Calculus on Strong Partition Cardinals

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In [HM] it was shown that if κ is a strong partition cardinal then every function from $[\kappa]^\kappa$ to $[\kappa]^\kappa$ is continuous almost everywhere. In this investigation, we explore whether such functions are differentiable or integrable in any sense. Some of them are.

In a paper with Adrian Mathias twenty-five years ago, an analogy was drawn between $[\kappa]^\kappa$ and the continuum. The analogy was partly topological and partly measure-theoretic. The goal of this paper is to extend the analogy into real analysis.

We present some background on strong partition cardinals in section 1. In section 2 we review the relevant results in [HM] on supercontinuity. In section 3 we develop some arithmetic—pseudo addition, subtraction, and multiplication. In section 4 we define derivatives. In section 5 we define integration. In section 6 we prove a "Fundamental Theorem."

The assumption of [HM] and our assumption here, the existence of a strong partition cardinal, is moderately special. On the one hand, it violates the Axiom of Choice and is not relatively consistent with ZF (unlike AC and its negation). On the other hand, under the Axiom of Determinacy (AD), such cardinals are abundant and consistent with countable choice and DC, the principle of Dependent Choices. $\aleph_1$, for example, is a strong partition cardinal and there are strong partition cardinals that are the limits of strong partition cardinals. AD itself, while once considered unimaginably powerful, seems fairly tame now by the yardstick of the large cardinal axiom hierarchy (well below supercompact cardinals in consistency strength). See [Ka] for details.

1 Background

If p is a set of ordinals, α an ordinal, we use $[p]^\alpha$ to denote the collection of all subsets of p of order-type α. We write $p(\alpha)$ for the αth element of p and $p \restriction \alpha$ for the first α elements of p.

Definition 1.1 *A cardinal κ satisfies $\kappa \rightarrow (\kappa)^\beta_\gamma$ if for all partitions $G : [\kappa]^\beta \rightarrow \gamma$ there exists $p \in [\kappa]^\kappa$ such that G is constant on $[p]^\kappa$. The set p is called "homogeneous" for*

G. When γ is 2, the subscript is usually omitted. Of course, any well-ordered set of type κ may be substituted here for κ itself.

Fact 1.1 *If κ satisfies $\kappa \to (\kappa)^\beta$ and $\alpha < \beta$, then κ satisfies $\kappa \to (\kappa)^\alpha$.*

Definition 1.2 *κ is a strong partition cardinal if κ satisfies $\kappa \to (\kappa)^\kappa_\lambda$ for all $\lambda < \kappa$.*

With much less than a strong partition cardinal, we can define a useful measure on κ:

Definition 1.3 *A subset p of κ is ω-closed if p contains the sups of all increasing ω-sequences from p. We write $(p)_\omega$ for the collection of all ω-sups of p. We say $\mu_\omega(p) = 1$ iff p contains an ω-closed, unbounded set.*

Given $p \subseteq \kappa$, we can define $F_p : [\kappa]^\omega \to 2$ by $F_p(s) = 0$ iff $\cup s \in p$. A homogeneous set q for F_p will have the property that either $(q)_\omega \subseteq p$ or $(q)_\omega \subseteq \kappa \setminus p$. This is the basis of the following fact:

Fact 1.2 (E. M. Kleinberg [K2]) *If κ satisfies $\kappa \to (\kappa)^\omega$, then μ_ω defines an ultrafilter on κ. If κ satisfies $\kappa \to (\kappa)^{\omega+\omega}$, then μ_ω is a κ-complete, normal measure on κ.*

Fact 1.3 *If κ satisfies $\kappa \to (\kappa)^{\omega \cdot \omega}$, then any partition $G : [\kappa]^n \to 2$ has a μ_ω-measure one homogeneous set.*

With a strong partition cardinal κ we can define a measure on $[\kappa]^\kappa$ analogous to μ_ω.

Definition 1.4 *For $p \in [\kappa]^\kappa$, we write $_\omega p$ for the successive ω-sups from p, that is, $_\omega p = \{\cup_n p(\lambda + n) : \lambda < k \text{ is a limit ordinal}\}$. We write $\langle p \rangle = \{_\omega r : r \in [p]^\kappa\}$. Note that if $x \in \langle p \rangle$, then $x \subseteq (p)_\omega$, but $x \cap (x)_\omega = \emptyset$.*

Definition 1.5 *For $A \subseteq [\kappa]^\kappa$ we define $\nu(A) = 1$ iff $\langle p \rangle \subseteq A$ for some $p \in [\kappa]^\kappa$.*

Fact 1.4 *If κ is a strong partition cardinal, then ν is a κ-additive ultrafilter on $[\kappa]^\kappa$.*

$\kappa \to (\kappa)^\kappa$ is used to prove that ν defines an ultrafilter. $\kappa \to (\kappa)^\kappa_\gamma$ is used to show γ-additivity. The full strong partition property is needed to show κ-additivity.

Definition 1.6 *For F, G functions from $[\kappa]^\kappa$ to $[\kappa]^\kappa$, we define*
$$F \sim G \iff \nu(\{p : \mu_\omega(\{\alpha : F(p)(\alpha) = G(p)(\alpha)\}) = 1\}) = 1.$$

It is easy to show $\sim$ is an equivalence relation.

Definition 1.7 *For $F : [\kappa]^\kappa \to [\kappa]^\kappa$, we denote by $[F]_\kappa$ the equivalence class of F modulo $\sim$.*

For more on strong partition cardinals, see [HM], [K1], or [Ka].

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2   Continuity and Supercontinuity

We assume for the remainder of this paper that κ is a strong partition cardinal.

The usual topology on $[\kappa]^\kappa$ features basic open sets based on initial segments, that is, sets of the form: $\{p \in [\kappa]^\kappa : p \restriction \alpha = s\}$, where $\alpha < \kappa$, $s \in [\kappa]^\alpha$. This leads to the following definition:

Definition 2.1 $F : [\kappa]^\kappa \to [\kappa]^\kappa$ *is continuous on* $A \subseteq [\kappa]^\kappa$ *iff for all* $p \in A$ *and* $\alpha < \kappa$ *there is a* $\beta < \kappa$ *such that* $p \restriction \beta = q \restriction \beta$ *implies* $F(p) \restriction \alpha = F(q) \restriction \alpha$, *for all* $q \in A$.

The motivation for this paper is the notion of "supercontinuity" introduced in [HM].

Definition 2.2 *Let* A *be a subset of* $[\kappa]^\kappa$ *and* F *a function from* $[\kappa]^\kappa$ *to* $[\kappa]^\kappa$. F *is supercontinuous on* A *iff for all* $p, q \in A$ *and all* $\alpha < \kappa$,

$$p \cap \alpha = q \cap \alpha \iff F(p) \cap \alpha = F(q) \cap \alpha.$$

In analogy with real analysis, a function is supercontinuous if we may take as the δ in the usual definition of continuous the given ϵ.

Proposition 2.1 *If* F *is any function from* $[\kappa]^\kappa$ *to* $[\kappa]^\kappa$, *then* F *is supercontinuous on a* ν-*measure one set.*

Proposition 3.1 is a deep combinatorial fact. To give the reader some idea of its proof, we show the following:

Proposition 2.2 *If* F *is any function from* $[\kappa]^\kappa$ *to* κ, *then* ν-*measure one many* $p \in [\kappa]^\kappa$ *satisfy* $\exists \alpha < \kappa \, \forall q \in [p]^\kappa \, q \cap \alpha = p \cap \alpha \Rightarrow F(q) = F(p)$.

Proof: We show first that there is a set $[y]^\kappa$ such that for all $p \in [y]^\kappa$ there exists $\alpha < \kappa$ such that $\forall q \in [p]^\kappa \, q \cap \alpha = p \cap \alpha \Rightarrow F(q) = F(p)$.

We begin by defining a partition $G : [\kappa]^\kappa \to 2$ by $G(p) = 0$ iff

$$\forall q \in [p]_\kappa \, F(p) \leq F(q).$$

Let $x \in [\kappa]^\kappa$ be homogeneous for G. Let $z \in [x]^\kappa$ be such that $F(z)$ is least in $\{F(p) : p \in [x]^\kappa\}$. Then $G(z) = 0$ and hence G is constantly 0 on $[x]^\kappa$. We describe this by saying that F is "monotonic" on $[x]^\kappa$.

Next, we define a partition $H : [x]^\kappa \to 2$ by $H(p) = 0$ iff

$$\exists \alpha < \kappa \, \forall q \in [p]^\kappa \, q \cap \alpha = p \cap \alpha \Rightarrow F(q) = F(p).$$

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Let $y \in [x]^\kappa$ be homogeneous for H. If the range of H on $[y]^\kappa$ is $\{0\}$ we have found our set y. To show that the range is $\{0\}$, we define a third partition, $K : [y]^\kappa \to 2$, by $K(q) = 0$ iff

$$F((y \cap q(0)) \cup (q \setminus \{q(0)\})) < q(0).$$

Let $z \in [y]^\kappa$ be homogeneous for K. Let z_0, z_1 be respectively, the even and odd halves of z, that is, z_0 consists of all $z(\alpha)$ where α is an even ordinal, and so on. Choose $\beta \in z_1$ greater than $F(z_0)$. Now form $q = \{\beta\} \cup (z_0 \setminus \beta)$. Then $F((y \cap q(0)) \cup (q \setminus \{q(0)\})) = F((y \cap \beta) \cup (z_0 \setminus \beta))$. This is no greater than $F(z_0 \cap \beta) \cup (z_0 \setminus \beta)$ by monotonicity. But $F(z_0 \cap \beta) \cup (z_0 \setminus \beta) = F(z_0) < \beta = q(0)$, hence we have that $K(q) = 0$, and so K is constantly 0 on $[z]^\kappa$.

We now define a fourth partition $I : [z \setminus \{z(0)\}]^\kappa \to z(0)$ by

$$I(r) = F((y \cap z(0)) \cup r).$$

Using $\kappa \to (\kappa)^\kappa_{z(0)}$, we can find $s \in [z \setminus \{z(0)\}]^\kappa$ homogeneous for I. It follows that $H((y \cap z(0)) \cup s) = 0$ and hence the range of H on $[y]^\kappa$ is $\{0\}$ and the set $[y]^\kappa$ has the promised property.

Extending this to a measure-one set with the same property requires the following lemma:

Lemma 2.1 *If F is any partition from $[\kappa]^\kappa$ to α, $\alpha < \kappa$, then we can find a set of ν-measure one on which F is constant.*

Proof of Lemma 3.1: Given F, we define an auxillary partition G_F defined by

$$G_F(p) = F(_\omega p).$$

Using $\kappa \to (\kappa)^\kappa_\alpha$, let $r \in [\kappa]^\kappa$ be homogeneous for G_F. Then F is constant on $\langle r \rangle$, a ν-measure one set. ∎

We apply Lemma 2.1 to the partitions G, H, I, and K. For G, H, and K, the range of the partition was forced to be 0. The same is true for F_G, F_H, and F_K since any set of the form $\langle r \rangle$ contains $[_\omega r]^\kappa$, an ordinary homogeneous set. Proposition 2.2 follows. ∎

3   Arithmetic

In this section we make some choices. They are motivated partly by the analogy we are building and partly by the results we will obtain in later sections.

The definition of continuity suggests that $p \in [\kappa]^\kappa$ is small if $p(0)$ is large. We consequently define:

Definition 3.1 *For $p, q \in [\kappa]^{\leq \kappa}$, we will say $p \prec q$ iff $(p \triangle q)(0) \in q$.*

Note that for $\alpha, \beta < \kappa$, $\alpha < \beta \Leftrightarrow \{\beta\} \prec \{\alpha\}$.

Fact 3.1 *$\prec$ is a linear order on $[\kappa]^{\kappa}$.*

Our continuum, $[\kappa]^{\kappa}$, has a greatest element, κ, but no least element. For simplicity, we will aim for an analogy between $[\kappa]^{\kappa}$ and the half-open interval, $(0, 1]$. Later, when integrating, we will expand to 2^{ω}, and by analogy, $[0, 1]$.

For reals r and s, r is close to s iff $|r - s|$ is small. In $[\kappa]^{\kappa}$ two members p and q are close if $p \triangle q$ is $\prec$-small. This suggests that the role of subtraction in $[\kappa]^{\kappa}$ should be played by $\triangle$, the symmetric difference. Since the inverse of $\triangle$ is $\triangle$ itself, $\triangle$ will serve as both $+$ and $-$. For example, the increment of a real function, $f : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$, is written $\Delta f = f(x + \Delta x) - f(x)$, we will write the increment of $F : [\kappa]^{\kappa} \to [\kappa]^{\kappa}$ with $x, \Delta x \in [\kappa]^{\kappa}$, as $\Delta F = F(x \triangle \Delta x) \triangle F(x)$.

Fact 3.2 *$\triangle$ is commutative and associative.*

Choosing multiplication is trickier. We can't expect the usual arithmetic laws to hold, but we will need the relationship with $\prec$ to make sense. In particular, we will want $p \prec q \Rightarrow rp \prec rq$ and $pq \prec p, q$. These are both accomplished by composition.

Notation 3.1 *For $p, q \in [\kappa]^{\kappa}$ we will write pq for $p \circ q$.*

Fact 3.3 *Composition is associative but not commutative. The distributive law holds one way, $p(q \triangle r) = (pq) \triangle (pr)$, but $(q \triangle r)p$ does not in general equal $qp \triangle rp$.*

Division is problematic. We'll finesse it when we can.

Fact 3.4 *For $a, b, c \in [\kappa]^{\kappa}$:*

1. $a(bc) = (ab)c$
2. $a \prec b$ implies $ac \prec b$
3. $a \prec b$ implies $ca \prec b$ and $ca \prec cb$
4. $(a \triangle b) \prec c$ implies $(ad \triangle bd) \prec c$.
5. $a, c \prec b$ implies $(a \triangle c) \prec b$.
6. $ab \prec a, b$

4 Differentiation

Given $F : [\kappa]^{\kappa} \to [\kappa]^{\kappa}$, we want to define $F' : [\kappa]^{\kappa} \to [\kappa]^{\kappa}$.

In real analysis, $f'(x)$ satisfies $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{h} = f'(x)$, that is,

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall \Delta x \; 0 < |\Delta x| < \delta \Rightarrow \left| \frac{f(x+\Delta x)-f(x)}{\Delta x} - f'(x) \right| < \varepsilon.$$

We rewrite this avoiding division:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall |\Delta x| < \delta \;\; |f(x+\Delta x) - f(x) - f'(x)\Delta x| < \varepsilon |\Delta x|.$$

In the context of $F : [\kappa]^\kappa \to [\kappa]^\kappa$ and $x \in [\kappa]^\kappa$, we might write instead

$$\forall \varepsilon \; \exists \delta \; \forall \Delta x \prec \delta \;\; F(x \triangle \Delta x) \triangle F(x) \triangle F'(x)\Delta x \prec \varepsilon \Delta x.$$

This will be our definition except that in the manner of Proposition 2.1 we will restrict Δx to a ν-measure one subset of subsets of x.

Definition 4.1 *For $F : [\kappa]^\kappa \to [\kappa]^\kappa$ and $x \in [\kappa]^\kappa$, a derivative of F at x is a set $D \in [\kappa]^\kappa$ with the property that for all $\varepsilon \in [\kappa]^\kappa$, there is an $\delta \in [\kappa]^\kappa$ such that for all $h \in \langle \delta \rangle$,*
$$F(x \triangle xh) \triangle F(x) \triangle Dxh \prec \varepsilon xh.$$

If $D_1 x \sim D_2 x \bmod \mu_\omega$ whenever D_1, D_2 are derivatives, we say that F is differentiable at x; we will write $F'(x)$ for any derivative of F at x.

Proposition 4.1 *If F has a derivative then F is differentiable.*

Proof: Suppose that D_1 and D_2 are derivatives of F at x. Let δ_1, δ_2 witness this fact, i.e., for all $h \in \langle \delta_i \rangle$, $F(x \triangle xh) \triangle F(x) \triangle D_i xh \prec \varepsilon xh$, $i = 1, 2$. Assume that $D_1 x$ and $D_2 x$ are not equivalent, that is, for some ω-closed $y \in [\kappa]^\kappa$, $D_1 x(\alpha) \neq D_2 x(\alpha)$ for all $\alpha \in y$. Choose $\varepsilon \in [\kappa]^\kappa$ so that $\varepsilon x(\alpha) > D_1 x(\alpha), D_2 x(\alpha)$ for all α. Then for $h \in \langle y \rangle \cap \langle \delta_1 \rangle \cap \langle \delta_2 \rangle$, $D_1 xh(0) \neq D_2 xh(0)$, so one of the sets $F(x \triangle xh) \triangle F(x) \triangle D_i xh$, $i = 1, 2$ contains an element below $\varepsilon xh(0)$. This contradicts that δ_1, δ_2 witness that D_1 and D_2 are derivatives. $\blacksquare$

Fact 4.1   *Let $c \in [\kappa]^\kappa$ be fixed.*

1. The derivative of $F(x) = c$ is $F'(x) = \emptyset$
2. The derivative of $F(x) = x$ is $F'(x) = \kappa$.
3. The derivative of $F(x) = cx$ is $F'(x) = c$.
2. The derivative of $G(x) = F(x) \triangle c$ is $G'(x) = F'(x)$.

For the last of these, note that $c(p \triangle ph) \triangle cp \triangle cph = cp \triangle cph \triangle cp \triangle cph = \emptyset$ by the one-way distributive law.

Proposition 4.2 *If F is differentiable and $c \in [\kappa]^\kappa$, then $G = cF$ is differentiable and $G'(x) = cF'(x)$.*

Proof: From $F(p \triangle ph) \triangle F(p) \triangle F'(p)ph \prec \varepsilon ph$, from 3.4 (3) and 3.3 we have $cF(p \triangle ph) \triangle cF(p) \triangle cF'(p)ph \prec \varepsilon ph$. ∎

We do not have that multiplication on the right by a constant preserves differentiability. For c, the set of the even ordinals, for example, $F(x) = xc$ is not differentiable.

And there is no sum rule. For $F(x) = x + 17 = \{\beta + 17\}_{\beta \in x}$, $F'(x) = \kappa + 17$.

$F(x) = \{x(\alpha), x(\alpha + 1)\}_{\alpha \in \kappa}$ is not differentiable at any $p \in [\kappa]^\kappa$. Suppose D is a possible derivative. Choose $\varepsilon \in [\kappa]^\kappa$ so that $\varepsilon(\alpha) > \alpha + 2$ for all α. Given $\delta \in [\kappa]^\kappa$, choose $h \in \langle \delta \rangle$ so that $Dxh(1) > \varepsilon xh(0)$. Let $\alpha = xh(0)$. Then below $\varepsilon xh(0)$, $F(x \triangle xh) \triangle F(x)$ will contain both α and $\alpha + 1$, but Dxh will have at most one element, $Dxh(0)$, so $F(x \triangle xh) \triangle F(x) \triangle Dxh$ will not be $\prec \varepsilon xh$.

Proposition 4.3 *For all $n < \omega$, $F(x) = x^n$ and $x \in \langle \kappa \rangle$, $F'(x) = x^{n-1}$.*

Proof: First, for $n = 2$, $F(x) = x^2$, let $D = x$ and suppose we are given $\varepsilon \in [\kappa]^\kappa$. Choose any $\delta \in [\kappa]^\kappa$. For any $h \in \langle \delta \rangle$, let $q = x \triangle xh$ and consider where x and q differ. Clearly, $x(\xi) \neq q(\xi)$ if $\xi = h(\alpha)$ or if $\xi = h(\alpha) + n$, $n < \omega$. But if $h(\alpha + 1) > h(\alpha) + \omega$, then $x(\xi + \omega) = q(\xi + \omega)$. Altogether, we have that x and q differ only at $\xi + n$ for $\xi \in h$.

Next, we claim that $xq = q^2$. Suppose x and q differ at $q(\alpha)$. Then $q(\alpha) = \xi + n$, $\xi \in h$, $n \in \omega$. But $h \subseteq (\delta)_\omega \subseteq (\kappa)_\omega$, $q \subseteq x$, and $x \cap (x)_\omega = \emptyset$ (see Def. 1.4), so q and h are disjoint, a contradiction.

From this it follows that $q^2 \triangle x^2 = xq \triangle x^2 = x(q \triangle x) = x(xh) = x^2h$ so that $F(x \triangle xh) \triangle F(x) \triangle Dxh = \emptyset \prec \varepsilon xh$, so $F'(x) = x$.

We can show by an easy induction that $x^k \triangle q^k = x^k h$, for $q = x \triangle xh$ and the result follows. ∎

Differentiability forces some homogeneity:

Proposition 4.4 *If F is supercontinous on A and differentiable at $x \in A$ then for almost every α, $F(x) \triangle F(x \triangle \{x(\alpha)\})$ contains at most one point.*

Proof: If not, say $\mu_\omega(X) = 1$ and $\alpha \in X$ implies there are $\xi_\alpha, \gamma_\alpha \in F(x) \triangle F(x \triangle \{x(\alpha)\})$, $\xi_\alpha \neq \gamma_\alpha$. Let D be the derivative of F at x. Take $\varepsilon \in [\kappa]^\kappa$ such that $\varepsilon x(\alpha) > \xi_\alpha, \gamma_\alpha$ for all $\alpha \in X$. Then no matter what δ is, a sufficiently thin h has the property that $h \subseteq X$, $Dxh(1) \geq xh(1) > \varepsilon xh(0)$. Then Dxh has only one point below $\varepsilon xh(0)$,

but $F(x) \triangle F(x \triangle \{x(h(0))\})$ will have two points below $\varepsilon x h(0)$, namely $\xi_{h(0)}$ and $\gamma_{h(0)}$. Furthermore, $x \triangle \{x(h(0))\}$ and $x \triangle xh$ are the same below $xh(1)$, hence $F(x \triangle \{x(h(0))\})$ and $F(x \triangle xh)$ are the same below $xh(1)$, hence they are the same below $\varepsilon x h(0)$, so $F(x \triangle xh) \triangle F(x) \triangle Dxh$ contains at least one of $\xi_{h(0)}$, $\gamma_{h(0)}$ and so fails to be less than $\varepsilon x h$. ■

Proposition 4.4 shows that $F(x) = (\kappa \triangle x)^2$ is not differentiable and so there is no simple Chain Rule. Consider:

$$
\begin{aligned}
& F(x) \triangle F(x \triangle \{x(\alpha)\}) \\
= \; & (\kappa \triangle x)^2 \triangle (\kappa \triangle x \triangle \{x(\alpha)\})^2 \\
= \; & (\kappa \triangle x)\kappa \triangle (\kappa \triangle x)x \triangle \\
& \quad (\kappa \triangle x \triangle \{x(\alpha)\})\kappa \triangle (\kappa \triangle x \triangle \{x(\alpha)\})x \triangle (\kappa \triangle x \triangle \{x(\alpha)\})\{x(\alpha)\} \\
= \; & \kappa \triangle x \triangle (\kappa \triangle x)x \triangle \\
& \quad \kappa \triangle x \triangle \{x(\alpha)\} \triangle (\kappa \triangle x \triangle \{x(\alpha)\})x \triangle (\kappa \triangle x \triangle \{x(\alpha)\})\{x(\alpha)\} \\
= \; & (\kappa \triangle x)x \triangle \{x(\alpha)\} \triangle (\kappa \triangle x \triangle \{x(\alpha)\})x \triangle (\kappa \triangle x \triangle \{x(\alpha)\})\{x(\alpha)\}
\end{aligned}
$$

For ν-measure-one many x, $\alpha + x(\alpha) = x(\alpha)$ and $|(\kappa \triangle x) \cap \alpha| = \alpha$, so we have $(\kappa \triangle x \triangle \{x(\alpha)\})\{x(\alpha)\} = \{x(\alpha)\}$, leaving us with

$$
F(x) \triangle F(x \triangle \{x(\alpha)\}) = (\kappa \triangle x)x \triangle (\kappa \triangle x \triangle \{x(\alpha)\})x.
$$

Then $(\kappa \triangle x)x = \{x(\delta) + 1\}_\delta$ and $(\kappa \triangle x \triangle \{x(\alpha)\})x = \{x(\delta) + 1\}_{\delta \neq \alpha} \cup \{x(\alpha)\}$, so $F(x) \triangle F(x \triangle \{x(\alpha)\}) = \{x(\alpha) + 1, x(\alpha)\}$. By 4.4 then, F is not differentiable.

While there is no Chain Rule, we do have the odd fact that for differentiable F, $(F(x)x)' = F(x)$.

We can prove the converse to Proposition 4.4:

Proposition 4.5 *Let F be supercontinuous on $\langle h \rangle$, $x \in \langle h \rangle$, $x \in [\kappa]^\kappa$. Then if $F(x) \triangle F(x \triangle \{x(\alpha)\})$ contains at most one point for μ_ω-measure-one many α then F is differentiable at x.*

Proof: Given $x \in [\kappa]^\kappa$, if for almost every α, $F(x) \triangle F(x \triangle \{x(\alpha)\})$ is empty, then $F'(x) = \emptyset$ as in Fact 4.1. Otherwise, let A be a μ_ω-measure 1 set of limit ordinals such that for all $\alpha \in A$ $F(x) \triangle F(x \triangle \{x(\alpha)\})$ is a singleton. Define $g : A \to \kappa$ by $\{g(\alpha)\} = F(x) \triangle F(x \triangle \{x(\alpha)\})$. Using Fact 1.3 we can find $B \subseteq A$, $\mu_\omega(B) = 1$, such that $\alpha_1, \alpha_2 \in B$, $\alpha_1 < \alpha_2$ imply that $x(\alpha_2) > g(\alpha_1)$. This allows the construction of $D \in [\kappa]^\kappa$ such that for all $\alpha \in B$ $D(x(\alpha)) = g(\alpha)$.

Claim: $F'(x) = D$. Proof of Claim: Given $\varepsilon \in [\kappa]^\kappa$, take $\delta \in [\kappa]^\kappa$ with $(\delta)_\omega \subseteq B$. Then for $h \in \langle \delta \rangle$, $xh(1) > \varepsilon x h(0)$. By supercontinuity, $F(x) \triangle F(x \triangle xh)$ and $F(x) \triangle F(x \triangle \{xh(0)\})$ are the same below $xh(1)$, hence they are the same below $\varepsilon x h(0)$. Thus, below $\varepsilon x h(0)$,

$$
\begin{aligned}
[F(x) \triangle F(x \triangle xh) \triangle Dxh] \cap \varepsilon x h(0) \; & = \; [\{gxh(0)\} \triangle Dxh] \cap \varepsilon x h(0) \\
& = \; [\{Dxh(0)\} \triangle Dxh] \cap \varepsilon x h(0) \\
& = \; \emptyset,
\end{aligned}
$$

again, since $Dxh(1) > \varepsilon xh(0)$. This gives us $F(x) \triangle F(x \triangle xh) \triangle Dxh \prec \varepsilon xh$.
∎

There is a function which is equal to its own derivative. Let $c = \{\alpha + 1 : \alpha < \kappa\}$ and consider $F(x) = x \triangle c$. From 4.1, $F'(x) = \kappa$, but we can also say $F'(x) = F(x)$ as follows: For any $\delta \in [\kappa]^\kappa$, $x \in \langle \delta \rangle$, x consists only of limit ordinals, so

$$F(x \triangle \{x(\alpha)\}) \triangle F(x) = \{x(\alpha)\}.$$

But below any limit ordinal λ, there are λ-many successors. Thus, $F(x)(x(\alpha)) = \{x(\alpha)\}$, so
$$F(x \triangle \{x(\alpha)\}) \triangle F(x) \triangle F(x)(x(\alpha)) = 0,$$

so $F'(x) = F(x)$.

5   Integration

What should be the area below a function F? For a constant function $F(x) = c$ on an interval $[a, b]$, $\int_a^b F(x)\,dx$ should be simply c times the difference between a and b, that is, $c(a \triangle b)$. This works well. We have easily, for example: $\int_a^b c\,dx + \int_b^d c\,dx = \int_a^d c\,dx$.

Our plan will be to define integrals of the form $\int_\emptyset^b F(x)\,dx$, then set $\int_a^b F(x)\,dx = \int_\emptyset^b F(x)\,dx \triangle \int_\emptyset^a F(x)\,dx$. We will use sums over subintervals delineated by consecutive elements of b (as when we defined F', we only used subsets of x for Δx):

$$[b \setminus b(\alpha + 1), b \setminus b(\alpha)].$$

Peculiar things happen, however, and any function less trivial than a constant function can produce surprises. Consider $F(x) = x$. The width of $[b \setminus b(\alpha + 1), b \setminus b(\alpha)]$ is $(b \setminus b(\alpha + 1)) \triangle (b \setminus b(\alpha)) = \{b(\alpha)\}$. On $[b \setminus b(\alpha + 1), b \setminus b(\alpha)]$ the greatest value of F is $b \setminus b(\alpha)$ and the least value of $b \setminus b(\alpha + 1)$. Assuming the ordinals of b are indecomposable (the indecomposable ordinals form a μ_ω-1 set—δ is indecomposable if $\gamma, \zeta < \delta \Rightarrow \gamma + \zeta < \delta$), we will have $(b \setminus b(\alpha))(b(\alpha)) = (b \setminus b(\alpha+1))(b(\alpha)) = b^2(\alpha)$. This suggests that

$$b^2(\alpha)) \leq \int_{b \setminus b(\alpha+1)}^{b \setminus b(\alpha)} x\,dx \leq b^2(\alpha).$$

But if we are composing a Riemann sum we could find $e \in [b \setminus b(\alpha+1), b \setminus b(\alpha)]$ with $e(b(\alpha)) < b^2(\alpha)$.

Indeed, nothing is secure in the usual sense here. The largest value of $F(c)(b \triangle a)$ for $c \in [a, b]$ may not involve using the largest $F(c)$. Other odd things can happen. Nonetheless, a respectable definition is feasible.

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For a first approximation we can use a $\triangle$ -sum using the right-endpoint rule:

$$\int_\emptyset^b F(x)\,dx \approx \triangle_{\delta<\kappa} F(b \setminus b(\delta))b(\delta).$$

We will need:

Definition 5.1 *For any collection,* $\{A_\delta\}_{\delta<\alpha},\ \alpha \leq \kappa,$

$$\triangle_{\delta<\alpha} A_\delta = \{\beta : |\{\delta : \beta \in A_\delta\}| \text{ is an odd ordinal}\}.$$

We could subdivide the intervals further, breaking each $[b \setminus b(\alpha+1), b \setminus b(\alpha)]$ into

$$\{[\{b(\alpha)\} \cup (b \setminus b(\delta+1)), \{b(\alpha)\} \cup (b \setminus b(\delta))]\}_{\alpha<\delta<\kappa}$$

and using the $\triangle$ -sum:

$$\triangle_{\alpha<\delta<\kappa} F(\{b(\alpha)\} \cup (b \setminus b(\delta)))b(\delta),$$

but this doesn't lead to a meaningful definition of the integral. The βth approximation would be

$$\triangle_{\substack{\sigma \subseteq \delta < \kappa \\ \text{o.t.}(\sigma) = \beta}} F(\sigma \cup (b \setminus b(\delta)))b(\delta),$$

and the limit of these sums is $\emptyset$ for any F since the least element of the βth approximation is greater than β. Consequently, we take the first approximation as the last approximation and define

Definition 5.2 *For any* $F : [\kappa]^\kappa \to [\kappa]^\kappa,\ b \in [\kappa]^\kappa,$

$$\int_\emptyset^b F(x)\,dx = \triangle_{\delta<\kappa} F(b \setminus b(\delta))b(\delta).$$

We let $\int_a^b F(x)\,dx = \int_\emptyset^b F(x)\,dx \triangle \int_\emptyset^a F(x)\,dx$. We will say G is a primitive of F if for some $r \in [\kappa]^\kappa$ and all $a, b \in \langle r \rangle$, $\int_a^b F(x)\,dx = G(b) \triangle G(a)$.

With this definition, every function is integrable; every function has a primitive. This is appropriate. In real analysis all continuous functions are integrable and from [HM], all our functions will be continuous.

If G is a primitive of F then so is $G \triangle C$ for any constant C. The reverse is true on a ν-measure one set so we can unambiguously write $\int F(x)\,dx = G(x) \triangle C$.

Fact 5.1

1. $\int h\,dx = hx \triangle C$
2. $\int x^n\,dx = x^{n+1} \triangle C$
3. $\int (x \triangle \kappa)\,dx = (x+1) \triangle C$

The function on page 7, $F(x) = \{x(\alpha), x(\alpha+1)\}_{\alpha \in \kappa}$, which was not differentiable has as integral, $\int F(x)\,dx = x^2 \triangle C$.

For functions with a stronger type of continuity, the integral has a special form.

Definition 5.3 *A function F is superdupercontinuous on A if for all $x, y \in A$, $\alpha \in x$,*

$$(x \triangle y) \subseteq \alpha \Rightarrow F(x)(\alpha) = F(y)(\alpha).$$

Proposition 5.1 *For every n, $F(x) = x^n$ is superdupercontinuous on a ν-measure one set.*

Proof: Choose $\langle r \rangle$ so that for all α, $r(\alpha+1) > r(\alpha) \cdot \omega$. Then it will be the case that $\beta < \alpha \in r$ implies that $\beta + \alpha = \alpha$. Then for any $x, y \in \langle r \rangle$, $(x \triangle y) \subseteq \alpha \in x$, we have $x \cap \alpha = x \restriction \beta$ for some $\beta < \alpha$, so $x(\alpha) = (x \setminus x(\alpha))(\alpha)$, and similarly for y, and since $(x \setminus x(\alpha)) = (y \setminus y(\alpha))$ we get $x(\alpha) = y(\alpha)$. We can extend this inductively to x^n. $\blacksquare$

All the functions discussed so far have been superdupercontinuous (and supercontinuous) but if we take a partition $\{A_\delta\}_{\delta < \kappa}$ of κ into disjoint members of $[\kappa]^\kappa$, then $F(x) = A_{x(\emptyset)}$ is not superdupercontinuous.

Proposition 5.2 *If F is superdupercontinuous, then*

$$\int F(x)\,dx = F(x)x \triangle C.$$

Proof: $\int_{\emptyset}^{b} F(x)\,dx = \underset{\delta < \kappa}{\triangle} F(b \setminus b(\delta))(b(\delta)) = \underset{\delta < \kappa}{\triangle} F(b)b(\delta) = F(b)b.$ $\blacksquare$

6 Antidifferentiation

We can prove a sort of "Fundamental Theorem," although that would be far too grand a title for it. First, we prove

Proposition 6.1 *If $\int F(x)\,dx = G(x) \triangle C$, then G is differentiable.*

Proof: We prove a more general statement. Any function of the form: $H(x) = \underset{\delta<\kappa}{\triangle}\, A_{x(\delta)}(x(\delta))$ is differentiable. This follows from the fact that $H(x) \,\triangle\, H(x \,\triangle\, \{x(\delta)\}) = \{H(A_{x(\delta)}(x(\delta)))\}$, a singleton. ■

Proposition 6.2 *If F is supercontinuous and superdupercontinuous on A, then F has an antiderivative and for all $x \in A$,*

$$\left(\int F(x)\,dx\right)' = F(x).$$

Proof: Define G to be the following primitive of F:

$$G(x) = \underset{\delta<\kappa}{\triangle}\, F(x \setminus x(\delta))(x(\delta)).$$

Let $r \in [\kappa]^{\kappa}$ witness the superdupercontinuity of F and take $x \in \langle r \rangle$. Choose $s \in [\kappa]^{\kappa}$ such that for $\alpha \in s$ $F(x) \cap x(\alpha)$ has order-type $x(\alpha)$. Let h be in $\langle s \rangle$. By superdupercontinuity, $F(x \setminus x(\delta))(x(\delta)) = F(x)(x(\delta))$. Thus, $G(x)(\delta) = F(x)(x(\delta))$. Then for $\beta < h(0)$,

$$
\begin{aligned}
G(x \,\triangle\, xh)(\beta) &= F(x \,\triangle\, xh)((x \,\triangle\, xh)(\beta)) \\
&= F(x)((x \,\triangle\, xh)(\beta)) && \text{(supercontinuity and choice of } s) \\
&= F(x)(x(\beta)) \\
&= G(\beta).
\end{aligned}
$$

And if $\beta = h(0)$,

$$
\begin{aligned}
G(x \,\triangle\, xh)(\beta) &= F(x \,\triangle\, xh)((x \,\triangle\, xh)(\beta)) \\
&= F(x \,\triangle\, xh)(x(\beta+1)) \\
&= F(x \setminus \{xh(0)\})(x(\beta+1)) && \text{(supercontinuity)} \\
&= F(x)(x(\beta+1)) && \text{(superdupercontinuity)} \\
&= G(\beta+1).
\end{aligned}
$$

Thus, the least element of $G(x \,\triangle\, xh) \,\triangle\, G(x)$ is $G(x)(h(0)) = F(x)(x(h(0))) = F(x)(xh(0))$.

We can similarly show that the next least element of $G(x \,\triangle\, xh) \,\triangle\, G(x)$ is $F(x)xh(1)$ and in general that

$$G(x \,\triangle\, xh) \,\triangle\, G(x) \,\triangle\, F(x)xh = \emptyset,$$

so $G' = F$. ■

We have one last result, one that mirrors real analysis and suggests that superdupercontinuous functions may be the analog of analytic functions.

Proposition 6.3 *If F is superdupercontinuous, then so is $\int F(x)\,dx$.*

Proof: Let $G(x) = \int F(x)\,dx = \underset{\delta<\kappa}{\triangle}\, F(x\setminus x(\delta))(x(\delta)) = \underset{\delta<\kappa}{\triangle}\, F(x)(x(\delta))$. Fix α and let $q = x\setminus x(\alpha)$ and $\beta = x(\alpha)$. Note that we can easily compel x to be such that $x(\alpha) + x(\beta) = x(\beta)$, so $q(\beta) = x(\beta) = x^2(\alpha) = q(x(\alpha))$. Note also that $x\setminus x^2(\alpha) = q\setminus x^2(\alpha) = q\setminus q(\beta)$. Then

$$
\begin{aligned}
G(x\setminus x(\alpha))(x(\alpha)) &= G(q)(x(\alpha)) \\
&= F(q)(q(x(\alpha)) \\
&= F(q)(q(\beta)) \\
&= F(q\setminus q(\beta))(q(\beta)) \\
&= F(x\setminus x^2(\alpha))(x^2(\alpha)) \\
&= F(x)(x^2(\alpha)) \\
&= G(x)(x(\alpha).
\end{aligned}
$$

∎

7 Questions

1. The definitions of derivative and integral are on probation. Are more appropriate or more fruitful definitions possible?

2. How special is superdupercontinuous? Can we prove a limited form, for example, for functions satisfying the Fundamental Theorem?

3. To what extent are differentiable functions locally linear? We can prove that for $F(x) = x^n$, F is linear at almost every p in the sense that $F(x) = F(p)\triangle F'(p)(p\triangle x)$, for $x \in [p]^\kappa$. In general, if F is differentiable on $\langle r\rangle$, $p \in \langle r\rangle$, we can prove $F(x) = F(p)\triangle F'(p)(p\triangle x)$ if $p\triangle x$ is a finite subset of x.

4. Is an interesting theory of series possible with this or any other definition of derivative?

8 References

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