Title:

Color may be the phenomenal dual aspect of two-state quantum systems

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Abstract

Panmicropsychism is the view that the fundamental physical ingredients of our universe are also its fundamental phenomenal ingredients. Since there is only a limited number of fundamental physical ingredients, panmicropsychism seems to imply that there exists only a small set (palette) of basic phenomenal qualities. How does this limited palette of basic phenomenal qualities give rise to our rich set of experiences? This is known as 'the palette problem'. One class of solutions to this problem, large-palette solutions, simply denies that the palette is limited. These solutions assume that all types of phenomenal qualities (color, sound, odor, taste, etc., and presumably also types not experienced by humans) were created *fully formed* at the birth of our universe. On this view, brains evoke conscious experiences by sampling primordial, preexisting phenomenal spaces. My main claim in this paper is that, by analogy with the mathematical description of the fundamental *physical* ingredients of our universe, which exhibits simplicity, symmetry, and beauty, panmicropsychists should expect the mathematical description of the fundamental *phenomenal* ingredients of our universe to exhibit similar features. The goal of this paper is to exemplify this claim using what is arguably the simplest of all types of phenomenal qualities—color. Specifically, I utilize phenomenological data on color to construct the maximally symmetric mathematical description of phenomenal color space. I then show that this mathematical description is isomorphic to the mathematical description of two-state quantum systems in a mixed state. Based on this isomorphism, I suggest that color may be the phenomenal dual aspect of two-state quantum systems.

Keywords:

color phenomenology, opponent-process theory, opponent-colors theory, panpsychism, qubit, two-state quantum system

1. Introduction

In recent decades, there has been a growing realization that a purely physicalist view of nature cannot account for phenomenal experience (Chalmers, 1995, 1996; Foster, 1991; Goff, 2017; Jackson, 1982; Levine, 1993; Nagel, 1974; Robinson, 1993). This failure compels us to adopt a non-physicalist view of phenomenal experience, namely, a view on which phenomenal experience is a fundamental feature of our universe, ontologically distinct from the universe's physical features. Non-physicalist views of phenomenal experience come in several forms and sub-forms. My interest here will be with panpsychism, which 'is the view that mentality is fundamental and ubiquitous in the natural world' (Goff et al., 2022). This view has several variants. This paper focuses on the most common variant, which is constitutive pan-*micro*-psychism. The two main premises of this view are (a) that the universe's physical ultimates (i.e., its fundamental, irreducible physical ingredients) are also phenomenal ultimates (i.e., fundamental, irreducible phenomenal elements) (Strawson, 2006), and (b) that these phenomenal ultimates constitute the macroexperiences of the sort humans (and presumably other organisms) have (Chalmers, 2015; Goff et al., 2022).[1] The greatest challenge to constitutive panmicropsychism is the notorious combination problem (Chalmers, 2017; Goff et al., 2022; Seager, 1995). This problem has many aspects. My concern here will be with an aspect that Chalmers (2017) dubbed *the palette problem*:

> There is a vast array of macroqualities, including many different phenomenal colors, shapes, sounds, smells, and tastes. There is presumably only a limited palette of microqualities. […] How can this limited palette of microqualities combine to yield the vast array of macroqualities? (p. 183)

Chalmers (*ibid*.) divides the possible solutions to the palette problem into *small-palette* and *large-palette* solutions. 'Small-palette solutions argue that all macroqualities can be generated from just a few microqualities' (*ibid*., p. 205). One problem that afflicts small-palette solutions is *the quality combination problem*: 'How do microqualities combine to yield macroqualities?' (*ibid*., p. 204). Both Roelofs (2014) and Coleman (2017) address this problem and conclude that it isn't insurmountable, each sketching a rough outline of how it can be solved. However, Chalmers (2017, footnote 11) is not convinced that either of these outlined solutions is valid. Be that as it may, small-palette solutions

[1] Note, however, that the phenomenal ultimates themselves do not have to be conscious; they can be merely precursors to conscious experience. This possibility is referred to as pan-*proto*-psychism (Chalmers, 2015; Goff & Coleman, 2020).

suffer from a much more serious problem, noted by many (Foster, 1991, p. 127; Lockwood, 1993; McGinn, 2006; Roelofs, 2014) and named *the incommensurability problem* by Coleman (2017):

> If ultimates have fixed qualities, just what set of microqualities *is* it that can be rearranged now as the smell of roses, now as an orgasm, now as a percept of the blue sky? These macroqualities seem so qualitatively *different*, it's hard to imagine generating them from some stable basic palette. (p. 252; italics in the original)

Coleman (*ibid*.) proposes the following solution to this problem:

> I think the answer [to the incommensurability problem] will require radical reconceptualization of our quality space: discarding the idea of discrete modalities, and coming to think of phenomenal qualities, of all kinds, as on a continuum, in the way we think of the colors. So just as it's possible to move across the color spectrum in tiny, almost undetectable steps, it must be possible to move from tastes to sounds, sounds to colors, and so on, via equally tiny steps. […] If the continuum hypothesis is correct, then there isn't any genuine incommensurability between different kinds of qualities—differences are always of degree rather than of kind. (pp. 264–265)

Roelofs's (2014) response to the incommensurability problem has some kinship with Coleman's response. He suggests that the qualitative discontinuity between different types of qualia might be illusory; perhaps they have 'shared features' (*ibid*., p. 66) that we are simply unable to detect.

These suggested solutions to the incommensurability aspect of the palette problem are unconvincing; they appear to be desperate attempts to salvage the hypothesis. Therefore, I find the other class of solutions to the palette problem, the large-palette solutions, to be much more plausible. These solutions stipulate that 'the full range of macroqualities are included among the microqualities. So there are microqualities associated with different colors, sounds, smells, tastes, and so on' (Chalmers, 2017, p. 205). One argument against large-palette solutions is that they lead to a 'bloated ontology' (Roelofs, 2014, p. 66). But this argument is weak unless one supposes that panmicropsychism assigns every particular color, sound, scent, etc., to a different physical ultimate (*ibid*., footnote 3). Yet, panmicropsychism is in no way committed to such an excessive approach (Coleman, 2017). Rather, a much more reasonable approach to panmicropsychism, which does not lead to a bloated ontology, is to assume that every *type* of phenomenal quality (i.e., color, sound, odor, etc.) is assigned a physical ultimate. Another potential objection to large-palette solutions is the seemingly *adaptive associations* between some types of qualia and their corresponding behavioral responses (a classic example being pain). The problem is that these associations seem to imply that qualia (which according to large-palette solutions must have been created billions of years ago, at the birth of our universe; see below) were prearranged to serve the needs of humans and other biological creatures on our planet. Advocates of large-palette solutions can address this objection by invoking the hypothesis suggested by Zietsch (2024). According to this hypothesis, the associations between qualia and behavior are acquired through associative learning and are therefore only *seemingly* adaptive.

Large-palette solutions to the palette problem assume that all types of phenomenal qualities were created *fully formed* at the birth of the universe. I'll refer to this hypothesis as the *Fully-Formed, Primordial Qualia (FFPQ) hypothesis*. My main argument in this paper will be that panmicropsychists who subscribe to the FFPQ hypothesis should expect phenomenal (quality) spaces, namely, the abstract spaces in which phenomenal qualities are organized relative to each other (e.g., phenomenal color space or phenomenal odor space), to exhibit simplicity, symmetry, and beauty. This argument is based on the fact that the mathematical description of the properties of the fundamental *physical* ingredients of our universe is characterized by these attributes (e.g., consider the mathematical description of spin-1/2 in quantum mechanics (Blum, 1981, chapter 1; Zwiebach, 2022, chapter 12)). Therefore, panmicropsychists should have every reason to believe that the mathematical description of the properties of the fundamental *phenomenal* ingredients of our universe will exhibit the same features.[2] An

[2] When I refer to the properties of the fundamental phenomenal ingredients, I am not discussing their intrinsic phenomenal properties, but rather their *relational properties* to one another. For example, the way color sensations are organized in phenomenal color space. (Notice that the same restriction applies to the properties of the physical ultimates since—as Russellian monists are fond of pointing out—we know nothing about their intrinsic nature; we only know how to describe their relational properties.)

elegant way of arriving at this prediction is through the *dual-aspect (or double-aspect) view* of panmicropsychism (Benovsky, 2016; Chalmers, 1995, 1996, chapter 8). On this view, which suggests that the physical and the phenomenal are two aspects of neutral ultimates or that the phenomenal is a dual aspect of physical ultimates,[3] the mathematical description of phenomenal qualities should be isomorphic to the mathematical description of their physical duals (Chalmers, 1995, 1996, chapter 8; Lockwood, 1989, chapter 11; also see G. E. Müller's famous psychophysical axioms (Boring, 1942, p. 89)). Therefore, the simplicity, symmetry, and beauty that characterize the mathematical description of physical ultimates should be mirrored in the mathematical description of phenomenal ultimates. For example, if spin-1/2 systems turn out to have phenomenal dual aspects, then the elegant mathematical structure of this physical system will, *ex hypothesi*, be reflected in the mathematical structure of its phenomenal dual states.

In summary, the FFPQ hypothesis predicts that the mathematical description of phenomenal spaces should exhibit simplicity, symmetry, and beauty. Of these three attributes, symmetry stands out as the only one that is objective. (After all, simplicity and beauty are in the eye of the beholder.) Therefore, the objective aspect of the prediction made by the FFPQ hypothesis is that phenomenal spaces should be symmetric. I will refer to this prediction as the *SymFFPQ hypothesis*. An immediate

[3] The former interpretation is a form of neutral monism (Stubenberg, 2014, section 9.4), whereas the latter is a form of property dualism (Van Gulick, 2022, section 8.1).

objection to the SymFFPQ hypothesis might arise from the observation that when we try to reconstruct our phenomenal spaces using psychophysical or psychochemical data, the resulting structure is in no way symmetric; instead, it is irregular and nonuniform (for example, psychophysical color space (Kuehni, 2003, chapters 2 and 7) or psychochemical odor space (Koulakov et al., 2011)). I attribute these deviations from the prediction of the SymFFPQ hypothesis to two main factors: (a) our brains are unable to fully, isotropically, and uniformly sample phenomenal spaces; and (b) it is inherently difficult to objectively measure subjectively perceived phenomena. *Subsection 2.5* provides further details.

The goal of this paper is to exemplify how the SymFFPQ hypothesis can be applied in the case of color, which is arguably the simplest of all types of phenomenal qualities. According to the SymFFPQ hypothesis, phenomenal color space should have a perfectly symmetric structure. Therefore, I utilize phenomenal data on color to construct the maximally symmetric mathematical description of phenomenal color space (*Section 2*). I then show that the mathematical model that gives this symmetric structure to phenomenal color space has a one-to-one correspondence with the mathematical description of two-state quantum systems in a mixed state (*Section 5*). This isomorphism between the two mathematical descriptions leads me to suggest that color is the phenomenal dual aspect of two-state quantum systems. *Sections 3* and *4* provide the necessary background for this suggestion: *Section 3* reviews the mathematical description of two-state quantum systems, while *Section 4* presents the hypothesis that the physical world results from a computation ('the computational universe hypothesis'). The latter section argues that adherents of the computational universe hypothesis must assume that Hilbert space, namely, the space in which quantum states exist, has a privileged basis—the basis in which the aforementioned computation is carried out.

2. Obtaining the maximally symmetric description of phenomenal color space

Following the rationale of the SymFFPQ hypothesis, the goal of this section is to obtain the maximally symmetric description of phenomenal color space. I start (*Subsection 2.1*) with a review of the phenomenal properties of color. I then show (*Subsection 2.2*) that this description is not maximally symmetric. Therefore, *Subsection 2.3* develops a color model that does comply with our goal, namely, that gives rise to the maximally symmetric phenomenal color space. *Subsection 2.4* provides an alternative formulation for the color model of *Subsection 2.3*. *Subsection 2.5* explains why psychophysical color spaces (e.g., the Munsell color space) do not exhibit the symmetry predicted by the SymFFPQ hypothesis.

2.1 The phenomenal properties of color

It was Leonardo da Vinci who first noticed that our color experience contains six *elementary* color sensations: red, yellow, green, blue, white, and black (Valberg, 2001). What makes these sensations elementary is that none of them is perceived as being composed of any other color sensation. For example, the elementary version of red (often referred to as *unique* red) is perceived as a purely red sensation that cannot be broken down to more basic color sensations. In contrast, an orangish red is perceived as a mixture of red and yellow, a purplish red as a mixture of red and blue, and a pinkish red as a mixture of red and white. All colors can be described as some combination of two, three, or four of the six elementary color sensations (Hård et al., 1996). The six elementary colors fall into two phenomenally distinct groups: one group contains two hueless (or 'achromatic') colors (white and black); the other group contains four hued (or 'chromatic') colors. The gamut of all hueless colors can be arranged in a one-dimensional phenomenal continuum that begins in black, continues to dark grays and then light grays, and ends in white. The gamut of all hues can also be arranged in a one-dimensional phenomenal continuum. However, in contrast to the gamut of the hueless colors, this one-dimensional continuum is *closed* (that is, if we start at an arbitrary hue and move continuously along the hue dimension, eventually we will return to the hue that we started with). This closed continuum is often portrayed as a circle known as the hue circle.

In the last quarter of the 19th century, the German physiologist Ewald Hering noticed that there are certain combinations of the four elementary hues that don't appear along the hue circle: red and green do not mix to yield intermediate hues (i.e., there are no greenish reds or reddish greens) and neither do yellow and blue (i.e., there are no bluish yellows or yellowish blues). By contrast, any hue from the red–green pair freely combines with any hue from the yellow–blue pair to yield phenomenal intermediates (reddish yellows, bluish greens, and so on). Based on these phenomenological observations, Hering proposed that our sensations of hue are produced by two opponent-colors (or opponent-processes) mechanisms: a red–green mechanism and a yellow–blue mechanism (Hering, 1878, pp. 118–119; Hurvich & Jameson, 1957; Palmer, 1999, pp. 108–114; Shevell & Martin, 2017). Each such mechanism consists of two *elementary-color processes* that operate in an opponent (or antagonistic) manner to each other. Thus, the output of each mechanism results from the difference between the activities of its two constituent processes. As its name implies, each elementary-color process is assumed to give rise to an elementary (i.e., unique) hue. For example, the red–green mechanism consists of one process that gives rise to unique red and another process that gives rise to unique green. The hue sensation that is produced by each mechanism is due to the elementary-color process whose activity is in excess relative to its opponent. Consequently, opponent hues are never perceived together in one color. In other words, opponent hues are mutually exclusive sensations. Thus, Hering's theory indeed explains the missing intermediate hues along the hue circle.

What about the pair of hueless elementary colors, white and black? Do they also form an opponent pair? The situation here is more complicated than for the hued elementary colors. On the one hand, because white and black—similarly to red and yellow or green and blue—combine to produce a phenomenal intermediate (gray), they don't seem to form an opponent pair. On the other hand, in the phenomena of afterimages and simultaneous color contrast, white and black behave analogously to the hued opponent pairs (Ladd Franklin, 1899; Titchener, 1910, p. 75). Thus, there is conflicting evidence as to whether white and black form an opponent pair. It is clear, however, that one cannot have the cake and eat it too: either white and black are opponent to each other, in which case they must be mutually exclusive sensations (namely, gray is *not* due to their mixture), or gray is taken to be a mixture of white and black, in which case white and black cannot be opponent to each other. Confusingly, however, Hering's approach to this dilemma was to hold on to both its horns (Heggelund, 1974a): he suggested that white and black are due to a third pair of opponent elementary-color processes (Hering, 1878, pp. 118–119), yet also contended that gray results from the mixture of white and black (Hering, 1878, pp. 58–62), which of course means that they are not mutually exclusive and hence not opponent.

The inconsistent treatment of the hueless colors in Hering's theory did not go unnoticed by his contemporaries or by the phenomenologists of the generation after him (Boring, 1942, p. 209; Ladd Franklin, 1899).[4] Here, for example, is Ernst Mach (1897, p. 35f) (whose ideas about color greatly influenced Hering):

> The only point that still dissatisfies me in Hering's theory is that it is difficult to perceive why the two opposed processes of black and white may be

[4] Quite perplexingly, however, this acute problem in Hering's theory is almost completely ignored in modern accounts of opponent-colors theory. For example, Palmer (1999), in his well-known textbook about vision, comments on the problematic status of the white–black mechanism in Hering's theory, but simply asserts that 'There is thus something qualitatively different about the achromatic dimension' (p. 110), without any attempt to explain why or how this difference comes about.

> simultaneously produced and simultaneously felt, while such is not the case with red–green and blue–yellow.

And here is Christine Ladd Franklin (1899, p. 78; italics in the original):

> A chief objection to the view of Hering, for those who have been interested in its theoretical aspect, is the inconsistency which meets us at the very beginning; why should black and white be regarded as an *antagonistic* sensation-pair, when they do not destroy each other, but give us, on the contrary, the whole series of grays?

There were two early attempts to fix this problem in Hering's theory. The earliest attempt, which was very influential at the time, was made by the prominent experimental psychologist G. E. Müller (Boring, 1942, p. 213; Ladd Franklin, 1899). The second (much less-known) attempt was by F. L. Dimmick (1929, 1948, 1962).[5] Although their theories differ in their details, both Müller and Dimmick solved the problem of the hueless colors in Hering's theory by positing that (a) white and black and opponents and are therefore mutually exclusive and (b) grayness is produced by a non-opponent mechanism that is separate from the white–black mechanism (see Boring (1949) for a review of both theories). To be tenable, both theories need gray to be an elementary color. This goes against the strong intuition that gray is an intermediate of white and black, and therefore not elementary. Indeed, experiments aimed at testing the

[5] It is noteworthy that relatively recently, Nayatani (2001, 2002) proposed a modification to Hering's theory that—to the best of this author's judgment—is virtually identical to the modification proposed long ago by Dimmick.

hypothesis that gray is an elementary color unequivocally refuted it (Quinn et al., 1985; also see Logvinenko & Beattie, 2011). Thus, both the Müller and the Dimmick theories are untenable.

A much more recent solution to the problem of the hueless colors in Hering's theory was proposed by Paul Heggelund in a series of papers starting in the 1970's (Heggelund, 1974a, 1974b, 1991, 1992, 1993). Based on systematic observations on the properties of hueless colors, Heggelund proposed that, in addition to black and white, there exists a third elementary hueless color—*luminous* (Heggelund, 1974a). This hueless sensation exists in colors that are perceived as emitting light. For example, this sensation is present in the color of stars in the night sky or in the color produced by light bulbs. On Heggelund's suggestion, the gamut of the hueless colors should be extended to end in a purely luminous sensation. That is, according to Heggelund, the hueless colors stretch from black, through grays, to white, and then continue through luminous whites all the way to a color that is purely luminous. Thus, the color positioned opposite to black on the continuum of hueless colors is luminous, not white. This, in turn, suggests that luminous, not white, is the opponent color to black (Evans, 1974, p. 100; Heggelund, 1974a, b; but see Vladusich et al. (2007), who claim that luminous and black are not mutually exclusive sensations). Hence, according to Heggelund's theory, white and black are *not* opponent to each other, which explains why they unproblematically combine to yield gray as an intermediate. Heggelund's theory of hueless colors therefore neatly solves the most serious flaw in Hering's theory.

The addition of the luminous sensation to the cadre of elementary color sensations means that there are seven elementary colors rather than six. (And since the luminous sensation is presumably produced by an elementary-color process, there is now a total of seven of those as well.) Importantly, there is independent evidence to support Heggelund's proposal that luminous is a third elementary hueless color. First, Evans (1959), based on experimental work that somewhat resembles that of Heggelund, emphasized the existence of a luminous attribute in hueless colors (he used the term '*fluorent*' rather than luminous). (Notably, however, Evans's overall model of the hueless colors was different from Heggelund's (Heggelund, 1974a).) Second, Izmailov and Sokolov (1991) conducted experiments where observers were asked to rank the perceptual distances between pairs of hueless colors. Multidimensional scaling analysis of the results showed that they could be best accounted for by adding a luminous attribute to the hueless colors (Izmailov and Sokolov used the term *bright* rather than luminous). Results consistent with those of Izmailov and Sokolov were later also obtained by Bimler et al. (2009).

When we combine Heggelund's model of hueless colors with Hering's model of hued colors we obtain a model wherein any color results from a mixture of one, two, three, or four of the following phenomenal components: a component of red *or* green, a component of yellow *or* blue, a component of luminous *or* black, a white component

(Heggelund, 1991, 1993). On Heggelund's extension of Hering's opponent-colors theory, we can represent every color by the following four-dimensional vector:

$$\boldsymbol{F} = W\hat{\boldsymbol{x}}_0 + (R - G)\hat{\boldsymbol{x}}_1 + (Y - B)\hat{\boldsymbol{x}}_2 + (L - Bk)\hat{\boldsymbol{x}}_3, \tag{1}$$

where the unit vectors $\hat{\boldsymbol{x}}_\mu$, $\mu = 0, 1, 2, 3$, are the standard basis of $\mathbb{R}^4$, namely, $\hat{\boldsymbol{x}}_0 = (1, 0, 0, 0)^\mathrm{T}$, $\hat{\boldsymbol{x}}_1 = (0, 1, 0, 0)^\mathrm{T}$, and so on (the superscript T stands for the transpose operation); $W \geq 0$ is the level of activity of the white elementary-color process; R, G, Y, B, L, and Bk (which are all ≥ 0) are, respectively, the levels of activity of the red, green, yellow, blue, luminous, and black elementary-color processes, and therefore $(R - G)$, $(Y - B)$, and $(L - Bk)$ are, respectively, the outputs of the red–green, yellow–blue, and luminous–black mechanisms.

Heggelund (1974a, 1991) completed his theory by suggesting that the *perceived intensity* of a color, which we will denote by I, is given by the Euclidean norm of $\boldsymbol{F}$, namely,

$$I^2 \equiv \|\boldsymbol{F}\|^2 = W^2 + (R - G)^2 + (Y - B)^2 + (L - Bk)^2. \tag{2}$$

Heggelund used the term 'color strength' rather than the term 'color intensity' that I use here.[6] Neither term, however, is commonly used. Nevertheless, there are two good reasons to prefer the term 'color intensity' (or 'color strength') over the much more commonly used term 'brightness', which refers to the sensation of the amount of light

[6] I prefer the term 'color intensity' over Heggelund's 'color strength' because it seems to me to generalize more naturally to other sensory modalities, e.g., sound intensity, odor intensity, and so on.

emitted or reflected from a colored area (Kuehni, 2003, p. 367). First, while brightness is usually taken to range from dim to bright (or dazzling) (Evans, 1974, p. 97; Kuehni, 2003, p. 367; Shevell, 2003), it is clear from the definition of color intensity in Eq. (2) that a patch of black color is perceived as having some intensity (for example, a deep black color is perceived as an *intense* black), yet a black patch is neither dim nor bright. Second, different authors define brightness differently thus leading to much confusion with respect to the meaning of the term (Evans, 1974, pp. 7–8; Heggelund, 1974b). I will therefore follow Heggelund's decision to use a straightforward, non-ambiguous term for the intensive aspect of color sensations.

Figure 1 provides a scheme of phenomenal color space according to Heggelund's extension of Hering's theory (Heggelund, 1991).

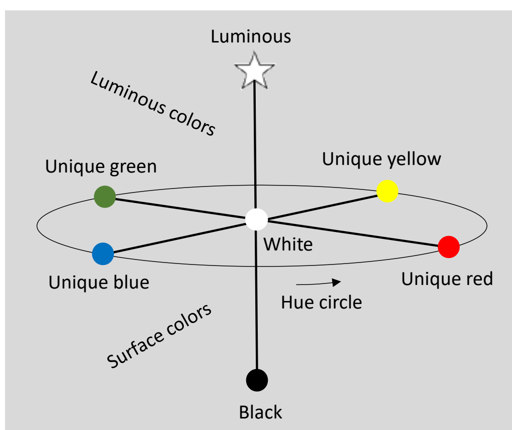

Figure 1 Phenomenal color space according to Heggelund's extension of Hering's opponent-colors theory. The outputs of the three opponent-colors mechanisms are projected into the three orthogonal opponent-colors axes shown in the figure. White is located at the intersection of these three axes. All colors in the depicted color space have the same color intensity.

Notice that even though the Heggelund theory requires four independent phenomenal attributes for the description of a color (see Eq. (1)), Fig. 1 shows phenomenal color space as only three-dimensional. This was achieved by requiring that all colors in the figure have the same color intensity, i.e., by holding the value of I fixed. Figure 1 locates pure white at the intersection of the three opponent-colors axes, namely, at the origin of phenomenal color space. (This is because Heggelund's theory suggests that when all three opponent-colors pairs are in equilibrium, the perceived color is purely white (see Eq. (1)).) Thus, as we move away from the origin of phenomenal color space, the color contains less and less white and more and more of the other six elementary colors.

The scheme of phenomenal color space in Fig. 1 underscores another advantage in Heggelund's version of opponent-colors theory over Hering's original theory. To understand this advantage, we need to acquaint ourselves with the two 'modes' of color perception:

> [C]olors may be perceived in two different modes often called the light and the object mode [...]. In the light mode the colors are perceived as a property of the light emitted from a field. These colors are called light colors. In the object mode the colors are perceived as a constant property of object surfaces. These colors are called surface colors […]. (Heggelund, 1993, p. 1709)

Here I will refer to what Heggelund called light colors (which are more often referred to as aperture colors) as *luminous colors*. The conditions under which each mode of color perception is evoked are discussed in Heggelund (1974a). As is evident from Fig. 1, the addition of luminous as an elementary color sensation seamlessly unifies surface and luminous colors into a single space. This unification does not exist in Hering's original theory, which, as a result, requires two separate color spaces to fully represent all colors, one for each mode (see Heggelund (1974a, 1991) for more details). Consequently, in the literature it is common to see color spaces in which the hueless axis is labeled as 'lightness *or* brightness' (e.g., Purves & Yegappan, 2017), where the lightness attribute is for surface colors while the brightness attribute is for luminous colors.[7]

In summary, our understanding of the structure of color experience started with Leonardo da Vinci's observation that there exist six elementary colors: red, yellow, green, blue, white, and black. Ewald Hering made the next advancement by arranging the six elementary colors into three opponent pairs: red–green, yellow–blue, and white–black. However, the hueless opponent pair in Hering's theory, white–black, is anomalous in that its two colors combine to give an intermediate. Paul Heggelund solved this problem in Hering's theory by suggesting that the gamut of hueless colors doesn't end at white, but rather continues to a pure, elementary luminous sensation. It is this elementary hueless sensation (which is the seventh elementary color sensation

[7] Here we see the confusion with the term 'brightness' that was mentioned above: some authors use it to refer to the luminousness of luminous colors, while others use it to mean the perceived intensity of a light stimulus.

overall) rather than white that is the opponent sensation to black. In addition, Heggelund's extension of Hering's theory also neatly unifies surface and luminous colors into a single framework.

2.2 The current description of phenomenal color space is not maximally symmetric

According to the SymFFPQ hypothesis, phenomenal color space should be maximally symmetric. Heggelund's extension of Hering's theory has gone a long way towards achieving this goal because the phenomenal color space that it suggests (see Fig. 1) is already highly symmetric. First, the two colors of each opponent pair are diametrically opposed to each other, i.e., they are positioned at the poles of an axis. Second, the three opponent-colors axes are orthogonal. Third, white, which is the only elementary color that doesn't have an opponent counterpart, is located at intersection of the three opponent-colors axes. We are left with only one more degree of freedom for phenomenal color space: the shape that colors assume in this space. Since the sphere is the most symmetric shape in three dimensions, the maximally symmetric phenomenal color space is one where the gamut of all colors with the same intensity fills a sphere. This space is schematically shown is Fig. 2.

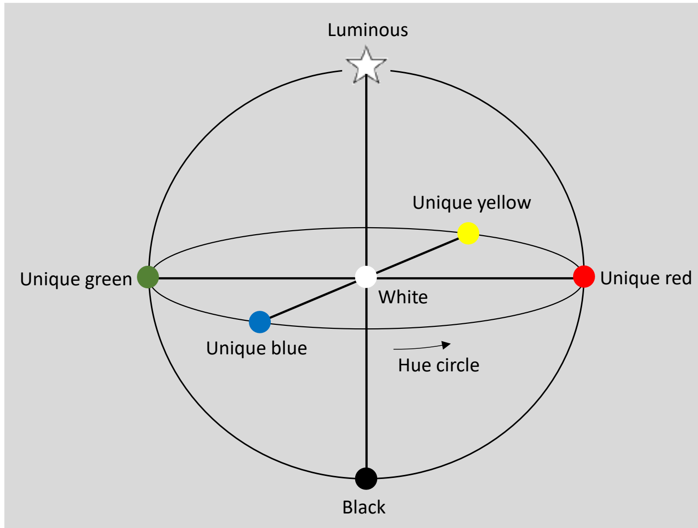

Figure 2 The maximally symmetric phenomenal color space. The gamut of all colors with the same intensity fills a three-dimensional sphere. This phenomenal color space accords with the SymFFPQ hypothesis.

Does Heggelund's model predict the phenomenal color space of Fig. 2? To get the answer, let us look at Eq. (2) of this model. Since to obtain a three-dimensional color space we held color intensity constant, we plug in a constant color intensity in this equation. If we denote this constant value by I_0, Eq. (2) becomes

$$I_0^2 = W^2 + (R - G)^2 + (Y - B)^2 + (L - Bk)^2. \tag{3}$$

Notice that if W, $(R - G)$, $(Y - B)$, and $(L - Bk)$ are all independent of each other, Eq. (3) describes a four-dimensional sphere with a radius of I_0. To relate this conclusion to the three-dimensional space of Fig. 1, we rearrange Eq. (3) in the following way:

$$I_0^2 - W^2 = (R - G)^2 + (Y - B)^2 + (L - Bk)^2. \tag{4}$$

Since $0 \le W \le I_0$,[8] Eq. (4) shows that *if* the variables W, $(R-G)$, $(Y-B)$, and $(L-Bk)$ are independent of each other, the gamut of all colors in the phenomenal color space of Fig. 1 fills a three-dimensional sphere with a radius of I_0.[9] Thus, if the variables W, $(R-G)$, $(Y-B)$, and $(L-Bk)$ are independent of each other, then Heggelund's theory indeed gives rise to the maximally symmetric phenomenal color space. However, Heggelund's observations showed that the value of W is *not* independent of the value of $(L-Bk)$; rather, the value of W is anticorrelated with the absolute value of $(L-Bk)$ (Heggelund, 1992, Eq. (6)). Consequently, Heggelund's model does not give rise to a spherical phenomenal color space,[10] and therefore does not give rise to the maximally symmetric phenomenal color space.

2.3 A color model that gives rise to the maximally symmetric phenomenal color space

As subscribers to the SymFFPQ hypothesis, our goal is to develop a mathematical model of color that provides the maximally symmetric description of phenomenal color space (Fig. 2). We notice that if we take the level of whiteness in a color, W, to be given by

$$W^2 = I^2 - (R-G)^2 - (Y-B)^2 - (L-Bk)^2, \tag{5}$$

and we demand that the variables I, $(R-G)$, $(Y-B)$, and $(L-Bk)$ are independent of each other, then we have attained our goal. That this is indeed the case is ascertained

[8] If this weren't the case, Eq. (4) would yield imaginary results for the opponent-colors components, which is impossible.
[9] For every value of W, the colors create a spherical shell whose radius is $(I_0^2 - W^2)^{1/2}$.
[10] Indeed, Heggelund concluded as follows with regard to the shape of phenomenal color space (he called it 'color quality body') predicted by his theory: 'The structure [i.e., shape] of such a color quality body is unknown. […] The actual structure probably depends to a large extent on the selected value for the constant color strength [i.e., color intensity]' (1991, p. 317).

by realizing that taking $I = I_0$ in Eq. (5) gives us Eq. (4), which, as we have already seen above, describes a spherical color space. In addition, in accordance with Heggelund's observations (1992; see previous subsection), W in Eq. (5) is anticorrelated with the absolute value of $(L - Bk)$. According to Eq. (5), whiteness 'fills the gap' between a color's intensity and the overall magnitude of the opponent-colors components. Thus, when all three opponent-colors components are zero, the entire intensity goes to white. This is why white is located at the intersection of the three opponent-colors axes in Fig. 2, namely, at the origin of phenomenal color space. As we move away from the origin in phenomenal color space, the color contains less and less white, i.e., it becomes more and more saturated (or more and more *pure*).[11] Colors located on the surface of the color sphere in Fig. 2 are fully saturated (i.e., pure), namely, they have no whiteness in them.[12]

By taking W to be given by Eq. (5) and demanding that I, $(R - G)$, $(Y - B)$, and $(L - Bk)$ are independent of each other we have achieved our goal of a perfectly symmetric phenomenal color space. However, Eq. (5) by itself doesn't constitute a full color model because we still need to find an expression for I that is independent of $(R - G)$, $(Y - B)$, and $(L - Bk)$. But before we do that, realize that because I is now

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[11] The terms saturation and purity are ordinarily used only with respect to hued colors. Here I am generalizing these terms to apply to the hueless colors luminous and black as well. Thus, a pure luminous color and a pure black color are taken to be fully saturated (with luminousness and blackness, respectively).

[12] For hued colors this is an idealization because even the most saturated hues (which are the spectral hues) have some whiteness in them (Gordon & Abramov, 1988; Gordon et al., 1994; Jacobs, 1967).

taken to be an independent variable, whereas W is taken to be a dependent variable, we need to replace the color vector $\boldsymbol{F}$ of Eq. (1) with a vector in which I takes the place of W as the $\widehat{\boldsymbol{x}}_0$-component:

$$\boldsymbol{C} = I\widehat{\boldsymbol{x}}_0 + (R - G)\widehat{\boldsymbol{x}}_1 + (Y - B)\widehat{\boldsymbol{x}}_2 + (L - Bk)\widehat{\boldsymbol{x}}_3. \tag{6}$$

We denote the new color vector by $\boldsymbol{C}$ to emphasize its distinctness from $\boldsymbol{F}$. Examining Eq. (5) for W and Eq. (6) for the color vector $\boldsymbol{C}$ we notice that there exists an elegant mathematical connection between them. If we take the vector $\boldsymbol{C}$ to be *a four-vector* in Minkowski space rather than a vector in $\mathbb{R}^4$, i.e., if we assume that $\boldsymbol{C} \in \mathbb{R}^{1,3}$,[13] then the magnitude of $\boldsymbol{C}$ is given by

$$\|\boldsymbol{C}\|^2 = I^2 - (R - G)^2 - (Y - B)^2 - (L - Bk)^2.$$

(For an introduction to four-vectors, see, e.g., Susskind and Friedman (2017).). But this is exactly the expression for W in Eq. (5). Thus, if we the take the color vector $\boldsymbol{C}$ to be a four-vector, we get the economical result that W is simply the magnitude of this four-vector, namely,

$$W^2 = \|\boldsymbol{C}\|^2 = I^2 - (R - G)^2 - (Y - B)^2 - (L - Bk)^2. \tag{7}$$

Taking stock, we see that currently our color model consists of Eqs. (6) and (7). What we are still missing is an expression for color intensity, I. There are a few constraints that assist us in finding this expression. First, we know that I must be independent of

[13] $\mathbb{R}^{1,3}$ is Minkowski space with a $(+,-,-,-)$ metric signature.

$(R-G)$, $(Y-B)$, and $(L-Bk)$, i.e., it cannot be a function of these variables. Second, an immediate implication of Eq. (7) is that the following inequality must hold:

$$(R-G)^2+(Y-B)^2+(L-Bk)^2 \leq I^2. \tag{8}$$

For if this inequality didn't hold, Eq. (7) would assign an imaginary value to W, which is impossible. Note that even though the value of I cannot be a function of $(R-G)$, $(Y-B)$, and $(L-Bk)$, it is clear that for Eq. (8) to be obeyed, its value must be a function of R, G, Y, B, L, and Bk. Otherwise, the visual system wouldn't be able to ensure that Eq. (8) *always* holds. And third, our commitment to the SymFFPQ hypothesis provides us with an additional constraint on the expression for I: this expression must be as *symmetric* as possible. An expression for I that obeys the three constraints above is the following:

$$I = R+G+Y+B+L+Bk. \tag{9}$$

Evidently, this is a very sensible expression for color intensity.[14] As required, I in this expression is evidently not a function of $(R-G)$, $(Y-B)$, and $(L-Bk)$. In addition, it can easily be shown that this expression obeys the inequality of Eq. (8).[15] Finally, the

[14] Indeed, this expression, with the obvious replacement of L with W, was suggested as the expression for brightness by Kaiser et al. (1971). It also seems to be the expression for brightness (again, with the replacement of L with W) according to Hering (Boring, 1949).

[15] To show this we begin with the following trivially true inequality:

$$(R-G)^2+(Y-B)^2+(L-Bk)^2 \leq (R+G)^2+(Y+B)^2+(L+Bk)^2.$$

Next, notice that the following inequality is also trivially true:

$$(R+G)^2+(Y+B)^2+(L+Bk)^2 \leq (R+G+Y+B+L+Bk)^2 = I^2,$$

where the expression for I in Eq. (9) was used for the equality on the right-hand side. Since the right-hand side of the first inequality equals the left-hand side of the second inequality, we can concatenate the two expression for I in Eq. (9) is perfectly *symmetric* with respect to the six elementary-color processes because they all contribute to I in the same way. The expression for I in Eq. (9) therefore completes our model.

In conclusion, in Eqs. (6)–(9) we have attained our goal, namely, a color model that gives rise to the maximally symmetric phenomenal color space. Here is a brief summary of this model. The model suggests that every color is determined by the values of four independent variables—I, $(R - G)$, $(Y - B)$, and $(L - Bk)$. Therefore, every color can be unequivocally described by the color vector $\boldsymbol{C}$ of Eq. (6). The amount of whiteness in a color, W, is given by Eq. (5). Since I, $(R - G)$, $(Y - B)$, and $(L - Bk)$ are independent variables, this equation gives rise to the spherical phenomenal color space of Fig. 2, which was our goal. It turns out that by taking $\boldsymbol{C}$ to be a vector in Minkowski space, i.e., to be a four-vector, we obtain the elegant result that W is simply the magnitude of this four-vector (Eq. (7)). Finally, by following the rationale of the SymFFPQ hypothesis, we concluded that the intensity of a color, I, is given by the summed activity of all six elementary-color processes (Eq. (9)).

2.4 An alternative formulation of the proposed model

A crucial argument in favor of adopting Eq. (9) as the expression for I was this expression's symmetry with respect to the elementary-color processes. Specifically, in

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inequalities above and immediately arrive at the following inequality, which is the desired result, namely, Eq. (8):

$$(R - G)^2 + (Y - B)^2 + (L - Bk)^2 \leq I^2.$$

this expression, all six elementary-color processes contribute to I in the same way. An expression for I that is different from the one in Eq. (9) but is nonetheless perfectly symmetric with respect to the elementary-color processes is the following:

$$I = R + G = Y + B = L + Bk. \tag{10}$$

Therefore, if we follow the rationale of the SymFFPQ hypothesis and use symmetry as our guideline in determining the form of the expression for I, Eq. (10) should be considered as an alternative to Eq. (9). This subsection will demonstrate, however, that even though the expressions for I in Eqs. (9) and (10) seem be totally different from each other, if the variables R, G, Y, B, L, and Bk are assigned a slightly different meaning from their original one as elementary-color processes, then these expressions turn out to be mathematically equivalent. The conclusion will be that Eqs. (9) and (10) give rise to two different, yet mathematically equivalent, formulations of the proposed model.

As was just noted, in the subsequent development, the variables R, G, Y, B, L, and Bk will be assigned a different meaning than their original one as elementary-color processes. To keep things clear and prevent confusion, in this subsection we will therefore denote the elementary-color processes by r, g, y, b, l, and bk. For example, in the new notation, Eq. (9) for color intensity will read:

$$I = r + g + y + b + l + bk. \tag{11}$$

Similarly, in terms of the newly named elementary-color processes, Eq. (6) for the color four-vector $\boldsymbol{C}$ will be written as

$$\boldsymbol{C} = (r + g + y + b + l + bk)\hat{\boldsymbol{x}}_0 + (r - g)\hat{\boldsymbol{x}}_1 + (y - b)\hat{\boldsymbol{x}}_2 + (l - bk)\hat{\boldsymbol{x}}_3,$$

where the explicit expression for I from Eq. (11) was used for the $\hat{\boldsymbol{x}}_0$-component.

We now define six *auxiliary variables*, R, G, Y, B, L, and Bk, in terms of the elementary-color processes (in their new notation) in the following way:

$$R = \frac{(r - g) + I}{2}, \qquad G = \frac{(g - r) + I}{2}, \tag{12a}$$

$$Y = \frac{(y - b) + I}{2}, \qquad B = \frac{(b - y) + I}{2}, \tag{12b}$$

$$L = \frac{(l - bk) + I}{2}, \qquad Bk = \frac{(bk - l) + I}{2}, \tag{12c}$$

where I is given in Eq. (11). It can easily be verified from Eq. (12) that

$$\begin{aligned} (R - G) &= (r - g), \\ (Y - B) &= (y - b), \\ (L - Bk) &= (l - bk). \end{aligned} \tag{13}$$

That is, the expressions for the opponent-colors components in terms of the auxiliary variables are equal to the expressions for these components in terms of the elementary-color processes. Next, notice that summing the terms for R and G in Eq. (12a) yields $I = R + G$, as required by Eq. (10). The same result is obtained for Y and B in Eq. (12b) and for L and Bk in Eq. (12c). We thus see that the expression for I in Eq. (10) holds true for the auxiliary variables R, G, Y, B, L, and Bk defined in Eq. (12).

We now observe that thanks to Eqs. (10) and (13), Eqs. (6)–(8) above are valid even when the variables R, G, Y, B, L, Bk are given by the auxiliary variables defined in Eq. (12). We have therefore obtained an alternative formulation for the proposed model. The two formulations share Eqs. (6)–(8), but differ in their expressions for color intensity—Eq. (9) in the original formulation, Eq. (10) in the new formulation. It should also be remembered that the meaning of the variables R, G, Y, B, L, and Bk is different in the two formulations. Later in the paper, when we discuss the duality of the proposed color model with the mathematical description of two-state quantum systems, the alternative formulation will turn out to be more useful.

2.5 Why psychophysical color spaces are not symmetrical

When psychophysicists try to organize our color percepts into three-dimensional solids (psychophysical color spaces or color order systems), the result is a far cry from the symmetrical phenomenal color space of Fig. 2. For example, consider the Munsell color order system. The shape of this color order system is irregular (even though Munsell initially attempted his system to be spherical), the opponent unique hues in this system are not diametrically opposed to each other, and the metric of the system is nonuniform, i.e., the perceptual distance between two adjacent color samples (as measured, for example, by the number of just noticeable differences) is different in different regions of the system (Judd, 1969; Kuehni, 2003, chapters 2 and 7). An even better example is given by the Swedish Natural Color System (NCS). This system 'is based on Hering's phenomenological analysis of the characteristic relationships of color percepts and on the postulate of the six elementary color sensations' (Hård et al., 1996, p. 187). The belief behind the NCS was that observers could reliably compare physical color samples to their internal phenomenal color space. Hence, the NCS was constructed using observers' assessments of the degree of similarity between various physical color samples and their 'inherent conceptions of pure white, black, yellow, red, blue, and green' (*ibid*., p. 186). These physical color samples were then embedded into the NCS color solid. This solid was *predetermined* to have a double-cone shape even before any measurement took place.[16] Thus, although the NCS color solid has a symmetric appearance, this symmetry is completely artificial. In fact, the NCS color solid could have been chosen to have any shape whatsoever. For example, '[w]ith minor changes [the NCS solid] could be fitted into a sphere' (Kuehni, 2003, p. 310). As a result of this artificial packing of colors into a predetermined shape, the NCS solid is grossly nonuniform, namely, pairs of adjacent color samples in different regions of the NCS solid have different perceptual distances between them (Kuehni, 2003, chapter 7, 2010).

[16] The double-cone shape arises from Hering's description of white, black, and each of the unique hues as located at the three vertices of an equilateral triangle (Hård et al., 1996, Fig. 7). By combining the four triangles created by the four unique hues with the hue circle, one obtains the double-cone shape.

According to the SymFFPQ hypothesis, our phenomenal color space is the one shown in Fig. 2. What could explain the glaring discrepancies between the properties of that space, which is perfectly symmetric and uniform, and the measured properties of psychophysical color spaces? At least two reasons come to mind. First, we have no *a priori* reason to think that the visual system is able to fully, isotropically, and uniformly sample the phenomenal color space of Fig. 2. Second, there is an inherent difficulty in trying to objectively measure the subjective properties of our perceptions. Color percepts are no exception. For example, Jacobs (1967) concluded from his experiments in which observers were asked to estimate the saturation level of various colors that 'It is, not surprisingly, a relatively difficult task for an observer to judge saturation for a series of heterochromatic stimuli' (p. 273). As another example, Gordon et al. (1994) found in their hue and saturation scaling experiments of spectral colors that their subjects' trial-by-trial variance was very large, indicating that the subjects found it hard to *consistently* estimate the relative portion of red, yellow, green, blue, and white (i.e., saturation) in their own subjective color sensations. More importantly, the variance in the subjects' trial-by-trial estimated values of hue and saturation differed markedly across various levels of saturation. This means that if one were to construct a psychophysical color space from the measurements of these subjects, the result would necessarily be nonuniform. Overall, then, it is not surprising that psychophysical color spaces provide a partial and distorted version of the perfectly symmetric phenomenal color space of Fig. 2.

3. The mathematical description of two-state quantum systems

The purpose of this section is to provide a brief review of the basic mathematical description of two-state quantum systems. The material covered here can be found in many textbooks on quantum mechanics (e.g., Blum, 1981; Zwiebach, 2022). Another excellent source for the mathematics of two-state quantum systems is textbooks on quantum computation and quantum information (e.g., Avron, 2023; Nielsen & Chuang, 2010), which refer to these systems as *qubits*.

3.1 Two-state quantum systems

Two-state quantum systems are quantum systems that exhibit two physically distinguishable states relative to some type of measurement. Some common examples of two-state quantum systems are the spin state of spin-1/2 particles, the polarization state of photons, and atomic systems that can be approximated as effectively having only two electronic levels (Altepeter et al., 2004). A two-state quantum system is fully described by a vector in two-dimensional Hilbert-space.[17] This vector is commonly referred to as a *state vector*. The state vector can be represented relative to any basis of two-dimensional Hilbert space. For example, suppose we pick the basis $\{|1\rangle, |2\rangle\}$, where $|1\rangle$ and $|2\rangle$ are two orthogonal vectors. We can expand the system's state vector, which we will denote by $|\psi\rangle$, as some linear combination (also called superposition) of $|1\rangle$ and $|2\rangle$:

$$|\psi\rangle = a|1\rangle + b|2\rangle, \tag{14}$$

[17] A Hilbert space is an inner-product space over the complex numbers (i.e., the result of the inner product defined in this vector space is in $\mathbb{C}$).

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$ (Nielsen & Chuang, 2010, chapter 1). Quantum mechanics tells us that a measurement conducted to determine the state of the system described by Eq. (14) has a probability of $|a|^2$ of finding the system in the state $|1\rangle$ and a probability of $|b|^2$ of finding it in the state $|2\rangle$.

Because there is an isomorphism between the set of Hilbert-space vectors $|\psi\rangle$ and the set of their *coordinate vectors* $\psi_{\{|1\rangle, |2\rangle\}} = (a, b)^{\mathrm{T}} \in \mathbb{C}^2$ (Lipschutz & Lipson, 2009, chapter 4),[18] it is common in the physics literature to disregard mathematical niceties and treat the vectors $|\psi\rangle$ as if they themselves were elements of $\mathbb{C}^2$ (e.g., Aerts & Sassoli de Bianchi, 2017). Here I will follow suit. The motivation behind this is that it allows one to directly operate on state vectors with *matrices* rather than with operators defined over Hilbert space.[19] This is very convenient during calculations.

3.2 The Pauli matrices

The following four 2×2 matrices are known as the Pauli matrices:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{15}$$

This representation of the Pauli matrices is referred to as their standard representation because it is given relative to the standard basis of $\mathbb{C}^2$, $\mathrm{S} = \{(1,0)^{\mathrm{T}}, (0,1)^{\mathrm{T}}\}$. That is, although this is not denoted explicitly, both input and output vectors of the Pauli

[18] The coordinate vector ψ is sometimes called the *wave function*.

[19] Thus, it is *very* common to see the *operator* $|\psi\rangle\langle\phi|$, where $|\psi\rangle$ and $|\phi\rangle$ are state vectors, treated as a matrix (e.g., Bertlmann & Krammer, 2008).

matrices in Eq. (15) are assumed to be represented relative to the basis S.[20] As can be verified from Eq. (15), all four Pauli matrices are *Hermitian*, i.e., $\sigma_\mu = \sigma_\mu^\dagger \equiv (\sigma_\mu^*)^{\mathrm{T}}$, $\mu = 0, 1, 2, 3$. This means that these matrices are *observables* of two-state quantum systems.[21] I will therefore sometimes refer to these matrices as the Pauli observables. The set $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ constitutes a basis for the vector space of 2×2 complex matrices (Aerts & Sassoli de Bianchi, 2017). Therefore, the Pauli basis necessarily also spans the vector space of all 2×2 Hermitian matrices, namely, the space of all observables of two-state quantum systems. Any 2×2 Hermitian matrix A can be expanded in the Pauli basis as $A = a\sigma_0 + b\sigma_1 + c\sigma_2 + d\sigma_3$, where $a, b, c, d \in \mathbb{R}$ (Zwiebach, 2022, chapter 12).

From Eq. (15) it is straightforward to show that the eigenvalues of the three Pauli matrices σ_i, $i = 1,2,3$, are ± 1. Therefore, if we denote the two eigenvectors of the Pauli matrix σ_i by $|\pm\hat{\boldsymbol{x}}_i\rangle \in \mathbb{C}^2$, we have the following eigenvector/eigenvalue equations:

$$\sigma_i|\pm\hat{\boldsymbol{x}}_i\rangle = \pm|\pm\hat{\boldsymbol{x}}_i\rangle$$

$i \in \{1, 2, 3\}$. Because the Pauli matrices are Hermitian, their eigenvectors, $|\pm\hat{\boldsymbol{x}}_i\rangle$, are orthogonal. From the explicit representation of the Pauli matrix σ_3 in Eq. (15), it is easy to verify that the eigenvectors of this matrix, i.e., $|\pm\hat{\boldsymbol{x}}_3\rangle$, are equal to the vectors of the

[20] To denote this explicitly we would write $[\sigma_\mu]_{\mathrm{S}\to\mathrm{S}}$, $\mu = 0, 1, 2, 3$, or more simply: $[\sigma_\mu]_{\mathrm{S}}$.

[21] A quantum observable of a quantum system is a Hermitian operator that corresponds to a measurable property of the system. For example, the spin-1/2 observable is a Hermitian operator, usually denoted $\hat{\boldsymbol{S}}$, that corresponds to the intrinsic angular momentum of a quantum system, which is a measurable property of the system. Notice that here, because we are treating vectors in two-dimensional Hilbert space as if they were vectors in $\mathbb{C}^2$, we can refer directly to the Pauli *matrices* as observables.

standard basis of $\mathbb{C}^2$. That is, $|+\hat{\boldsymbol{x}}_3\rangle = (1,0)^{\mathrm{T}}$ and $|-\hat{\boldsymbol{x}}_3\rangle = (0,1)^{\mathrm{T}}$. (Notice that this means that the representation of the Pauli matrices in Eq. (15) is relative to the basis $\{|+\hat{\boldsymbol{x}}_3\rangle, |-\hat{\boldsymbol{x}}_3\rangle\}$.) It is therefore common to represent the vectors $|\pm\hat{\boldsymbol{x}}_1\rangle$ and $|\pm\hat{\boldsymbol{x}}_2\rangle$ relative to the basis $\{|+\hat{\boldsymbol{x}}_3\rangle, |-\hat{\boldsymbol{x}}_3\rangle\}$. It is easily shown that the expansion of the vectors $|\pm\hat{\boldsymbol{x}}_1\rangle$ and $|\pm\hat{\boldsymbol{x}}_2\rangle$ relative to this basis is given by

$$\begin{aligned} |\pm\hat{\boldsymbol{x}}_1\rangle &= \frac{1}{\sqrt{2}}(|+\hat{\boldsymbol{x}}_3\rangle \pm |-\hat{\boldsymbol{x}}_3\rangle), \\ |\pm\hat{\boldsymbol{x}}_2\rangle &= \frac{1}{\sqrt{2}}(|+\hat{\boldsymbol{x}}_3\rangle \pm i|-\hat{\boldsymbol{x}}_3\rangle). \end{aligned} \tag{16}$$

We can employ the spectral decomposition theorem (Zwiebach, 2022, chapter 15) to obtain the following expression for three Pauli matrices σ_i, $i = 1, 2, 3$:

$$\sigma_i = |+\hat{\boldsymbol{x}}_i\rangle\langle+\hat{\boldsymbol{x}}_i| - |-\hat{\boldsymbol{x}}_i\rangle\langle-\hat{\boldsymbol{x}}_i|, \tag{17}$$

where the vectors $\langle+\hat{\boldsymbol{x}}_i|$ and $\langle-\hat{\boldsymbol{x}}_i|$ are, respectively, the transposed complex conjugates of the vectors $|+\hat{\boldsymbol{x}}_i\rangle$ and $|-\hat{\boldsymbol{x}}_i\rangle$, i.e., $\langle\pm\hat{\boldsymbol{x}}_i| = (|\pm\hat{\boldsymbol{x}}_i\rangle)^{\dagger} \equiv (|\pm\hat{\boldsymbol{x}}_i\rangle^{*})^{\mathrm{T}}$. Thus, Eq. (17) shows that each one of the three Pauli matrices σ_i is an operator whose output results from the *antagonistic* operation of two matrices, $|+\hat{\boldsymbol{x}}_i\rangle\langle+\hat{\boldsymbol{x}}_i|$ and $|-\hat{\boldsymbol{x}}_i\rangle\langle-\hat{\boldsymbol{x}}_i|$. By utilizing the linearity of the matrices $|\pm\hat{\boldsymbol{x}}_i\rangle\langle\pm\hat{\boldsymbol{x}}_i|$ in Eq. (17), it is easy to show that the expectation values of the Pauli observables σ_i (i.e., the mean values of these observables gleaned from many measurements on a two-state quantum system) are given by

$$\langle\sigma_i\rangle = \langle|+\hat{\boldsymbol{x}}_i\rangle\langle+\hat{\boldsymbol{x}}_i|\rangle - \langle|-\hat{\boldsymbol{x}}_i\rangle\langle-\hat{\boldsymbol{x}}_i|\rangle \tag{18}$$

(Altepeter et al., 2004), where the brackets $\langle\cdot\rangle$ denote expectation value.[22]

Just as we did above for the three Pauli matrices σ_i, $i = 1, 2, 3$, we can use the spectral decomposition theorem on the Pauli matrix σ_0 (which, as will be recalled, is the identity matrix). Since *any* two-dimensional vector is an eigenvector of the identity matrix with a corresponding eigenvalue of 1, the result of eigenvalue decomposition in this case is the following:

$$\sigma_0 = |+\hat{\boldsymbol{n}}\rangle\langle+\hat{\boldsymbol{n}}| + |-\hat{\boldsymbol{n}}\rangle\langle-\hat{\boldsymbol{n}}|, \qquad (19)$$

where $|+\hat{\boldsymbol{n}}\rangle$ and $|-\hat{\boldsymbol{n}}\rangle$ are any two orthogonal unit vectors in two-dimensional Hilbert space. Equation (19) is known as the *completeness relation* (Zwiebach, 2022, chapter 15). In our case, we will be mostly interested in using the completeness relation with the pairs of orthogonal vectors $|+\hat{\boldsymbol{x}}_i\rangle$ and $|-\hat{\boldsymbol{x}}_i\rangle$, $i = 1, 2, 3$, namely,

$$\sigma_0 = |+\hat{\boldsymbol{x}}_i\rangle\langle+\hat{\boldsymbol{x}}_i| + |-\hat{\boldsymbol{x}}_i\rangle\langle-\hat{\boldsymbol{x}}_i|.$$

From the linearity of the matrices $|+\hat{\boldsymbol{x}}_i\rangle\langle+\hat{\boldsymbol{x}}_i|$ and $|-\hat{\boldsymbol{x}}_i\rangle\langle-\hat{\boldsymbol{x}}_i|$ we obtain that the expectation value of the Pauli observable σ_0 is given by

$$\langle\sigma_0\rangle = \langle|+\hat{\boldsymbol{x}}_i\rangle\langle+\hat{\boldsymbol{x}}_i|\rangle + \langle|-\hat{\boldsymbol{x}}_i\rangle\langle-\hat{\boldsymbol{x}}_i|\rangle, \qquad (20)$$

$i \in \{1, 2, 3\}$.

[22] When we measure the expectation value $\langle\sigma_i\rangle$, it is assumed that we have at our disposal a large population of the two-state systems in the *same state*. This allows us to carry out many experiments on this population and obtain the mean value (i.e., expectation value) of the observable σ_i.

3.3 The isomorphism between 2×2 Hermitian matrices and four-vectors

We saw in the previous subsection that the Pauli basis spans the vector space of 2×2 Hermitian matrices. Thus, any 2×2 Hermitian matrix A can be expanded in the Pauli basis as $A = a\sigma_0 + b\sigma_1 + c\sigma_2 + d\sigma_3$, where $a, b, c, d \in R$. It is easy to show that there is an isomorphism between the set of all 2×2 Hermitian matrices and the set of all *four-vectors* in Minkowski space:

$$A = a\sigma_0 + b\sigma_1 + c\sigma_2 + d\sigma_3 \leftrightarrow v^{\mu} = (a, b, c, d), \tag{21}$$

where $\boldsymbol{v} \in \mathbb{R}^{1,3}$ (Schulten, 2000, chapter 11). In fact, the vector $\boldsymbol{v}$ is simply the coordinate vector of A relative to the Pauli basis, i.e., $\boldsymbol{v} = [A]_{\{\sigma_0,\sigma_1,\sigma_2,\sigma_3\}}$. Notice that Eq. (21) gives the contravariant representation of this coordinate vector. (For an introduction to contravariant and covariant representations of vectors, see Fleisch (2012).) The *covariant* form of Eq. (21) is given by

$$\tilde{A} = a\sigma_0 - b\sigma_1 - c\sigma_2 - d\sigma_3 \leftrightarrow v_{\mu} = (a, -b, -c, -d)$$

(Schulten, 2000, chapter 11). The matrix $\tilde{A}$ is therefore the ‘covariant form’ of the matrix A.

The vector space of 2×2 Hermitian matrices is equipped with the following inner product:

$$\langle A, B \rangle \equiv \mathrm{tr}\big(\tilde{A}B\big), \tag{22}$$

where A and B are any 2×2 Hermitian matrices and $\mathrm{tr}(\cdot)$ is the trace operation (*ibid*.). Given the inner product in Eq. (22) we can complete the isomorphism given in Eq. (21) with a correspondence between the inner products in the space of 2×2 Hermitian matrices and Minkowski space:

$$\langle A, B\rangle \equiv \mathrm{tr}\big(\tilde{A}B\big) \leftrightarrow \langle \boldsymbol{v}, \boldsymbol{u}\rangle \equiv \sum_{\mu=0}^{3} v_\mu u^\mu,$$

where $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^{1,3}$ and $\tilde{A} \leftrightarrow v_\mu, B \leftrightarrow u^\mu$.

3.4 Two-state quantum systems in a mixed state and the density matrix

A *mixture* of quantum systems is an ensemble of quantum systems in which the quantum states do not coherently interfere with each other (Altepeter et al., 2004; Blum, 1981, chapter 1; Nielsen & Chuang, 2010, chapter 2). The ensemble as a whole is then said to be in an incoherent state, or, more commonly, in a *mixed state*. By contrast, quantum systems whose component states *do* coherently interfere with each other, are said to be in a *pure state* (for example, the two-state quantum system described by Eq. (14) above is in a pure state). A second physical situation where mixed states arise is in composite quantum systems whose constituent subsystems are entangled with each other (Altepeter et al., 2004; Nielsen & Chuang, 2010, chapter 2). In this case, it is not the composite system that is in a mixed state, but rather each of the entangled subsystems. (Note: since most of the material covered in this subsection is standard, henceforth I will cite references only when some nonstandard issue is discussed. The sources cited in the paragraph above, and many other standard sources, can be used to find information on the properties of two-state quantum systems in a mixed state.)

Quantum systems in a mixed state cannot be described by a state vector. Rather, they are described by a *density matrix*.[23] This matrix provides the *full* physical description of a system in a mixed state.[24] Density matrices have several distinct mathematical properties. Most important for our purposes will be the fact that these matrices are *Hermitian*. Here we will focus on density matrices of two-state quantum systems. It turns out that these 2×2 matrices can be expanded in the Pauli basis, $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, in the following manner:

$$\rho = \frac{1}{2}\sum_{\mu=0}^{3} \langle\sigma_\mu\rangle\sigma_\mu \tag{23}$$

(ρ is the standard notation for density matrices).[25] Notice that the expectation values $\langle\sigma_\mu\rangle$ in Eq. (23) are computed from the two-state system that the density matrix ρ describes. It is noteworthy that $\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle \in \mathbb{R}$, while $\langle\sigma_0\rangle \in \mathbb{N}^+$. The latter expectation value, $\langle\sigma_0\rangle$, carries a special meaning—it is the number of two-state systems in the mixture (Blum, 1981, chapter 1). This number is sometimes referred to as

[23] It would be more precise to say that a mixture is described by a density *operator*. The density matrix is a representation of this operator relative to some basis, whose identity is generally understood from the physical context. Here, however, I follow most textbooks and refer directly to the density matrix. The reason textbooks often take this approach is that *in calculations* one must use a specific representation of the density operator, namely, one must use the density matrix. (This issue of committing to a representation of vectors and operators relative to a specific basis when performing calculations will figure prominently in *Section 4* below.)

[24] This means the following: the density matrix allows the calculation of the probability of finding the system in *any* particular pure state upon measurement of *any* observable (Altepeter et al., 2004; Blum, 1981, chapter 1).

[25] Equation (23) gives the *unnormalized* version of the density matrix. It is more common to see this matrix with all its values normalized by $\langle\sigma_0\rangle$ (Altepeter et al., 2004).

the *intensity* of the mixture (*ibid*.).[26] The expectation values appearing in Eq. (23) obey the following inequality:

$$\langle\sigma_1\rangle^2 + \langle\sigma_2\rangle^2 + \langle\sigma_3\rangle^2 \leq \langle\sigma_0\rangle^2 \tag{24}$$

(*ibid*.). By substituting the standard representations of the four Pauli matrices (Eq. (15)) into the expression for the density matrix in Eq. (23) we obtain the explicit form of this matrix:

$$\rho = \frac{1}{2}\begin{bmatrix} \langle\sigma_0\rangle + \langle\sigma_3\rangle & \langle\sigma_1\rangle - i\langle\sigma_2\rangle \\ \langle\sigma_1\rangle + i\langle\sigma_2\rangle & \langle\sigma_0\rangle - \langle\sigma_3\rangle \end{bmatrix}. \tag{25}$$

Recall that because ρ is a 2×2 Hermitian matrix, it can be viewed as a vector in the space of 2×2 Hermitian matrices. The magnitude of this 'vector', $\|\rho\|$, is given by $\sqrt{\langle\rho,\rho\rangle}$, where the inner product $\langle\cdot,\cdot\rangle$ is defined in Eq. (22). From Eq. (25) it is easy to calculate this magnitude to be

$$\|\rho\|^2 = \frac{\langle\sigma_0\rangle^2 - \langle\sigma_1\rangle^2 - \langle\sigma_2\rangle^2 - \langle\sigma_3\rangle^2}{2}. \tag{26}$$

We saw in the preceding subsection that there is an isomorphism between the vector space of 2×2 Hermitian matrices and Minkowski space. Therefore, the density matrix ρ of Eq. (23), which is a 2×2 Hermitian matrix, can be assigned a corresponding four-vector $\boldsymbol{B} \in \mathbb{R}^{1,3}$:

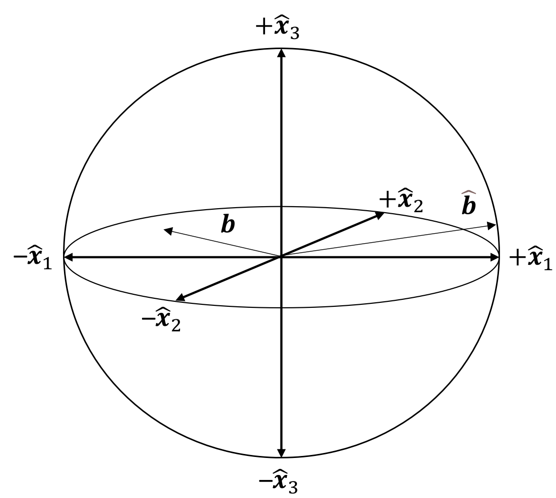
[26] This term is usually used in the context of particle *beams* (e.g., a beam of silver atoms (these are spin-1/2 particles)).

$$\rho = \frac{1}{2}\sum_{\mu=0}^{3}\langle\sigma_\mu\rangle\sigma_\mu \leftrightarrow \boldsymbol{B} = \sum_{\mu=0}^{3}\langle\sigma_\mu\rangle x^\mu. \tag{27}$$

Often, the number of two-state systems in the mixture is immaterial to us. In such cases, we can ignore the zeroth component of the four-vector $\boldsymbol{B}$ and represent any mixture of two-state quantum systems by the following $\mathbb{R}^3$ vector:

$$\boldsymbol{b} = \langle\sigma_1\rangle\widehat{\boldsymbol{x}}_1 + \langle\sigma_2\rangle\widehat{\boldsymbol{x}}_2 + \langle\sigma_3\rangle\widehat{\boldsymbol{x}}_3. \tag{28}$$

These $\mathbb{R}^3$ vectors are known as Bloch vectors. Since $\|\boldsymbol{b}\| \leq \langle\sigma_0\rangle$ (see Eq. (24)), the set of all Bloch vectors is contained within a sphere whose radius is $\langle\sigma_0\rangle$. This sphere is known as the Bloch sphere.[27] Figure 3 provides an illustration of this sphere. Note that Bloch vectors for which $\|\boldsymbol{b}\| = \langle\sigma_0\rangle$, namely, those that lie on the surface of the Bloch sphere, represent two-state quantum systems in a pure state, like the one described in Eq. (14). Six special cases are the Bloch vectors $\pm\widehat{\boldsymbol{x}}_i$, $i = 1, 2, 3$, which correspond to the six eigenvectors of the Pauli matrices σ_i, i.e., to $|\pm\widehat{\boldsymbol{x}}_i\rangle$.



[27] As was noted above, density matrices are usually given after a normalization by the expectation value $\langle\sigma_0\rangle$. Consequently, the Bloch sphere is usually shown as a *unit* sphere.

Figure 3 The Bloch sphere. There is a one-to-one correspondence between the set of all density matrices describing a mixture of two-state quantum systems (Eq. (23)) and the set of vectors contained within a sphere in $\mathbb{R}^3$ whose radius is given by the number of systems in the mixture, $\langle\sigma_0\rangle$ (Eq. (28)). This sphere is known as the Bloch sphere and the vectors contained within it are known as Bloch vectors. Thus, the Bloch vector $\boldsymbol{b}$ shown in the figure ($\|\boldsymbol{b}\| < \langle\sigma_0\rangle$) represents some mixture of two-state quantum systems. The vector $\widehat{\boldsymbol{b}}$, which resides on the surface of the Bloch sphere (i.e., $\|\widehat{\boldsymbol{b}}\| = \langle\sigma_0\rangle$), represents a mixture of two-state systems that are all in the same state, namely, the mixture is in a pure state. Six special cases are the Bloch vectors $\pm\widehat{\boldsymbol{x}}_i$, $i = 1, 2, 3$. These correspond to the eigenvectors of the Pauli matrices σ_i, i.e., to the Hilbert=space vectors $|\pm\widehat{\boldsymbol{x}}_i\rangle$.

4. The computational universe hypothesis and the universal computational basis of Hilbert space

In recent decades, several notable researchers have advocated the hypothesis that the physical world is the result of a gigantic computation (Fredkin, 2003; Lloyd, 2007, 2013; ‘t Hooft, 2016; Wolfram, 2002; Zuse, 1982). A similar hypothesis, which elegantly explains the mysterious success that mathematics has had in describing the physical world (Wigner, 1960), is that the universe is an abstract mathematical model (Carroll, 2022; Tegmark, 2008; Tipler, 2005; Woit, 2015). Evidently, these two hypotheses are very closely related.[28] This is because mathematical models are inherently atemporal and, therefore, for a mathematical model to constitute an implementation of the physical universe, it must be run in time.[29] But this turns it into a computation.[30]

In this paper I will adopt the general metaphysical framework suggested by the above hypotheses, namely, that the physical world results from a computation. In addition, I

[28] And both have some kinship with the ‘simulation hypothesis’ (Bostrom, 2003).
[29] As pointed out by Tegmark (2008), the time dimension in which the computation supposedly takes place is not the same time dimension that is simulated by the computation.
[30] Initial conditions will also be required.

will assume that quantum mechanics provides a correct description of the physical world. Taking these two assumptions together, we arrive at *the computational universe hypothesis*, which suggests that the physical world results from a computation that either directly follows the rules of quantum mechanics (Lloyd 2007, 2013; Carroll, 2022) or is classical but gives rise to quantum mechanics (‘t Hooft, 2016).

In quantum theory it is often emphasized that a quantum state, i.e., a Hilbert-space vector, can be represented in any basis of Hilbert space (e.g., position space, momentum space, etc.). However, when one carries out *computations* on Hilbert-space vectors, one must commit to a definite representation of these vectors, namely, one must choose a particular basis for Hilbert space. Indeed, in the context of quantum computation and quantum information, physicists define a *computational basis*, in which all computations are carried out (Avron, 2023, chapter 1; Nielsen & Chuang, 2010, chapter 1). Therefore, since in the metaphysical worldview adopted here the physical world results from a computation, we must assume that there exists a computational basis for universal Hilbert space. *Ex hypothesi*, the universe represents all quantum states relative to this *universal computational basis* (UCB).

Two-state quantum systems are represented in a two-dimensional subspace of the universal Hilbert space. I will denote the basis of this subspace by $\mathrm{UCB}(2)$. The two orthogonal vectors that constitute this basis will be denoted $|C_1\rangle$ and $|C_2\rangle$, i.e.,

$\mathrm{UCB}(2) = \{|C_1\rangle, |C_2\rangle\}$. As an example, consider a two-state quantum system in the pure state $|\psi\rangle$ of Eq. (14). This state can be represented relative to *any* basis of two-dimensional Hilbert space. In Eq. (14) we chose to represent it in the basis $\{|1\rangle, |2\rangle\}$. However, on the computational universe hypothesis, *the universe itself* represents this state in the universal computational basis of two-dimensional Hilbert space, $\mathrm{UCB}(2)$:

$$|\psi\rangle_{\mathrm{UCB}(2)} = p|C_1\rangle + q|C_2\rangle,$$

where $p, q \in \mathbb{C}$ and $|p|^2 + |q|^2 = 1$.

Since quantum observables (i.e., Hermitian operators) take quantum states in Hilbert space as input and produce quantum states in Hilbert space as outputs, consistency requires that if quantum states are represented relative to some basis, observables must also be represented relative to the same basis. Thus, in the quantum computational universe hypothesized here, quantum observables are represented as matrices relative to the universal computational basis. For example, the Pauli matrices σ_μ, $\mu = 0, 1, 2, 3$, will be represented as $[\sigma_\mu]_{\mathrm{UCB}(2)}$. Notice that the matrices $[\sigma_\mu]_{\mathrm{UCB}(2)}$ are simply the Pauli matrices appearing in Eq. (15), but we now understand the input and output vectors of these matrices to be given relative to the privileged basis $\mathrm{UCB}(2)$.

Since the Pauli matrices constitute a basis for all two-dimensional quantum observables, their representation relative to $\mathrm{UCB}(2)$ constitutes *a universal computational basis* for two-dimensional quantum observables, $\left\{[\sigma_\mu]_{\mathrm{UCB}(2)}\right\}$. This allows us to represent observables of two-state quantum systems either through $\mathrm{UCB}(2)$ or through $\left\{[\sigma_\mu]_{\mathrm{UCB}(2)}\right\}$. As an example, let us consider the density matrix ρ. Its representation relative to the basis $\mathrm{UCB}(2)$ is given by

$$[\rho]_{\mathrm{UCB}(2)} = \begin{bmatrix} \langle C_1|\rho|C_1\rangle & \langle C_1|\rho|C_2\rangle \\ \langle C_2|\rho|C_1\rangle & \langle C_2|\rho|C_2\rangle \end{bmatrix}.$$

However, we can also represent ρ relative to the basis $\left\{[\sigma_\mu]_{\mathrm{UCB}(2)}\right\}$. We do this using the expansion of Eq. (23):

$$[\rho]_{\mathrm{UCB}(2)} = \frac{1}{2}\sum_{\mu=0}^{3} \langle [\sigma_\mu]_{\mathrm{UCB}(2)}\rangle \, [\sigma_\mu]_{\mathrm{UCB}(2)}.$$

Before we go on, there is one last issue that is worthwhile pointing out now since it will become important later. We saw in *Subsection 3.2* that the representation of the Pauli matrices in Eq. (15) is, in fact, given relative to the basis formed from the eigenvectors of σ_3, i.e., relative to $\{|+\hat{\boldsymbol{x}}_3\rangle, |-\hat{\boldsymbol{x}}_3\rangle\}$. Therefore, it must be the case that

$$\begin{aligned} |+\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)} &= |C_1\rangle, \\ |-\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)} &= |C_2\rangle. \end{aligned} \tag{29}$$

That is, the basis $\left\{|+\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)}, |-\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)}\right\}$ is identical to $\mathrm{UCB}(2)$.

5. Color is the phenomenal dual aspect of two-state quantum systems in a mixed state

5.1 An isomorphism between the mathematical description of two-state quantum systems and the proposed model of color phenomenology

The goal of this subsection is to show that there exists a one-to-one correspondence (isomorphism) between the mathematical description of two-state quantum systems in a mixed state and the model of color phenomenology developed in *Section 2*. We begin by positing the following set of correspondences between the six expectation values $\langle|\pm\hat{\boldsymbol{x}}_i\rangle\langle\pm\hat{\boldsymbol{x}}_i|\rangle$, $i = 1, 2, 3$, and the six processes R, G, Y, B, L, and Bk:

$$\begin{aligned} &\langle|+\hat{\boldsymbol{x}}_1\rangle\langle+\hat{\boldsymbol{x}}_1|\rangle \leftrightarrow R, \qquad \langle|-\hat{\boldsymbol{x}}_1\rangle\langle-\hat{\boldsymbol{x}}_1|\rangle \leftrightarrow G,\\ &\langle|+\hat{\boldsymbol{x}}_2\rangle\langle+\hat{\boldsymbol{x}}_2|\rangle \leftrightarrow Y, \qquad \langle|-\hat{\boldsymbol{x}}_2\rangle\langle-\hat{\boldsymbol{x}}_2|\rangle \leftrightarrow B,\\ &\langle|+\hat{\boldsymbol{x}}_3\rangle\langle+\hat{\boldsymbol{x}}_3|\rangle \leftrightarrow L, \qquad \langle|-\hat{\boldsymbol{x}}_3\rangle\langle-\hat{\boldsymbol{x}}_3|\rangle \leftrightarrow Bk. \end{aligned} \tag{30}$$

It immediately follows that

$$\begin{aligned} \langle\sigma_0\rangle &= \langle|+\hat{\boldsymbol{x}}_1\rangle\langle+\hat{\boldsymbol{x}}_1|\rangle + \langle|-\hat{\boldsymbol{x}}_1\rangle\langle-\hat{\boldsymbol{x}}_1|\rangle \leftrightarrow R + G = I,\\ \langle\sigma_0\rangle &= \langle|+\hat{\boldsymbol{x}}_2\rangle\langle+\hat{\boldsymbol{x}}_2|\rangle + \langle|-\hat{\boldsymbol{x}}_2\rangle\langle-\hat{\boldsymbol{x}}_2|\rangle \leftrightarrow Y + B = I,\\ \langle\sigma_0\rangle &= \langle|+\hat{\boldsymbol{x}}_3\rangle\langle+\hat{\boldsymbol{x}}_3|\rangle + \langle|-\hat{\boldsymbol{x}}_3\rangle\langle-\hat{\boldsymbol{x}}_3|\rangle \leftrightarrow L + Bk = I, \end{aligned} \tag{31}$$

where the equalities on the left-hand side are from Eq. (20) and the equalities on the right-hand side are from Eq. (10). Thus, quite satisfyingly, Eq. (31) shows a correspondence between a mixture's intensity, $\langle\sigma_0\rangle$, and color intensity, I. Another immediate result from Eq. (30) is the following:

$$\begin{aligned}\langle\sigma_1\rangle &= \langle|+\hat{\boldsymbol{x}}_1\rangle\langle+\hat{\boldsymbol{x}}_1|\rangle - \langle|-\hat{\boldsymbol{x}}_1\rangle\langle-\hat{\boldsymbol{x}}_1|\rangle \leftrightarrow R - G,\\ \langle\sigma_2\rangle &= \langle|+\hat{\boldsymbol{x}}_2\rangle\langle+\hat{\boldsymbol{x}}_2|\rangle - \langle|-\hat{\boldsymbol{x}}_2\rangle\langle-\hat{\boldsymbol{x}}_2|\rangle \leftrightarrow Y - B,\\ \langle\sigma_3\rangle &= \langle|+\hat{\boldsymbol{x}}_3\rangle\langle+\hat{\boldsymbol{x}}_3|\rangle - \langle|-\hat{\boldsymbol{x}}_3\rangle\langle-\hat{\boldsymbol{x}}_3|\rangle \leftrightarrow L - Bk,\end{aligned} \tag{32}$$

where the equalities on the left-hand side are from Eq. (18). Thus, the antagonistic operation between the two matrices that comprise each of the Pauli matrices σ_i, i.e., $|+\hat{\boldsymbol{x}}_i\rangle\langle+\hat{\boldsymbol{x}}_i|$ and $|-\hat{\boldsymbol{x}}_i\rangle\langle-\hat{\boldsymbol{x}}_i|$, $i = 1, 2, 3$, corresponds to the opponent operation of the two processes that comprise each of the opponent-colors pairs.

From Eqs. (31) and (32) we see that there exists an exact correspondence between the density matrix ρ in Eq. (23) and the color vector $\boldsymbol{C}$ in Eq. (6):

$$\rho = \frac{1}{2}\sum_{\mu=0}^{3}\langle\sigma_\mu\rangle\sigma_\mu \leftrightarrow \boldsymbol{C} = I\hat{\boldsymbol{x}}_0 + (R-G)\hat{\boldsymbol{x}}_1 + (Y-B)\hat{\boldsymbol{x}}_2 + (L-Bk)\hat{\boldsymbol{x}}_3, \tag{33}$$

where, it will be noticed, the correspondences $\sigma_\mu \leftrightarrow \hat{\boldsymbol{x}}_\mu$, $\mu = 0, 1, 2, 3$, are implicitly assumed. From Eq. (33) it is evident that the magnitude of ρ (when treated as a vector in the vector space of 2×2 Hermitian matrices; see Eq. (26)) should correspond to the magnitude of the four-vector $\boldsymbol{C}$. But since the latter magnitude gives the amount of whiteness in a color, W, (Eq. (7)), we see that $\|\rho\|$ (Eq. (26)) corresponds to W:

$$\|\rho\|^2 = \frac{\langle\sigma_0\rangle^2 - \langle\sigma_1\rangle^2 - \langle\sigma_2\rangle^2 - \langle\sigma_3\rangle^2}{2} \leftrightarrow W^2 = I^2 - (R-G)^2 - (Y-B)^2 - (L-Bk)^2. \tag{34}$$

Another correspondence that is easily obtained from Eqs. (31) and (32) is between Eq. (24) and Eq. (8), namely,

$$\langle\sigma_1\rangle^2 + \langle\sigma_2\rangle^2 + \langle\sigma_3\rangle^2 \le \langle\sigma_0\rangle^2 \leftrightarrow (R-G)^2 + (Y-B)^2 + (L-Bk)^2 \le I^2. \tag{35}$$

A more direct way to see the correspondence of Eq. (33) is through the four-vector $\boldsymbol{B}$ that is isomorphic to ρ (see Eq. (27)). Using this isomorphism, Eq. (33) can be expressed as

$$\boldsymbol{B} = \sum_{\mu=0}^{3} \langle\sigma_\mu\rangle x^\mu \leftrightarrow \boldsymbol{C} = I\hat{\boldsymbol{x}}_0 + (R-G)\hat{\boldsymbol{x}}_1 + (Y-B)\hat{\boldsymbol{x}}_2 + (L-Bk)\hat{\boldsymbol{x}}_3, \tag{36}$$

where $x^\mu \leftrightarrow \hat{\boldsymbol{x}}_\mu$. An implication of Eq. (36) is that there is a one-to-one correspondence between the Bloch sphere of Fig. 3 and the phenomenal color space of Fig. 2. Hence,

$$\begin{array}{ll} -\hat{\boldsymbol{x}}_1 \leftrightarrow \text{Unique green} & +\hat{\boldsymbol{x}}_1 \leftrightarrow \text{Unique red,} \\ -\hat{\boldsymbol{x}}_2 \leftrightarrow \text{Unique blue} & +\hat{\boldsymbol{x}}_2 \leftrightarrow \text{Unique yellow,} \\ -\hat{\boldsymbol{x}}_3 \leftrightarrow \text{Black} & +\hat{\boldsymbol{x}}_3 \leftrightarrow \text{Luminous,} \end{array} \tag{37}$$
$$\boldsymbol{0} \leftrightarrow \text{White,}$$

where $\boldsymbol{0}$ is the zero vector.

Overall, Eqs. (30)–(37) show that the mathematical description of two-state quantum systems in a mixed state and the color model of Eqs. (6)–(8) and (10) are isomorphic to each other.[31] (Since the formulation of the color model in Eqs. (6)–(8) and (10) is

[31] It is noteworthy that even though color scientists and quantum physicists were unaware of the isomorphism established here, and even though they worked totally independently of each other, both communities converged on an identical terminology to describe their subjects of study. Thus, in color

equivalent to the formulation of Eqs. (6)–(9) (*Subsection 2.4*), this isomorphism also applies to the latter formulation of the model.)

5.2 The hypothesis $\mathcal{CQ}$: Color qualia are the phenomenal dual aspects of the mixed states of a two-state quantum system

The preceding subsection established that there exists an isomorphism between the mathematical description of two-state quantum systems in a mixed state and the mathematical description of the structure of the phenomenal qualities of color. Dual-aspect theory predicts that such an isomorphism between physical states and phenomenal states should exist if the latter are phenomenal duals of the former (see the *Introduction*). We therefore seem to be in a position to suggest that color is the phenomenal dual aspect of two-state quantum systems. There is a snag here, however: the values on the left-hand sides of the correspondences in Eqs. (30)–(37) change with the choice of basis for Hilbert space. Thus, from the perspective of dual-aspect theory, the correspondences in Eqs. (30)–(37) are underdetermined. To clearly see the problem, consider, for example, Eq. (30). The expectation values $\langle|\pm\hat{\boldsymbol{x}}_i\rangle\langle\pm\hat{\boldsymbol{x}}_i|\rangle$, $i = 1, 2, 3$, appearing in this equation *depend on the specific basis that is* chosen for two-dimensional Hilbert space. That is, different bases for two-dimensional Hilbert space will lead to different expectation values. But according to Eq. (30), these expectation values correspond to the fundamental color sensations, R, G, Y, B, L, and Bk. Thus, if we take

science *pure* colors (i.e., fully saturated colors, which are the colors on the surface of the sphere in Fig. 2) give rise to a color *mixture* when combined. Analogously, in quantum physics, *pure* states, which are the states located on the surface of the Bloch sphere of Fig. 3, combine to give a *mixture*. We have also seen that a mixture's intensity corresponds to color intensity.

Eq. (30) to mean that our elementary color sensations R, G, Y, B, L, and Bk are phenomenal duals of the expectation values $\langle|\pm\hat{\boldsymbol{x}}_i\rangle\langle\pm\hat{\boldsymbol{x}}_i|\rangle$, $i = 1, 2, 3$, we will have to conclude that our color sensations should depend on the particular basis chosen for two-dimensional Hilbert space. This, of course, is absurd.

To solve this problem, we invoke the computational universe hypothesis suggested in *Section 4*. Recall that on this hypothesis, the universe represents the quantum states of all the systems in the universe relative to a privileged basis, which is the computational basis of the universal Hilbert space. This solves the problem of underdeterminism pointed out above because from nature's point of view, the mathematical description of two-state quantum systems given in Eqs. (30)–(37) exists in a *specific* representation—the representation relative to the universal computational basis of two-dimensional Hilbert space, $\mathrm{UCB}(2)$. Therefore, following the principles of dual-aspect theory, we reach the main hypothesis of this paper:

> $\mathcal{CQ}$: Color qualia are the phenomenal dual aspects of the mixed states of a two-state quantum system. (These mixed states are represented in the universal computational basis of two-dimensional Hilbert space, $\mathrm{UCB}(2)$.)

5.3 The hypothesis $\mathcal{CQ}$ explains several fundamental phenomenal properties of color

The goal of this subsection is to show how the hypothesis $\mathcal{CQ}$ accounts for several fundamental phenomenal properties of color.

1. $\mathcal{CQ}$ explains why there exist *seven* special colors in phenomenal color space, i.e., the seven elementary colors. The reason is that two-state quantum systems have seven special states. The first six of these special states occur when the *entire mixture* occupies one of the six eigenvectors of the three privileged Pauli matrices, $[\sigma_i]_{\mathrm{UCB}(2)}$, $i = 1, 2, 3$, namely, the six Hilbert-space vectors $|\pm\hat{\boldsymbol{x}}_i\rangle_{\mathrm{UCB}(2)}$.[32] The seventh special state occurs when the mixture is in the *fully mixed state*:

$$[\rho]_{\mathrm{UCB}(2)} = \frac{N}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where N is the number of systems in the mixture (i.e., $N = \langle[\sigma_0]_{\mathrm{UCB}(2)}\rangle = \langle\sigma_0\rangle$). As Eq. (37) shows, the seven special states of two-state quantum systems correspond to the seven special colors of phenomenal color space.

2. As can be easily realized from Eq. (31), $\mathcal{CQ}$ neatly explains why color sensations result from the operation of three pairs of opponent-colors processes.
3. $\mathcal{CQ}$ provides an explanation for why the sensations of luminous and black (essentially light and dark) are perceived as more fundamental than the four elementary hues. To see this, recall from Eq. (29) that the vectors $|\pm\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)}$ are the two vectors of the universal computational basis of two-dimensional Hilbert space, UCB(2). Thus, the colors that are the dual aspects of the vectors $|\pm\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)}$ should be perceived as the most fundamental colors in phenomenal color space. As Eq. (37) shows, these colors are luminous and black. Contrast this with the four

[32] Written in terms of density matrices, the six pure states $|\pm\hat{\boldsymbol{x}}_i\rangle_{\mathrm{UCB}(2)}$ are given by $[\rho]_{\mathrm{UCB}(2)} = \frac{\langle\sigma_0\rangle}{2}\left(\sigma_0 \pm [\sigma_i]_{\mathrm{UCB}(2)}\right)$. (Notice that because σ_0 is the identity matrix, $\sigma_0 = [\sigma_0]_{\mathrm{UCB}(2)}$.)

vectors $|\pm\hat{\boldsymbol{x}}_1\rangle_{\mathrm{UCB}(2)}$ and $|\pm\hat{\boldsymbol{x}}_2\rangle_{\mathrm{UCB}(2)}$, which, as Eq. (16) shows, are due to linear combinations of the two privileged basis vectors, $|\pm\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)}$. This explains why the four elementary hues, which are the phenomenal dual aspects of $|\pm\hat{\boldsymbol{x}}_1\rangle_{\mathrm{UCB}(2)}$ and $|\pm\hat{\boldsymbol{x}}_2\rangle_{\mathrm{UCB}(2)}$ (Eq. (37)), are perceived as less fundamental than luminous and black.

4. $\mathcal{CQ}$ explains the age-old puzzle (Purves & Yegappan, 2017; Shepard, 1994) of why the hues can be ordered in a *closed* continuum (the hue circle). As is evident from Eq. (16), the four vectors $|\pm\hat{\boldsymbol{x}}_1\rangle_{\mathrm{UCB}(2)}$ and $|\pm\hat{\boldsymbol{x}}_2\rangle_{\mathrm{UCB}(2)}$, whose dual aspects are the four fundamental hues, can all be converted to each other by varying the *relative phase* between the basis vectors $|\pm\hat{\boldsymbol{x}}_3\rangle_{\mathrm{UCB}(2)}$. Thus, on $\mathcal{CQ}$, the continuum of hues between the four elementary hues is closed because this continuum reflects a phase angle.

6. Discussion

The starting point of this paper was the FFPQ (fully-formed, primordial qualia) hypothesis, which posits that the qualia experienced by humans (and other organisms) were created fully-formed at the creation of the universe. It was then argued that the FFPQ hypothesis naturally leads to the prediction that the phenomenal spaces of the fully-formed, primordial qualia should exhibit simplicity, symmetry, and beauty. The objective part of this prediction is that phenomenal spaces should be perfectly symmetric. This prediction was dubbed the SymFFPQ hypothesis. The goal of this paper was to apply this hypothesis to the (arguably) simplest type of phenomenal experience—color. The result is the hypothesis $\mathcal{CQ}$, which suggests that color qualia are the phenomenal dual aspects of the mixed states of two-state quantum systems.

Since the universe contains a mind-boggling number of two-state quantum systems, a straightforward reading of $\mathcal{CQ}$ seems to imply that color qualia should be practically everywhere. This predicted uniquity of phenomenal experience is, of course, a feature common to all panmicropsychist theories. Whether one sees this wild proliferation of phenomenal experience as a problem or not probably depends on one's philosophical predilections. Thus, Chalmers (1996) is 'not sure that [the ubiquity of phenomenal experience] is such a bad prospect' (p. 154),[33] whereas Coleman (2012) asserts that 'If the cost of solving the mind–body problem is that there are subjects everywhere, it is not a cost most philosophers will ever want to pay, nor is it a cost that we should pay' (p. 149). My own response to this issue is to adopt a form of panprotopsychism known as *panqualityism*. Panqualityism rejects the standard panmicropsychist suggestion that phenomenal ultimates are minds, namely, that they are microsubjects entertaining experiences; instead, it posits that phenomenal ultimates carry *unexperienced* qualia (Coleman, 2012, 2014, 2017; Chalmers, 2015, 2017; Lockwood, 1989). Thus, panqualityism distinguishes between phenomenal *qualities* (i.e., qualia) and phenomenal *experience*. Panqualityists argue that to close this gap between quality and experience, a conscious subject is required. Hence, the hard problem for panqualityism is to explain how *conscious* subjects arise.[34] The fact that panqualityism does not

[33] It should be stressed, however, that Chalmers is in no way committed to panmicropsychism (e.g., Chalmers, 1996, p. 299).

[34] To solve this problem, panqualityists can choose one of two options. One option is to argue that conscious subjects can be reduced to structural or functional properties of the phenomenally-qualitied

consider qualitied ultimates to be subjects of experience means that it avoids the excessive proliferation of conscious subjects that afflicts standard panmicropsychism.[35] Specifically, panqualityists will interpret $\mathcal{CQ}$ as suggesting that color qualia are 'painted', so to speak, on two-state quantum systems, but that these qualia are unexperienced until they are (somehow) incorporated into a conscious subject.

In the remainder of the *Discussion*, I address three issues. In *Subsection 6.1*, I propose that 'color-blind' individuals offer a possible way to test $\mathcal{CQ}$. In *Subsection 6.2*, I tackle the possible skepticism regarding the plausibility of two-state quantum systems existing in the brain. I argue that two-state ion channels (or, more likely, a component in them) can be the two-state quantum systems whose phenomenal dual aspect is color. Finally, in *Subsection 6.3*, I generalize $\mathcal{CQ}$ to all types of qualia. The generalized hypothesis, denoted $\mathcal{Q}$, suggests that *all* types of qualia, not only color, are dual aspects of quantum systems (each type is assigned a quantum system with a certain dimensionality).

ultimates (e.g., Coleman, 2012, 2014, 2017). Not everyone is convinced that this deflationary approach to conscious subjects succeeds in closing the quality–experience gap (Blamauer, 2013; Chalmers, 2015; Mihálik, 2022; Shani, 2021). The second option (which happens to be the option I prefer) is to suggest that conscious subjects are *irreducible entities* (e.g., Lockwood, 1989, 1993). On this view, a natural law prescribes when conscious subjects arise. For example, Lockwood suggested that quantum measurement has an *inherent* 'phenomenal perspective' and thus gives rise to a conscious subject (Lockwood, 1989, p. 215). More specifically, Lockwood's hypothesis was that subjective awareness in humans arises from quantum measurement performed by the brain on its own quantum state (*ibid*., p. 213).

[35] An additional advantage that panqualityism has over standard forms of panmicropsychism is that it evades the hardest aspect of the combination problem, which is the subject combination problem (Coleman, 2014; Goff, 2009; Roelofs, 2020).

6.1 Color-blind individuals can be used to test the hypothesis $\mathcal{CQ}$

Dichromats and (the much rarer) monochromats respectively lack one or two of the three types of photoreceptors that normal human trichromats have in their retinas (Sharpe et al., 1999). There is no question that such individuals are color-deficient, meaning that the color gamut that they perceive is partial relative to the color gamut perceived by trichromatic individuals (*ibid*.). However, the question of exactly what colors such individuals *experience* has been contentious ever since Dalton realized that his color vision was different from those around him (for a thorough historical review of this controversy, see Broackes (2010)).

According to $\mathcal{CQ}$, all our color sensations result from an ensemble of two-state quantum systems somewhere in our brains. Since, *ex hypothesi*, the presence of these two-state quantum systems is required for *any* color experience, they should also exist in the brains of dichromats and monochromats. Therefore, $\mathcal{CQ}$ predicts that dichromats and monochromats should have the *potential* to experience the full gamut of colors. For example, protanopes and deuteranopes, i.e., dichromats who lack L- and M-cones, respectively, are usually described as not perceiving the sensations of red and green (Sharpe et al., 1999). However, according to $\mathcal{CQ}$, if we 'shake', so to speak, such individuals vigorously enough, they should be able to experience these color sensations nonetheless. Remarkably, from his review of the many experiments that were conducted to determine what colors the color-blind see (including fascinating experiments on individuals whose one eye is trichromatic but the other is dichromatic), Broackes (2010) concluded that (under the appropriate conditions) dichromats probably *experience* all the color sensations that trichromats do: 'as for the broad structures of the color space of the normal trichromat—hues (forming a circle), saturation, and brightness—I would be very surprised if the majority of dichromats did not have all of that' (p. 375).[36] This conclusion corroborates the prediction of $\mathcal{CQ}$. However, an unequivocal verification of this prediction will require more systematic experiments on the color experience of color-deficient individuals (admittedly, these experiments will need to be quite ingenious).[37]

6.2 Two-state ion channels may be the two-state quantum systems that give rise to color

According to the hypothesis $\mathcal{CQ}$, our color sensations arise from an ensemble of two-state quantum systems that exists somewhere in the brain. Since discussions on ensembles of two-state quantum systems in the physics literature usually involve beams of spin-1/2 particles or photons, one might get the impression that $\mathcal{CQ}$ predicts such beams should exist in the brain—a prediction that seems ludicrous. The goal of this subsection is to dispel such suspicions and show that the biochemical machinery of the brain is rich enough to (theoretically) contain an ensemble of two-state quantum systems.

[36] There is no doubt that given sufficiently large color stimuli or long exposure times, dichromats can discriminate between colors that theoretically they should confuse (see Broackes (2010) for a survey of studies). However, the question of what colors they experience is more subtle.
[37] Broackes (2010) proposed a series of such experiments.

Where, then, can we find two-state quantum systems in the brain? Since we are looking for a quantum system, we should probably aim our searches at the molecular level. I suggest that *two-state ion channels*—or, more likely, some component in these channels—are the two-state quantum systems whose phenomenal dual aspect is color.

These channels, which lie at the very basis of neuronal activity, are large membrane-spanning proteins that transition between two discrete molecular conformations in response to the binding/release of a ligand or the presence/absence of voltage (Siegelbaum & Koester, 2000). The two molecular conformations of two-state ion channels correspond to two functional states: an open state, in which channel-specific ions are free to cross from the extracellular side to the intracellular side (or vice versa), and a closed state, in which ions cannot pass through the channel. Since ion channels are huge protein molecules, and since the transition from one molecular conformation to the other entails many structural changes in the amino acids that comprise the channels (see DaCosta & Baenziger, 2013), it is unlikely that a two-state ion channel *as a whole* can be considered to be a two-state quantum system. Rather, it is more likely that the sought-for two-state quantum system is some *component* of the ion channel. Specifically, I suggest that the two-state quantum system that gives rise to color is the molecular component that resides at the orthosteric site, namely, the site where the channel's agonist operates (Changeux & Christopoulos, 2016). The interaction with the channel's agonist (e.g., the docking of a ligand) causes some conformational change in the orthosteric site which, in turn, cascades into the many structural changes that constitute the conformational change of the entire ion-channel molecule.

A well-established example of a conformational change in the orthosteric site of a receptor protein is given by rhodopsin, which is the photoreceptor molecule of rod cells in the retina (Tessier-Lavigne, 2000).[38] The rhodopsin molecule is composed of a large protein, opsin, that is covalently bonded to a small, light-absorbing molecule—retinal. 'In its nonactivated form rhodopsin contains the 11-*cis* isomer of retinal. Absorption of light by 11-*cis* retinal causes a rotation around the 11-*cis* double bond. As retinal returns to its more stable all-*trans* configuration, it brings about a conformational change in the opsin portion of rhodopsin, which triggers the other events of visual transduction' (*ibid*., p. 511). If the *cis* and *trans* configurations of retinal constitute the only two physical configurations of this molecule, then the configuration state of the retinal molecule is a two-state quantum system. Therefore, on the hypothesis suggested here, the quantum states of the retinal molecule have color qualia as phenomenal dual aspects. (Of course, I do not claim that our color sensations arise from retinal molecules in our retinas (rather, these sensations likely arise from somewhere in the visual cortex (see next).[39])

[38] Notably, rhodopsin is not an ion channel, but rather a G-protein-coupled receptor. However, the biophysical principles behind its conformational changes are similar to those in ion channels (Changeux & Christopoulos, 2016).

[39] Panqualityists (see the beginning of the *Discussion*) will argue that while retinal molecules indeed carry color qualia, these qualia are not experienced, because (presumably) they are not part of a subject.

Where can we expect to find the hypothesized two-state ion channels that give rise to color? The recent work by Li et al. (2022) has shown that there exist cone-opponent functional domains in the primary visual cortex (V1). The hue preferences in these functional domains are geometrically organized into so-called 'pinwheels'. It is therefore natural to suggest that the first place to look for two-state ion channels that give rise to color experience is in the neurons within these cone-opponent functional domains.

6.3 The hypothesis $\mathcal{Q}$: Generalizing $\mathcal{CQ}$ to all types of qualia

According to $\mathcal{CQ}$, color qualia are the phenomenal dual aspects of two-state quantum system in a mixed state. Nothing in this hypothesis indicates that the fact that two-state quantum systems have phenomenal dual aspects depends on their two-dimensionality. It is therefore natural to generalize $\mathcal{CQ}$ to all types of qualia and suggest that:

> $\mathcal{Q}$: All types of qualia are the phenomenal dual aspects of the mixed states of quantum systems. Each specific type of qualia (color, sound, odor, taste, etc.) is the dual aspect of a quantum system with a certain dimensionality $D = 2, 3, \ldots$.. (The mixed states are represented in the universal computational basis of D-dimensional Hilbert space, $\mathrm{UCB}(D)$.)

Thus, color is merely the simplest example of a general rule. Notice that $\mathcal{Q}$ is consistent with the FFPQ hypothesis: since quantum systems were created at the birth of our universe, their phenomenal dual aspects were born fully formed with them. As is shown next, $\mathcal{Q}$ is also consistent with the SymFFPQ hypothesis.

The mixed states of a D-state quantum system can be represented by vectors contained within a $(D^2 - 1)$-dimensional hypersphere (Aerts & Sassoli de Bianchi, 2017; Bertlmann & Krammer, 2008). For the case $D = 2$, namely, for two-state quantum systems, this hypersphere is a three-dimensional sphere, i.e., the Bloch sphere of Fig. 3. Notably, when $D > 2$, the vectors representing the mixed states *do not fill* the $(D^2 - 1)$-dimensional hypersphere, but rather create a complex shape within it. Two-state quantum systems and their Bloch-sphere representation are therefore an exception in this regard. According to Q, each D-state quantum system has a specific type of phenomenal quality as its dual. Therefore, Q predicts that the phenomenal space of each type of phenomenal quality will be a $(D^2 - 1)$-dimensional hypersphere (again, we saw an example for this in the case of the phenomenal space that is dual to two-state quantum systems, namely, the spherical phenomenal color space of Fig. 2). This prediction accords with the SymFFPQ hypothesis.

A boon of the prediction that the dimensionalities of all types of phenomenal spaces should be *quantized* to $D^2 - 1$ is that it can be used to test Q. Let us consider phenomenal odor space as an example. Mamlouk and Martinetz (2004) concluded that odor space is at least 32-dimensional but is no more than 68-dimensional. Since according to Q the dimensionalities of phenomenal spaces are quantized to $D^2 - 1$, the only dimensionalities that are in line with the results of Mamlouk and Martinetz are 35, 48, and 63 (for $D = 6, 7, 8$, respectively). However, since Mamlouk and Martinetz showed that the quality of fit of their model does not rise appreciably beyond 32 dimensions, we remain with a prediction of 35-dimensional odor space (i.e., $D = 6$). Interestingly, Weiss et al. (2012) showed that different odorant mixtures all smell alike once the number of molecular components in them exceeds ~30. They concluded that 'a common olfactory percept, "olfactory white," is associated with mixtures of $\sim$30 or more equal-intensity components that span stimulus space' (p. 19959). Since a minimum of $\sim$30 odorants that span odor space was required to reach 'olfactory white', we can conclude that odor space is ~30-dimensional. A more detailed look at the results of Weiss et al. shows that for a mixture of odorants to be identified as 'olfactory white' with a probability of 50%, the mixture needs to contain between 35 and 40 components (see their Fig. 3). These results are consistent with the prediction made above of a 35-dimensional odor space. In summary, then, when we combine the hypothesis that odor sensations are the phenomenal dual aspects of some D-state quantum system with experimental results on the dimensionality of odor space, we arrive at a quantitative prediction on this space: it should be 35-dimensional. To test this prediction, the procedure described by Meister (2015, p. 9) may be used.

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