Localization in Relativistic Quantum Theories

L. O. Herrmann*

Abstract

Although the concept of localization in non-relativistic quantum mechanics is mathematically well-defined there is no obvious and unambiguous way to generalize it to relativistic quantum theory. After a brief review of localization in quantum mechanics the Newton-Wigner localization scheme is introduced and it is shown by the example of a massive spinless system how it leads to a position operator with seemingly unphysical properties. This motivated the development of several theorems which claim to rule out the existence of localizable particles. A particularly important one of them is presented and its validity discussed. In the last section however it is shown that if localization is considered with respect to spacelike hyperplanes, the properties of the Newton-Wigner position operators are not necessarily unphysical and localizable particles can exist.

Contents

I would like to thank my supervisor, Dr Jeremy Butterfield, for his helpful comments and support during the writing of this essay.

1 Introduction

By the end of the nineteenth century many scientists believed the universe to be totally deterministic, i.e. that the state of the universe at one instant would entirely define its future and past evolution. This point of view was primarily motivated by the progress in classical mechanics and the discovery of Maxwell's equations. These achievements made it tempting to model the universe as a collection of massive bodies which evolve under gravitational and electromagnetic forces, the

*Robinson College, Grange Road, Cambridge CB3 9AN, UK, email: loh20@cam.ac.uk

former determined by the masses of the bodies and the later by fields obeying Maxwell's equations. Knowledge of the initial position and momentum of all bodies and their masses as well as the initial field configuration would then be sufficient to calculate the future development of the universe.

But already during the nineteenth century several observations were made which indicated that this can not be the whole story. As an example one may think of the photoelectric effect, namely the emission of electrons from matter under an incident electromagnetic field. Although this effect can be explained without quantization of the electromagnetic field [1, p. 11] it was historically important for substantiating the idea of photons. But it was mainly early twentieth century observations of small-scale phenomena which required a new theory, viz. quantum mechanics. The notion of a particle with a well-defined position and momentum had to be replaced by a wave-function $|\psi\rangle$ which obeys Schrödinger's equation (setting $\hbar = 1$; cf. the appendix for notational conventions)

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle \tag{1}$$

and therefore evolves deterministically. But the square of the wave-function merely gives the probability density to find the particle in a certain position or state of motion. And clearly this probabilistic description torpedoes the idea that the universe could be deterministic. Moreover in quantum mechanics position and momentum have lost their fundamental status as dynamical variables. Nevertheless they are still observables which are measured in the laboratory and therefore need to be represented in the theory. In the case of the position observable this is accomplished by associating with it a hermitian operator X with a purely continuous spectrum $\sigma_c(X) = \mathbb{R}^3$ whose (improper) eigenfunctions form an orthonormal basis $\{|\vec{x}\rangle : \vec{x} \in \mathbb{R}^3\}$ of the state space of the system, i.e.

$$\langle\vec{x}|\vec{x}'\rangle = \delta^{(3)}(\vec{x} - \vec{x}'), \qquad \int d^3x\,|\vec{x}\rangle\langle\vec{x}| = 1. \tag{2}$$

The action of this position operator on an arbitrary wave-function in position space lying in the domain of X is then defined by

$$\langle\vec{x}|X|\psi\rangle := \vec{x}\,\langle\vec{x}|\psi\rangle\,. \tag{3}$$

Similarly other observables such as momentum and angular momentum are represented by associating with them hermitian operators P and L respectively.

What happens with the wave-function if such an observable, say the position, is being measured? Before the measurement the position space wave-function $\langle\vec{x}|\psi\rangle$ allows us to predict the probabilities of obtaining the various possible outcomes. However, once having measured the state in a certain volume $V \subset \mathbb{R}^3$ the wave-function immediately after the measurement needs to have compact support in this volume, and thus has to have collapsed to a different wave-function given by the measurement postulate

$$\langle\vec{x}|\psi'\rangle = \frac{\langle\vec{x}|P_V|\psi\rangle}{\sqrt{\langle\psi|P_V|\psi\rangle}}, \tag{4}$$

where P_V is the projection operator onto the eigenspace associated with the volume V, i.e.

$$P_V = \int_V d^3x\,|\vec{x}\rangle\langle\vec{x}|\,. \tag{5}$$

A further remarkable difference between classical mechanics and quantum mechanics are the canonical commutation relations between the components of the position and momentum operators

$$[X^i, P_j] = i\delta^i_j, \qquad [X^i, X^j] = 0, \qquad [P_i, P_j] = 0. \tag{6}$$

which are needed to account for the observation that certain observables take on only a discrete number of values; e.g. the energy of an electron in a hydrogen atom and its angular momentum are both quantized. Heisenberg has shown that these commutation relations lead to an uncertainty principle for the position and momentum operators

$$\langle X \rangle \langle P \rangle \gtrsim 1, \tag{7}$$

where $\langle X \rangle$ and $\langle P \rangle$ denote the standard deviation of position and momentum respectively. As a consequence it is impossible to know both position and momentum of a particle at the same time to an arbitrary precision which abolishes once and for all the special role these two observables have had in classical mechanics.

At this point it has to be mentioned that quantum mechanics can technically be divided into a well-defined mathematical framework and an interpretation which connects the mathematical formulation with the experiment. The interpretation of quantum mechanics given above is part of what is known as the Copenhagen interpretation. Although it is nowadays the most widely-accepted interpretation of quantum mechanics, other interpretations have been developed, not least because the measurement postulate (4) with its predicted wave-function collapse remains controversial. But the important point is that within the mathematical framework of quantum mechanics the concept of position is well-defined and unambiguous, although its consequences are admittedly not very intuitive for us macroscopic beings.

At the beginning of the twentieth century a second major revolution in physics took place: the birth of the theory of relativity. However, quantum mechanics as described above is not compatible with relativity, mainly because the Schrödinger equation (1) is not relativistically invariant. Consequently there were several attempts to combine concepts from relativity with quantum mechanics out of which relativistic quantum mechanics and quantum field theory grew, the later one pushed forward by the need to find a quantum theory for the electromagnetic field. But unfortunately it was exactly the concept of localization which proved very difficult to carry over to a relativistic quantum theory; a concept so heavily and successfully used in experimental physics.

In 1949 Newton and Wigner tried to tackle this problem systematically by writing down the postulates which in their eyes were necessary and sufficient to characterize localization. On the one hand their postulates turned out to be very compelling in the sense that they give rise to a unique position operator for every massive system of arbitrary spin and for every massless system of either spin 0 or $\frac{1}{2}$. On the other hand the eigenstates of these position operators have strange and unpleasant properties, i.e. they propagate superluminally and are only localized for special inertial observers. Whilst the former property raised concern that these states could be used to signal superluminally and thus generate acausal behaviour, the later property interferes with the principle of relativity which requires the physical laws to be equivalent in all inertial frames. But this would certainly not be the case if the wave-function describing a particle could have compact support in one inertial frame but extend to infinity in another. Moreover the Newton-Wigner postulates do not lead to any position operator for massless systems with spin 1 or higher and thereby miss such important particles as the photon.

Out of all these concerns two fundamentally different points of view developed.

- The difficulties can be considered as evidence that strict localization does not exist, and particles are a pure illusion.

- Despite their strange properties, the Newton-Wigner position operators and their eigenstates make physical sense.

In fact there have also been attempts to downplay the issues by claiming that the whole problem is confined to systems with a fixed number of particles, but following Fleming and Butterfield it needs to be emphasized that this is not true: the aforementioned strange properties are equally present in a theory of variable particle numbers such as quantum field theory [2, p. 110–111].

The structure of this essay is as follows. At the beginning it is shown that the superluminal propagation of the position operator eigenstates is already present in non-relativistic quantum mechanics, but not the delocalization under certain spacetime symmetry transformations. Subsequently the localization concept due to Newton and Wigner is introduced and the strange properties of the eigenstates of their position operator are explicitly demonstrated in the example of a massive spinless particle. Thereafter a theorem is presented which supports the point of view that particles are a pure illusion. Finally, the concept of hyperplane-dependent localization is described which shows that, against all the odds, localizable particles are not necessarily unphysical.

2 Localization in non-relativistic Quantum Mechanics

In section 1 the position operator for a particle was introduced by its action on the wave-function of the particle in position space representation. For later convenience it is worth recalling its action in momentum space representation, viz.

$$\begin{aligned}\langle\vec{p}|X|\psi\rangle &= \int d^3x\,\langle\vec{p}|\vec{x}\rangle\,\langle\vec{x}|\vec{X}|\psi\rangle = \int d^3x e^{-i\vec{p}\cdot\vec{x}}\vec{x}\psi(\vec{x})\\ &= i\nabla_{\vec{p}}\int d^3x e^{-i\vec{p}\cdot\vec{x}}\psi(\vec{x}) = i\nabla_{\vec{p}}\psi(\vec{p}).\end{aligned} \tag{8}$$

The probability amplitude for a free particle initially located at $\vec{x}_0$ to propagate to $\vec{x}$ within a time t is

$$\langle\vec{x}|e^{-iH_0t}|\vec{x}_0\rangle = \int\frac{d^3p}{(2\pi)^3}e^{-it\vec{p}^2/2m}e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)}, \tag{9}$$

which after substitution of $\vec{q} := \vec{p} - m(\vec{x}-\vec{x}_0)/t$ becomes

$$= \int\frac{d^3q}{(2\pi)^3}e^{-it\vec{q}^2/2m}e^{im(\vec{x}-\vec{x}_0)^2/2t} = \left(\frac{m}{2\pi it}\right)^{3/2}e^{im(\vec{x}-\vec{x}_0)^2/2t}, \tag{10}$$

an oscillating wave, spread out over all space [3, p. 1989]. Because it does not vanish for arbitrary separations $|\vec{x}-\vec{x}_0|$ the particle can propagate superluminally.

In the following it is shown that a localized wave-function remains localized under the action of the Galilean group which is the largest symmetry group of non-relativistic quantum mechanics leaving scales invariant. As mentioned above this behaviour can not be taken for granted anymore in the Newton-Wigner scheme, and so it may well be worth verifying explicitly that it is true in this case. For this purpose consider two Galilean inertial frames $\mathscr{O}$ and $\mathscr{O}'$ equipped with coordinates $(t,\vec{x})$ and $(t',\vec{x}')$ respectively. Assume a particle localized in $\mathscr{O}$ at position $\vec{x}_0$, i.e. with a wave-function in position space $\langle\vec{x}|\vec{x}_0\rangle = \delta^{(3)}(\vec{x}-\vec{x}_0)$. It is obvious that the localization of the wave-function is not affected by spatial rotations nor by spacetime translations. Under a Galilean boost

$$t \to t' = t, \qquad\qquad \vec{x} \to \vec{x}' = \vec{x} - \vec{v}t. \tag{11}$$

the wave-function transforms into

$$\begin{aligned}\langle\vec{x}'|\vec{x}'_0\rangle &= \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}'}\,\langle\vec{p}+m\vec{v}|\vec{x}_0\rangle = \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{v}t-\vec{x}_0)}e^{-im\vec{v}\cdot\vec{x}_0}\\ &= e^{-im\vec{v}\cdot\vec{x}_0}\delta^{(3)}(\vec{x}-\vec{v}t-\vec{x}_0).\end{aligned} \tag{12}$$

Up to a phase this is the same wave-function as before. Because every wave-function can be expanded in terms of δ-functions the localization regime of an arbitrary wave-function is indeed unaffected by a Galilean transformation [4, p. 104]. It is now time to introduce Newton and Wigner's attempt to reconcile localization with the special theory of relativity.

3 Newton-Wigner Localization

The requirements Newton and Wigner considered as necessary for a system to be localized are summarized in the following postulates.

Newton-Wigner postulates [5, p. 401], [6, p. 1093]
Let $\mathcal{S}$ denote the set of localized states at the origin of a spacetime coordinate system with the following properties

(a) $\mathcal{S}$ is linear, i.e. $a\,|\psi\rangle + b\,|\varphi\rangle \in \mathcal{S}$ for all $|\psi\rangle\,, |\varphi\rangle \in \mathcal{S}$ and for all $a, b \in \mathbb{C}$.

(b) $\langle\psi|T_{\vec{a}}|\psi\rangle = 0$ for all $|\psi\rangle \in \mathcal{S}$ and for all $\vec{a} \neq 0$ where $T_{\vec{a}}$ is the translation operator defined by $T_{\vec{a}}\,|\vec{x_0}\rangle := |\vec{x_0} + \vec{a}\rangle$.

(c) $\mathcal{S}$ is invariant under rotations $R \in O(3)$ and time reflections.

(d) The states $|\psi\rangle \in \mathcal{S}$ obey a mathematical regularity condition which essentially eliminates discontinuous functions from $\mathcal{S}$.

These postulates alone cannot entirely determine the localized states as they do not contain any information about the internal structure of the system. Consequently a requirement on the state space of the system needs to be imposed, namely that it be the carrier space of a single irreducible and unitary representation of the Poincaré group [2, p. 114], [7, p. 524–525]. The state space of a system containing an arbitrary number of particles can always be decomposed into such carrier spaces and the physical system associated with a carrier space is called an elementary system. An elementary particle is then defined to be an elementary system whose states cannot be connected by physical interactions to the states of other systems. As an example the neutron is not an elementary particle because it can be connected to the proton by β-decay. From these definitions it follows that an elementary system is a more general concept than an elementary particle since, for example, the ground state $1s$ of a hydrogen atom forms an elementary system [5, p. 400] but not an elementary particle as it can be connected to other states of the hydrogen atom by photon absorption.

3.1 Newton-Wigner States and their Properties

In the following the Newton-Wigner operator for a massive spin zero system is introduced and its most important properties are discussed. Massive spin zero systems are described by the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0. \tag{13}$$

Writing $\phi(x)$ as a Fourier decomposition one obtains the Klein-Gordon equation in momentum space

$$\left(\partial_t^2 + \omega_{\vec{p}}^2\right)\phi(t, \vec{p}) = 0, \tag{14}$$

where $\omega_{\vec{p}}^2 := \vec{p}^2 + m^2$. The set of positive energy solutions is defined as $\mathfrak{U}_+ := \{\phi(t, \vec{p}) : \omega_{\vec{p}} \geq 0\}$ and a Lorentz-invariant inner product on this set is given by

$$\langle \varphi | \psi \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \, \varphi(\vec{p})^* \psi(\vec{p}), \qquad \varphi, \psi \in \mathfrak{U}_+. \tag{15}$$

The reason for restriction to positive energy solutions is that the negative energy solutions have eigenvalues which are unbounded from below. Therefore an arbitrary amount of energy could be extracted from the system by lowering its energy state further and further. This problem is resolved in quantum field theory by reinterpreting the negative energy solutions as positive energy states of an antiparticle. But by making this restriction to positive energy solutions the consequences of the Newton-Wigner localization derived below will then be present also in a quantum field theory.

The factor of $1/2\omega_{\vec{p}}$ in the definition of the inner product (15) is necessary to make the integration measure Lorentz-invariant. At the same time it prevents the non-relativistic position operator (8) from being used because this one is not hermitian with respect to the Lorentz-invariant inner product

$$\begin{aligned} \langle X\psi | \varphi \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left[-i \nabla_{\vec{p}} \psi^*(\vec{p}) \right] \varphi(\vec{p}) \\ &= \int \frac{d^3 p}{(2\pi)^3} \psi^*(\vec{p}) i \nabla_{\vec{p}} \left[\frac{\varphi(\vec{p})}{2\sqrt{\vec{p}^2 + m^2}} \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \psi^*(\vec{p}) i \left[\nabla_{\vec{p}} - \frac{\vec{p}}{\vec{p}^2 + m^2} \right] \varphi(\vec{p}), \end{aligned} \tag{16}$$

and does therefore not correspond to an observable. From the above calculation however it is not hard to see how it can be turned into a hermitian operator with respect to the Lorentz-invariant inner product, namely by setting

$$X_{nw} := i \left(\nabla_{\vec{p}} - \frac{\vec{p}}{2\omega_{\vec{p}}^2} \right). \tag{17}$$

This is indeed the position operator Newton and Wigner derived from their postulates. The commutation relations of its components then follow from (6)

$$[X_{nw}^i, p_j] = \left[i \frac{\partial}{\partial p_i}, p_j \right] - i \left[\frac{p^i}{2\omega_{\vec{p}}^2}, p_j \right] = i \delta_j^i \tag{18}$$

$$[X_{nw}^i, X_{nw}^j] = -\left[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right] + \left[\frac{\partial}{\partial p_i}, \frac{p^j}{2\omega_{\vec{p}}^2} \right] + \left[\frac{p^i}{2\omega_{\vec{p}}^2}, \frac{\partial}{\partial p_j} \right] - \left[\frac{p^i}{2\omega_{\vec{p}}^2}, \frac{p^j}{2\omega_{\vec{p}}^2} \right] = 0, \tag{19}$$

where in the last line the third term cancels the second and the other two terms vanish individually. A general eigenstate of the Newton-Wigner position operator in momentum space at position $\vec{x}_0$ and time $t = 0$ is

$$\langle \vec{p} | \vec{x}_0 \rangle = \sqrt{2\omega_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}_0}, \tag{20}$$

which can be verified by acting with the Newton-Wigner position operator (17) on this state

$$\langle \vec{p} | X_{nw} | \vec{x}_0 \rangle = i \left(\nabla_{\vec{p}} - \frac{\vec{p}}{2\omega_{\vec{p}}^2} \right) \sqrt{2\omega_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}_0} = \vec{x}_0 \sqrt{2\omega_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}_0} = \vec{x}_0 \langle \vec{p} | \vec{x}_0 \rangle. \tag{21}$$

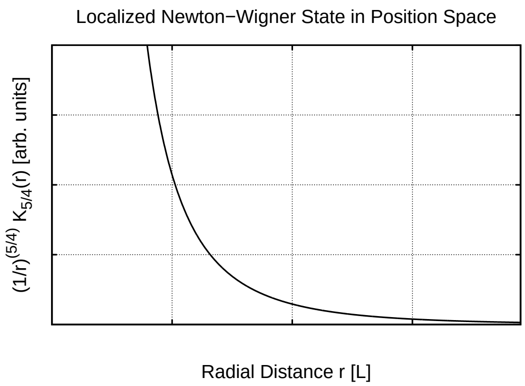


Figure 1: Qualitative form of the Newton-Wigner eigenfunction in position space (22) as a function of radial distance. Because this function is not square integrable it is not normalizable and the square of the amplitude cannot be interpreted as a probability density. Hence the tail as $r \to \infty$ does not have a physical meaning.

The wave-function in position space is obtained from (20) by using the inverse Fourier-transform

$$\begin{aligned} \langle \vec{x} | \vec{x_0} \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \langle \vec{x} | \vec{p} \rangle \langle \vec{p} | \vec{x_0} \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{x_0})}}{\sqrt{2\omega_{\vec{p}}}} \\ &= \text{const} \left(\frac{m}{|\vec{x}-\vec{x_0}|} \right)^{5/4} K_{5/4} \left(\frac{|\vec{x}-\vec{x_0}|}{\lambda_0} \right), \end{aligned} \tag{22}$$

where $\lambda_0 = 1/m$ is the Compton wave length and $K_\nu(z)$ is the modified Bessel function of the second kind [5, p. 402], [8, p. A253]. The qualitative form of this wave-function is plotted in Fig. 1. At first glance it seems as if (22) would not represent a localized particle because it does not have compact support. However, the function is not square integrable and therefore can not be interpreted as a probability density. It is rather the fact that the Newton-Wigner eigenstates satisfy the above postulate (b) which justifies their interpretation as describing localized states. Indeed,

$$\begin{aligned} \langle \vec{x}_0 | \vec{x}_0 + \vec{a} \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \langle \vec{x}_0 | \vec{p} \rangle \langle \vec{p} | T_{\vec{a}} | \vec{x}_0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-i\vec{p}\cdot\vec{a}} \langle \vec{x}_0 | \vec{p} \rangle \langle \vec{p} | \vec{x}_0 \rangle = \delta^{(3)}(\vec{a}) = 0 \quad \forall\, \vec{a} \neq 0, \end{aligned} \tag{23}$$

where $T_{\vec{a}}$ is the translation operator. Nevertheless it would be convenient to have an orthonormal basis of the state space which allows the definition of a position dependent probability density for a state $|\psi\rangle$ in this state space. Fortunately the Newton-Wigner eigenstates $|\vec{x}_0\rangle$ form exactly such a basis.

Proof Orthonormality follows from

$$\langle \vec{x}_0 | \vec{x}_0' \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \langle \vec{x}_0 | \vec{p} \rangle \langle \vec{p} | \vec{x}_0' \rangle = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}_0 - \vec{x}_0')} = \delta^{(3)}(\vec{x}_0 - \vec{x}_0')$$

and closure from

$$\int d^3x_0 \, |\vec{x}_0\rangle \langle \vec{x}_0|\psi\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{d^3x_0}{2\omega_{\vec{p}}\, 2\omega_{\vec{p}'}} \, |\vec{p}\rangle \langle \vec{p}|\vec{x}_0\rangle \langle \vec{x}_0|\vec{p}'\rangle \langle \vec{p}'|\psi\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}\, 2\omega_{\vec{p}'}} (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \sqrt{2\omega_{\vec{p}}} \sqrt{2\omega_{\vec{p}'}} \, |\vec{p}\rangle \langle \vec{p}'|\psi\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \, |\vec{p}\rangle \langle \vec{p}|\psi\rangle = 1\, |\psi\rangle \, . \qquad \square$$

Using the completeness every normalized state $|\psi\rangle$ can then be expanded as [9, p. 64]

$$1 = \langle \psi|\psi\rangle = \int d^3x_0 \, \langle \psi|\vec{x}_0\rangle \langle \vec{x}_0|\psi\rangle = \int d^3x_0 |\psi(\vec{x}_0)|^2 \tag{24}$$

which allows to interpret $|\psi(\vec{x}_0)|^2$ as a probability density.

3.1.1 Superluminal Propagation

How does a Newton-Wigner state evolve in time? Consider a Newton-Wigner state initially localized at $\vec{x}_0'$. The probability amplitude for this state to propagate within a time t to $\vec{x}_0$ is then [10, p. 14]

$$\langle \vec{x}_0|e^{-ip_0t}|\vec{x}_0'\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-it\sqrt{\vec{p}^2+m^2}} e^{i\vec{p}\cdot(\vec{x}_0-\vec{x}_0')}. \tag{25}$$

Rewriting the above integral in spherical coordinates using $\mathfrak{p} := |\vec{p}|$

$$\frac{1}{(2\pi)^3} \int_{\mathfrak{p}=0}^{\infty} d\mathfrak{p}\mathfrak{p}^2 \int_{\varphi=0}^{2\pi} d\varphi \int_{\vartheta=0}^{\pi} d\vartheta \sin\vartheta e^{-it\sqrt{\mathfrak{p}^2+m^2}} e^{i\mathfrak{p}|\vec{x}_0-\vec{x}_0'|\cos\vartheta}. \tag{26}$$

Substituting $f := \cos\vartheta$ and carrying out the integration over df and $d\varphi$ gives

$$\frac{1}{(2\pi)^2 i} \frac{1}{|\vec{x}_0-\vec{x}_0'|} \int_0^{\infty} d\mathfrak{p}\mathfrak{p} e^{-it\sqrt{\mathfrak{p}^2+m^2}} \left(e^{i\mathfrak{p}|\vec{x}_0-\vec{x}_0'|} - e^{-i\mathfrak{p}|\vec{x}_0-\vec{x}_0'|} \right). \tag{27}$$

Using the symmetries of the integrand, the region of integration can be extended to the entire real axis

$$\frac{1}{(2\pi)^2 i} \frac{1}{|\vec{x}_0-\vec{x}_0'|} \int_{-\infty}^{\infty} d\mathfrak{p}\mathfrak{p} e^{i\Phi(\mathfrak{p})} \tag{28}$$

where $\Phi(\mathfrak{p}) := -t\sqrt{\mathfrak{p}^2+m^2} + \mathfrak{p}|\vec{x}_0-\vec{x}_0'|$. Well outside the light cone $|\vec{x}_0-\vec{x}_0'| \gg t$ this integral can be approximated using the method of stationary phase [10, p. 14]. In the following the abbreviation $|\vec{x}| := |\vec{x}_0-\vec{x}_0'|$ is used. The phase Φ has a stationary point at $\mathfrak{p}_S = im|\vec{x}|/\sqrt{|\vec{x}|^2-t^2}$ where it takes on the value $\Phi(\mathfrak{p}_S) = im\sqrt{|\vec{x}|^2-t^2}$. The second derivative of Φ with respect to $\mathfrak{p}$ is

$$\frac{d^2\Phi}{d\mathfrak{p}^2} = -\frac{t}{\sqrt{\mathfrak{p}^2+m^2}} \left(1 - \frac{\mathfrak{p}^2}{\mathfrak{p}^2+m^2} \right), \tag{29}$$

and

$$\left| \frac{d^2\Phi(\mathfrak{p})}{d\mathfrak{p}^2} \right|^2_{\mathfrak{p}=\mathfrak{p}_S} = \frac{|\vec{x}|^2-t^2}{m^2} \left[1 - \frac{|\vec{x}|^2}{t^2} \right]^2 > 0. \tag{30}$$

Hence the matrix element well outside the light cone is apart from a phase approximated by [11, p. 307]

$$\langle \vec{x}_0 | e^{-ip_0 t} | \vec{x}_0' \rangle \simeq \frac{1}{\sqrt{(2\pi)^2}} \frac{mt}{\sqrt{|\vec{x}|^2 - t^2}} \sqrt{\frac{2\pi m}{(|\vec{x}|^2 - t^2)^{3/2}}} e^{-m\sqrt{|\vec{x}|^2 - t^2}}, \tag{31}$$

where $|\vec{x}| := |\vec{x}_0 - \vec{x}_0'|$. The propagation amplitude is therefore dominated by a term of the form

$$\langle \vec{x}_0 | e^{-ip_0 t} | \vec{x}_0' \rangle \propto e^{-m\sqrt{|\vec{x}_0 - \vec{x}_0'|^2 - t^2}}. \tag{32}$$

Although damped by an exponential term proportional to the mass m of the system, the amplitude is non vanishing and superluminal propagation is therefore possible.

3.1.2 Delocalization under Lorentz Boosts

Let $\mathscr{O}$ and $\mathscr{O}'$ be two inertial frames with associated Newton-Wigner eigenbases $|\vec{x}_0\rangle$ and $|\vec{x}_0'\rangle$ of the state space of the system. For simplicity assume an eigenstate localized at the origin of $\mathscr{O}$ denoted by $|\vec{x}_0^o\rangle$. According to equation (20) its momentum space representation at time $t = 0$ is

$$\langle \vec{p} | \vec{x}_0^o \rangle = \sqrt{2\omega_{\vec{p}}}. \tag{33}$$

Furthermore assume $\mathscr{O}'$ is moving along the x-axis of $\mathscr{O}$ with relative velocity v, thus the two inertial frames are related by a Lorentz boost

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{34}$$

The corresponding transformation induced on the state space of the system is implemented by a unitary representation $U[\Lambda]$ of the Lorentz group. Because (34) is a pure Lorentz boost this representation is entirely determined in terms of the infinitesimal boost generators $\vec{K}$

$$U[\Lambda] = e^{-i\vec{K}\cdot\vec{v}}. \tag{35}$$

The boosted state in the momentum space representation then becomes [10, p. 23], [12, p. 65]

$$\langle \vec{p} | \vec{x}_0^{o,B} \rangle = \langle \vec{p} | e^{-i\vec{K}\cdot\vec{v}} | \vec{x}_0^o \rangle = \langle \Lambda\vec{p} | \vec{x}_0^o \rangle = \sqrt{2\omega_{\Lambda\vec{p}}} \tag{36}$$

where $\omega_{\Lambda\vec{p}} = (\Lambda p)^0 = \gamma(\omega_{\vec{p}} - vp_1)$ as can be checked by acting with (34) on the momentum p. Thus, in terms of the Newton-Wigner eigenbasis $|\vec{x}_0'\rangle$ of $\mathscr{O}'$ the boosted state is

$$\begin{aligned} \langle \vec{x}_0' | \vec{x}_0^{o,B} \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \langle \vec{x}_0' | \vec{p} \rangle \langle \vec{p} | U[\Lambda] | \vec{x}_0^o \rangle = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\Lambda\vec{p}}}{\omega_{\vec{p}}}} e^{i\vec{x}_0'\cdot\vec{p}} \\ &= \int \frac{d^3p}{(2\pi)^3} \sqrt{\gamma \left(1 - \frac{vp_1}{\sqrt{\vec{p}^2 + m^2}}\right)} e^{i\vec{x}_0'\cdot\vec{p}}. \end{aligned} \tag{37}$$

The Paley-Wiener-Schwartz theorem [13, ch. 7] states that the Fourier transform of a compactly supported tempered distribution on $\mathbb{R}^n$ is an entire function on $\mathbb{C}^n$, i.e. a function which is analytic at all finite points of $\mathbb{C}^n$. The space of tempered distributions $\mathscr{S}^*$ is defined as the continuous dual of the Schwartz space $\mathscr{S}$ and the state space spanned by the Newton-Wigner basis of $\mathscr{O}'$ is $\mathscr{H} = L^2(\mathbb{R}^3, d^3x_0')$. Together $\mathscr{S}, \mathscr{H}$ and $\mathscr{S}*$ form what is known as a Gelfand triple [14, p. 383]

$$\mathscr{S} \subset L^2(\mathbb{R}^3, d^3\vec{x}_0') \subset \mathscr{S}^*. \tag{38}$$

But the integrand in (37) is not an entire function since the square root can not be analytically continued to all complex values and it follows by Paley-Wiener-Schwartz that the integral is in general non-vanishing for arbitrary $\vec{x}_0$, i.e. the Newton-Wigner state is completely delocalized in $\mathscr{O}'$.

Therefore the Newton-Winger eigenstates, although arising from seemingly reasonable postulates in a unique way, have the aforementioned strange properties; they

1. propagate superluminally and

2. are delocalized by Lorentz boosts.

This has attracted criticism in different forms which can roughly be divided into two categories.

On the one hand the strange properties can be taken as evidence that a concept of strictly localizable particles is not adequate to describe a relativistic quantum theory and although the notion of particles is successfully used in the macroscopic or non-relativistic limit, on a fundamental level particles are nothing but illusion.

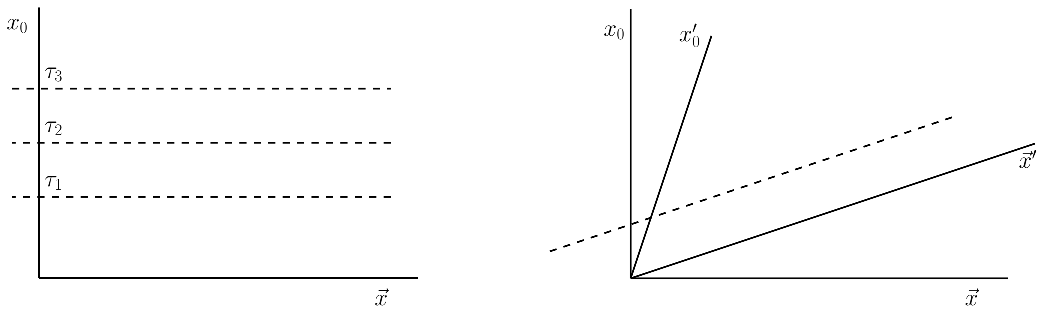
On the other hand several objections were raised against the postulates nurtured by the hope that a suitable modification of them would make the strange properties vanish. For example Newton and Wigner are treating localization only in the limit of perfectly localized states since they assume that every non-zero spatial displacement of a localized state renders it orthogonal to the original state; cf. postulate (b) and equation (23). But confining a physical particle to an infinitesimal spatial region would require an infinite amount of energy and it could well be that the strange properties of the localized states are merely a manifestation of this unphysical assumption. Accordingly modified postulates might then resolve the problems. Another objection brought up was that Newton and Wigner only consider localization on instantaneous hyperplanes [2, p. 114], [15, p. 237]. Whereas the former objection ended in smoke after Wightman carried out the analysis for partially localized states and found himself confronted with the same strange properties, the latter proved more promising; the generalization of the Newton-Wigner localization to arbitrary hyperplanes resolves the problem that initially localized states are delocalized under Lorentz boosts and will be topic of the last section. Before that, a theorem is introduced which supports the point of view of all those who deny the existence of localizable particles.

4 Particles – a pure Illusion?

In more recent years several theorems have been proven which seem to rule out the existence of localizable particles in a relativistic quantum theory. But obviously the statement of each such theorem depends crucially on its assumptions and it is almost impossible to remove all doubts that they might be unjustified. In this section the focus is laid on Malament's theorem whose soundness has been discussed extensively in the literature [16, p. 5–7]. In order to introduce this theorem and later the concept of hyperplane-dependent localization, some remarks about hyperplanes are required.

4.1 Spacetime structure and Hyperplanes

Consider an inertial frame equipped with Minkowski coordinates $x = (t, \vec{x})$.



(a) Foliation of spacetime into instantaneous hyperplanes obtained by setting $\eta = (1,0,0,0)$. Each hyperplane is then determined by the equation $x_0 = \tau$.

(b) Given an arbitrary hyperplane in $\mathscr{O}$ there always exists an inertial frame $\mathscr{O}'$ in which this hyperplane is instantaneous.

Figure 2: Minkowski diagrams illustrating two remarks made in the text.

Def. A spacelike hyperplane is defined to be the set of points

$$\Sigma_{(\eta,\tau)} := \{x \mid \eta \cdot x = \tau \text{ with } \eta^2 = 1 \text{ and } \eta^0 \geq 1\} \tag{39}$$

From this definition it immediately follows that

(i) every ordered pair (η, τ) defines a unique hyperplane and

(ii) any two distinct points on the hyperplane (η, τ) are separated by a spacelike interval.

Proof (i) Assume $(\eta, \tau) \neq (\eta', \tau')$ define the same hyperplane, i.e. $\Sigma_{(\eta,\tau)} = \Sigma_{(\eta',\tau')}$. Consider $x_1 := \tau\eta \in \Sigma_{(\eta,\tau)}$ and $x_1' := \tau'\eta' \in \Sigma_{(\eta',\tau')}$. But by assumption they have to be elements of both hyperplanes and consequently $\eta \cdot \eta' = \tau'/\tau = \tau/\tau'$. This implies $\tau' = -\tau$ since τ and τ' are assumed to be distinct. However, $x_2 = (\tau/\eta^0, 0, 0, 0) \in \Sigma_{(\eta,\tau)}$ has to be an element of $\Sigma_{(\eta',\tau')}$ as well and thus $\tau\eta'^0 = \tau'\eta^0 = -\tau\eta^0$ in contradiction with the requirement that both $\eta'^0, \eta^0 \geq 1$. A similar argumentation works for the cases $\eta \neq \eta', \tau = \tau'$ and $\eta = \eta', \tau \neq \tau'$.
(ii) Assume x, x' are two distinct points on $\Sigma_{(\eta,\tau)}$, hence $\eta \cdot (x - x') = 0$. But this is equivalent to $\eta^0(x_0 - x_0') = \eta^i(x_i - x_i')$. From $\eta^0 \geq 1$ and $\eta \cdot \eta = 1$ it follows $\eta^i\eta_i < (\eta^o)^2$. Hence $(x_0 - x_0') < (x_i - x_i')$.

Every fixed η thus defines a foliation $\mathfrak{S}$ of spacetime into spacelike hyperplanes parametrized by τ. For the special case $\eta = (1, \vec{0})$ the spacetime of $\mathscr{O}$ is foliated into instantaneous hyperplanes. Such a foliation is shown in Fig. 3a.

Moreover for every spacelike hyperplane $\Sigma_{(\eta,\tau)}$ there exists an inertial frame in which this hyperplane is instantaneous. In order to show this consider an arbitrary Lorentz boost connecting two inertial frames

$$\Lambda(\theta, \vec{a}) = \begin{pmatrix} \cosh\theta & \sinh\theta\, \vec{a}^T \\ \sinh\theta\, \vec{a} & I_3 + (\cosh\theta - 1)\, \vec{a}\, \vec{a}^T \end{pmatrix}, \tag{40}$$

where $\vec{a}$ determines the direction of the relative velocity of the inertial frames and $\tanh\theta = |\vec{v}|$ its magnitude. A boost therefore has a total of four degrees of freedom $\theta, \vec{a}$ which can be chosen such that $\eta \to \eta' = (1, \vec{0})$ under Λ, cf. Fig. 3b. It is now possible to introduce Malament's theorem.

4.2 Malament's Theorem

Let $\mathcal{M}$ be an affine spacetime manifold equipped with a foliation $\mathfrak{S}$ into spacelike hyperplanes Σ and $\mathscr{H}$ the state space of the quantum system under consideration. Assume the following structure:

(i) For all bounded subsets $\Delta \subset \Sigma \in \mathfrak{S}$ there exists a map $h : \Delta \mapsto P_\Delta$, where P_Δ is a projection operator on $\mathscr{H}$.

(ii) Let G be the translation group of $\mathcal{M}$ and d a homomorphism from G into the unitary representations $U(g \in G)$ such that $\langle\psi|U(g)|\psi\rangle \to 1$ as $g \to 0$ for all $|\psi\rangle \in \mathscr{H}$ with $\langle\psi|\psi\rangle = 1$.

Then $(\mathscr{H}, h, d)$ defines a *localization system* on $\mathcal{M}$ [16, p. 3].

One can interpret $\langle\psi|P_\Delta|\psi\rangle$ as the probability amplitude of finding the state $|\psi\rangle$ within the region $\Delta \subseteq \Sigma$. Assume the localization system has the following properties.

Malament's postulates [16, p. 3–5], [17, p. 3–4]

(a) The energy of all states $|\psi\rangle \in \mathscr{H}$ is bounded from below, i.e. $\exists\ E_0$ such that $\langle\psi|H|\psi\rangle \geq E_0$ for all $|\psi\rangle$ in the domain of the Hamiltonian H of the system.

(b) A state can not be found in two disjoint spatial regions of the same hyperplane: $\Delta_1 \cap \Delta_2 = \emptyset \Rightarrow P_{\Delta_1} P_{\Delta_2} = 0$.

(c) Projection operators associated with two spacelike separated regions Δ and Δ' do not influence the statistics of each other: $[P_\Delta, P_{\Delta'}] = 0$.

(d) The statistics of the projection operators are invariant under spacetime translations: $P_{\Delta+\vec{a}} = U(\vec{a}) P_\Delta U^\dagger(\vec{a})$, where $\vec{a} \in G$ and $\Delta + \vec{a}$ denotes the set obtained by translating every point in Δ by the vector $\vec{a}$.

It is certainly worth seeing how these postulates compare to the Newton-Wigner postulates. The first postulate simply assures that only a finite amount of energy can be extracted from the particle. The same assumption has been made for the Newton-Wigner localization of a massive spin zero system by the restriction to the positive energy solutions of the Klein Gordon equation and is generally contained implicitly in the Newton-Wigner localization scheme. The three remaining postulates however differ substantially from the Newton-Wigner postulates. They are valid for all possible localized states and not only for perfectly localized ones and localization is considered on arbitrary hyperplanes and not only on instantaneous ones. Moreover the third postulate imposes an explicit requirement on causality. However, it follows

Thm. Malament [17, p. 6] A localization system satisfying Malament's postulates also satisfies $P_\Delta = 0$ for all bounded subsets Δ and for all times.

Consequently a state can never be detected within a bounded region of space and acceptance of Malament's postulates would lead to a world without localizable particles. This is reason enough to find good arguments against them and indeed there is room for criticism.

One objection is that Malament's theorem only applies to a flat spacetime and its statement could therefore be an artefact of the Minkowskian spacetime. Although Halvorson and Clifton have proven a theorem [16, p. 13] which entails Malament's theorem and only relies on a globally hyperbolic spacetime it is not entirely accepted that the present universe is globally hyperbolic [18, p. 9] and there remains the possibility that a suitably curved spacetime could save the concept of localizable particles. But this would not be completely satisfactory since no concept of localizable particles would exist in a flat spacetime and it would be better to find another way to prove

Malament wrong. For example one could argue that the solution of the measurement problem might incorporate an abolition of unitary dynamics [19, p. 170] and thereby invalidate postulate (d). However, Halvorson and Clifton [16, p. 7] point out that

> ... it would be quite another thing to provide a model [with non unitary dynamics] ... which is also capable of reproducing the well-confirmed quantum interference effects at the micro-level. Until we have such a model, pinning our hopes for localizable particles on a failure of unitary dynamics is little more than wishful thinking.

Various other objections have been raised. Some of them turned out to be unfounded, but many remain controversial and without definite answers.

In addition to Malament's theorem there are several other theorems which claim to rule out the existence of localizable particles. Some among them seem quite powerful in the sense that they only rely on a very limited number of assumptions, but certainly none of them is free of all doubts. In fact many of the objections against a world without localizable particles are fueled by a very promising theory developed by Fleming, Butterfield et al. whose basic ideas are presented in the next section.

5 Hyperplane-dependent Localization

This section relies heavily on [2, esp. sec. 9–11]. Newton-Wigner localization as introduced above is always with respect to an instantaneous hyperplane $x^0 = t$. Due to the superluminal propagation (32) a Newton-Wigner state localized at time t is not localized anymore at any later time. Bearing in mind the above discussion of hyperplanes the delocalization of such a state under a Lorentz boost does no longer come as a surprise since a Newton-Wigner state localized in the $x^0 = 0$ hyperplane of observer $\mathscr{O}$ is in general not localized in the $x'^0 = 0$ hyperplane of observer $\mathscr{O}'$. But by restriction of localization to a certain hyperplane these issues are immediately resolved as all observers – no matter what their state of motion – can always refer to this specific hyperplane. Whether a state is localized with respect to this hyperplane or not is then well-defined.

Consider two parametrizations (η, τ) and (η', τ') of a given hyperplane in the coordinate systems of inertial observers $\mathscr{O}$ and $\mathscr{O}'$ which are connected by a Poincaré transformation (Λ, a) such that $\eta' = \Lambda\eta$ and $\tau' = \tau + a \cdot \Lambda\eta$. In the Heisenberg picture a position operator (e.g. the center of spin or the center of energy position operator) then has the two different parametrizations $X^\mu(\eta, \tau)$ and $X^\mu(\eta', \tau')$ which for consistency need to be related by a Poincaré transformation

$$\langle\psi'|X^\mu(\eta', \tau')|\psi'\rangle = \Lambda^\mu_\nu \langle\psi|X^\mu(\eta, \tau)|\psi\rangle + a^\mu \langle\psi|\psi\rangle \qquad \forall\,|\psi\rangle \tag{41}$$

where $|\psi'\rangle$ is obtained by acting with the unitary representation of the Poincaré group $U(\Lambda, a)$ on $|\psi\rangle$. The hyperplane-dependent version of the Newton-Wigner position operator $X^\mu(\eta, \tau)$ for a massive spinless system, which in this special case coincides with the center of energy operator [2, p. 149], is given in terms of the symmetric product by

$$\frac{1}{2}(X^\mu H + HX^\mu)(\eta, \tau) := \int d^4x\delta(\eta x - \tau)x^\mu\theta^{\nu\rho}(x)\eta_\nu\eta_\rho, \tag{42}$$

where $\theta^{\nu\rho}$ is the stress-energy-momentum tensor and

$$H(\eta, \tau) := P^\mu\eta_\mu := \int d^4x\delta(\eta x - \tau)\theta^{\nu\rho}(x)\eta_\rho. \tag{43}$$

is the hyperplane-dependent energy. The δ-function ensures that the integration takes place only on the hyperplane. The spatial components of this operator on an instantaneous hyperplane are

$$X^iP^0 = \int d^3xx^i\theta^{00}(\tau, \vec{x}), \tag{44}$$

where the factor of $1/P^0$ is the total energy and serves as a normalization factor. The θ^{00}-component of the stress-energy-momentum tensor corresponds to the energy density and is weighted with the position on the instantaneous hyperplane. Thus X^i indeed corresponds to the center of energy. It needs to be added that there always exists an inertial frame in which the center of energy position operator takes the form (44).

If the system under consideration carries spin the hyperplane-dependent center of energy operator differs from the hyperplane-dependent Newton-Wigner position operator which then measures the center of spin. Certainly other localizable properties require other operators. In contrast to the Newton-Wigner case, it can then happen that the components of such an operator $X^\mu(\eta,\tau)$ do not commute. Localization is then only possible with respect to a chosen component of the position operator, i.e. within a subset $\Delta \times \mathbb{R}^2$ of the hyperplane $\Sigma_{(\eta,\tau)}$.

5.1 Lorentz Boosts and Delocalization

It is now time to see how the problem of delocalization under Lorentz boosts is naturally resolved in the formalism of hyperplane-dependent localization. Consider the intersection of two distinct hyperplanes which defines a two-dimensional subset of spacetime and associate with each of these hyperplanes a position operator. The sets of eigenvectors of these position operators lying in the intersection are then given by

$$\begin{aligned} \eta' \cdot X(\eta,\tau)\,|\alpha,\tau';\eta,\tau\rangle &= \tau'\,|\alpha,\tau';\eta,\tau\rangle\,, \\ \eta \cdot X(\eta',\tau')\,|\alpha,\tau;\eta',\tau'\rangle &= \tau\,|\alpha,\tau;\eta',\tau'\rangle\,, \end{aligned} \tag{45}$$

where α denotes the additional parameters needed to uniquely define the state. But there is no common set of eigenstates since the components of operators associated with different hyperplanes in general do not commute. Therefore it is possible to have a state $|\psi\rangle$ such that

$$\langle\alpha,\tau';\eta,\tau|\psi\rangle = 0 \;\text{ but }\; \langle\alpha,\tau;\eta',\tau'|\psi\rangle \neq 0, \tag{46}$$

i.e. on $\Sigma_{(\eta',\tau')}$ the state $|\psi\rangle$ can be found within the intersection, but on $\Sigma_{(\eta,\tau)}$ it cannot be found within the same intersection. In fact this property occurs for any hyperplane-dependent position operator and is not specific for the Newton-Wigner case. Consequently localization always needs to be considered with respect to a certain hyperplane whose specification requires three additional parameters and Butterfield and Fleming conclude that

> [...] quantum localization [thus] takes place in a seven-dimensional manifold, rather than in four dimensional Minkowski spacetime [2, p. 131].

One of the strange properties of the conventional Newton-Wigner localization concept, namely the subjectivity of localization is therefore nothing but a manifestation of the three unspecified degrees of freedom and is no longer worrying once one has introduced the hyperplane-dependent formulation.

5.2 Superluminal Propagation and Causality

Unfortunately the superluminal propagation of the Newton-Wigner states still persists, but it can be divided into two categories. On the one hand for an open system the superluminal propagation of certain position operators is not surprising. As an example one may consider a perfect vacuum tube containing a single massive and spinless particle at one end of the tube. The center of energy of the content of the tube therefore coincides with the position of this particle. But injection of additional particles at the other end of the tube can easily cause the center of energy to move superluminally.

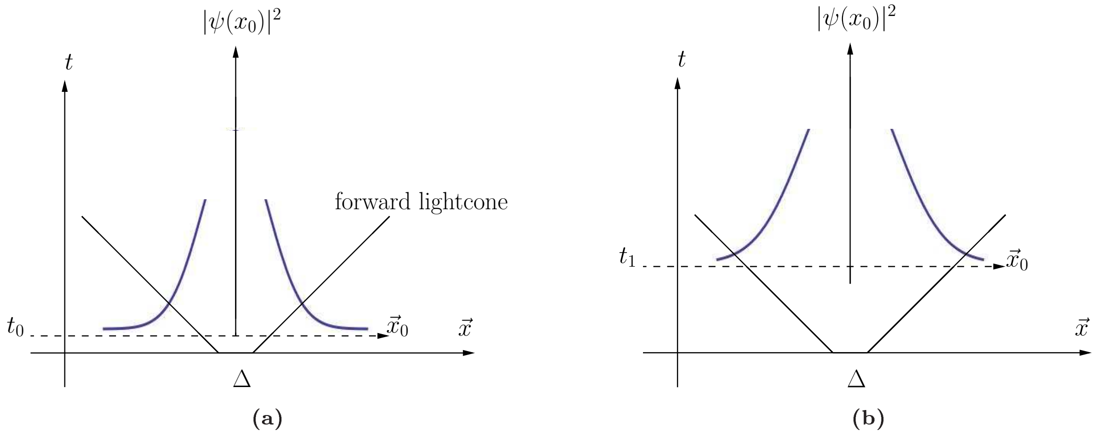


Figure 3: Superluminal propagation of Newton-Wigner state with respect to instantaneous hyperplanes. The probability density outside the forward lightcone diminishes with increasing spacelike separation and the probability to find the state inside the lightcone rapidly tends towards 1.

On the other hand superluminal propagation occurs also in closed systems and is in fact a general feature of hyperplane-dependent position operators. To characterize the superluminal propagation let $|\alpha, x; \eta, \tau\rangle$ be a basis of eigenfunctions of the hyperplane-dependent Newton-Wigner position operator and imagine a wave-function $\langle\alpha, x; \eta, \tau|\psi\rangle$ expressed in this basis with compact support on $\Sigma_{(\eta,\tau)}$. Although this wave-function spreads out instantaneously

- the probability density outside the forward lightcone diminishes with increasing spacelike separation and
- the integrated probability density inside the forward lightcone rapidly tends toward unity with increasing time, see Fig. 3.

However, the physically relevant question is whether this superluminal propagation can be used to signal superluminally and hence to create causal anomalies. To date there is no proof that such anomalies are avoided in the hyperplane-dependent formulation, but in the following an argument due to Fleming [20, p. 123–124] is presented which may allay these fears. Fleming considers the setup shown in Fig. 4 which at first sight serves to abuse the superluminal propagation of a Newton-Wigner state so as to generate a contradiction. Initially, two remarks need to be made.

(i) Both confinement and detection of a particle are always with respect to a certain hyperplane which here for simplicity is assumed to be the instantaneous hyperplane in the corresponding reference frame.

(ii) In the framework of hyperplane-dependent localization the state reduction due to a measurement occurs only on hyperplanes in the future of the state reducing region.

On the one hand the state released in (A) is not confined to any hyperplane and can be measured by the detector (B) with a non-vanishing probability, thus being relocalized. But from (ii) it follows that the effect of this relocalization only manifests itself on hyperplanes lying in the future of (B). The instantaneous hyperplane on which the detector (C) is sensitive is not among them and therefore the mechanism which prevents the box from being opened cannot be triggered. On the other hand the confinement of the particle in the box is only with respect to the instantaneous

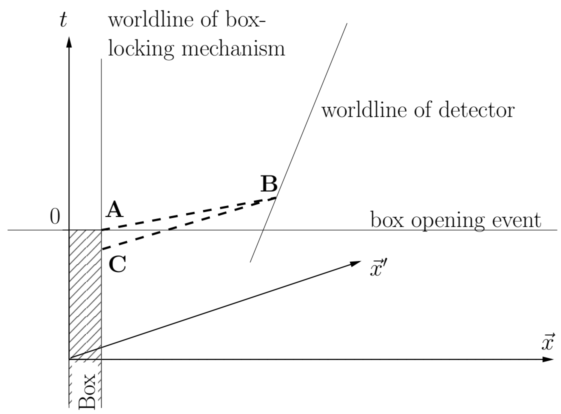


Figure 4: A box containing a localized Newton-Wigner state at time $t = 0$ is opened (A). Due to the superluminal propagation the state can possibly be measured by a spacelike related detection measurement (B). Assume that when this happens, the state collapses to another Newton-Wigner state which can then superluminally propagate to a detection apparatus (C) in the past of (A). If this apparatus detects the state it triggers a mechanism which prevents the box from being opened in the future. But then the released Newton-Wigner state prevents itself from being released – a contradiction [20, p. 123].

hyperplane in the inertial frame of the box and there is nothing which hinders the state to propagate on other hyperplanes. The state can then propagate on the hyperplane on which (B) is sensitive and by doing so prior to the box opening event (A) it is possible to trigger the box locking mechanism before (A). However, this is not a contradiction because the triggering does not occur as a consequence of the box being opened, but rather is the result of an earlier propagation of the state on a hyperplane on which the state has never been confined.

6 Conclusion

Although it is not obvious how to introduce the concept of localization in relativistic quantum theory the hyperplane-dependent formulation is a very promising attempt which indicates that superluminal propagation does not inevitably lead to causal loops: though there is no proof for that and further investigation is needed. But certainly hyperplane-dependent localization shows that it would be premature to appeal to Malament's theorem, so as to rule out localizable particles.

A Appendix

A.1 Conventions

Vectors in three dimensional space are denoted by an arrow $(\vec{x}, \vec{p}, \dots)$ whereas 4-vectors are written without $(x, p, \dots)$. All calculations are carried out in natural units $(\hbar = c = 1)$ and the signature of the metric tensor of flat space-time is chosen to be $(+, -, -, -)$. The Fourier-transform $\tilde{f}(k)$ in n dimensions is defined as

$$\tilde{f}(k) := \int d^n x \; f(x) e^{-ik \cdot x} \tag{47}$$

and its inverse as

$$f(x) := \int \frac{d^n k}{(2\pi)^n} \tilde{f}(k) e^{ik\cdot x}. \tag{48}$$

For wave-functions and operator-fields the convention $\phi(\vec{p}) := \tilde{\phi}(\vec{p})$ is used.

Bibliography

1. E. W. Lamb Jr. and M. O.Scully. *The Photoelectric Effect without Photons*. Defense Technical Information Center OAI-PMH Repository, United States, 1968.

2. G. N. Fleming and J. Butterfield. Strange positions. In J. Butterfield and C. Pagonis, editors, *From Physics to Philosophy*. Cambridge University Press, The Edinburgh Building, 1999.

3. R. de la Madrid. Localization of Non-Relativistic Particles. *International Journal of Theoretical Physics*, 46:1986–1997, 2007.

4. L. E. Ballentine. *Quantum Mechanics: A Modern Development*. World Scientific Publishing Co. Pte. Ltd., 5 Toh Tuck Link, Singapore 596224, 1998.

5. T. Newton and E. Wigner. Localized states for elementary systems. *Reviews of Modern Physics*, 21:400–406, 1949.

6. A. J. Kálnay. Lorentz-invariant localization for elementary systems. ii. *Physical Review D*, 1:1092–1104, 1969.

7. A. O. Barut and R. Rączka. *Theory of Group Representations and Applications*. World Scientific Publishing Co. Pte. Ltd., 5 Toh Tuck Link, Singapore 596224, 1986.

8. R. M. Mir-Kasimov. The Newton-Wigner Localization Concept and noncommutative Space. *Czechoslovak Journal of Physics*, 55:A251–A260, 2005.

9. Th. W. Ruijgrok. On Localisation in Relativistic Quantum Mechanics. In *Theoretical Physics Fin de Siècle*, volume 539 of *Lecture Notes in Physics*. Springer-Verlag, Heidelberg, 2000.

10. M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press, Perseus Books Group, Boulder, Colorado 80301, 1995.

11. E. Zauderer. *Partial Differential Equations of Applied Mathematics*. Wiley-Interscience, New York, NY 10158-0012, 3 edition, 2006.

12. S. Weinberg. *The Quantum Theory of Fields*, volume 1, Foundations. Press Syndicate of the University of Cambridge, The Pitt Building, Trumpington Street, Cambridge CB2 1RP, 1995. p. 65–66, see equation (2.5.11) but with N(p) = 1 due to the different normalization used here.

13. L. Hörmander. *The analysis of linear partial differential operators I (Grundlehren der mathematischen Wissenschaften)*, volume 1. Springer Verlag, Berlin, 1983.

14. W. Appel. *Mathematics for Physics and Physicists*. Princeton University Press, Princeton, New Jersey 08540, 2007.

15. G. N. Fleming and H. Bennett. Hyperplane dependence in relativistic quantum mechanics. *Foundations of Physics*, 19(3):231–267, 1989.

16. H. Halvorson and R. Clifton. No Place for Particles in Relativistic Quantum Theories? *Philosophy of Science*, 69(1):1–28, March 2002. Reprinted in: J. Butterfield and H. Halvorson (eds.), *Quantum Entanglements: Selected Papers of Rob Clifton*, Oxford University Press, 2004.

17. D. Malament. In defence of dogma: Why there cannot be a relativistic quantum mechanics of (localizable) particles. In R. Clifton, editor, *Perspectives on Quantum Reality: Non-Relativistic, Relativistic, and Field-Theoretic*, pages 181–201. Kluwer, Dordrecht and Boston, 1996.

18. S. W. Hawking and R. Penrose. *The nature of space and time*. Princeton University Press, Princeton, New Jersey 0854, USA and Woodstock, Oxfordshire OX20 1SY, UK, 1996.

19. J. A. Barrett. On the nature of measurement records in relativistic quantum field theory. In M. Kuhlmann, H. Lyre, and A. Wayne, editors, *Ontological aspects of quantum field theory*, chapter 9. World Scientific Publishing, Singapore 912805, 2002.

20. G. N. Fleming. Lorentz invariant state reduction, and localization. *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*, 1988:112–126, 1988.