THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

ON THE EQUATIONAL THEORY OF PROJECTION LATTICES OF FINITE VON-NEUMANN FACTORS

CHRISTIAN HERRMANN

Abstract. For a finite von-Neumann algebra factor $\mathbf{M}$, the projections form a modular ortholattice $L(\mathbf{M})$. We show that the equational theory of $L(\mathbf{M})$ coincides with that of some resp. all $L(\mathbb{C}^{n\times n})$ and is decidable. In contrast, the uniform word problem for the variety generated by all $L(\mathbb{C}^{n\times n})$ is shown to be undecidable.

§1. Introduction. Projection lattices $L(\mathbf{M})$ of finite von-Neumann algebra factors $\mathbf{M}$ are continuous orthocomplemented modular lattices and have been considered as logics resp. geometries of quantum meachnics cf. [25]. In the finite dimensional case, the correspondence between irreducible lattices and algebras, to wit the matrix rings $\mathbb{C}^{n\times n}$, has been completely clarified by Birkhoff and von Neumann [5]. Combining this with Tarski's [27] decidability result for the reals and elementary geometry, decidability of the first order theory of $L(\mathbf{M})$ for a finite dimensional factor $\mathbf{M}$ has been observed by Dunn, Hagge, Moss, and Wang [7].

The infinite dimensional case has been studied by von Neumann and Murray in the landmark series of papers on 'Rings of Operators' [23], von Neumann's lectures on 'Continuous Geometry' [28], and in the treatment of traces resp. transition probabilities in a ring resp. lattice-theoretic framework [20, 29].

The key to an algebraic treatment is the coordinatization of $L(\mathbf{M})$ by a $*$-regular ring $U(\mathbf{M})$ derived from $\mathbf{M}$ and having the same projections: $L(\mathbf{M})$ is isomorphic to the lattice of principal right ideals of $U(\mathbf{M})$ (cf. [8] for a thorough discussion of coordinatization theory). For finite factors this has been achieved in [23], more generally for finite AW*-algebras and certain Baer-*-rings by Berberian in [2, 3].

In the present note we show that the equational theory of $L(\mathbf{M})$ coincides with that of $L(\mathbb{C}^{n\times n})$ if $L(\mathbf{M})$ is $n+1$- but not n-distributive for some n; and with that of all $L(\mathbb{C}^{n\times n})$, $n<\infty$, otherwise - which applies to the type II_1 factors. In the latter case, the equational theory is decidable, but the theory of quasi-identities is not.

2000 *Mathematics Subject Classification.* 06C20, 16E50, 16W10, 46L10.
Key words and phrases. Von Neumann algebra, projection lattice, continuous geometry, equational theory .

The author is deeply obliged to Luca Giudici for manifold contributions, discusssions, and encouragement.

§2. Modular ortholattices: Equations and representations. An algebraic structure $(L, \cdot, +, ', 0, 1)$ is an *ortholattice* (shortly OL) if there is a partial order $\leq$ on L such that, for all $a, b \in L$, $0 \leq a \leq 1$, $a \cdot b = ab = \inf\{a, b\}$, $a + b = \sup\{a, b\}$, $a'' = a$, and $a \leq b$ iff $b' \leq a'$. It is a *modular ortholattice* (shortly: MOL) if, in addition, $a \geq b$ implies $a(b + c) = b + ac$. One can define this class by a finite set of equations, easily ([4, 5]).

If V is a unitary space then the subspaces of finite dimensions together with their orthogonal complements form an MOL $L(V)$ - a sublattice of the lattice of all subspaces. For V of finite dimension n, we have $L(V) \cong L(\mathbb{C}^n)$ for $\mathbb{C}^n$ endowed with the canonical scalar product. A lattice is *n-distributive* if and only if it satsifies

$$x \sum_{i=0}^{n} y_i = \sum_{i=0}^{n} x \sum_{j \neq i} y_j.$$

Lemma 2.1. *$L(\mathbb{C}^k)$ is n-distributive if and only if $k \leq n$.*

Proof. Huhn [18, p. 304] cf. [13]. ⊣

For a class $\mathcal{C}$ of algebraic structures, e.g. ortholattices, let $\mathsf{V}\mathcal{C}$ denote the smallest equationally definable class (variety) containing $\mathcal{C}$ cf. [6]. By Tarski's version of Birkhoff's Theorem, $\mathsf{V}\mathcal{C} = \mathsf{HSP}\mathcal{C}$ where $\mathsf{H}\mathcal{C}$, $\mathsf{S}\mathcal{C}$, and $\mathsf{P}\mathcal{C}$ denote the classes of all homomomorphic images, subalgebras, and direct products, resp., of members of $\mathcal{C}$. Define

$$\mathcal{N} = \mathsf{V}\{L(\mathbb{C}^k) \mid k < \infty\}.$$

Clearly, $L(\mathbb{C}^k) \in \mathsf{SH}L(\mathbb{C}^n)$ for $k \leq n$. Within the variety of MOLs, each ortholattice identity is equivalent to an identity $t = 0$ (namely, $a = b$ if and only if $a(ab)' + b(ab)' = 0$). If L is an MOL and $u \in L$ then the section $[0, u]$ is naturally an MOL with orthocomplement $x \mapsto x^u = x'u$.

Lemma 2.2. *An ortholattice identity $t = 0$ with m occurences of variables holds in a given atomic MOL L if any only if it holds in all sections $[0, u]$ of L with $\dim u \leq m$.*

Proof. As usual, we write $\overline{x}$ for sequences $(x_1, \ldots, x_n)$ with n varying according to the context. We show by induction on complexity: if $f(\overline{x})$ is a lattice term with each variable occuring exactly once and if p is an atom of L and a_i in L with $p \leq f(\overline{a})$ in L then there are $p_i \leq a_i$ in L which are atoms or 0 such that $p \leq f(\overline{p})$. Indeed, if $f = x_1$ let $p_1 = p$. Now, let $\overline{x} = \overline{y}\,\overline{z}$ and $\overline{a} = \overline{b}\,\overline{c}$, accordingly. If $f(\overline{x}) = f_1(\overline{y}) \cdot f_2(\overline{z})$ then $p \leq f_1(\overline{b})$ and $p \leq f_2(\overline{c})$ and we may choose the $q_i \leq b_i$ and and $r_j \leq c_j$ by inductive hypothesis and put $\overline{p} = \overline{q}\,\overline{r}$. On the other hand, consider $f(\overline{x}) = f_1(\overline{y}) + f_2(\overline{z})$. If $f_2(\overline{c}) = 0$ then $p \leq f_1(\overline{b})$ and we may choose $q_i \leq b_i$ by induction and $r_j = 0$. Similarly, if $f_1(\overline{b}) = 0$. Otherwise, there are atoms p^i such that $p^1 \leq f_1(\overline{b})$, $p^2 \leq f_2(\overline{c})$ and $p \leq p^1 + p^2$ (cf. [1]). Applying the inductive hypothesis, we may choose $q_i \leq b_i$ and $r_j \leq c_j$, atoms or 0, such that $p^1 \leq f_1(\overline{q})$ and $p^2 \leq f_2(\overline{r})$ whence $p \leq f(\overline{p})$ where $\overline{p} = \overline{q}\,\overline{r}$.

Now, consider an identity $t(\overline{x}) = 0$. By de Morgan's laws, we may assume that t is in so called negation normal form, i.e. there is a lattice term $f(\overline{y}\,\overline{z})$ with each variable occuring exactly once from which $t(\overline{x})$ arises substituting the variable $x_{\alpha i}$ for y_i, the negated variable $x'_{\beta j}$ for z_j (with suitable functions α and β).

Assume $t(\overline{a}) > 0$ in L. Since L is atomic, there is an atom p with $p \leq t(\overline{a})$. With $b_i = a_{\alpha i}$ and $c_j = a'_{\beta j}$ one obtains $t(\overline{a}) = f(\overline{b}\,\overline{c})$. As shown above, there are $q_i \leq b_i$ and $r_j \leq c_j$ such that $p \leq f(\overline{q}\,\overline{r})$. Put

$$u_k = \sum_{\alpha i = k} q_i, \;\; v_k = \sum_{\beta j = k} r_j, \;\; w = \sum_{k=1}^{n} u_k + v_k.$$

Then $u_k \leq a_k \leq w$ and $v_k \leq a'_k \leq w$. Thus, $a'_k \leq u'_k$ and $v_k \leq u^w_k$. For the MOL $[0, w]$ it follows by monotonicity that $0 < p \leq f(\overline{q}\,\overline{r}) \leq t(\overline{a})$. ⊣

A *unitary representation* of an MOL L is a 0-lattice embedding $\varepsilon : L \to L(V)$ into the lattice of all subspaces of a unitary space such that

$$\varepsilon(a') = \varepsilon(a)^{\perp} \text{ for all } a \in L.$$

This means that ε is an embedding of the ortholattice L into the orthostable lattice associated with the unitary space V in the sense of Herbert Gross [10].

Corollary 2.3. *$L \in \mathcal{N}$ for any MOL admitting a unitary representation.*

Proof. By [14, Thm.2.1]) L embeds into an atomic MOL $\hat{L}$ such that the sections $[0, u]$, $\dim u < \infty$ are subspace ortholattices of finite dimensional unitary spaces (namely, if L is represented in V then $\hat{L}$ consists of all closed subspaces X such that $\dim[X \cap \varepsilon a,\, X + \varepsilon a] < \infty$ for some $a \in L$). By Lemma 2.2, $\hat{L}$ whence also L belong to the variety $\mathcal{N}$ generated by these. ⊣

Corollary 2.4. *$\mathcal{N} = \mathsf{V}L(V)$ for any unitary space of infinite dimension.*

§3. Regular rings with positive involution. An associative ring (with or without unit) R is (von Neumann) *regular* if for any $a \in R$ there is a *quasi-inverse* $x \in R$ such that $axa = a$ cf. [28, 22, 9]. A $*$-ring is a ring with an involution * as additional operation:

$$(x + y)^* = x^* + y^*, \;\; (xy)^* = y^* x^*, \;\; x^{**} = x.$$

A $*$-ring is $*$-*regular* if it is regular and, moreover, *positive*: $xx^* = 0$ only for $x = 0$. Equivalently, for any $a \in R$ there is a (unique) *projection* e (i.e. $e = e^* = e^2$) such that $aR = eR$. Particular examples are the rings $\mathbb{C}^{n \times n}$ of all complex $n \times n$-matrices with r^* the adjoint matrix, i.e. the transpose of the conjugate.

The projections of a $*$-regular ring with unit form a modular ortholattice $L(R)$ where $e \leq f \Leftrightarrow e = ef$ and $e' = 1 - e$. Now, $e \mapsto eR$ is an isomorphism of $L(R)$ onto the ortholattice of principal right ideals of R and we may use the same notation for both. Observe that $L(\mathbb{C}^n) \cong L(\mathbb{C}^{n \times n})$, canonically, where a subspace X corresponds to the set of all matrices with columns in X cf. the following Proposition.

Proposition 3.1. (*Giudici*). *Let M be a right module over a ring S and let R be a regular subring of the endomorphism ring $\mathsf{End}(M_S)$. Then $L(R)$ embeds into the lattice of submodules of M_S via $\varepsilon(\phi R) = \mathsf{Im}\phi$.*

PROOF. This is (1) in the proof of [8, Thm.4.2.1] in the thesis of Luca Giudici, cf. [15, Prop.9.1]. ⊣

COROLLARY 3.2. *If R and S are $*$-regular rings, R a $*$-subring of S, then $L(R)$ is a sub-OL of $L(S)$.*

PROOF. R embeds into $\mathsf{End}S_S$ via $r \mapsto \hat{r}$ where $\hat{r}(x) = rx$ for $x \in S$. By Prop.3.1 this yields an embedding of $L(R)$ into $L(S)$ with $eR \mapsto \mathsf{Im}\hat{e} = eS$ for $e \in L(R)$. Since $e' = 1 - e$ in both OLs, we have $L(R)$ a sub-OL of $L(S)$. ⊣

COROLLARY 3.3. *For any $*$-regular ring S,*

$$\mathsf{V}L(S) = \mathsf{V}\{L(R) \mid R \text{ at most countable, regular } *\text{-subring of } S\}$$

PROOF. '$\supseteq$' follows from Cor.3.2. Conversely, $L(S)$ belongs to the variety generated by its finitely generated sub-OLs L. Endow S with a unary operation q such that $a\mathsf{q}(a)a = a$ for all a in S. Now, for any such L there is an at most countable $*$-subring R of S containing L and also closed under the operation q. Observe that for $e, f \in L(R)$ one has $e \leq f$ if and only if $ef = e$, i.e. $e \leq f$ in $L(S)$. Thus L is also a sublattice of $L(R)$: assume we have join $e \vee f = g$ in L and $h \in L(R)$ with $h \geq e$, f in $L(R)$. Then $h \geq g$ in $L(S)$ whence $h \geq g$ which means $e \vee f = g$ also in $L(R)$. Similarly for meets. Also, since L is closed under the orthocomplement $e \mapsto 1 - e$ in $L(S)$, the same is true in $L(R)$. It follows, that L is a sub-OL of $L(R)$. ⊣

Let V be a unitary space. Denote by ϕ^* the adjoint of ϕ - if it exists. A *unitary representation* of a $*$-ring R is a ring embedding $\iota : R \to \mathsf{End}(V)$ such that $\iota(r^*) = \iota(r)^*$ for any $r \in R$.

COROLLARY 3.4. *If $\iota : R \to \mathsf{End}(V)$ is a unitary representation of the $*$-regular ring R, then*

$$\varepsilon(eR) = \mathsf{Im}\iota(e)$$

is a unitary representation of the MOL $L(R)$ in V.

PROOF. The lattice embedding follows from Prop.3.1. Now, observe that

$$\varepsilon(eR)^\perp = \mathsf{Im}(\mathsf{id} - \iota(e)) = \varepsilon((1-e)R) = \varepsilon(eR)')$$

since e and $\iota(e)$ are selfadjoint idempotents. ⊣

§4. Finite von-Neumann algebras. A *von-Neumann algebra* (cf. [17]) $\mathbf{M}$ is an unital involutive $\mathbb{C}$-subalgebra of the algebra $\mathcal{B}(H)$ of all bounded operators of a separable Hilbert space H with $\mathbf{M} = \mathbf{M}''$ where $\mathbf{A}' = \{\phi \in \mathcal{B}(H) \mid \phi\psi = \psi\phi \ \ \forall\phi \in \mathbf{A}\}$ is the *commutant* of $\mathbf{A}$. $\mathbf{M}$ is *finite* if $rr^* = 1$ implies $r^*r = 1$. For such, the *projections* e of $\mathbf{M}$, i.e. the $e = e^2 = e^*$, form a (continuous) modular ortholattice $L(\mathbf{M})$. Here, the order is given by $e \leq f \Leftrightarrow e = ef$ and one has $e' = 1 - e$. A finite von-Neumann algebra is a *factor* if its center is $\mathbb{C} \cdot 1$. Particular examples of a finite factors are the algebras $\mathbb{C}^{n\times n}$ of all complex n-by-n-matrices.

THEOREM 4.1. (*Murray-von-Neumann.*) *Any finite von-Neumann algebra factor is either isomorphic to $\mathbb{C}^{n\times n}$ for some $n < \infty$* (type I_n) *or contains for any $n < \infty$ a subalgebra isomorphic to $\mathbb{C}^{n\times n}$* (type II_1).

PROOF. [23, 14.1] and [24, Thm. XIII]. ⊣

For any operator ϕ defined on some linear subspace of H, write $\phi\eta\mathbf{M}$ if $\psi\phi\psi^{-1} = \phi$ for all unitary $\psi \in \mathbf{M}'$ (cf [23, Def.4.2.1]). Let $U(\mathbf{M})$ consist of all closed linear operators ϕ with $\phi\eta\mathbf{M}$ and having dense linear domain. Consider the following operations with domain $U(\mathbf{M})$

$$(\phi, \psi) \mapsto [\phi + \psi],\ (\phi, \psi) \mapsto [\phi \circ \psi],\ \phi \mapsto [\phi^*]$$

where $[\chi]$ denotes the closure of the linear operator χ.

THEOREM 4.2. (*Murray-von-Neumann.*) *For every finite factor* $\mathbf{M}$*,* $U(\mathbf{M})$ *is a* $*$*-regular ring having* $\mathbf{M}$ *as* $*$*-subring and such that* ϕ^* *is adjoint to* ϕ*. Moreover,* $\mathbf{M}$ *and* $U(\mathbf{M})$ *have the same projections.*

PROOF. This is trivial for type I_n. For II_1 factors this is [23, Thm. XV] together with [28, Part II, Ch.II, App 2.(VI)] and [29, p.191] for $*$-regularity. Now, consider $\pi : D \to H$ in $U(\mathbf{M})$ such that $\pi = \pi^* = \pi^2$. Then $U = \mathsf{Im}\pi \subseteq D$ so π is a projection of D, i.e. $D = U \oplus^\perp V$. By density of D it follows $U^{\perp\perp} \oplus^\perp V^{\perp\perp} = H$ and π extends to a projection $\hat{\pi}$ of H onto $U^{\perp\perp}$. From $\pi\eta\mathbf{M}$ it follows $\hat{\pi}\eta\mathbf{M}$, whence $\hat{\pi} \in U(\mathbf{M})$ and $\pi = \hat{\pi} \in \mathbf{M}$ by [23] Lemmas 16.4.2 and 4.2.1. ⊣

An important concept in the Murray-von-Neumann construction is that of an *essentially dense* linear subspace X of H (w.r.t. $\mathbf{M}$). Here, we need only the following properties:

1. Essentially dense X is dense in H [23, Lemma 16.2.1].
2. The domains of members of $U(\mathbf{M})$ are essentially dense [23, Lemma 16.4.3].
3. For any $\phi \in U(\mathbf{M})$ and essentially dense X, the preimage $\phi^{-1}(X)$ is essentially dense [23, Lemma 16.2.3].
4. Any finite or countable intersection of essentially dense X_n is essentially dense [23, Lemma 16.2.2].

THEOREM 4.3. (*Luca Guidici.*) *Any countable* $*$*-subring of* $U(\mathbf{M})$ *is representable.*

PROOF. Consider any countable $*$-subring R of $U(\mathbf{M})$. A representation of R is constructed from the given Hilbert space H. Let H_0 be the intersection of all domains of operators $\phi \in R$. By (2), H_0 is essentially dense. Define, recursively, H_{n+1} as the intersection of H_n and all preimages $\phi^{-1}(H_n)$ where $\phi \in R$. By (3) and (4), H_{n+1} is essentially dense. By (4), the intersection $H_\omega = \bigcap_{n<\omega} H_n$ is essentially dense and, by (1), dense in H. By construction, H_ω is invariant under R.

Now, for $\phi \in R$ define $\varepsilon(\phi) = \phi|H_\omega$. Then $\varepsilon : R \to \mathsf{End}_{\mathbb{C}}(H_\omega)$ is a $*$-ring homomorphism. Indeed, e.g. $[\phi + \psi]|H_\omega$ is an extension of $\phi|H_\omega + \psi|H_\omega$ and equality holds since both are maps with the same domain. Also $\varepsilon(\phi^*)$ is the restriction of the adjoint ϕ^* in H, whence the adjoint in H_ω. If $\varepsilon(\phi) = 0$, then H_ω is contained in the closed subspace $\ker\phi$ and it follows $\phi = 0$ by density. Thus, ε is a representation. ⊣

§5. Equational theory of projection lattices.

Theorem 5.1. *For any class $\mathcal{M}$ of finite von-Neumann algebra factors, and $\mathcal{V} = V\{L(\mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\}$ one has $\mathcal{V} = \mathsf{V}L(\mathbb{C}^n)$ if and only if $\mathcal{V}$ satisfies the $n+1$-distributive law but not the n-distributive law. Moreover, $\mathcal{V} = \mathcal{N}$ if and only if $\mathcal{V}$ satisfies no n-distributive law. In any case, the equational theory of $\mathcal{V}$ is decidable.*

Proof. Let $\mathbf{M}$ be a finite von-Neumann algebra factor. In view of Thm.4.2 and Cor.3.3, we have to consider countable regular $*$-subrings R of $U(\mathbf{M})$. By Thm.4.3, each such R is representable. By Cor.3.4 and Cor.2.3 we have $L(R) \in \mathcal{N}$ and it follows $L(\mathbf{M}) \in \mathcal{N}$.

By Lemma 2.1, Cor.3.2, and Thm.4.1, $\mathcal{M}$ contains factors of arbitrarily large finite dimensions or a type II_1 factor if and only if $\mathcal{V}$ is n-distributive for no n. In this case, $\mathcal{V} = \mathcal{N}$. Otherwise, there is a maximal n such that $\mathcal{V}$ is n-distributive, in particular all members of $\mathcal{M}$ are of the form $\mathbb{C}^{k\times k}$ with $k \leq n$ and $k = n$ occurs, so $\mathcal{V} = \mathsf{V}L(\mathbb{C}^{n\times n})$.

Recall that according to Tarski [27] the (ordered) field $\mathbb{R}$ has a decidable first order theory. This extends to the field $\mathbb{C}$ endowed with the unary operation of conjugation and then (uniformly) to the involutive $\mathbb{C}$-algebras $\mathbb{C}^{n\times n}$. Encoding the geometry in Tarski style into $\mathbb{C}$ or von-Neumann style into $\mathbb{C}^{n\times n}$, it follows, that there is a uniform decision procedure for the first order theories of the $L(\mathbb{C}^n) \cong L(\mathbb{C}^{n\times n})$. This settles the case of $\mathcal{V} = \mathsf{V}L(\mathbb{C}^{n\times n})$. To decide whether an identity $t = 0$ holds in $\mathcal{N}$, by Lemma 2.2 it suffices to decide validity in $L(\mathbb{C}^{m\times m})$, m the number of occurences of variables in t. ⊣

§6. Von-Neumann frames. Let $n \geq 3$ fixed. An *n-frame*, in the sense of von-Neumann [28], in a lattice L is a list $\overline{a} : a_i, a_{ij}, 1 \leq i,j \leq n, i \neq j$ of elements of L such that for any 3 distinct j,k,l

$$a_j \sum_{i\neq j} a_i = 0 = a_j a_{jk}, \quad \sum_i a_i = 1$$

$$a_j + a_{jk} = a_j + a_k, \quad a_{jl} = a_{lj} = (a_j + a_l)(a_{jk} + a_{kl}).$$

If L is modular and $n \geq 4$ then

$$R(L,\overline{a}) = \{r \in L \mid ra_2 = 0, \quad r + a_2 = a_1 + a_2\}$$

can be turned into a ring, the *coordinate ring*. For the present purpose it suffices to know that $R(L,\overline{a})$ is a semigroup under the multiplication

$$s \otimes r = [(r + a_{23})(a_1 + a_3) + (s + a_{13})](a_2 + a_3)](a_1 + a_2)$$

cf. [21] where $R(L,\overline{a})$ is denoted by L_{12} and $r = r_{12}$ replaced by the array of the r_{ij} obtained via the prespectivities provided by the a_{kl}. Thus, for each multiplicative term $t(\overline{x}) = x_n \cdot ((\ldots \cdot x_2) \cdot x_1)$ there is a lattice polynomial

$$\hat{t}(\overline{a},\overline{x}) = x_n \otimes ((\ldots \otimes x_2) \otimes x_1)$$

such that $\hat{t}(\overline{a},\overline{r}) = t(\overline{r})$ for all substitutions $\overline{r}$ in $R(L,\overline{a})$.

In the sequel, orthocomplementation is no longer an issue and we write $L(V)$ for the lattice of all subspaces of V, $L(R)$ for the lattice of all right ideals of R. If $R^{n\times n}$ is the $n\times n$-matrix ring of some ring R with unit and $L = L(R^{n\times n})$ with the canonical n-frame $\overline{a}$ then $R(L,\overline{a})$ is isomorphic to R - here $\overline{a}$ consists of the $E_{jj}R^{n\times n}$ and $(E_{jj}-E_{ij})R^{n\times n}$ where the E_{ij} form the canonical basis of the R-module $R^{n\times n}$. Indeed, one has a 1-1-correspondence between R, $R(L,\overline{a})$, and certain right submodules of R^n given by

$$r \leftrightarrow (E_{11}-rE_{21})R^{n\times n} \leftrightarrow (e_1-re_2)R$$

where $e_1,\dots,e_n$ is the canonical basis of R^n. Using the notations $(rx,sx,tx) = (e_1r+e_2s+e_3t)R$ and $\tilde{r}=(e_1-re_2)R$ we compute

$$\begin{array}{lclclcl} (\tilde{r}+a_{23})(a_1+a_3) & = & (x,y-rx,-y) & \cap & (u,0,v) & = & (x,0,-rx) \\ (\tilde{s}+a_{13})(a_2+a_3) & = & (x-y,-sx,y) & \cap & (0,u,v) & = & (0,-sy,y) \\ \tilde{s}\otimes\tilde{r} & = & (x,-sy,y-rx) & \cap & (u,v,0) & = & (x,-srx,0). \end{array}$$

This translates back into $L(R^{n\times n})$ and shows that $r\mapsto\tilde{r}$ is an isomorphism between the semigroups R and $R(L,\overline{a})$.

§7. Quasivarieties and word problems. A *quasi-identity* is a sentence

$$\forall\overline{x}.\ \bigwedge_{j=1}^{k} s_j(\overline{x})=t_j(\overline{x}) \Rightarrow s(\overline{x})=t(\overline{x})$$

where the $s_j(\overline{x})$ and so on are terms. A *quasivariety* is a class of algebraic structures defined by quasi-identities, equivalently an axiomatic class closed under substructures and direct products.

A solution of the *uniform word problem* for a class $\mathcal{C}$ consists in a decision procedure for quasi-identities (i.e. a solution for all finite presentations). The *restriced word problem* is unsolvable for $\mathcal{C}$ if for some fixed premise the associated set of quasi-identities is undecidable within $\mathcal{C}$. In other words, within the quasivariety $\mathsf{Q}\mathcal{C}$ generated by $\mathcal{C}$ there is a finitely presented member having unsolvable word problem.

Unsolvability of the restricted word problem has been established by Hutchinson [19] and Lipshitz [21] for any class $\mathcal{C}$ of modular lattices with $L(V)\in\mathsf{Q}\mathcal{C}$ for some infinite dimensional vector space V. Also, based on analoguous results of Gurevich [11] for semigroups, Lipshitz has shown unsolvability for classes $\{L(F^n)\mid F\in\mathcal{F},\, n<\infty\}$, $\mathcal{F}$ any class of fields, and for $\mathcal{C}$ the class of finite (complemented) modular lattices. These results extend to classes having the appropriate lattice reducts.

For sufficiently large classes of modular ortholattices (e.g. containing all 14-distributives) unsolvability in 3 generators has been shown by M.S. Roddy [26] and this has been used in [16] to prove undecidabilty of the equational theory for the class of all n-distributives for fixed $n\geq 14$.

Let $\mathcal{S}$ ($\mathcal{S}_{fin}$) denote the class of all (finite) semigroups, and $\mathcal{S}_p$ the set of semigroups $F_p^{n\times n}$ $(n\geq 1)$ where F_p is the prime field of characteristic p, prime or 0. Let $\mathcal{M}$ denote the class of all modular lattices, $\mathcal{M}_p$ the set of lattices $L(F_p^n)\cong L(F_p^{n\times n})$ $(n\geq 1)$. For a class $\mathcal{C}$ denote by $\mathsf{R}_S\mathcal{C}$ and $\mathsf{R}_L\mathcal{C}$ the class of all semigroup resp. lattice reducts of structures in $\mathcal{C}$ and by $\mathsf{Th}_q\mathcal{C}$ the set of all quasi-identities valid in $\mathcal{C}$.

THEOREM 7.1. *A quasivariety $\mathcal{Q}$ has unsolvable uniform word problem if $\mathcal{S}_p\subseteq \mathsf{SR}_S\mathcal{Q}\subseteq\mathcal{S}$ or $\mathcal{M}_p\subseteq\mathsf{SR}_L\mathcal{Q}\subseteq\mathcal{M}$ for some p.*

PROOF. Given a finite semigroup S, one may consider the semigroup ring $F_p[S]$ as an F_p-vector space V and thus embed S into $\mathsf{End}_{F_p}(V) \cong F_p^{n\times n}$ where $n = |S|$. It follows $\mathsf{Th}_q\mathcal{S}_p \subseteq \mathsf{Th}_q\mathcal{S}_{fin}$ for all p and equality for $p > 0$. Since $\mathbb{Q}^{n\times n} \in \mathsf{SP}_u\{F_p^{n\times n} \mid p \text{ prime}\}$, one has

$$\mathsf{Th}_q\mathcal{S}_p = \mathsf{Th}_q\mathcal{S}_{fin} \text{ for all } p.$$

This is contained in Lipshitz [21, Lemma 3.5]. The claim in the semigroup case follows from the result of Gurevich and Lewis [12] that there is no recursive Γ such that $\mathsf{Th}_q\mathcal{S} \subseteq \Gamma \subseteq \mathsf{Th}_q\mathcal{S}_{fin}$.

According to the preceeding section and again following Lipshitz [21], one may associate with each quasi-identity ϕ as above in the semigroup language a quasi-identity $\hat{\phi}$ in the lattice language

$$\forall\overline{a}\forall\overline{x}\ \alpha(\overline{a}) \wedge \bigwedge_i(x_i a_a = 0 \wedge x_i + a_2 = a_1 + a_2) \\ \wedge \bigwedge_j \hat{s}_j(\overline{a},\overline{x}) = \hat{t}_j(\overline{a},\overline{x}) \Rightarrow \hat{s}(\overline{a},\overline{x}) = \hat{t}(\overline{a},\overline{x})$$

where $\alpha(\overline{a})$ states that $\overline{a}$ is a 4-frame. Since $R(L,\overline{a})$ is a semigroup for any modular lattice L, it follows that $\hat{\phi} \in \mathsf{Th}_q\mathcal{M}$ for all $\phi \in \mathsf{Th}_q\mathcal{S}$. On the other hand, if $\hat{\phi}$ holds in $L(R^{4\times 4})$, substituting the canoncial 4-frame for $\overline{a}$, then ϕ holds in R. In particular, for the ring $R = F_p^{n\times n}$ we encode equality of products of $n \times n$-matrices over F_p into equality of particular lattice elements. Thus, considering all $R = F_p^{n\times n}$, $n \geq 1$, it follows $\phi \in \mathsf{Th}_q\mathcal{S}_p$ for $\hat{\phi} \in \mathsf{Th}_q\mathcal{M}_p$. This proves that $\phi \in \mathsf{Th}_q\mathcal{S}_p$ if and only if $\hat{\phi} \in \mathsf{Th}_q\mathcal{M}_p$.

Now, given $\mathsf{Th}_q\mathcal{M} \subseteq \Delta \subseteq \mathsf{Th}_q\mathcal{M}_p$ define Γ as the set of those quasi-identities ϕ in semigroup language with $\hat{\phi} \in \Delta$. Then

$$\mathsf{Th}_q\mathcal{S} \subseteq \Gamma \subseteq \mathsf{Th}_q\mathcal{S}_p$$

and if Δ is recursive then so is Γ. ⊣

COROLLARY 7.2. *$\mathcal{N}$ as well as the class of projection lattices of finite factors have an undecidable uniform word problem. The quasivariety $\mathcal{Q}$ generated by all ortholattices $L(\mathbb{C}^{n\times n})$ $(n < \omega)$ has an undecidable restricted word problem and is not a variety.*

PROOF. The undecidability claim is immediate by Thm.7.1 resp. the quoted result of Lipshitz [21, Thm.3.6]. By decidability of the $L(\mathbb{C}^{n\times n})$, the complement of $\mathsf{Th}_q\mathcal{Q}$ within the set of quasi-identities is recursively enumerable. If $\mathcal{Q}$ were a variety, then by Thm.5.1 it would coincide with $\mathcal{N}$ and be recursively axiomatizable. Thus $\mathsf{Th}_q\mathcal{Q}$ would be recursively enumerable, too, and this would imply solvability of the uniform word problem. ⊣

PROBLEM 7.3. *Is the restricted word problem solvable for (a) $\mathcal{N}$ resp. (b) the class of projection lattices of finite factors ?*

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TECHNISCHE UNIVERSITÄT DARMSTADT
FB MATHEMATIK
SCHLOSSGARTENSTR. 7, D 64289 DARMSTADT, GERMANY
E-mail: herrmann@mathematik.tu-darmstadt.de