, May 5-9, 2014*, 2014, pp. 341–348.
URL http://dl.acm.org/citation.cfm?id=2615788

[20] van der Hoek, W., P. Iliev and B. P. Kooi, *On the relative succinctness of two extensions by definitions of multimodal logic*, in: *How the World Computes - Turing Centenary Conference and 8th Conference on Computability in Europe, CiE 2012, Cambridge, UK, June 18-23, 2012. Proceedings*, 2012, pp. 323–333.
URL http://dx.doi.org/10.1007/978-3-642-30870-3_33

[21] van Ditmarsch, H., J. Fan, W. van der Hoek and P. Iliev, *Some exponential lower bounds on formula-size in modal logic*, in: *Advances in Modal Logic 10, invited and contributed papers from the tenth conference on "Advances in Modal Logic," held in Groningen, The Netherlands, August 5-8, 2014*, 2014, pp. 139–157.
URL http://www.aiml.net/volumes/volume10/Ditmarsch-Fan-Hoek-Iliev.pdf

[22] Wilke, T., *Ctl^+ is exponentially more succinct than CTL*, in: *Foundations of Software Technology and Theoretical Computer Science, 19th Conference, Chennai, India, December 13-15, 1999, Proceedings*, 1999, pp. 110–121.
URL http://dx.doi.org/10.1007/3-540-46691-6_9