FREGE'S THEOREM IN PLURAL LOGIC

SIMON HEWITT

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In the *Foundations of Arithmetic* Frege advances the thesis that numbers[1] are objects. Specifically he claims that they are the extensions of equinumerosity concepts, knowledge about which may be obtained by logical means alone. Yet the first example of a number term he offers when clarifying his account of the relationship between numbers and their associated concept is striking, 'the number of the moons of Jupiter'[Frege, 1950, 69e].[2] This is noteworthy because 'the moons of Jupiter' doesn't look like a noun phrase denoting a *concept* at all, but rather a *plural* noun phrase denoting some physical objects, the moons taken together.[3] Moreover when latter day neo-Fregeans offer a natural language interpretation of Hume's Principle,

(HP) $$\forall F \forall G\, (\#F = \#G) \leftrightarrow (F \approx G)$$

they almost universally do so in a manner similar to the following: 'the number of Fs is the same as the number of Gs iff the Fs are bijectable onto the Gs'. But again 'the Fs' and 'the Gs' are plural definite descriptions, and *prima facie* at least do not denote concepts at all, but rather denote collectively the objects falling under the relevant concepts.[4] That this plural usage is well advised may be confirmed by comparing Frege's own more prolix alternative: 'the number which belongs to the concept F is the same as that which belongs to the concept G . . .'. When concerns about ease of reading are disregarded, however, it is arguably the case that Frege captures better the second-order variables in the formalised version. Second- order quantification is, after all, quantification into predicate position, and there is not obviously anything plural about the semantic contributions of predicates.[5] Yet an intriguing possibility is opened by reflection on the use of plurals by logicists in articulating claims about number. What if we took seriously the suggestion that numbers are plural properties, properties exemplified by one or more things together?[6] Might the resulting picture give rise to a tenable version of logicism about arithmetic?

Any logicism deserving of consideration as part of the Fregean tradition will contain at least two distinct, but of necessity related, strands:

Thanks to Keith Hossack, Martin Pleitz, Ian Rumfitt, Robbie Williams, Jack Woods, and an audience at the Reasoning Club in Turin for discussion.

[1]Here and in what follows 'numbers' abbreviates 'natural numbers' unless otherwise stated. As we will see, it is important for the project outlined here that zero is not considered one of the natural numbers.

[2]German: *die Zahl der Jupitersmonde.*

[3]On plurals, see for instance [Oliver and Smiley, 2013], [Linnebo, 2012], [McKay, 2006] and [Boolos, 1998d].

[4]On the collective nature of plural designation I side with [Hossack, 2000] against [Oliver and Smiley, 2013]. For the view that attributions of numbers are made of objects, rather than of concepts under which those objects fall, see [van Inwagen, 2014, 61].

[5]For arguments to this effect see, for example, [Williamson, 2013, 243].

[6]It is important for clarity to distinguish between a *plural property*, that is a property with one argument place saturatable by some n entities for $n \geq 1$, and a *singular relation*, that is a relation with n argument places saturable by single entities. Nothing said here need to turn on the nature, or metaphysical depth, of this distinction. For an argument that numbers are plural properties on the basis of considerations from the semantics of number talk see [Moltmann, 2016]. Note, however, that Moltmann takes numbers to be tropes, rather than universals. Arguing the contrary position is work for elsewhere.

(1) A **metaphysical** account of arithmetical truth: that is to say an account of what (if anything) number theoretic propositions concern. This will typically proceed by analysis of arithmetical language.
(2) A **logico-epistemological** account of how it is that the axioms of arithmetic can be known in such a way that makes their status appropriately described in terms of analyticity, or a similar notion. In neo-Fregean writing from recent decades this status is understood to be secured if a suitable collection of axioms can be derived by logical means alone (possibly including higher-order logical means) from definitions (possibly including implicit definitions). A constraint on this account is that it should be compatible with the account offered of arithmetic truth.[7]

The metaphysical account is business for a future paper, and no illusions should be sown about the philosophically exacting nature of the task of constructing such an account. Not least amongst the obstacles that lie in the way here are Frege's own animadversions on the doctrines that 'Numbers are properties of external things' and that 'The number word "one" stand[s] for a property of objects' [Frege, 1950, 29]. Something should be said, however, about the understanding of existence and its relation to quantification that informs what follows. We allow, indeed it is vital for our purposes, that first-order variables include numbers in their range(or to avoid begging any questions at this stage, would do were numbers to exist). This is in spite of the insistence that numbers are properties. We are therefore committed, in a very non-Fregean fashion, to a *flat ontology*. It is possible to say, univocally, of properties and of the entities that exemplify properties that they *exist*, and this is appropriately formalised using the singular existential quantifier of first-order logic.[8] It is an an immediate corrolary, on pain of a version of Cantor's paradox, that the account of properties appealed to in defence of our logicism must be relatively sparse in David Lewis' sense [Lewis, 1997]. The availibility of a univocal quantifier of this sort is not incompataible with the suggestion that reality has a more fine-grained structure than is evident in the correct use of this quantifier, perhaps hinted at by restricted uses of 'exists' in ordinary contexts – McDaniel provides one model for how things might proceed here [McDaniel, 2009]. Nothing said in the present paper, therefore, forecloses on the position, Aristotelian in feel, that the number n depends for its existence on the objects in the n-membered pluralities, just so long as we can say together of entities that stand in asymmetric relations of ontological dependence that they *exist*.

The paper proceeds as follows. Section One notes existing work of relevance, examining some comments of George Boolos and the more substantial plural logicist project of Fracesca Boccuni. Boccuni's work, which provides much of the background for the logicism developed here is criticised on the basis of its logical and metaphysical commitments, and it is observed that bi-interpretability results for plural and second-order logics suggest that a derivation of an implementation of arithmetic in a plural logic should be possible which avoids the problems we identify for Boccuni. Section Two develops the plural logic, **PFO**. Section Three introduces an abstraction principle for ordered pairs, and comments briefly on the ontological implications of this. Section Four motivates and lays out what we term *Boolos' Principle* – an implicit definition of number appealing to only the logical vocabulary of the language of **PFO** and terms for our pair abstracts. Section Five proves a version of Frege's theorem for **PFO** on the basis of Boolos' Principle and suitable definitions. Section Six concludes the paper with some philosophical commentary.

[7]Compatibility here is intended as a more exacting constraint than mere logical consistency – the logico-epistemological strand to the project should shed some light on how arithmetic knowledge is possible, given that numbers are the kind of things the metaphysical strand proposes them to be, assuming that the metaphysical strand claims that numbers are *any* kind of thing. Alternatively, if the metaphysical strand disavows an ontology of numbers, and therefore offers a semantics of number talk which comes apart from the apparent surface form of mathematical language, this should be reflected in the account of mathematical knowledge placed on the table. Either way, the challenge is to respond to the dilemma posed by [Benacerraf, 1973] and its successors
[8]For motivation see [van Inwagen, 2009].

1. Plural Logicism

George Boolos addressed the question how the higher-order locutions of Frege's *Bergriffsschrift* should be interepreted. If the formalism is available for discourse about any subject matter whatsoever, as its presumed status as logic suggests, then there is a profound difficulty in interpreting higher-order quantification as over sets or properties of objects of whatever is in the first-order theory; a commitment to the Russell set or a similarly problematic property[9] is quickly forthcoming. The moral Boolos draws is that 'we must find another way to interpret the formalism of the *Bergriffsschift*' [Boolos, 1998c, 163] There is, Boolos believes, no one intended interpretation of the formalism, so the task simply is to provide a means of understanding it that is both lucid and avoids unwanted commitments. He proposes that translating monadic second-order existential quantification into natural language plurals[10] serves these purposes. Elsewhere he develops a plural interpretation of monadic second-order logic (**MSOL**) in some detail [Boolos, 1998d] [Boolos, 1998b].

Since Boolos' pioneering work, formal logics of plurals have been developed. A natural thought is that a version of logicism along the lines of [Wright, 1983] might be developed in one of these logics. This is reinforced by the fact that an intuitive and fairly minimal logic of plurals, which we will call **PFO**, is bi-interpretable with **MSOL**.[11] Plural logic, then, seems to have a close relationship with second-order logic and might reasonably be supposed to be available for many of its usual applications. Unfortunately for those with ambitions in this direction, things are far from straightforward once we consider irreducibly polyadic second-order formulae.[12] There is no obvious plural interpretation of these; [13] and the limitation is far from being of mere formal interest. One irreducibly polyadic second-order formula is (HP), central to modern logicism of the neo-Fregean variety.[14] Boolos' own work-around, appealing to the existence of pair functions on mathematically interesting domains to simulate polyadic quantification by plural quantification over tuples, is suggestive, but without explication insufficient for logicist purposes. For if pairs are understood as set theoretic constructs, or are implemented in number-theory itself, then the quantificational resources used to prove Frege's theorem are mathematically entangled in a manner that fails to satisfy the epistemic desiderata of neo-logicism. It seems clear that any execution of the logicist project that makes use of a plural logic will have to draw on additional resources, and will need to provide reassurance that these do not interfere with either of the strands of logicism described above.

Important work in this area by Francesca Boccuni proposes that a theory of extensions plus predicative second-order logic (interpreted substituionally) be deployed along with a plural logic to derive an interpretation of **PA2**. Assertions about the existence of extensions are licensed by a schematic version of Basic Law V:

(V) $$\{x : \phi x\} = \{x : \psi x\} \leftrightarrow \forall x(\phi x \leftrightarrow \psi x)$$

Where ϕ and ψ contain neither free plural variables, nor bound second-order variables. The same restrictions, plus the requirement that 'F' not occur free in ϕ, apply to the second-order comprehension schema:

(PRC) $$\exists F \forall x\,(Fx \leftrightarrow \phi x)$$

9 The property exemplified by all and only those properties that are not self-exemplifying.

10 A 'fix' is required to ensure that every instance of monadic second-order comprehension is validated, owing to the unavailibility of quantification over an 'empty' plurality.

11 To show this, we prove the existence of an effective translation in each direction by routine induction on complexity of formulae, and then check the theoremhood of the translations of axioms and that inferences licensed by rule applications continue to be licensed with respect to translations.

12 ϕ in the language of second-order logic is irreducibly polyadic iff there is no ψ such that: ψ contains at most only monadic predicates or second-order variables and ψ is logically equivalent to ϕ.

13 For one suggestion see [Hewitt, 2012].

14 The point is made by Hale and Wright, who also note that, since Boolos only translates existential quantification he 'provides no resources for the construal of universal higher-order quantification for one not inclined to accept the classical interdefinability of the quantifiers' [Hale and Wright, 2005, 197]. The present author is so inclined, but the point is well made.

Plural comprehension is also a schema; I have modified Boccuni's notation for reasons of continuity with what follows:

(PLC) $$\exists xx \forall x\, (x \prec xx \leftrightarrow \phi x)$$

where ϕ does not contain $\ulcorner xx \urcorner$ free. It's worth observing that this fails to enforce the requirement that at least one x be amongst any given xx, which is surely natural if xx are supposed to be *some entities*.[15] After all, there are no things such that nothing is one of those things. The system that results from **V**, **PRC**, and **PLC** plus an appropriate proof system is called *Plural Grundesetze* (PG). Given appropriate definitions, an interpretation of second-order Peano Arithmetic can be derived in PG. A model theoretic consistency proof is readily forthcoming [Boccuni, 2011]. The details of the formal development of arithmetic within PG are not our immediate concern – except where they matter for purposes of critical commentary, in which case I will highlight them – and the reader is referred to Boccuni's own work [Boccuni, 2010] [Boccuni, 2013].

Suggestive of the possibilities for a plural logicism about arithmetic though PG is, a number of philosophical difficulties arise. Not least amongst these is the appeal to naive extensions. Consistency is preserved in spite of this appeal by the restrictions placed on ϕ and ψ in **V** and **PRC**. Yet consistency doesn't suffice for acceptance of an abstraction principle to be philosophically well-motivated. For familiar reasons, not every individually consistent abstraction principle can be accepted, on pain of contradiction[16]. Why suppose that **V** is one of those that demands acceptance?

To my mind, there is at least one powerful reason that the logician who admits plural resources ought *not* to admit **V**. For such a logician has available a neat diagnosis of the pull towards acceptance of naive sets that was present at the dawn of modern set theory, and continues to manifest itself in contemporary authors [Linnebo, 2010] [Priest, 1987].[17] She can argue as follows:

> We think it is obvious, something like a basic law of thought, that given a condition ϕ there is a *collection* of all and only the x such that $\phi(x)$. What on earth could prevent this from being the case; surely for the collection to exist just is for each of the x s.t. $\phi(x)$ to exist? But now, for as long as we constrain our resources in a singularist fashion, we have a problem. For consider the conditions 'being a set' or 'being an ordinal', or – even worse – 'being a collection'. Paradox lurks not far away. But this just shows how mistaken we are to tie our hands by allowing only a singularist logic. For once we bring plurals within our purview, we can see that the word 'collection' (and similar words, including perhaps some uses of 'set'[18]) are ambiguous. There is one sense of 'collection', in which a collection is an entity, the value of a variable. Such collections are the subject matter of set theory, and their existence is far from trivial: on pain of paradox, some purported sets do not exist. On the other hand, 'collection' can simply mean 'some things'. And in this sense the existence of collections is entirely trivial given the existence of the collected objects, for the collection is not an entity over and above those objects, rather it simply *is* those objects. The collection, in this sense, of the non self-membered sets exists, and 'the collection of the non self-membered sets' denotes plurally the non self-membered sets.

In other words, once we disambiguate our collection concepts, we come to see that the pull towards naive *sets*, where sets are entities, is illusory. What is compelling is the idea, captured by the comprehension schema of any reasonable plural logic, that any satisfied[19] condition whatsoever corresponds to a plurality. This diagnosis is implicit in Boolos' writing on plurals:

[15]Rather than, say, a *set* of entities (or possibly of none).

[16]Consider, for instance, principles A and B such that each of A and B has a model, but that they jointly put incompatible cardinality demands on the domain.

[17]I develop here thoughts from my ANONYMISED.

[18]When I talk about a set of golf clubs, am I really committed, on pain of contradiction, to denying that the sets – in the mathematician's sense of that word – are all and only the pure sets?

[19]There is, surely, no empty plurality. The contrary belief arises, again, from a confusion of pluralities with sets.

> We cannot always pass from a predicate to the extension of the predicate, a set of things satisfying the predicate. We can, however, always pass to the things satisfying the predicate (if there is at least one), and therefore we cannot always pass from the things to a set of them. [Boolos, 1998c, 168]

It would be better, then, for the plural logicist to do without naive extensions if she can.

Another worry about PG relates to the logico-epistemological desideratum for logicism set out above. The logicist seeks to show that arithmetic enjoys an epistemologically basic status, akin to that of logic, perhaps suitably described in terms of analyticity. To do this, she shows that from logic and definitions alone, an interpretation of arithmetic can be derived. The operative thought here is that analyticity (as we shall now routinely call the target epistemological status[20]) is preserved over logical consequence. To secure the desired result, it is important therefore, that any definitions used not only genuinely *are* defintions (albeit possibly implicit ones), but that they clearly possess the requisite epistemological status; likewise it is important that the logic within which the derivation is undertaken genuinely *is* logic, or at least enjoys whatever cognitively basic status is taken to be a necessary condition for logicality.[21] It is furthermore surely desirable what is derived is an interpretation of arithmetic, and *no more*, or at least as little more as possible. For if our concern is epistemological score keeping, then the more extraneous information that is forthcoming from a logicist derivation, the more purchase is given to the opponent who would question the status of the resulting interpretation. Anyhow, logicism is of philosophical interest because it claims to show how knowledge of the basic principles of arithmetic[22] can be arrived at in a manner that exhibits its status as analytic. It seems important therefore that the output of the logicist endeavour be recognisable as (an interpretation of) the basic principles of arithmetic.

How should we assess success with respect to these epistemological aims? A useful formal surrogate for epistemological fundamentality is consistency strength.[23] In terms of this, we can formulate a new desideratum for the logicist:

> (**CON-EP**) The theory resulting from the derived principles of arithmetic, resulting from the logicist derivation, when closed under (second-order) logical consequence should have a consistency strength close to PA2 (ideally, the same). Each resource deployed for the derivation (abstraction principles/ system of logic) should have as low a consistency strength as possible.

There are reasons to doubt (CON-EP) is satisfied by PG, which is at least as strong as *third order* Peano Arithmetic. Arguably the theory is better viewed as a form of weak set theory, rather than a formalisation of *arithmetic*. This in no way robs it of mathematical interest but does provide reason to do that it demonstrates the epistemically basic character of arithmetic.

A final worry regarding Boccuni's execution of plural logicism is that it does not proceed via Hume's principle, or a suitable plural equivalent. To see this as a failing is not simply to cry apostasy on behalf of Scottish neo-logicism. Rather, the concern is that the use of Hume's principle in the Hale/Wright version of neo-logicism secures the satisfaction of what we can call, following Wright [Wright, 2000], *Frege's constraint*:

> (**FC**) A satisfactory foundation for a branch of mathematics should explain its basic concepts so that their applications are immediate.

The foundation offered by neo-Fregeanism for arithemtic satisfies (FC) admirably, explicating the concept of cardinal number through an implicit defintion which ties the concept to its characteristic application, numbering. By contrast, within PG, number is defined explicitly and internally to pure

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[20]Quinean sceptics are invited to substitute uniformly a more acceptable term throughout what follows.

[21]For some discussion of this, see [Linnebo, 2003]. Note that logicists may well want to resist the insistence that logic possesses some features often taken to be necessary for logicality, notably ontological innocence.

[22]Note we speak of *principles*, not *truths*. The more ambitious logicist project whose target was all the truths of arithmetic died with Gödel's proof of the First Incompleteness Theorem.

[23]This is, to deploy Benacerraf's term, the 'Princess Margaret' premise governing this paragraph [Benacerraf, 1998].

mathematics: some x is a number iff x is one of every inductive plurality. (FC) is not satisfied.

Why does this matter? On the one hand it sits uncomfortably with the identification of meaning acquisition with coming to be able to use the acquired expressions. The neo-Fregean programme ties my acquisition of singular numerical terms (and the sortal 'number') to what is often taken to be the canonical application of numbers, namely numbering. In determining that some statement asserting the existence of a one-one correlation is true, a language user may determine that a numerical equality is similarly true and thereby that numbers exist, introducing numerical expressions to her language by the abstraction principle that facilitates this transition. Epistemology, semantics, and metaphysics cohere tightly; this coherence is lost in PG. On the other hand, failure of (FC) looks likely to undermine a key aim of many logicist programmes. We want to be able to answer the nominalist who denies that reference to numbers is so much as even possible. One way of answering this character is to show that such reference is indeed possible, by recourse to the case of Crispin Wright's *Hero*, who possesses a higher-order language but no arithmetical concepts. In order to answer the nominalist satisfactorily, we need to show that the numerical concepts Hero acquires through (HP) are indeed *our* numerical concepts, and so that he refers to *numbers* (as opposed to things that imitate numbers for formal purposes). Insisting that the meaning of Hero's expressions be established in a way that maps *our* usage may provide reassurance here.

The task remains then to develop a plural logicism that does not appeal to naive sets, that gets PA2 (and no more) on the basis of minimal resources, and that satisfies (FC). To that task our attention now turns.

2. THE LOGIC PFO

We develop the logic **PFO** (for 'Plural First Order'[24]). First we specify a formal language $\mathcal{L}_{PFO}$. The lexicon of $\mathcal{L}_{PFO}$ consists of. :

(1) Denumerably many singular variables 'x_0', 'x_1',...
(2) Denumerably many plural variables 'xx_0', 'xx_1,...
(3) Denumerably many individual constants 'a_0', 'a_1',...
(4) Denumerably many plural constants 'aa_0', 'aa_1',...
(5) Denumerably many function terms '$f_0(x_0)$', '$f_1(x_0)$',... for each singular variable and plural variable, singular constant and plural constant.
(6) The dyadic predicate '='.
(7) The dyadic predicate '$\prec$'.
(8) The logical constants '$\perp$', '$\wedge$' and '$\neg$'.
(9) The existential quantifier '$\exists$'.
(10) The parentheses '(' and ')'.
(11) n-adic predicates, for $n \in \mathbb{N} > 2$, from the signature of the theory being formulated. Without loss of generality, take these to be $R^1_n, R^2_n \ldots$
(12) Nothing else.

As usual we admit the other standard logical constants and the universal quantifier as abbreviations. We also allow the informal use of 'y', 'zz' *etc.* as variables. The wffs of $\mathcal{L}_{PFO-}$ are defined recursively. In what follows t_0 and t_1 are metavariables ranging over singular variables and individual constants; tt_0 and tt_1 are metavariables ranging over plural variables and constants;P^n is a metavariable ranging over n-adic predicates. First the atomic wffs:

(1) '$\perp$' is a wff.
(2) $\ulcorner P^n t_1 \ldots t_n \urcorner$ is a wff
(3) $\ulcorner t_0 = t_1 \urcorner$ is a wff.
(4) $\ulcorner t_0 \prec t_1 \urcorner$ is a a wff.

[24]The title is owing to [Rayo, 2006] and isn't entirely happy, since on the standard semantics **PFO** lacks characteristic first-order metalogical properties: compactness, completeness, and upward and downward Löwenheim-Skolem properties. It is first-order purely in a *syntactic* sense, in that it permits only quantification into (albeit plural) *name* position. Regardless of the merits of the title, '**PFO**' has become established in the literature, and we are stuck with it. That which they call a rose...

Next we specify the rules for molecular wffs. Here ϕ and ψ range over wffs, v over singular variables, and vv over plural variables:

(1) $\ulcorner(\phi \wedge \psi)\urcorner$ is a wff.
(2) $\ulcorner\neg(\phi)\urcorner$ is a wff.
(3) $\ulcorner\exists v\, \phi\urcorner$ is a wff, so long as, if v has occurences in ϕ, they are not all bound.
(4) $\ulcorner\exists vv\, \phi\urcorner$ is a wff, so long as, if vv has occurences in ϕ, they are not all bound.

Nothing other than a string on the lexicon formed in accordance with these rules is a wff. In common usage, however, brackets are ommitted or added for the sake of clarity, where this is appropriate.

2.1. **An axiom system for PFO.** In what follows we will allow ourselves free use of defined vocabulary, and omit brackets where clarity of reading favours this. Every axiom, however, corresponds to an 'official' formula of $\mathcal{L}_{PFO}$. To an axiomatisation of first-order logic with identity, we add the following axioms:

(PLUR-IN) $$\forall xx\, \phi(xx) \rightarrow \phi(tt)$$

where tt is a plural term (variable or constant) free for 'xx' in ϕ.

(N-EMPT) $$\forall xx \exists x\, x \prec xx$$

And the following axiom schema:

(COMP) $$\exists y\, \phi(y) \rightarrow \exists xx \forall x\, (x \prec xx \leftrightarrow \phi(x))$$

where ϕ does not contain any free occurences of 'xx'.[25]

We also admit the following rule:

- **Plural generalisation:** $(\phi \rightarrow \psi(xx)) \Longrightarrow (\phi \rightarrow \forall xx\, \psi(xx))$, provided that 'xx' has no free occurence in either ψ or in any premise of the deduction in which the rule is invoked.

In what follows, however, we won't insist on formalising derivations as axiomatic proofs.

2.2. **Intuitive semantics.** First-order variables range over, and individual constants refer to, entities in some domain. Plural constants refer to pluralities on the domain. Plural variables range over pluralities on the domain. If the range of the plural variables encompass *all* the pluralities on the domain, every combinatorial possibility (equivalently: for every non-empty element x of the full classical power-set of the domain: the plurality consisting of all and only the elements of x is in the range of the plural variables), then we call the semantics *standard*. A plurality is to be understood as *some things*[26], rather than a reified collection – 'plurality' being a grammatically singular convenience.

If we can guarantee that the language $\mathcal{L}_{PFO}$ be interpreted in accordance with a standard semantics then the system of arithmetic resulting from what follows will be categorical. Such a guarantee seems reasonable, although has been questioned recently by Florio and Linnebo [Florio and Linnebo, 2015]. However, as befits a logicist project, we will proceed entirely proof theoretically, and leave semantic considerations for another occasion.

3. Abstraction for Pairs

PFO affords us resources that are provably equivalent to monadic second-order logic. However, in discussing Boccuni's logicism, we noted that it would be desirable to execute a logicist route to arithmetic via Hume's Principle, or something similar, in order to satisfy Frege's constraint. Yet Hume's Principle is irreducibly polyadic. How can we afford ourselves additional resources whilst

[25]In many developments of plural logic a version of the axiom of choice is included. We do not require this for our purposes.
[26]One thing can count as a limiting case of *some* things.

remaining within the bounds of the logicist acceptability?

The most familiar way of lending plural quantification the power of polyadic quantification is to simulate quantification into n-adic predicate position by means of plural quantification over pluralities of n-tuples. For most mathematically interesting theories, a pair function is available on the domain, and so for many purposes we can straightforwardly avail ourselves of plural quantification over tuples. For neo-logicist purposes, however, appeal to an arithmetical pairing function (say, using prime exponentiation) would introduce a fatal circularity to the derivation of an implementation of arithmetic. The existence of numbers is foremost amongst the things that neo-logicism is supposed to *establish*; the neo-logicist cannot, then, help herself to a pairing operation premised on the existence of the very numbers whose reality it is her purpose to demonstrate.

An alternative strategy is to introduce *sui generis* ordered pairs (hereafter, pairs) by abstraction, and this will be the path we take here. Pairs, after all, occur throughout mathematics and logic,[27] and it could be argued that we have a grasp of them independent of mathematics: consider talk of a pair of shoes.[28]. Pairs are certainly governed by a simple criterion of identity:

(PAIR) $$(< x_1, x_2 >=< x_3, x_4 >) \leftrightarrow (x_1 = x_3 \wedge x_2 = x_4)$$

This looks like an abstraction principle, although it is different in form from (HP), taking – in Shapiro's terminology – the variables 'two at a time' [Shapiro, 2000, 337]. This can be expressed using a quarternary predicate 'E' such that (PAIR) is equivalent to:

(1) $$(< x_1, x_2 >=< x_3, x_4 >) \leftrightarrow E(x_1, x_2, x_3, x_4)$$

It is routine to verify that E is an equivalence relation. We can understand (PAIR) then as introducing singular terms for pairs by abstraction on this equivalence. The principle has, moreover, a good claim to analyticity in Wright and Hale's sense, as an implicit definition of the sortal term 'pair'. It is ontologically inflationary (as of course is Boccuni's naive set comprehension, and as will need to be at least one of the foundational principles for a successful logicism about arithmetic), and has no non-trivial finite models[29]. Yet it is conservative in the sense that its application to an antecedently existing domain does not affect the cardinality of any sort of entity already in that domain[30]. It is intuitive and has consistency strength no greater than that of $I\Pi^0_1$. It is well suited for our purposes.[31]

4. Boolos' Principle

We noted in our discussion of Boccuni that it is desirable for a plural version of logicism to proceed *via* something along the lines of Hume's Principle, since this would satisfy the application constraint by tying the introduction of the concept of (cardinal) number to the canonical application of that concept in *counting*. What we want is to say that the number of xx is the same as the number

[27] Note that we do not need to decide the question whether the pairs intoduced by abstraction are identical with pairs implemented in one of the customary set theoretic fashions.

[28] I myself do not find this consideration very persuasive, for reasons related to the earlier discussion of naive sets. I think it probable that talk of pairs of shoes should be understood as plural talk about shoes, rather than as talk about more exotic entities (*pairs*) By contrast mathematical pair talk does seem to be reifying in import: consider, say, a construction of the integers in terms of equivalence classes of pairs of natural numbers. Having *introduced* pairs as sui generis, it is of course open to us *subsequently* to identify these pairs with objects indentified in a distinct theory, perhaps their Weiner-Kuratowski implementations in ZFC. This avenue does duty to the relatively epistemologically basic nature of pairs whilst streamlining ontological commitments.

[29] Trivially it has the model where the domain contains the sole object $< a, a >= a$, note that (PAIR) does not require well-foundedness. This raises issues noted in [Tennant, 2010] and [Pleitz, 2017]. I discuss these in [].

[30] *Modulo* a solution to the problem identified in the previous footnote.

[31] Martin Pleitz has drawn my attention to the worry that once we have found a way of ruling out trivial interpretations, (PAIR) already secures us a countable infinity of objects: won't this suffice for the ontology of arithmetic? Not from a neo-Fregean perspective, no: for we are not in the business of gross mass-tonnage ontology, but rather of explicating the metaphysics of mathematics in a manner that makes intelligible our reference to and knowledge of mathematical objects. To do *this* we must proceed by means of something like (HP).

of *yy* just in case there are some pairs, such that every one of *xx* is the first co-ordinate of exactly one pair and every one of *yy* is the second co-ordinate of exactly one pair. This captures the basic idea that we test pluralities for sameness of number by attempting to pair up their members without remainder. We can indeed state this is the language of **PFO**, enriched with term forming operators for pairs. We will call it *Boolos' principle*:

$$\text{(BP)}\quad \forall xx, yy\, (\mathbf{N}xx = \mathbf{N}yy \leftrightarrow \exists zz\, (\forall x \prec zz \exists z_1 \exists z_2 x =< z_1, z_2 >) \wedge \\ (\forall x \prec xx \exists! z \prec zz \exists! y \prec yy\, z =< x, y >) \wedge \\ (\forall y \prec yy \exists! z \prec zz \exists! x \prec xx\, x =< x, y >))$$

Here '**N**(xx)', and so on for other plural variables, is a functional term, intended to be interpreted as referring to the number of the plurality picked out by the variable. Note that 'N' properly extends the lexicon of **PFO** since it is a term-forming operator of singular type taking a plural term as its argument. This does not seem controversial, so we will admit it without further comment.[32].

The bounded quantifiers and uniqueness are defined as abbreviations in the obvious way.

5. A plural version of Frege's theorem

We proceed now to show that an interpretation of PA2 can be derived in **PFO** from definitions, (BP), and pair abstraction. In the spirit of Wright's original proof sketch in second-order logic we work informally rather than by axiomatic proof, the intention being to persuade that such a proof is to be had [Wright, 1983, 162]. The bi-interpretability of **MSOL** and **PFO**, extendable to full second-order logic since we have pairs provides reassurance here, although as we will see in the sequel there are differences between the executions of the proofs of the respective versions of Frege's theorem, and these are of philosophical interest.

Definition 1. *Nx iff $\exists xx\; x = \boldsymbol{N}xx$*

Informally, we read 'Nx' as 'x is a cardinal number. We will sometimes use the variables 'm', 'n' etc. to denote quantification restricted to cardinal numbers, and to natural numbers once we have defined these. We can quickly show:

Lemma 1. $\forall xx \exists n\; n = \mathbf{N}(xx)$

Proof. Consider an arbitrary plurality *aa* In (BP) instantiate both the prenex quantifiers with *aa*. We verify the RHS of the biconditional using the identity map from the members of *aa* to themselves. This gives us that $\mathbf{N}(aa) = \mathbf{N}(aa)$, and we now apply existential and universal instantion to get our lemma. □

We'll make free use of this lemma without explicit mention in what follows. Now let *a* be an object, chosen arbitrarily. We are assured that there is such an object by existential instantiation on $\exists x\; x = x$, which is a theorem of **PFO**. However, we will strengthen this to the assumption that *a* is a non-number ($\neg Na$ in the notation introduced below). Philosophical issues arising here will be addressed in the sequel.

Definition 2. $1 = \boldsymbol{N}[x : x = a]$

Where $\ulcorner[\phi]\urcorner$ abbreviates a term for the plurality consisting of all and only the x such that $\phi(x)$ (which exists by (COMP)). Now we have:

Theorem 1. *$N1$*

Proof. We have $\forall x\, x = x$ as a theorem of **PFO**. Instantiating $\forall$, $a = a$. By (COMP) $\exists xx \forall y\, y \prec xx \leftrightarrow y = a$. Instantiating $\exists$, $y \prec aa \leftrightarrow y = a$. By definition 2, then $\mathbf{N}aa = 1$. So, by definition 1, $N1$. □

[32]Slightly more carefully such term-forming operators do not in themselves seem controversial. There will have to be some constraint on their introduction and interpretation since, in particular, we cannot allow any one to express a bijection between pluralities and the singular domain, under pain of the plural version of Cantor's paradox (sometimes called Bernays' theorem). The *number of* operation does not, however, pose any problems in this respect.

In due course we will see that this theorem can be strengthened to an explicit assertion that 1 is a natural number. To proceed we must define predecession:

Definition 3.

(P) $\quad Pmn \leftrightarrow_{df} \exists xx(m = \mathbf{N}xx \wedge (\exists yy \exists y \nprec yy \forall x \prec yy[x \prec xx \vee x = y] \wedge n = \mathbf{N}yy))$

Now we can show:

Theorem 2. $\neg\exists n Pn1$

Proof. Suppose otherwise. By definition 1 and (BP) we can see that for any such n, $n = \mathbf{N}(xx)$ iff $\neg\exists x\, x \prec xx$, contradicting (N-EMPT). □

Also:

Theorem 3. $\forall m, n(\exists y[Pym \wedge Pyn] \rightarrow m = n)$.

Proof. Let $m = \mathbf{N}(mm)$ and $n = \mathbf{N}(nn)$. By definition 3 and the fact that Ny we can find such mm, nn such that the sole difference between mm and nn is that $a \prec mm$ and $b \prec nn$, $a \neq b$, whilst $a \nprec nn$ and $b \nprec mm$. But now by (BP) $\mathbf{N}(nn) = \mathbf{N}(mm)$, so $m = n$. □

In order to proceed we need to define the ancestral:

Definition 4.

(2) $\quad P^*xy \leftrightarrow_{df} \forall xx(\forall a \forall b([a = x \vee a \prec xx) \wedge Pab] \rightarrow b \prec xx) \rightarrow y \prec xx)$

This in turn allows us to define *natural number*:

Definition 5. $Nat(n) \leftrightarrow_{df} n = 1 \vee P^*1n$

We are now in a position to demonstrate induction. With $\forall nn$ abbreviating quantification restricted to natural numbers, as defined above:

Theorem 4. $\forall nn\, ((1 \prec nn \wedge \forall n(n \prec nn \rightarrow \forall m[Pnm \rightarrow m \prec nn])) \rightarrow \forall n\, n \prec nn)$

Proof. We assume the antecedent. Now let aa be a plurality, instantiating the variable 'nn'. In order to show that every natural number p is one of aa it suffices, by the previous definition, to consider two cases:

(1) $p = 1$. Immediate by the first conjunct.
(2) P^*1p. We know that $1 \prec aa$, by the first conjunct, but now by the second conjunct and (ANC), we have $p \prec aa$ as required.

□

It remains to prove:

Theorem 5. $\forall n \exists m Pnm$

This is done *via* two lemmas, the first conditional on the assumption $\exists x \neg Nx$, and instantiating 'x' with 'a':

Lemma 2. $\forall n\, n = \mathbf{N}[x : (Nat(x) \wedge P^*xn) \vee x = a]$

Note here the explicit appeal to the existence of the non-number a. This is required, by contrast with the proof of Frege's theorem from (HP) in second-order logic, because we have not established that zero is a natural number and therefore cannot identify natural numbers with the number of their ancestral predecessors. Rather each number n is the number of that plurality consisting of all of n's ancestral predecessors, together with a.

Lemma 3. $\forall n\, Nat(n) \rightarrow \neg P^*nn$

The proof is lengthy but straightforward and proceeds as per. the standard proof of Frege's theorem from (HP) in second-order logic, *modulo* the modification of Lemma 2 to appeal to the existence of a non-number, noted above. The theorems to this point plus bi-interpretability give us:

Theorem 6. ***(Plural Frege's Theorem)****: PA2 can be interpreted in plural logic plus pair abstraction and Boolos' principle, on the assumption that there exists at least one non-number.*

6. COMMENTARY

Once we have availed ourselves of pair abstraction, a version of Frege's theorem can be derived in plural logic. The *number of* operator, '**N**()' is suggestive of an understanding of numbers as *plural properties*, following a suggestion of Hossack [Hossack, 2000]. Whilst the formal result does not force acceptance of this metaphysical picture, the latter seems to me an attractive situation of arithmetical reality in relation to other components of reality that may well provide occasion for progress with respect to, for example, offering an account of the applicability of mathematics. That, however, is work for elsewhere; it remains here to comment on some aspects of the proof of Plural Frege's Theorem.

We have had to appeal to the logical theorem $\exists x\ x = x$ during our proof of Frege's theorem. Moreover we had to strengthen this to the assumption that there exists at least one non-number in order to prove that every natural number has a successor. Both moves will move some to worry about the logicist credentials of our result. Oliver and Smiley have, in another context, objected to the suggestion that it could be a truth of *logic* that anything whatsoever exists (and so, for example, object to the invocation of the theorem $\exists x\, x = x$ plus Separation to prove the existence of the empty set) [Oliver and Smiley, 2013]. This is, of course, a highly non-Fregean worry: Frege's intention with repsect to arithmetic was to show that the natural numbers are *logical objects*, and *inter alia* therefore that logic is not ontologically neutral. Nevertheless it is a common enough thought in the contemporary philosophy of logic that logic is devoid of ontological commitments. Why should we believe this? The thought is presumably that logic is concerned with everything in general, and nothing in particular, and that this generality means that its distinctive claims would apply regardless of how things turned out to be. Indeed so, but ontological neutrality doesn't follow: the claim that logic remains applicable regardless of subject matter is weaker than the claim that logic remains applicable in the *absence* of a subject matter. It is this latter claim – that, as it were, anything deserving the honorific 'logic' remains correct in the empty world – which Oliver and Smiley need. I myself can make no genuine sense of talk of an empty world, but I am happy to allow that were such a thing conceivable logic would fail there. I don't see this as a significant concession. Recall, after all, the function of logic in the neo-Fregean programme: the appeal to logical status is *epistemological*. It is not demanding anything significant from an agent that she allow that something exist: the *cogito* should persuade her of this if nothing else does.

A more difficult case is provided by the appeal to the existence of a non-number. Here the claim that the existence in question obtains as a matter of logic is more difficult to sustain. Whilst I do not want to rule out that a case can be made to this effect, neither do I want to rely on logicality at this point. Instead what we have is a result that secures arithmetic in plural logic *given the existence of at least one non-number*. Yet surely this is all we need as a matter of epistemology: if there are things in the concrete world to count, we are assured of referential and epistemic access to numbers with which to count them. Moreover if we hold – in an Aristotelian fashion – that numbers depend for their existence on there being concreta to count[33] then the conditionality of the theorems of arithmetic on the existence of a non-number is the *right result*.

Mathematically, the reason that appeal to a non-number is needed is that the absence of zero means that we cannot identity each number with the number of its ancestral predecessors and have to 'import' a non-number to make up the difference. The existence of zero cannot be proved because there is no empty plurality: there are no things such that nothing whatsoever is amongst those things.[34] Some authors have appealed to an empty plurality during work in the philosophical foundations of mathematics; I think that this does not take the distinction between pluralities and sets

[33] Or, in a more constructivist vein, on the practice of counting.

[34] Of course it is going to be mathematically possible to appeal to bi-interpretability and recover the original proof of Frege's theorem (including the existence of zero), appealing to the Boolos 'trick' for rendering the empty second-order case plurally. What is mathematically possible here, however, lacks epistemological or metaphysical attractiveness from a neo-Fregean perspective.

seriously enough. [Burgess, 2004] [Linnebo, 2013] The notion of an empty set is coherent; a box without contents is still a box. A plurality however just *is* its members; if there are no members, there is no plurality.

An alternative approach here is that of Ian Rumfitt, who [Rumfitt, 2017] proposes a version of logicism against the background of Smiley and Oliver's free plural logic [Oliver and Smiley, 2013]. Following Boolos' criticism of Scottish neo-Fregeanism, Rumfitt proposes that an explicit existence claim needs to be made on behalf of each number; thus the metaphysical and logico-epistemological strands of logicism come apart. Compared to this, the ontological presuppositions of the present project are modest; the success of our derivation of laws for arithmetic requires only the existence of one object, and that a non-number. Of course the question which approach is to be preferred cannot be settled on the basis of gross tonnage ontology, but rather will gravitate about the relationship between language and reality, and in particular whether the occurence of a singular term denoting a purported object in a non-intensional context in a truth sentence is sufficient for the existence of that object. Much has been written on this and I refer the interested reader to the literature. My point here is merely that there is a sharp contrast between the kind of 'bootstrapping' appeal to extra-numerical existence required to justify belief in the existence of numbers above and the altogether stronger existential claims required once the metaphysical strand of logicism is abandoned.

We are still left, however, without zero. Isn't this a worry if logicism is supposed to trace an epistemic route to possible knowledge of arithmetical truths and grasp of arithmetical concepts? For isn't the concept *zero* one such concept, and aren't there arithmetical truths concerning the number zero? In other words, doesn't the absence of zero from our system reveal that it is not a system of *arithmetic* in the usual sense? Here my response is that the absence of zero is a feature rather than a fault. The concept of zero is more advanced than those attached to greater numbers, as is evident both from the historical development of arithmetic and from the usual order of acquistion. The fact that our approach elucidates this advanced status by relating it to the absence of any plurality zero can be used to number is a positive advantage. In any case, once we have $\mathbb{N} > 0$, it is easy to recover zero as part of a construction of the integers. Following what Wright terms the Dedekindian Way, we can identify integers with differences, defined over pairs of naturals [Wright, 2000, 318]:

(Diff) $$\mathrm{Dif}(\langle x, y\rangle) = \mathrm{Dif}(\langle v, w\rangle) \leftrightarrow x + w = v + y$$

There are, then, no defeating objections to plural logicism as this has been pursued here. Given how easily the project sits with our everyday practice of numbering pluralities, I conclude that it has much to commend it.

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SCHOOL OF PHILOSOPHY, RELIGION, AND THE HISTORY OF SCIENCE,, UNIVERSITY OF LEEDS, UK
E-mail address: `s.hewitt@leeds.ac.uk`