Clifford Algebras and the Dirac-Bohm Quantum Hamilton-Jacobi Equation.

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Abstract

In this paper we show how the dynamics of the Schrödinger, Pauli and Dirac particles can be described entirely within the hierarchy of Clifford algebras, $\mathcal{C}_{1,3}, \mathcal{C}_{3,0}$, and $\mathcal{C}_{0,1}$. There is no need to introduce vectors in Hilbert space, but that option is always available. The state of the quantum process is characterised by algebraic bilinear invariants of the first and second kind. We show the bilinears of the second kind emerge from the energy-momentum tensor of standard quantum field theory and are identical to the energy and momentum used in the Bohm model. In our approach there is no need to appeal to classical mechanics at any stage. Thus we are able to obtain a complete relativistic version of the Bohm model and derive an expression for the quantum potential for the Dirac particle.

1 Introduction

In this paper I want to report some recent results of Hiley and Callaghan [1] who have obtained a complete relativistic generalisation of the Bohm interpretation [2], [3]. By complete I mean we have derived expressions for the Bohm energy-momentum, the Dirac quantum potential energy, and the evolution of the components of the spin of a Dirac particle. Since the Dirac theory introduces a Clifford algebra in an essential way, we show how not only the Dirac particle, but also the earlier work on the Pauli [4] *and*

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the Schrödinger particles [5] emerge from a nested series of Clifford algebras, namely, $\mathcal{C}_{1,3}, \mathcal{C}_{3,0}$, and $\mathcal{C}_{0,1}$. This provides a mathematical hierarchy in which to embed naturally the successive physical approximations, relativistic particle with spin, non-relativistic particle with spin, non-relativistic particle without spin.

The use of Clifford algebras for the Dirac and Pauli particles will come as no surprise as the mathematical significance of the γ-matrices and the σ Pauli matrices are well know. That the Schrödinger particle can be included within a Clifford algebra might come as a bit of a surprise. However once one realises the isomorphism $\mathcal{C}_{0,1} \cong \mathbb{C}$, the surprise vanishes.

Our use of Clifford algebras is different from but related to the work of Hestenes [6], Gull *et al* [7] and Doran and Lasenby [8]. In all of these approachs there is no need to use a Hilbert space, rather the information normally encoded in the wave function is contained in an element of a minimum left ideal [10].

Although in this paper we apply our techniques to the Bohm approach, they have a much wider implication, namely, that quantum phenomena *per se* can be entirely described in terms of Clifford algebras taken over the reals without the need to appeal to a specific representation in terms of wave functions in a Hilbert space. This removes the *necessity* of using Hilbert space and all the physical imagery that goes with the use of the wave function. This could have important consequences for the interpretation of the quantum formalism.

2   Use of the Clifford Algebra.

My original interest in Clifford algebras arose in a very different context. In the 1960s at Birkbeck, I had many discussions with Roger Penrose as he was pioneering his twistor theory [9] and David Bohm who was developing his more general ideas of the implicate order and arguing that one should be looking for a more algebraic approach in which to describe quantum phenomena, an idea that has gained prominence, although from a different perspective, in the work of Connes [11], Haag [12] and many others.

Frescura and Hiley [10] soon realised that if one focused attention on the Clifford algebra, a common element in both these approaches, we could do quantum mechanics completely within this algebraic structure. What we showed in this earlier paper was that all the information contained in the wave function, or the spinor in this case, is already contained within the algebra itself and there was no need to introduce a representation through

an external Hilbert space. In fact we showed that all this information was encoded within an element, Φ_L, of a minimum left ideal within the Clifford algebra itself.

2.1 The Clifford Bundle.

Let us recall that a Clifford algebra, $\mathcal{C}_{i,j}$ is characterised by a distinguished sub-space, V, together with a quadratic form Q. For physicists this vector space is represented by a set of γ^μ and the quadratic form by a metric tensor $g^{\mu\nu}$ with signature (i,j). We first construct a vector Clifford bundle $E \xrightarrow{\pi} M$ where the fibres are isomorphic to $\mathcal{C}_{i,j}$ and the structure group is the Clifford group. Here, of course, M is the base manifold with metric g. What we are effectively using is a vector sub-bundle whose fibre is a minimal left ideal of $\mathcal{C}_{i,j}$ together with a dual sub-bundle whose fibre is the dual minimal right ideal [13].

The way to characterise the minimal ideals within the algebra is to focus on a set of primitive idempotents, ϵ_i. This set has the properties

$$\sum \epsilon_i = 1$$
$$\epsilon_i^2 = \epsilon_i \quad \epsilon_i \epsilon_j = 0 \quad i \neq j$$
$$\text{The rank of } \epsilon_i \text{ is minimal} \neq 0$$

The next question is to choose our primitive idempotent. The choice is determined by the physics of the situation we wish to discuss. For example if we are considering a relativistic situation, then we choose the idempotent [1] $(1+\gamma^0)/2$. This chooses the Lorentz frame within which we wish to work. It effectively defines the time axis in the rest frame [14]. If we are considering a non-relativistic particle with spin, we choose $(1+\sigma^3)/2$, using σ^3 to define a particular axis in space. Physically this direction is often defined by a uniform magnetic field. The Schrödinger case is trivial in the sense that there is only one idempotent, namely, 1.

In terms of the chosen primitive idempotent, we can write an element of the minimal left ideal in the form $\Phi_L(x^\mu) = \phi_L(x^\mu)\epsilon$. Here the x^μ are the co-ordinates of the base manifold, M. It should be noted here that ϕ_L is an element of the algebra and not a wave function. We will consider the following elements in each case.

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[1] We use the physicists notation for the generators, γ and σ, of the algebras $\mathcal{C}_{1,3}$ and $\mathcal{C}_{3,0}$ respectively. These should be thought of as elements of the algebra that are independent of any specific representation.

Dirac: $\quad \phi_L^D = R(g_0 + g_1\gamma_{23} + g_2\gamma_{13} + g_3\gamma_{12} + g_4\gamma_{01} + g_5\gamma_{02} + g_6\gamma_{03} + g_7\gamma_5)$

Pauli: $\quad \phi_L^P = R(g_0 + g_1\sigma_{23} + g_2\sigma_{13} + g_3\sigma_{12})$

Schrödinger: $\quad \phi_L^S = R(g_0 + g_1 e)$

Were the $g_i = g_i(x^\mu)$ real functions with the restriction $\sum g_i(x^\mu) = 1$. We have used the three primitive idempotents

$$\epsilon_D = (1 + \gamma^0)/2, \quad \epsilon_P = (1 + \sigma^3)/2 \quad \text{and} \quad \epsilon_S = 1 \quad \text{respectively.}$$

There is an advantage of writing ϕ_L in the form we have. Firstly we see that in the Dirac ϕ_L^D, the first four terms and the final four terms are isomorphic to the quaternions, explaining its biquaternion nature, first pointed out by Eddington [15]. For a slow moving Dirac particle, the first four terms correspond to what is normally called the large component, while the rest are referred to as the small component. In the non-relativistic limit the small component is neglected and we immediately see that we are left with an expression that is isomorphic to ϕ_L^P. If we further reduce ϕ_L^P by deleting two of the σ_{ij} and writing the third as e, we arrive at ϕ_L^S. Thus we see how the ϕ_L are nested one inside the other, showing that the Schrödiner theory is an integral part of the hierarchy of Clifford algebras. This, of course, matches what we see in the standard Hilbert space approach using complex functions expressed as column matrices, $1 \times 4, 1 \times 2$ and 1×1 respectively.

We have already indicated that the real functions g_i contain all the information that is normally contained in the wave function. In fact one can show that the relation between these functions and the complex wave functions, ψ_i, are

Dirac:

$$
\begin{aligned}
g_0 &= (\psi_1^* + \psi_1)/2 & g_1 &= i(\psi_2 - \psi_2^*)/2 \\
g_2 &= -(\psi_2^* + \psi_2)/2 & g_3 &= i(\psi_1 - \psi_1^*)/2 \\
g_4 &= (\psi_4^* + \psi_4)/2 & g_5 &= i(\psi_4^* - \psi_2)/2 \\
g_6 &= (\psi_3^* + \psi_3)/2 & g_7 &= i(\psi_3 - \psi_3^*)/2
\end{aligned}
\tag{2.1}
$$

Here the ψ_i are the four components of the Dirac spinor.

Pauli:

$$
\begin{aligned}
g_0 &= (\psi_1^* + \psi_1)/2 & g_1 &= i(\psi_2^* - \psi_2)/2 \\
g_2 &= (\psi_2^* + \psi_2)/2 & g_3 &= i(\psi_1^* - \psi_1)/2
\end{aligned}
\tag{2.2}
$$

Here the ψ_i are the two components of a Pauli spinor.

Schrödinger: $2g_0 = \psi + \psi^*$ and $2eg_1 = \psi - \psi^*$. $\qquad\qquad$ (2.3)

Here ψ is the usual wave function.

In the approach we adopt here, we also need the Clifford conjugate [2] to Φ_L, namely, an element of the dual right ideal which we denote by $\Phi_R = \widetilde{\Phi}_L$. The dual elements to those defined above are

Dirac: $\widetilde{\phi}_L^D = R(g_0 - g_1\gamma_{12} - g_2\gamma_{23} - g_3\gamma_{13} - g_4\gamma_{01} - g_5\gamma_{02} - g_6\gamma_{03} + g_7\gamma_5)$

Pauli: $\widetilde{\phi}_L^P = R(g_0 - g_1e_{23} - g_2e_{13} - g_3e_{12})$

Schrödinger: $\widetilde{\phi}_L^S = R(g_0 - g_1e)$

One can quickly check that this corresponds to the conjugate wave function in each case.

To provide a complete description of the state of our quantum system. we need to form the Clifford density element, $\rho_c = \Phi_L\widetilde{\Phi}_L = \phi_L\epsilon\widetilde{\phi}_L$. This is the algebraic equivalent of the density matrix.

3   Bilinear Invariants of the First Kind.

3.1   General Construction.

In order to find the physical properties of the system, we need to form the bilinear invariants of the first kind. These are expression of the form

$$\langle B \rangle = tr(\epsilon\widetilde{\phi}_L B\phi_L\epsilon) = tr(B\rho_c) \qquad\qquad (3.1)$$

where B is an element of the algebra corresponding to a physical aspect of the quantum process.

However every element can be written in the form

$$B = b^s + \sum b^i e_i + \sum b^{ij} e_{ij} + \sum b^{ijk} e_{ijk} + \dots$$

so that the mean value of any dynamical element in the algebra can be expressed as

$$tr(B\rho_c) = b^s tr(\rho_c) + \sum b^i tr(e_i\rho_c) + \sum b^{ij} tr(e_{ij}\rho_c)/2 \dots$$

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[2]See Porteous [16] for formal definition.

This shows that the state of our system is specified by a set of bilinear invariants

$$tr(1\rho_c) \rightarrow \text{scalar} \qquad tr(e_j\rho_c) \rightarrow \text{vector} \qquad tr(e_{ij}\rho_c) \rightarrow \text{bivector} \qquad tr(\dots) \rightarrow \dots$$

It is these bilinear invariants that characterise the physical properties of the quantum process in the algebraic approach.

3.2 Example in Schrödinger case.

We start with the Clifford algebra, $\mathcal{C}_{0,1}$, taken over the reals. This algebra is generated by the elements $\{1, e\}$ where $e^2 = -1$. As we have indicated above $\phi_L = R(g_0 + g_1 e)$, while $\widetilde{\phi_L} = R(g_0 - g_1 e)$. Combining these results we find that that the CDE is $\rho_c = \Phi_L \widetilde{\Phi}_L = \widetilde{\phi_L}\phi_L = R^2$.

In order to see what the CDE corresponds to in the standard approach we use the relations

$$2g_0 = \psi + \psi^* \qquad \text{and} \qquad 2eg_1 = \psi - \psi^*.$$

Here ψ is the ordinary wave function. This means that

$$\rho_c = \psi^*\psi = R^2 = \rho$$

Thus in this case, the CDE is simply the probability. At first sight it seems we have gained no advantage over the conventional approach. However in the case for the Pauli and Dirac particles, we find an essential difference between the CDE and the probability as we will show in the next two sections.

Notice we have replaced i by e. As we have remarked before it is this replacement that enables us to embed the Schrödinger formalism in the algebra $\mathcal{C}_{0,1}$ taken over the *reals*. Again the reason why this works is because $\mathcal{C}_{0,1} \cong \mathbb{C}$.

Notice we only have one parameter to specify the state of the system, whereas we actually need two. (The wave function is complex number.) It is not difficult to see what is missing. We have no information about the phase. We need one more invariant and this leads us to invariants of the second kind. A full discussion of these invariants in the standard approach has already been presented by Takabayasi [17]. Before going on to discuss these additional invariants let us show that an additional invariant is also needed in both the Pauli and Dirac cases.

3.3 Bilinear Invariants of the First Kind for Pauli Case.

In the Pauli case, we use the Clifford algebra generated by the elements $\{1, \sigma_1, \sigma_2, \sigma_3\}$ with the usual multiplication rule $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. This forms the Pauli Clifford, $\mathcal{C}_{3,0}$. As we have seen, an element of the minimal left ideal is $\phi_L^P = R(g_0 + g_1\sigma_{23} + g_2\sigma_{13} + g_3\sigma_{12}) = RU$. We can immediately write down CDE

$$\rho_c = \Phi_L \widetilde{\Phi}_L = \psi_L \epsilon \widetilde{\phi}_L = \rho U \epsilon \widetilde{U}.$$

where $\rho = R^2$ is the probability density. We now choose a z-direction in space and choose the idempotent $\epsilon = (1 + \sigma_3)/2$. The CDE now becomes

$$\rho_c = \rho(1 + U\sigma_3\widetilde{U})/2$$

Here $Ue_3\widetilde{U}/2 = \mathbf{s}$ is the spin vector which can also be written in the form $\mathbf{s} = (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)/2$. It is not difficult to show that the a_i are given by

$$a_1 = 2(g_1g_3 - g_0g_2) \quad a_2 = 2(g_0g_1 + g_2g_3) \quad a_3 = g_0^2 - g_1^2 - g_2^2 + g_3^2.$$

Using the relations between the gs and the components of the wave function ψ_i given above in equation (2.2), we find these coefficients to be

$$a_1 = \psi_1\psi_2^* + \psi_2\psi_1^* \quad a_2 = i(\psi_1\psi_2^* - \psi_2\psi_1^*) \quad a_3 = |\psi_1|^2 - |\psi_2|^2$$

This is just the usual well known expressions for the spin vector when written in standard form. Now we can write the CDE in the form

$$\rho_c = \rho(1 + \mathbf{s}.\boldsymbol{\sigma})/2$$

When normalised with $\rho = 1$, and the $\{\sigma_i\}$ replaced by the Pauli matrices, this operator becomes the standard expression for the density matrix [19].

In the case of the Pauli particle there are four bilinear invariants. They are

$$\begin{aligned} \rho &= \phi_L \widetilde{\phi}_L = R^2. \\ V &= \phi_L \sigma_3 \widetilde{\phi}_L = 2\mathbf{s} \end{aligned}$$

ρ is clearly the probability, while V gives the three components of spin $\{s_1, s_2, s_3\}$. It appears that we have our four real parameters to completely specify the state of the Pauli particle, but they are not linearly independent since $s^2 = 1/4$. Thus we, in fact, only have three independent real parameters, so once again one parameter is missing. Again we have no information about the phase so we need an other kind of bilinear invariant.

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3.4 Bilinear Invariants of the First Kind for the Dirac particle.

To construct the bilinear invariants of the Dirac particle in a form that can be compared with the standard approach, we need to chose a different, but equivalent, idempotent given by

$$\epsilon = (1 + \gamma_0 + i\gamma_{12} + i\gamma_{012})/2$$

Here we have introduced $i^2 = -1$ to compare with the standard Dirac matrix approach. The bilinear invariants are then

$$
\begin{aligned}
\rho &= \phi_L \widetilde{\phi}_L = R^2. \\
V &= \phi_L \gamma_0 \widetilde{\phi}_L \\
S &= i\phi_L \gamma_{12} \widetilde{\phi}_L \\
A &= i\phi_L \gamma_{012} \widetilde{\phi}_L
\end{aligned}
$$

Here V is the velocity of the particle, S its spin, while A is the axial vector related to the Proca current. Takabayasi [17] has shown that these invariants provide only seven linearly independent parameters and we need eight so once again an additional parameter is needed.

4 Bilinear Invariants of the Second kind.

4.1 The Appearance of Energy and Momentum.

If we want the get information about the phase in the conventional approach we would need to introduce a differential operator into the expression for a bilinear invariant. In fact that is exactly what Takabayasi [17] has done. We do the same in the algebra. Of course when we use derivatives like $\partial/\partial t$ and $\partial/\partial x$ we might expect some connection with the energy and the momentum and this is exactly what happens.

As we show in [20], we can introduce a momentum defined by

$$\rho P^j(t) = -i\alpha \Phi_L \overleftrightarrow{\partial}^j \widetilde{\Phi}_L = -i\alpha \left[(\partial^j \Phi_L) \widetilde{\Phi}_L - \Phi_L (\partial^j \widetilde{\Phi}_L) \right] \qquad (4.1)$$

and an energy defined by

$$\rho E(t) = i\alpha \Phi_L \overleftrightarrow{\partial}^0 \widetilde{\Phi}_L = i\alpha [(\partial^0 \Phi_L) \widetilde{\Phi}_L - \Phi_L (\partial^0 \widetilde{\Phi}_L)] \qquad (4.2)$$

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Here $\alpha = 1/2$ for the Schrödinger case, while $\alpha = 1$ for the Pauli and the Dirac cases[3]. Our choice of notation should already give a clue as to the identity of these formulae. They are the algebraic equivalents of the the components, $T^{\mu 0}$, of the energy-momentum tensor used in standard quantum field theory.

What we also showed in [20] was the P^j and E were identical to the Bohm momentum and the Bohm energy used in the Bohm interpretation [3] [18]. I personally found this a somewhat surprising result since one is led to believe that the Bohm model was an attempt to return to classical determinism. Clearly it is not the case and it actually lies at the heart of low energy quantum field theory.

4.2 The Bohm Energy and Momentum for the Schrödinger Particle.

Let us start by applying equation (4.1) to the Schrödinger case. Here $\epsilon = 1$, so that

$$2\rho P^j(t) = -e[(\partial^j(RU)) R\widetilde{U} - RU\partial^j(R\widetilde{U})]$$

and $U = g_0 + eg_1$. This equation becomes

$$P^j = -e[(\partial^j U)\widetilde{U} - U\partial^j\widetilde{U}].$$

If we write $U = \exp(eS)$, where S is the phase of the wave function, then $g_0 = \cos S$ and $g_1 = \sin S$, from which we find

$$P^j(t) = \partial^j S \qquad \text{or} \qquad \mathbf{P}(t) = \boldsymbol{\nabla} S.$$

Similarly it is not difficult to show, using equation (4.2), that

$$E(t) = -\partial_t S. \tag{4.3}$$

Thus we see that we can get the expressions for the Bohm momentum and energy directly from the energy-momentum tensor of quantum field theory. There is no need to appeal to an analogy with classical Hamilton-Jacobi as was done in Bohm's original work [5]. Furthermore we find that these expressions are necessary to complete the description of the Schrödinger particle.

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[3] The reason for this difference is because the primitive idempotent for the Schrödinger case is 1, whereas the Pauli and Dirac primitive idempotents are of the form $(1 + \gamma)/2$ where γ is an element of their respective algebras.

4.3 The Bohm Energy and Momentum for the Pauli Particle.

In the usual discussion of the Pauli theory, we generally assume the particle is charged and coupled to the electromagnetic field through the vector potential. In order to keep the formalism as simple as possible so that we can bring out clearly the quantum aspects of our approach, we examine the behaviour of the particle in the absence of an electromagnetic field. Once the principles involved in our approach have been brought out clearly, it is easy then to introduce the electromagnetic coupling through the minimal coupling $\nabla \to \nabla - eA$.

We start by considering the momentum, P^j, defined by equation (4.1) with $\alpha = 1$. Thus substituting $\rho_c = \phi_L \epsilon \widetilde{\phi}_L$ into this equation, we find

$$P^j(t) = -\frac{i}{2}[\Omega^j \Sigma + \Sigma \Omega^j] = -i\Omega^j \cdot \Sigma \tag{4.4}$$

We are using a more succinct notation with $\Sigma = U\epsilon\widetilde{U}$, and $\Omega^j = 2(\partial^j U)\tilde{U}$. We now choose $\epsilon = (1 + \sigma_3)/2$ and substitute this into equation (4.4). This gives

$$P^j(t) = -\Omega^j \cdot S - i\Omega^j/2 \tag{4.5}$$

where $S = i(U\sigma_3\tilde{U})/2$ is the spin bivector. In the Pauli case, we write $i = \sigma_{123}$ since it commutes with all the elements of the real Pauli algebra and $(\sigma_{123})^2 = -1$. The Bohm momentum is the scalar part of this expression so that we can write

$$P_B(t) = -\Omega \cdot S \tag{4.6}$$

where $\Omega = \sum \Omega^j$.

We can now turn to the energy equation (4.2) and find in this case

$$E(t) = \frac{i}{2}[\Omega_t \Sigma + \Sigma \Omega_t] = i\Omega_t \cdot \Sigma. \tag{4.7}$$

The corresponding scalar part of the energy becomes

$$E_B(t) = \Omega_t \cdot S \tag{4.8}$$

It should be noted that equations (4.4) and (4.7) are in a sufficiently general form to be applied in all three cases, Schrödinger, Pauli and Dirac. The only difference lies in the choice of ϵ.

If we now use the conversion equation (2.2) in (4.6) we can show that

$$2\rho P_B(t) = i[(\nabla\psi_1)\psi_1^* - (\nabla\psi_1^*)\psi_1 + (\nabla\psi_2)\psi_2^* - (\nabla\psi_2^*)\psi_2] \tag{4.9}$$

If we write $\psi_1 = R_1 e^{iS_1}$ and $\psi_2 = R_2 e^{iS_2}$, then we find

$$\rho P_B(t) = (\nabla S_1)\rho_1 + (\nabla S_2)\rho_2 \tag{4.10}$$

where $\rho_i = R_i^2$. The meaning becomes more transparent if we write $P_i = \nabla S_i$ when the expression for the momentum becomes

$$\rho P_B(t) = P_1\rho_1 + P_2\rho_2 \tag{4.11}$$

Thus we see that in terms of the usual approach $P_B(t)$ is the weighted mean of the momentum that can be attributed to each component of the spinor acting by itself. This result was already noted in Bohm and Hiley [3].

Similarly the energy becomes

$$\rho E_B(t) = -[(\partial_t S_1)\rho_1 + (\partial_t S_2)\rho_2] \tag{4.12}$$

This can also be written in the form

$$\rho E_B = E_1\rho_1 + E_2\rho_2 \tag{4.13}$$

which is just the weighted mean of the energy associated with each component of the spinor.

We have remaining the vector part for both P and E defined by equations (4.1) and (4.2). We see from the earlier work that $\Omega^j = 2(\partial^j U)\widetilde{U} = -2U(\partial^j \widetilde{U})$ and $\Omega_t = 2(\partial_t U)\widetilde{U} = -2U(\partial_t \widetilde{U})$ which implies that Ω appears to be a form of angular velocity. Indeed if we express the components ψ_1 and ψ_2 in terms of Euler angles as explained in the next section, we find Ω is exactly the expression for the angular velocity of a rotating frame. This result suggests that we can describe the spinning electron in terms of Cartan's moving frames [21], a feature that Hestenes [22] exploits.

4.4   The Bohm Energy-Momentum for the Dirac Particle.

We have seen in the previous sections that the Bohm energy-momentum can be calculated from the energy-momentum tensor, $T^{\mu\nu}$. For the Dirac particle, this tensor when written in algebraic form becomes

$$2iT^{\mu\nu} = tr\left\{\gamma^\mu[(\partial^\nu \phi_L)\epsilon\widetilde{\phi}_L - \phi_L\epsilon(\partial^\nu \widetilde{\phi}_L)]\right\} = tr[\gamma^\mu(\phi_L\epsilon\overleftrightarrow{\partial^\nu}\widetilde{\phi}_L)]$$

Still to be chosen is the primitive idempotent ϵ. Again in order to make contact with the standard results used in the Dirac representation, we again choose the idempotent

$$\epsilon = (1 + \gamma_0 + i\gamma_{12} + i\gamma_{012})/4.$$

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Since the only non-vanishing trace is a Clifford scalar, and since γ^μ is a Clifford vector, we must find the Clifford vector part of $\phi_L \epsilon \overleftrightarrow{\partial^\nu} \widetilde{\phi}_L$. We find the only term in ϵ that gives a Clifford vector is γ_{012}, so that we need only consider

$$2T^{\mu\nu} = tr[\gamma^\mu(\phi_L \overleftrightarrow{\partial^\nu} \gamma_{012} \widetilde{\phi}_L)]. \tag{4.14}$$

To proceed, we must first evaluate $(\phi_L \overleftrightarrow{\partial_\nu} \gamma_{012} \widetilde{\phi}_L)$ in terms of the functions $\{g_i(x^\rho)\}$. After some straight forward but tedious calculations we find

$$\phi_L \overleftrightarrow{\partial^\nu} \gamma_{012} \widetilde{\phi}_L = A_i^\nu(x^\mu)\gamma_i \tag{4.15}$$

where the A_i^ν are given by

$$
\begin{aligned}
A_0^\nu &= -(g_0 \overleftrightarrow{\partial^\nu} g_3 + g_1 \overleftrightarrow{\partial^\nu} g_2 + g_4 \overleftrightarrow{\partial^\nu} g_5 + g_6 \overleftrightarrow{\partial^\nu} g_7) \\
A_1^\nu &= -(g_0 \overleftrightarrow{\partial^\nu} g_5 + g_3 \overleftrightarrow{\partial^\nu} g_4 + g_1 \overleftrightarrow{\partial^\nu} g_6 + g_2 \overleftrightarrow{\partial^\nu} g_7) \\
A_2^\nu &= (g_0 \overleftrightarrow{\partial^\nu} g_4 - g_3 \overleftrightarrow{\partial^\nu} g_5 - g_1 \overleftrightarrow{\partial^\nu} g_7 + g_2 \overleftrightarrow{\partial^\nu} g_6) \\
A_3^\nu &= (g_0 \overleftrightarrow{\partial^\nu} g_7 - g_3 \overleftrightarrow{\partial^\nu} g_6 + g_1 \overleftrightarrow{\partial^\nu} g_4 - g_2 \overleftrightarrow{\partial^\nu} g_5)
\end{aligned}
$$

Now let us first consider the energy density T^{00} and calculate it in terms of the relations defined in equation (2.1) so that we can see what it looks like in terms of wave functions. After some work we find

$$T^{00} = i\sum_{j=1}^{4}(\psi_j^* \partial^0 \psi_j - \psi_j \partial^0 \psi_j^*) = -\sum R_j^2 \partial_t S_j = \rho E_B,$$

giving us an expression for the Bohm energy for the Dirac particle. In the final stages of the calculation we have written $\psi_j = R_j \exp iS_j$ with R_j and S_j real functions.

We then see that this result reduces to the non-relativistic E_B for the Pauli particle since $\sum_{j=1}^{j=4} R_j^2 \partial_t S_j \rightarrow \sum_{j=1}^{j=2}(R_j^2 \partial_t S_j)$. Furthermore this, in turn, reduces to the well known expression for the Bohm energy for the Schrödinger particle, namely, $E_B = -\partial_t S$.

Similarly we can also show that the momentum density can be written in the form

$$T^{k0} = -i\sum_{j=1}^{3}(\psi_j^* \partial_k \psi_j - \psi_j \partial_k \psi_j^*) = \sum R_j^2 \nabla S_j = \rho P_B^k.$$

which is obviously the relativistic extension of the Pauli Bohm momentum. Thus in more general terms we find

$$2\rho P_B^\mu = 2T^{\mu 0} = tr[\gamma^0(\phi_L \overleftrightarrow{\partial_\mu} \gamma_{012} \widetilde{\phi}_L)]. \tag{4.16}$$

5 The Time Evolution Equations.

5.1 Basic Equations.

Let us now turn to consider how time evolution can be described in the algebraic theory. We have constructed a Clifford bundle and on this bundle we can define two derivatives, one acting from the left and the other acting from the right. They are the so called generalised Dirac derivatives [23], defined by

$$\overrightarrow{D} = \sum e_i \partial_{x_i} \quad \text{and} \quad \overleftarrow{D} = \sum \partial_{x_i} e_i. \tag{5.1}$$

Here e_i are the generators of the Clifford algebra, while x_i are are the co-ordinates of the base manifold.

Since we are interested in the time evolution of $\rho_c = \Phi_L \widetilde{\Phi}_L$, we must consider derivatives of the form $(\overrightarrow{D} \Phi_L)\widetilde{\Phi}_L$ and $\Phi_L(\widetilde{\Phi}_L \overleftarrow{D})$, where the Ds, are defined in equation (5.1). Rather than treat these two derivatives separately, we will consider expressions like

$$(\overrightarrow{D} \Phi_L)\widetilde{\Phi}_L \pm \Phi_L(\widetilde{\Phi}_L \overleftarrow{D})$$

Thus in the case of the time derivatives, we have

$$(\partial_t \Phi_L)\widetilde{\Phi}_L + \Phi_L(\partial_t \widetilde{\Phi}_L) = \partial_t \rho_c, \quad \text{and} \quad (\partial_t \Phi_L)\widetilde{\Phi}_L - \Phi_L(\partial_t \widetilde{\Phi}_L) = \Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L$$

To discuss the dynamics we need to introduce the Hamiltonian which will include the Dirac derivatives, the external potentials and the mass of the particle. Thus we introduce two forms of the Hamiltonian, $\overrightarrow{H} = \overrightarrow{H}(\overrightarrow{D}, V, m)$ and $\overleftarrow{H} = \overleftarrow{H}(\overleftarrow{D}, V, m)$. Our defining dynamical equations will now read

$$i[(\partial_t \Phi_L)\widetilde{\Phi}_L + \Phi_L(\partial_t \widetilde{\Phi}_L)] = i\partial_t \rho_c = (\overrightarrow{H} \Phi_L)\widetilde{\Phi}_L - \Phi_L(\widetilde{\Phi}_L \overleftarrow{H}) \tag{5.2}$$

and

$$i[(\partial_t \Phi_L)\widetilde{\Phi}_L - \Psi_L(\partial_t \widetilde{\Phi}_L)] = \Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L = (\overrightarrow{H} \Phi_L)\widetilde{\Phi}_L + \Phi_L(\widetilde{\Phi}_L \overleftarrow{H}) \tag{5.3}$$

The equations (5.2) and (5.3) can be written in the more compact form by writing the RHS as

$$[H, \rho_c]_\pm = (\overrightarrow{H} \Phi_L)\widetilde{\Phi}_L \pm \Phi_L(\widetilde{\Phi}_L \overleftarrow{H})$$

Then equation (5.2) becomes

$$i\partial_t \rho_c = [H, \rho_c]_- \tag{5.4}$$

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While equation (5.3) can be written in the form

$$i\Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L = [H, \rho_c]_+ \tag{5.5}$$

Using equation (4.2), we find

$$\rho E(t) = \alpha[H, \rho_c]_+ \tag{5.6}$$

We find that a complete specification of the dynamics is contained in the two equations, (5.4) and (5.6). We will now show that (5.4) is a generalized Liouville equation giving rise to the conservation of probability and, in the case of a particle with spin, conservation of the components of spin. Equation (5.6) gives rise to what we have called the quantum Hamilton-Jacobi equation [3], which now generalises to the relativistic domain.

5.2   The Time Evolution of the Schrödinger Particle.

Let us now apply equation (5.6) to the Schrödinger particle, remembering to put $\alpha = 1/2$. Then using equation (4.3), we find immediately

$$-2\partial_t S = [H, \rho_c]_+ \tag{5.7}$$

If we use the Hamiltonian $H = p^2/2m + V$ and use $p = \nabla S$, we find

$$\partial_t S + (\nabla S)^2/2m + Q + V = 0 \tag{5.8}$$

where $Q = -\nabla^2 R/2mR$ which is immediately recognised as the quantum potential. This equation is what we have previously called the quantum Hamilton-Jacobi equation [3]. Thus we see there is no need to appeal to classical mechanics and then identify the classical action with the phase to obtain the so-called 'guidance' equation $p = \nabla S$. This result immediately follows from the standard energy-momentum tensor of quantum field theory.

Now let us turn to equation (5.4), using the same Hamiltonian, we find

$$e\partial_t \rho_c + [\rho_c, H]_- = 0 \tag{5.9}$$

This is immediately recognised as the Liouville equation which shows that the probability is conserved as required.

5.3 The Time Evolution of the Pauli Particle.

To obtain an expression for the Pauli quantum Hamilton-Jacobi equation, we need to examine equation (5.6) with $\rho_c = \phi_L \epsilon \widetilde{\phi}_L$ where now $\epsilon = (1 + \sigma_3)/2$. In this case $\Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L$ splits into two parts, a scalar part and a vector part. A detailed examination shows that these two parts produce identical equations (see [20]) so we need only consider the scalar part which is

$$2\langle \Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L \rangle_s = (\partial_t \phi_L)\sigma_{12}\widetilde{\phi}_L - \phi_L \sigma_{12}(\partial_t \widetilde{\phi}_L) \tag{5.10}$$

Similarly we can split the RHS of equation (5.6) into two parts, the scalar part is

$$2\langle [H, \rho_c] \rangle_s = (H\phi_L)\widetilde{\phi}_L + \phi_L(H\widetilde{\phi}_L) \tag{5.11}$$

Combining the scalar equations (5.10) and (5.11) we obtain

$$(\partial_t \phi_L)\sigma_{12}\widetilde{\phi}_L - \psi_L \sigma_{12}(\partial_t \widetilde{\phi}_L) = (H\phi_L)\widetilde{\phi}_L + \phi_L(H\widetilde{\phi}_L) \tag{5.12}$$

However we have already evaluated the LHS of this equation in working out the Bohm energy, namely

$$\rho E_B(t) = (\partial_t \phi_L)\sigma_{12}\widetilde{\phi}_L - \phi_L \sigma_{12}(\partial_t \widetilde{\phi}_L) = \rho \Omega_t \cdot S. \tag{5.13}$$

To evaluate the RHS we assume a free particle Hamiltonian[4] $H = -\nabla^2/2m$. We find

$$2m[(H\phi_L)\widetilde{\phi}_L + \phi_L(H\widetilde{\phi}_L)] = \rho[S \wedge \nabla P + S \cdot \nabla W + P \cdot P + W \cdot W] \tag{5.14}$$

where we have introduced the shorthand $A \wedge B = (AB - BA)/2$. Here $W = \rho^{-1}\nabla(\rho S)$. It is not difficult to show that $S \wedge \nabla = 0$ so that we end up with the equation

$$2mE(t) = P^2 + [2(\nabla W \cdot S) + W^2] \tag{5.15}$$

This is the Pauli quantum Hamilton-Jacobi equation where the quantum potential is

$$Q = (\nabla W \cdot S)/m + W^2/2m \tag{5.16}$$

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[4]The inclusion of an interaction with the electromagnetic field is straight forward. However we again omit the details in this presentation to keep the everything as short and simple as possible.

Now let us express equation (5.16) purely in terms of P, ρ and the spin bivector S. After some straight forward but tedious work, we find

$$Q = \{S^2[2\nabla^2 \ln \rho + (\nabla \ln \rho)^2] + S \cdot \nabla^2 S\}/2m = Q_1 + Q_2 \qquad (5.17)$$

To compare this expression with the quantum potential of the Schrödinger particle we must use the relations (2.2) in their polar form, we then find

$$Q_1 = -\frac{1}{2m}\frac{\nabla^2 R}{R} \qquad (5.18)$$

which is immediately recognised the quantum potential contribution to the Schrödinger particle. This leaves the spin dependent part

$$Q_2 = \frac{1}{2m}S \cdot \nabla^2 S \qquad (5.19)$$

This expression was derived in a different method using the wave function expressed in Caley-Klein parameters [4], [24]. If we use the Caley-Klien representation in (5.17) we find

$$Q_2 = [(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2]/8m \qquad (5.20)$$

Finally, in the same representation, we can write the Pauli quantum Hamilton-Jacobi equation in the form

$$(\partial_t\psi + \cos\theta\partial_t\phi)/2 + P_B^2/2m + Q_1 + Q_2 = 0 \qquad (5.21)$$

which agrees exactly with the expression given in Dewdney *et al.* [24].

The method we have used here is thus a general approach in which the previous model fits once one chooses a specific representation. Our approach also removes the necessity of identifying the phase of the wave function with a particular Euler angle, an identification that can not be justified by the physics. Our method completely removes this difficulty.

5.4   The Conservation Equations for the Pauli Particle.

Now we turn to equation (5.4) which we have called the generalised Liouville equation. This equation (5.4) also contains a scalar part and a bivector part. We will now show that the scalar equation gives us a conservation of probability equation, a genuine Liouville equation. The other gives us an equation involving the spin of the particle.

We find the LHS of equation (5.6) becomes

$$i\partial_t\rho_c = i\partial_t[\rho + \phi_L\sigma_3\widetilde{\phi_L}] = i\partial_t\rho + 2\partial_t(\rho S) \qquad (5.22)$$

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where ρ is the usual probability. In this expression, the term $i\partial_t\rho$ corresponds to the scalar part, while $\partial_t(\rho S)$ is the bivector part.

The RHS of equation (5.4), $[H, \rho_c]_-$ also splits into a scalar and a bivector part, the scalar part being

$$\langle [H, \rho_c]_-\rangle_s = (H\phi_L)\sigma_3\widetilde{\phi}_L - \phi_L\sigma_3(H\widetilde{\phi}_L) \qquad (5.23)$$

We will again, for simplicity, use the free particle Hamiltonian and after some manipulation we find

$$2m\langle [H, \rho_c]_-\rangle_s = -i\rho\{2\nabla P - [S(P \cdot W) + (P \cdot W)S]\} \qquad (5.24)$$

Using the expression for $W = \rho^{-1}\nabla(\rho S)$ given above and after some work, we find

$$\partial_t\rho + \nabla(\rho P/m) = 0 \qquad (5.25)$$

This will immediately be recognised as the Liouville equation for the conservation of probability.

Now let us turn to the bivector part of the equation (5.6). We need to combine $\partial_t(\rho S)$ with the bivector part of $[H, \rho_c]_-$. We find

$$4m\partial_t(\rho S) = 2m\langle [H, \rho_c]_-\rangle_B = 4\rho[(\nabla P \cdot S) + (S \wedge \nabla W) + (P \cdot W)] \quad (5.26)$$

Since

$$2P \cdot W = (\nabla \ln \rho)P \cdot S + 2(P \cdot \nabla)S \qquad (5.27)$$

we find

$$\left[\partial_t + \frac{P \cdot \nabla}{m}\right]S = 2(\nabla \wedge S) \qquad (5.28)$$

However simplifying the RHS of equation (5.28) we find

$$\nabla W \wedge S = (\nabla \ln \rho)(\nabla S \wedge S) + \nabla^2 S \wedge S \qquad (5.29)$$

so that finally we get

$$\left[\partial_t + \frac{P \cdot \nabla}{m}\right]S = \frac{dS}{dt} = \frac{1}{m}\left[\nabla^2 S + (\nabla \ln \rho)\nabla S\right] \wedge S \qquad (5.30)$$

The LHS of this equation shows that the spin experiences an internal 'torque' which was exploited by Dewdney *et al.* [24]. To connect up with their work,

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we now write this equation in terms of the spin vector $\mathbf{s}$ rather than the spin bivector $\mathbf{S}$. We can do this because $\mathbf{S} = i\mathbf{s}$ so that equation (5.30) becomes

$$\frac{d\mathbf{s}}{dt} = \frac{\mathbf{s}}{m} \times [\nabla^2\mathbf{s} + (\nabla \ln \rho)\nabla\mathbf{s}]. \tag{5.31}$$

where we have used the identity $A \wedge B = i(A \times B)$. This shows that a quantum torque acting on the components of the spin. Equation (5.31) then ensures the total spin is conserved during the time development. This equation was exploited numerically by Dewdney *et al* [24] to show how the spin turned as it passed through an inhomogenious magnetic field.

5.5 The Time Evolution of the Dirac Particle.

In order to produce time evolution equations for the Dirac particle, we have to modify our approach to the two key equations (5.4) and (5.5). The reason for this lies in the fact that the energy constraint insists that we must have $(\partial_\mu\partial^\mu + m^2)\psi = 0$ satisfied for both Φ_L and $\widetilde{\Phi}_L$. Thus two second order derivatives are involved, namely, $(\partial_\mu\partial^\mu\Phi_L)\widetilde{\Phi}_L$ and $\Phi_L(\partial_\mu\partial^\mu\widetilde{\Phi}_L)$. These are used to produce two equations by taking their sum and their difference as before. The sum gives the energy conservation equation

$$(\partial_\mu\partial^\mu\Phi_L)\widetilde{\Phi}_L + \Phi_L(\partial_\mu\partial^\mu\widetilde{\Phi}_L) + 2m^2\Phi_L\widetilde{\Phi}_L = 0. \tag{5.32}$$

While the difference produces the following equation

$$\Phi_L(\partial_\mu\partial^\mu\widetilde{\Phi}_L) - (\partial_\mu\partial^\mu\Phi_L)\widetilde{\Phi}_L = 0 \tag{5.33}$$

which, as we will show below, describes the time evolution of the spin and its components, the relativistic generalisation of the corresponding Pauli equation (5.30).

5.6 The Dirac Quantum Hamilton-Jacobi Equation.

Now we will use equation (5.32) to investigate energy conservation. To analyse this equation further we need to see where the Bohm energy-momentum as defined in equation (4.16) fits in as was done for the Pauli particle. To proceed let us first introduce a more general variable P^μ defined by

$$2\rho P^\mu = \left[(\partial^\mu\phi_L)\gamma_{012}\widetilde{\phi}_L - \phi_L\gamma_{012}(\partial^\mu\widetilde{\phi}_L)\right] \tag{5.34}$$

Let us also introduce a quantity

$$2\rho W^\mu = -\partial^\mu(\phi_L\gamma_{012}\widetilde{\phi}_L)$$

Combining these two equations, we obtain

$$(\partial^\mu \phi_L)\gamma_{012}\widetilde{\phi}_L = \rho[P^\mu - W^\mu]$$
$$\text{and} \quad -\phi_L\gamma_{012}(\partial^\mu\widetilde{\phi}_L) = \rho[P^\mu + W^\mu]$$

which can then be written as

$$-\partial^\mu \phi_L = [P^\mu - W^\mu]\phi_L\gamma_{012} \tag{5.35}$$

and

$$\partial^\mu \widetilde{\phi}_L = \gamma_{012}\widetilde{\phi}_L[P^\mu + W^\mu] \tag{5.36}$$

Since we are going to use equation (5.32), we can use equation (5.35), to form

$$-\partial_\mu\partial^\mu \phi_L = [\partial_\mu P^\mu - \partial_\mu W^\mu]\,\phi_L\gamma_{012} + [P^\mu - W^\mu]\,\partial_\mu\phi_L\gamma_{012}$$

After some algebra and finally multiplying from the right by $\widetilde{\phi}_L$, this equation can be written in the form

$$-(\partial_\mu\partial^\mu \phi_L)\widetilde{\phi}_L = \rho\,[P_\mu P^\mu + W_\mu W^\mu - (P^\mu W_\mu + W_\mu P^\mu)]$$
$$+ [\partial_\mu P^\mu - \partial_\mu W^\mu]\,(\phi_L\gamma_{012}\widetilde{\phi}_L) \tag{5.37}$$

This gives us the first term in equation (5.32). Now we must consider the second term in this equation. Repeating an analogous set of steps but now using equation (5.36), we find

$$\phi_L(\partial_\mu\partial^\mu \widetilde{\phi}_L) = \rho\,[P_\mu P^\mu + W_\mu W^\mu + (P_\mu W^\mu + W_\mu P^\mu)]$$
$$+ (\phi_L\gamma_{012}\widetilde{\phi}_L)[\partial_\mu P^\mu + \partial_\mu W^\mu] \tag{5.38}$$

Substituting both these equations in equation (5.32), we finally find

$$P^2 + W^2 + [J\partial_\mu P^\mu - \partial_\mu P^\mu J] + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0 \tag{5.39}$$

Here we have used the relation $2\rho J = \phi_L\gamma_{012}\widetilde{\phi}_L$, where J is essentially the axial current. This term reduces to the spin of the Pauli particle in the non-relativistic limit. Equation (5.39) can be further simplified by splitting it into its Clifford scalar and pseudoscalar parts. The scalar part is

$$P^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0 \tag{5.40}$$

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This is to be compared with the energy equation

$$p_\mu p^\mu - m^2 = 0$$

Thus we see that the extra two terms must be related to the quantum potential in some way. Before we arrive at an exact expression for the quantum potential, we must first note that the momentum, P^μ, as defined in equation (5.34) is not yet the Bohm momentum defined in equation (4.16). This equation tells us that P_B^μ is the γ^0 coefficient in the expression for P^μ. However it is not difficult to extract the Bohm momentum from the P^2 term in equation (5.40). To do this we need to recall equation (5.34) and use equation (4.16) to find

$$4\rho^2 P^2 = \sum_{i=0}^{3} A_{i\nu} A_i^\nu$$

Using the definition of P_B^μ given in equation (4.16), we find

$$4\rho^2 P^2 = 4\rho^2 P_B^2 + \sum_{i=1}^{3} A_{i\nu} A_i^\nu$$

If we write

$$\sum_{i=1}^{3} A_{i\nu} A_i^\nu = 4\rho^2 \Pi^2$$

we then find equation (5.40) can be written in the form

$$P_B^2 + \Pi^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0$$

Then we see that the quantum potential for the Dirac particle is

$$Q_D = \Pi^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] \tag{5.41}$$

In the non-relativistic limit, $\Pi = 0$, and equation (5.41) reduces to the quantum potential for the Pauli particle, [20],

$$Q_P = W^2 + [S(\nabla W) + (\nabla W)S] \tag{5.42}$$

where $2\rho S = \phi_L e_{12} \widetilde{\phi}_L$ is the non-relativistic spin limit of J. W is the non-relativistic limit of W^μ.

The pseudoscalar part of equation (5.39) is simply $[J\partial_\mu P^\mu - \partial_\mu P^\mu J] = 0$. This puts a constraint on the relation between the spin and the momentum of the particle. In the non-relativistic limit this term vanishes.

5.7 The Time Development of the Dirac Spin.

Finally let us turn our attention to equation (5.33) and show that it leads to an equation for the time development of the spin of the Dirac particle. By substituting equations (5.37) and (5.38) into equation (5.33) we find

$$J \cdot \partial_\mu P^\mu - P \cdot W + J \wedge \partial_\mu W^\mu = 0$$

where we have written

$$
\begin{aligned}
2J \cdot \partial_\mu P^\mu &= J\partial_\mu P^\mu + \partial_\mu P^\mu J \\
2P \cdot W &= PW + WP \\
2J \wedge \partial_\mu W^\mu &= J\partial_\mu W^\mu - \partial_\mu W^\mu J.
\end{aligned}
$$

All of these terms are Clifford bivectors so that equation (5.33) gives just one equation. We can now simplify this equation since

$$\rho(P \cdot W) = -\partial^\mu \rho(P_\mu \cdot J) - \rho(P_\mu \cdot \partial^\mu J)$$

so that

$$\partial_\mu(\rho P^\mu) \cdot J + \rho(P_\mu \cdot \partial^\mu J) + \rho(J \wedge \partial_\mu W^\mu) = 0$$

However since $2\rho P^\mu = T^{\mu 0}$, the conservation of the energy-momentum tensor implies

$$\partial_\mu(T^{\mu 0}) = 2\partial_\mu(\rho P^\mu) = 0$$

so that we have finally

$$P_\mu \cdot \partial^\mu J + J \wedge \partial_\mu W^\mu = 0 \tag{5.43}$$

This equation describes the quantum torque experienced by the spin of the particle in the absence of any external field. Coupling to an external field is achieved in the usual manner by replacing ∂^μ by $\partial^\mu - ieA_\mu$. The equation (5.43) reduces to the quantum torque equation for the Pauli particle [20]

$$(\partial_t + \frac{P \cdot \nabla}{m})S = \frac{2}{m}(\nabla W \wedge S)$$

Here P is the three-momentum and S and W have the same meanings as in equation (5.42).

6 Conclusion.

In this paper we have shown that we can describe the behaviour of quantum particles entirely within the hierarchy of Clifford algebras, $\mathcal{C}_{0,1}, \mathcal{C}_{3,0}$ and $\mathcal{C}_{1,3}$, corresponding to the Schrödinger, Pauli and Dirac particle. Thus we have a natural mathematical hierarchy for the successive generalisations non-relativistic particle without spin, non-relativistic particle with spin and a relativistic particle with spin. Furthermore there is no need to appeal to wave functions as one can use algebraic elements abstracted from an appropriately chosen minimal left ideal. However one can always use a representation in terms of wave functions should one wish.

The state of a particle is described by bilinear invariants constructed in the algebra. As Takabayasi [17] has shown, we need to consider two kinds of bilinear invariants to obtain a complete description of the state of the particle. The first kind involve terms like $\langle B \rangle = tr(B\phi_L \epsilon \widetilde{\phi}_L) = tr(B\rho_c)$ where ρ_c is the Clifford density element. However there are not enough of these invariants to completely define the system. We must therefore introduce invariants of the second kind viz, $\Phi_L \overleftrightarrow{\partial}^\mu \widetilde{\Phi}_L$. These invariants are essentially part of the standard energy-momentum tensor used in quantum field theory. Indeed the four momentum is given by $\rho P^\mu(t) = -i\Phi_L \overleftrightarrow{\partial}^\mu \widetilde{\Phi}_L$. Thus we can use these bilinears to complete the specification of the state of the particle.

However these P^μ turn out to be the Bohm energy-momentum. Thus rather than the Bohm approach being some *ad hoc* addition to a misguided attempt to return to a classical determinism, they are essential parameters to completely specify the *quantum* state of the system.

We then show how the time evolution of the states of a particle is discussed in this theory. We have two evolution equations.The first, $i\Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L = [H, \rho_c]_+$, is an energy conservation equation which we have previously called the quantum Hamilton-Jacobi equation. The other is a generalised conservation equation, $i\partial_t \rho_c = [H, \rho_c]_-$, conserving probability and spin.

We have shown how these equations can be applied to the Schrödinger, Pauli and the Dirac particles. In all three case we find that the first equation always produces an additional energy term that has traditionally been called the 'quantum potential'. Thus rather than being something arbitrary, it is an essential feature of a quantum process, ensuring that energy is conserved.

7   Acknowledgements

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