Ronnie Hermens

Weakly Intuitionistic Quantum Logic

Abstract. In this article von Neumann's proposal that in quantum mechanics projections can be seen as propositions is followed. However, the quantum logic derived by Birkhoff and von Neumann is rejected due to the failure of the law of distributivity. The options for constructing a distributive logic while adhering to von Neumann's proposal are investigated. This is done by rejecting the converse of the proposal, namely, that propositions can always be seen as projections. The result is a weakly Heyting algebra for describing the language of quantum mechanics.

Keywords: Quantum logic, Intuitionistic logic, Weakly Heyting algebras.

1. Introduction

Empirical investigation of a scientific theory requires a rigorous formulation of the propositions about possible outcomes for possible experiments that play a role in the theory. The possible experiments within the theory usually concern the possible measurements of observables. In quantum mechanics, every observable is identified with a self-adjoint operator A acting on a Hilbert space $\mathcal{H}$ with domain dense in $\mathcal{H}$. The set of possible outcomes for a measurement of A is given by the spectrum $\sigma(A)$ of A. I will denote the proposition that a measurement of A will yield a result in Δ with probability one for some Borel set $\Delta \subset \sigma(A)$ with "$A \in \Delta$". It then follows from the axioms that this proposition is true iff the state of the system (described by a non-zero element of $\mathcal{H}$) lies in the subspace $\mu_A(\Delta)\,\mathcal{H}$, where μ_A is the projective measure associated with the operator A.

The above observations led von Neumann [12, §III.5] to the idea that the projection $\mu_A(\Delta)$ may be directly associated with the proposition $A \in \Delta$. Since every projection is of this form the credo 'projections as propositions' is readily established. Some years later von Neumann together with Birkhoff [1] extrapolated this credo to identify projections with propositions. An investigation of how such propositions should behave with respect to logical connectives led to the introduction of the quantum logic $L(\mathcal{H})$. Explicitly,

Presented by **Robert Goldblatt**; *Received* January 18, 2012

Studia Logica (2013) 101: 901–913
DOI: 10.1007/s11225-012-9401-3

they showed that the set $L(\mathcal{H})$ of projections on $\mathcal{H}$ is turned into a complete bounded lattice by the partial order

$$P_1 \leq P_2 \Longleftrightarrow P_1\mathcal{H} \subset P_2\mathcal{H}\,. \tag{1}$$

Besides introducing the logical connectives 'and' and 'or' (as the meet and join in $L(\mathcal{H})$) also an operation for the negation was introduced:

$$\neg P := 1 - P, \tag{2}$$

where 1 denotes the unit operator and top element of $L(\mathcal{H})$.

The lattice $L(\mathcal{H})$ is almost a Boolean algebra, except for the fact that the laws of distributivity

$$P_1 \vee (P_2 \wedge P_3) = (P_1 \vee P_2) \wedge (P_1 \vee P_3) \tag{3}$$

and

$$P_1 \wedge (P_2 \vee P_3) = (P_1 \wedge P_2) \vee (P_1 \wedge P_3) \tag{4}$$

do not hold in general. Consequently, it is hard to interpret the meet and join as the logical connectives "and" and "or" (cf. [6]). Needless to say, quantum logic has struggled with interpretational problems ever since it was conceived.

The source of the counter-intuitive aspect of Birkhoff and von Neumann's result may be sought in the assumption that *all* propositions can be identified with projections. This idea indeed seems somewhat reckless. The credo 'projections as propositions' was based on considerations of propositions of the form $A \in \Delta$. However, there is of course no guarantee that new propositions formed from such propositions using logical connectives should again be of this form. In the next section I discuss two perspectives that lead to an expansion of the set of propositions. The first is an intuitionistic perspective, criticizing the disjunction in $L(\mathcal{H})$ for being too weak, and the second is a more classical perspective, criticizing the negation for being too strong. It is then shown that both result to the same Boolean logic for quantum mechanics.

In the final section I argue that this Boolean logic is actually unsatisfactory from the intuitionistic perspective. I show that upon defining the new stronger disjunction, the meaning of the negation has shifted. I then propose to keep both the new disjunction and the old negation and show that this can be done consistently within the framework of weakly Heyting algebras.

2. Two Perspectives

The peculiarity of quantum logic becomes explicit when focusing on the law of excluded middle. The proposition $P \vee \neg P$ is a tautology for every $P \in L(\mathcal{H})$. Consequently, for every $P' \in L(\mathcal{H})$ one has $P' \wedge (P \vee \neg P) = P'$. But it is not hard to find a pair P, P' with $P' \neq \bot$ such that $P' \wedge P = \bot$ and $P' \wedge \neg P = \bot$.

This can be traced back to Heisenberg's uncertainty principle [7]. If P is a proposition about the position of some particle and P' about the momentum of that particle, then this principle suggests that no certainty can be obtained about $P' \wedge P$ or $P' \wedge \neg P$. It has become consensus that this uncertainty is not just epistemic. It was shown by Kochen and Specker [10] that one cannot consistently attribute definite values to all observables. This result raises the question how one can meaningfully maintain that a disjunction $A \in \Delta \vee A \in \Delta^c$ can be true while rejecting that either of the disjuncts is true. A more extensive discussion on this issue may be found in [11]. Either way, for someone who upholds that a disjunction is only true if at least one of the disjuncts is true, the disjunction introduced by Birkhoff and von Neumann is unsatisfactory. Consequently, the law of excluded middle has to fail.[1] This is also roughly the viewpoint Coecke expresses in [4], and he argues that for an intuitionistic view on quantum mechanics

> "we formally need to introduce those additional propositions that express disjunctions of properties and that do not correspond to a property in the property lattice."

Note that this conclusion is obtained under the assumption that $A \in \Delta^c$ is the correct reading of the negation of $A \in \Delta$. This form of negation is somewhat intuitionistic in nature; it transforms a 'positive' proposition (concerning something happening with certainty) into another 'positive' proposition. The classical logician will want to uphold the law of excluded middle and thus opt for a new negation. I will return to this point later on.

In [4] the additional disjunctions are introduced by making use of Bruns and Lakser's theory of injective hulls [2]. Concretely, this means that the quantum lattice $L(\mathcal{H})$ is replaced by the lattice of distributive ideals of the quantum lattice:

$$\mathcal{DI}(L(\mathcal{H})) := \{I \subset L(\mathcal{H})\,;\ I \text{ is a distributive ideal}\}, \tag{5}$$

[1] A more extensive motivation for the use of intuitionistic logic in quantum mechanics may be found in [8, Ch. 5].

where a distributive ideal is a non-empty subset I such that

(i) If $P \in I$ and $P' \leq P$, then $P' \in I$.

(ii) If $\mathcal{K} \subset I$ and $\forall P' \in L(\mathcal{H})$: $\left(\bigvee_{P\in\mathcal{K}} P\right) \wedge P' = \bigvee_{P\in\mathcal{K}} (P \wedge P')$, then $\bigvee_{P\in\mathcal{K}} P \in I$.

This new set is turned into a lattice with the partial order

$$I_1 \leq I_2 \Longleftrightarrow I_1 \subset I_2, \tag{6}$$

where the join and meet are given by

$$\bigwedge_{I\in\mathcal{I}} I = \bigcap_{I\in\mathcal{I}} I,\ \mathcal{I} \subset \mathcal{DI}(L(\mathcal{H})), \tag{7}$$

$$\bigvee_{I\in\mathcal{I}} I = \bigwedge\{I' \in \mathcal{DI}(L(\mathcal{H}))\ ;\ I \leq I' \forall I \in \mathcal{I}\},\ \mathcal{I} \subset \mathcal{DI}(L(\mathcal{H})). \tag{8}$$

With these definitions, $\mathcal{DI}(L(\mathcal{H}))$ is a complete distributive lattice. The propositions of the original lattice $L(\mathcal{H})$ are identified with elements of $\mathcal{DI}(L(\mathcal{H}))$ by the injection

$$i : L(\mathcal{H}) \to \mathcal{DI}(L(\mathcal{H})), \quad P \mapsto \downarrow P := \{P' \in L(\mathcal{H})\ ;\ P' \leq P\}. \tag{9}$$

As such, the construction of $\mathcal{DI}(L(\mathcal{H}))$ meets the desires; the new disjunction $\downarrow P_1 \vee \downarrow P_2$ is not of the form $\downarrow P$ whenever $P_1P_2 \neq P_2P_1$ and thus does not correspond to any element in the original lattice. Because the new lattice is complete and the infinite laws of distributivity hold, it is also a complete Heyting algebra if one introduces the relative pseudo-complement

$$I_1 \to I_2 := \bigvee\{I_3 \in \mathcal{DI}(L(\mathcal{H}))\ ;\ I_3 \wedge I_1 \leq I_2\}. \tag{10}$$

Complementary to the approach above, instead of introducing a new disjunction that is more intuitionistic in nature than the one in quantum logic, one may want to define a new negation that is more classical in nature than the one in quantum logic. Indeed, if one reads the negation literally, the proposition $\neg(A \in \Delta)$ should be identified with the proposition that a measurement of A will not yield a result in Δ with probability one; i.e. one is not entirely certain that the measurement of A will yield a result in Δ. This statement is true for all the states in the set $(\mathcal{H} \setminus \mu_A(\Delta)\,\mathcal{H}) \cup \{0\}$.

To make this more formal, it is convenient to identify states with rays in the Hilbert space. The ray space is defined as

$$R(\mathcal{H}) := \{[\psi]\ ;\ \psi \in \mathcal{H}_0\},\ [\psi] := \{\lambda\psi\ ;\ \lambda \in \mathbb{C}\},\ \mathcal{H}_0 := \mathcal{H} \setminus \{0\}. \tag{11}$$

Propositions may then be identified with elements of the power set $\mathcal{P}(R(\mathcal{H}))$. Indeed, the proposition $A \in \Delta$ is now identified with the set

$$\{[\psi] \in R(\mathcal{H})\,;\, \psi \in \mu_A(\Delta)\,\mathcal{H}_0\} \tag{12}$$

and its negation, $\neg(A \in \Delta)$, with the complement of this set. The set $\mathcal{P}(R(\mathcal{H}))$ is turned into a Boolean algebra in the usual way:

$$S_1 \leq S_2 \Longleftrightarrow S_1 \subset S_2, \tag{13}$$

$$\bigwedge_{S \in \mathcal{S}} S = \bigcap_{S \in \mathcal{S}} S, \; \bigvee_{S \in \mathcal{S}} S = \bigcup_{S \in \mathcal{S}} S. \tag{14}$$

Although this approach differs strongly from the intuitionistic approach, it is remarkable that both constructions are in fact identical:

THEOREM 2.1. *The lattices $\mathcal{DI}(L(\mathcal{H}))$ and $\mathcal{P}(R(\mathcal{H}))$ are isomorphic (as complete bounded lattices). Consequently, the Heyting algebra $\mathcal{DI}(L(\mathcal{H}))$ is Boolean.*[2]

PROOF. For the proof the following definition is useful:

$$\forall S \in \mathcal{P}(R(\mathcal{H})): \; S^0 := \{\psi \in \mathcal{H}_0\,;\, [\psi] \in S\} \cup \{0\}. \tag{15}$$

Define the following function $f : \mathcal{P}(R(\mathcal{H})) \to \mathcal{P}(L(\mathcal{H}))$:

$$f(S) := \left\{P \in L(\mathcal{H})\,;\, P\,\mathcal{H} \subset S^0\right\}. \tag{16}$$

This function satisfies

$$f(R(\mathcal{H})) = L(\mathcal{H}), \quad f(\varnothing) = \{0\} \quad \text{and} \quad f(\{[\psi]\}) = \{P_\psi\} \;\; \forall [\psi] \in R(\mathcal{H}), \tag{17}$$

where P_ψ is the one-dimensional projection on the subspace spanned by ψ.

Now, for every $S \in \mathcal{P}(R(\mathcal{H}))$, $f(S)$ is a distributive ideal. This is proven by showing that $f(S)$ satisfies the properties (i) and (ii). Suppose $P \in f(S)$ and $P' \leq P$. Then

$$P'\,\mathcal{H} \subset P\,\mathcal{H} \subset S^0 \tag{18}$$

and thus $P' \in f(S)$. To show property (ii) assume that $S \neq R(\mathcal{H})$ (for $S = R(\mathcal{H})$ (ii) is trivially satisfied). Suppose $\mathcal{K} \subset f(S)$ such that for every $P' \in L(\mathcal{H})$: $\left(\bigvee_{P \in \mathcal{K}} P\right) \wedge P' = \bigvee_{P \in \mathcal{K}} (P \wedge P')$. It then has to be shown that $\bigvee_{P \in \mathcal{K}} P \in f(S)$.

Suppose this isn't the case. Then there exists a non-zero vector $\psi \in \left(\bigvee_{P \in \mathcal{K}} P\right) \mathcal{H}$ such that $[\psi] \notin S$ and $P_\psi \wedge P = 0$ for all $P \in \mathcal{K}$. It then follows that

[2] This second statement is in fact a consequence of a more general result in [4] where it is shown that $\mathcal{DI}(L)$ is Boolean whenever L is atomic.

$$P_\psi = \left(\bigvee_{P\in\mathcal{K}} P\right) \wedge P_\psi = \bigvee_{P\in\mathcal{K}} (P \wedge P_\psi) = 0 \tag{19}$$

which is a contradiction since $\psi \neq 0$ being an element of the complement of S^0. This proves that $f : \mathcal{P}(R(\mathcal{H})) \to \mathcal{DI}(L(\mathcal{H}))$.

Next, consider the map

$$g : \mathcal{DI}(L(\mathcal{H})) \to \mathcal{P}(R(\mathcal{H})), \quad g : I \mapsto \bigcup_{P\in I} \{[\psi] \in R(\mathcal{H})\,;\, \psi \in P\mathcal{H}_0\}. \tag{20}$$

It will be shown that this is the inverse of f. For every set $S \in \mathcal{P}(R(\mathcal{H}))$ one has

$$g(f(S)) = \bigcup_{P\in f(S)} \bigcup_{\psi\in P\mathcal{H}_0} \{[\psi]\} = S. \tag{21}$$

Indeed, for every $[\psi] \in S$ it holds that $P_\psi \in f(S)$ and thus $[\psi] \in g(f(S))$. Conversely, if $[\psi] \in g(f(S))$ then $\exists P \in f(S)$ such that $\psi \in P\mathcal{H}$. Then, because $P\mathcal{H} \subset S^0, [\psi] \in S$.

The other way around one has that for every $I \in \mathcal{DI}(L(\mathcal{H}))\, f(g(I)) = I$. This can be shown directly:

$$\begin{aligned}
f(g(I)) &= \{P \in L(\mathcal{H})\,;\, P\mathcal{H} \subset g(I)^0\}\\
&= \{P \in L(\mathcal{H})\,;\, P\mathcal{H}_0 \subset \{\psi \in \mathcal{H}_0\,;\, [\psi] \in g(I)\}\}\\
&= \{P \in L(\mathcal{H})\,;\, P\mathcal{H}_0 \subset \{\psi \in \mathcal{H}_0\,;\, [\psi] \in \bigcup_{P'\in I} \{[\psi']\,;\, \psi'{\in}P'\mathcal{H}_0\}\}\}\\
&= \{P{\in}L(\mathcal{H})\,;\, P\mathcal{H}_0 \subset \{\psi{\in}\mathcal{H}_0\,;\, \exists P'{\in}I \text{ such that } \psi{\in}P'\mathcal{H}_0\}\}\\
&= \{P \in L(\mathcal{H})\,;\, P\mathcal{H}_0 \subset \{\psi \in \mathcal{H}_0\,;\, P_\psi \in I\}\}\\
&= \{P \in L(\mathcal{H})\,;\, P_\psi \in I\ \forall\psi \in P\mathcal{H}\} = I
\end{aligned} \tag{22}$$

where it has been used that I is a distributive ideal and that $P \in I$ iff $P_\psi \in I$ for all $\psi \in P\mathcal{H}$.

This shows that $\mathcal{DI}(L(\mathcal{H}))$ and $\mathcal{P}(R(\mathcal{H}))$ are isomorphic as sets. However, since both f and g respect the partial order structure, it follows that $\mathcal{DI}(L(\mathcal{H}))$ and $\mathcal{P}(R(\mathcal{H}))$ are also isomorphic as complete lattices. ■

3. Weakly Intuitionistic Quantum Logic

The fact that the application of Bruns and Lakser's theory to the quantum lattice results in the construction of a Boolean algebra may be explained in the following way. The introduction of a new disjunction forces the introduction of a new negation. Indeed, the new negation in $\mathcal{DI}(L(\mathcal{H}))$ is defined as $\neg I := I \to \downarrow 0$ and it is much weaker than the negation in quantum logic because one has

$$\downarrow \neg P \leq \neg \downarrow P, \quad \forall P \in L(\mathcal{H}) \tag{23}$$

with equality iff $P = 0$ or $P = 1$. From the perspective of $\mathcal{P}(R(\mathcal{H}))$ it is clear to see that the negation in $\mathcal{DI}(L(\mathcal{H}))$ behaves classical rather than intuitionistic. This is made more explicit by introducing the embedding $r : L(\mathcal{H}) \to \mathcal{P}(R(\mathcal{H}))$ given by $r(P) := \{[\psi] \in R(\mathcal{H})\ ;\ \psi \in P\mathcal{H}_0\}$. Theorem 2.1 then shows that the diagram

$$\begin{tikzcd}
L(\mathcal{H}) \arrow[r, hook, "i"] \arrow[d, hook, "r"] & \mathcal{DI}(L(\mathcal{H})) \arrow[dl, shift left, "f^{-1}"] \\
\mathcal{P}(R(\mathcal{H})) \arrow[ur, shift left, "f"] &
\end{tikzcd} \tag{24}$$

commutes.

It would seem more intuitionistic if one could generalize the negation of the quantum lattice to a negation in the lattice $\mathcal{DI}(L(\mathcal{H}))$. That is, by introducing a function $\sim: \mathcal{DI}(L(\mathcal{H})) \to \mathcal{DI}(L(\mathcal{H}))$ such that $\sim\downarrow P = \downarrow \neg P$ for all $P \in L(\mathcal{H})$. In such a scheme, the negation of $A \in \Delta$ would coincide with $A \in \Delta^c$ like in quantum logic, but the disjunction of $A \in \Delta$ and $A \in \Delta^c$ would not be a triviality.

The introduction of $\sim$ is actually straightforward. First note that

$$r(\neg P) = \{[\psi] \in R(\mathcal{H})\ ;\ \langle \psi, \phi \rangle = 0 \quad \forall \phi \in P\mathcal{H}\}. \tag{25}$$

This suggests the definition

$$\sim S := \{[\psi] \in R(\mathcal{H})\ ;\ \langle \psi, \phi \rangle = 0 \quad \forall \phi \text{ with } [\phi] \in S\}. \tag{26}$$

Indeed, this satisfies $\sim r(P) = r(\neg P)$ for all $P \in L(\mathcal{H})$. The 'pseudo-negation' $\sim$ also behaves typically intuitionistic since one has

$$S \vee \sim S = R(\mathcal{H}) \text{ iff } S = \varnothing \text{ or } S = R(\mathcal{H}), \tag{27}$$

$$\sim S \vee \sim\sim S = R(\mathcal{H}) \text{ iff } S = \varnothing \text{ or } S = R(\mathcal{H}), \tag{28}$$

while maintaining

$$\sim\sim (S \vee \sim S) = R(\mathcal{H}), \quad \forall S \in \mathcal{P}(R(\mathcal{H})). \tag{29}$$

One may also show that of the De Morgan laws only

$$\sim S_1 \wedge \sim S_2 = \sim (S_1 \vee S_2), \quad \forall S_1, S_2 \in \mathcal{P}(R(\mathcal{H})) \tag{30}$$

holds, and the other only holds in one direction:

$$\sim S_1 \vee \sim S_2 \leq \sim (S_1 \wedge S_2), \quad \forall S_1, S_2 \in \mathcal{P}(R(\mathcal{H})). \tag{31}$$

The pseudo-negation also relates the 'intuitionistic' disjunction of $\mathcal{P}(R(\mathcal{H}))$ to the 'classical' disjunction of $L(\mathcal{H})$ by the equality

$$\sim\sim \left(\bigvee_{P \in \mathcal{K}} r(P) \right) = r \left(\bigvee_{P \in \mathcal{K}} P \right) \quad \forall \mathcal{K} \subset L(\mathcal{H}). \tag{32}$$

So for any subset S of $R(\mathcal{H})$, its double pseudo-negation coincides with the closed linear subspace spanned by all the elements of S.

Although the pseudo-negation appears to behave intuitionistic, there is no trivial way to incorporate the lattice $(\mathcal{P}(R(\mathcal{H})), \vee, \wedge, \sim)$ in a Heyting algebra. This is because the relative pseudo-complement for the lattice $(\mathcal{P}(R(\mathcal{H})), \vee, \wedge)$ is uniquely defined. There may however still be the possibility that a satisfactory implication relation $\rightarrow$ (that is not a relative pseudo-complement) may be defined on this lattice such that $S \rightarrow \perp = \sim S$ for all $S \in \mathcal{P}(R(\mathcal{H}))$. It turns out that this is indeed possible within the theory of weakly Heyting algebras, where a weakly Heyting algebra is defined as follows.

Definition 3.1. A weakly Heyting algebra $(L, \vee, \wedge, \rightarrow)$ is a bounded distributive lattice with an implication relation that satisfies

(i) $S_1 \rightarrow S_1 = \top$,

(ii) $S_1 \rightarrow (S_2 \wedge S_3) = (S_1 \rightarrow S_2) \wedge (S_1 \rightarrow S_3)$,

(iii) $(S_1 \vee S_2) \rightarrow S_3 = (S_1 \rightarrow S_3) \wedge (S_2 \rightarrow S_3)$,

(iv) $(S_1 \rightarrow S_2) \wedge (S_2 \rightarrow S_3) \leq S_1 \rightarrow S_3$,

for all $S_1, S_2, S_3 \in L$.

The merit of this framework is that if an implication relation satisfies these rules one immediately acquires some more properties one would expect to hold for such a relation e.g.[3]

(a) If $S_1 \leq S_2$, then for all S_3 $S_3 \rightarrow S_1 \leq S_3 \rightarrow S_2$ and $S_2 \rightarrow S_3 \leq S_1 \rightarrow S_3$.

(b) If $S_1 \leq S_2$, then $S_1 \rightarrow S_2 = \top$.

(c) For all S_1, S_2, S_3 $(S_1 \rightarrow S_2) \wedge (S_1 \rightarrow S_3) \leq S_1 \rightarrow (S_2 \vee S_3)$.

[3] Proofs may be found in [3].

A straightforward approach to finding an implication relation on the lattice $(\mathcal{P}(R(\mathcal{H})), \vee, \wedge)$ would be to generalize the Sasaki arrow on $L(\mathcal{H})$ given by

$$P_1 \overset{S}{\rightarrow} P_2 := \neg P_1 \vee (P_1 \wedge P_2). \tag{33}$$

However, this approach is bound to fail. The rules (i), (ii) and (iv) would then have to hold also for the Sasaki arrow on $L(\mathcal{H})$ (because the injection r preserves order and meets). But although $\overset{S}{\rightarrow}$ does satisfy (i) and (ii), a counter example for (iv) is found for taking $\mathcal{H} = \mathbb{C}^2$ and $P_1 = P_x, P_3 = P_{x+y}$ and P_2 the unit matrix. (iv) then reads $P_3 \leq \neg P_1$ which is clearly false. On the other hand this may be a merit because of the criticism the Sasaki arrow has received c.f. [5].

That there does exist an implication relation with the desired properties is shown in the following theorem:

Theorem 3.2. *There exists an implication relation on* $(\mathcal{P}(R(\mathcal{H})), \vee, \wedge)$ *such that* $(\mathcal{P}(R(\mathcal{H})), \vee, \wedge, \rightarrow)$ *is a weakly Heyting algebra and*

(v) $S \rightarrow \bot = \sim S$ *for all* $S \in \mathcal{P}(R(\mathcal{H}))$.

Proof. The implication relation will be defined in steps. Because of (b) one requires that $\bot \rightarrow S = \top$ for every $S \in \mathcal{P}(R(\mathcal{H}))$. Now let $[\psi] \in R(\mathcal{H})$. If $[\psi] \in S$ then because of (b) $\{[\psi]\} \rightarrow S = \top$. On the other hand, if $S = \bot$, then because of (v) $\{[\psi]\} \rightarrow \bot = \sim \{[\psi]\}$. In all other cases one has

$$\{[\psi]\} \rightarrow S \overset{\text{(i)}}{=} (\{[\psi]\} \rightarrow S) \wedge (\{[\psi]\} \rightarrow \{[\psi]\}) \overset{\text{(ii)}}{=} \{[\psi]\} \rightarrow \bot = \sim \{[\psi]\}. \tag{34}$$

To sum up:

$$\{[\psi]\} \rightarrow S = \begin{cases} \top, & [\psi] \in S, \\ \sim \{[\psi]\}, & [\psi] \notin S. \end{cases} = \sim\sim (\sim \{[\psi]\} \vee (\{[\psi]\} \wedge S_2)). \tag{35}$$

Now the general case can be defined by assuming (iii) to hold for arbitrary joins and meets:

$$\begin{aligned} S_1 \rightarrow S_2 &:= \bigwedge_{[\psi] \in S_1} \{[\psi]\} \rightarrow S_2 = \bigwedge_{[\psi] \in S_1} \sim\sim (\sim \{[\psi]\} \vee (\{[\psi]\} \wedge S_2)) \\ &= \bigwedge_{[\psi] \in S_1 \setminus S_2} \sim \{[\psi]\}, \end{aligned} \tag{36}$$

where the empty meet is identified with $\top$.

It will now be shown that (36) indeed satisfies (i)–(v). In fact, (i) is trivial. (ii),(iii) and (iv) follow by writing out.

$$
\begin{aligned}
S_1 \to (S_2 \wedge S_3) &= \bigwedge_{[\psi]\in S_1\setminus(S_2\wedge S_3)} \sim\{[\psi]\} = \bigwedge_{[\psi]\in(S_1\setminus S_2)\vee(S_1\setminus S_3)} \sim\{[\psi]\} \\
&= \left(\bigwedge_{[\psi]\in S_1\setminus S_2} \sim\{[\psi]\}\right) \wedge \left(\bigwedge_{[\psi]\in S_1\setminus S_3} \sim\{[\psi]\}\right) \\
&= (S_1 \to S_2) \wedge (S_1 \to S_3).
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
(S_1 \vee S_2) \to S_3 &= \bigwedge_{[\psi]\in(S_1\vee S_2)\setminus S_3} \sim\{[\psi]\} = \bigwedge_{[\psi]\in(S_1\setminus S_3)\vee(S_2\setminus S_3)} \sim\{[\psi]\} \\
&= \left(\bigwedge_{[\psi]\in S_1\setminus S_3} \sim\{[\psi]\}\right) \wedge \left(\bigwedge_{[\psi]\in S_2\setminus S_3} \sim\{[\psi]\}\right) \\
&= (S_1 \to S_3) \wedge (S_2 \to S_3).
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
(S_1 \to S_2) \wedge (S_2 \to S_3) &= \left(\bigwedge_{[\psi]\in S_1\setminus S_2} \sim\{[\psi]\}\right) \wedge \left(\bigwedge_{[\psi]\in S_2\setminus S_3} \sim\{[\psi]\}\right) \\
&= \bigwedge_{[\psi]\in(S_1\setminus S_2)\vee(S_2\setminus S_3)} \sim\{[\psi]\} \leq \bigwedge_{[\psi]\in S_1\setminus S_3} \sim\{[\psi]\} \\
&= (S_1 \to S_3).
\end{aligned} \tag{39}
$$

Finally, it remains to be shown that (v) holds:

$$
S \to \bot = \bigwedge_{[\psi]\in S} \{[\psi]\} \to \bot = \bigwedge_{[\psi]\in S} \sim\{[\psi]\} =\sim S. \tag{40}
$$

■

The proof almost shows that the defined implication is also unique. At least it is the unique one satisfying (iii) for arbitrary joins and meets. A more formal and general proof of uniqueness follows from the following theorem by observing that $(\mathcal{P}(R(\mathcal{H})), \vee, \wedge)$ is a Boolean lattice.

Theorem 3.3. *Suppose $(A, \vee, \wedge)$ is a Boolean lattice and suppose both $\to$ and $\to'$ turn $(A, \vee, \wedge)$ into a weakly Heyting algebra. If*

$$
\forall a \in A: \ a \to \bot = a \to' \bot, \tag{41}
$$

then $\to = \to'$.

Proof. It was shown in [3] that every weakly Heyting algebra $(A, \vee, \wedge, \to)$ with $(A, \vee, \wedge)$ a Boolean lattice is derived from a normal modal algebra $(A, \vee, \wedge, ^c, \Diamond)$ by the rule $a_1 \to a_2 = (\Diamond(a_1 \wedge a_2^c))^c$, where ^{c} denotes the complement in the Boolean lattice $(A, \wedge, \vee)$. The reverse rule is given by $\Diamond a = (\top \to a^c)^c$.

Now let $(A, \vee, \wedge, ^c, \Diamond)$ be the normal modal algebra corresponding to the weakly Heyting algebra $(A, \vee, \wedge, \to)$ and $(A, \vee, \wedge, ^c, \Diamond')$ the one corresponding to $(A, \vee, \wedge, \to')$. Because $a \to \bot = a \to' \bot$ for all a, it follows that

$$\forall a \in A : (\Diamond(a \wedge \bot^c))^c = (\Diamond'(a \wedge \bot^c))^c. \tag{42}$$

Consequently, $\Diamond a = \Diamond' a$ for all a and thus $\to = \to'$. ■

The relation between weakly Heyting algebras with a Boolean lattice and normal modal algebras is also useful for investigating the weakly intuitionistic quantum logic defined by (36). For one, it allows the introduction of the modal operator $\Diamond$ given by

$$\Diamond S := (\top \to S^c)^c = \left(\bigwedge_{[\psi] \in S} \sim \{[\psi]\} \right)^c = (\sim S)^c. \tag{43}$$

In the simple case where S is of the form $A \in \Delta$ (i.e. $S = r(\mu_A(\Delta))$, see section 1) $\Diamond S$ corresponds to the set of all rays given by states ψ for which the probability of obtaining a value in Δ upon a measurement of A is greater than zero. This coincides nicely with the interpretation of 'possibility' for $\Diamond$.

The simple propositions of the form $A \in \Delta$ are also the most convenient for revealing the main features of the implication relation. One may show that for any pair of observables A_1, A_2 one has

$$A_1 \in \Delta_1 \to A_2 \in \Delta_2 = \begin{cases} \top, & \text{if } \mu_{A_1}(\Delta_1) \leq \mu_{A_2}(\Delta_2), \\ \sim (A_1 \in \Delta_1), & \text{else.} \end{cases} \tag{44}$$

This is to be contrasted with the original Sasaki implication

$$\mu_{A_1}(\Delta_1) \xrightarrow{S} \mu_{A_2}(\Delta_2) = \neg \mu_{A_1}(\Delta_1) \vee (\mu_{A_1}(\Delta_1) \wedge \mu_{A_2}(\Delta_2)). \tag{45}$$

They coincide iff $\mu_{A_1}(\Delta_1) \leq \mu_{A_2}(\Delta_2)$ or $\mu_{A_1}(\Delta_1) \wedge \mu_{A_2}(\Delta_2) = \bot$ and these are the situations in which they both behave reasonably. It is a peculiar phenomenon of the weakly intuitionistic implication in other situations that it is true precisely when the antecedent is false (or not possible in the classical sense: $A_1 \in \Delta_1 \to A_2 \in \Delta_2 = (\Diamond A_1 \in \Delta_1)^c$). This has an explanation in the case where A_1 and A_2 are incompatible observables (i.e. they cannot be measured simultaneously) for then it makes sense that a proposition about A_1 has no bearing on any proposition about A_2 except when reasoning from a contradiction $(A_1 \in \Delta_1 \wedge \sim (A_1 \in \Delta_1))$. The Sasaki arrow on the other hand completely ignores the incompatibility between antecedent and consequent in the case of incompatible observables. But in the case where A_1 and A_2 are compatible (44) seems wrong. For example, if $A_1 = A_2$ and $\Delta_2 \subset \Delta_1$ one would want that at least $A_2 \in \Delta_2 \leq A_1 \in \Delta_1 \to A_2 \in \Delta_2$.

In conclusion, the weakly Heyting algebra $(\mathcal{P}(R(\mathcal{H})), \vee, \wedge, \to)$ solves some of the interpretational problems that arise in the standard quantum logic of Birkhoff and von Neumann. One of the main advantages is that the laws of distributivity are recovered. Also the failure of the law of excluded middle may be seen as merit in connection with the impossibility of assigning definite values to all observables. However, some problems remain, especially when it comes to the implication relation. The source of these problems may be sought in the fact that the weakly intuitionistic logic (like the standard quantum logic) does not distinguish propositions concerning incompatible observables from propositions concerning compatible observables. It may be interesting to investigate the possibilities for making this distinction. At least the results in this paper may serve as a source for inspiration.

Acknowledgements. The author would like to thank C. Heunen, B. Coecke and one anonymous referee for very useful comments on earlier versions.

References

[1] Birkhoff, G., and J. von Neumann, The logic of quantum mechanics, *Annals of Mathematics* 37(4):823–843, 1936. Reprinted in [9], pp. 1–26.

[2] Bruns, G., and H. Lakser, Injective hulls of semilattices, *Canadian Mathematical Bulletin* 13:115–118, 1970.

[3] Celani, S., and R. Jansana, Bounded distributive lattices with strict implication, *Mathematical Logic Quarterly* 51(3):219–246, 2005.

[4] Coecke, B., Quantum logic in intuitionistic perspective, *Studia Logica* 70(3):411–440, 2002.

[5] Coecke, B., and S. Smets, The Sasaki hook is not a [static] implicative connective but induces a backward [in time] dynamic one that assigns causes, *International Journal of Theoretical Physics* 43(7/8):1705–1736, 2004.

[6] Dummett, M., Is logic empirical?, in H.D. Lewis (ed.), *Contemporary British Philosophy Volume IV*, George Allen and Unwin, London, 1976.

[7] Heisenberg, W., Über den anschauliche Inhalt der quantentheoretischen Kinematik und Mechanik, *Zeitschrift für Physik* 43:172–198, 1927.

[8] Hermens, R., Quantum mechanics: from realism to intuitionism, Master's thesis available at http://philsci-archive.pitt.edu/id/eprint/5021, 2010.

[9] Hooker, C.A., *The Logico-Algebraic Approach to Quantum Mechanics*, Volume I, D. Reidel, Boston, 1975.

[10] Kochen, S., and E.P. Specker, The problem of hidden variables in quantum mechanics, *Journal of Mathematics and Mechanics* 17:59–67, 1967. Reprinted in [9], pp. 293–328.

[11] Stairs, A., Quantum logic, realism, and value definiteness, *Philosophy of Science* 50:578–602, 1983.

[12] von Neumann, J., *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955. Translated from the German by R.T. Beyer. Original title: *Mathematische Grundlagen der Quantenmechanik*, 1932.

Ronnie Hermens
Department of Theoretical Philosophy
University of Groningen
Oude Boteringestraat 52
Groningen, The Netherlands
r.hermens@rug.nl