MEET-COMPLETIONS AND REPRESENTATIONS OF ORDERED DOMAIN ALGEBRAS

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Abstract. We apply the well known equivalence between meet-completions of posets and standard closure operators to construct a meet-completion for ordered domain algebras which simulateously serves as the base of a representation for such algebras, thereby proving that ordered domain algebras have the finite representation property. We show that many of the equations defining ordered domain algebras are preserved in this completion but associativity, (**D2**) and (**D6**) can fail in the completion.

§1. Introduction. When considering algebras of binary relations, it is generally not the case that a finite representable algebra has a representation on a finite base. Indeed, in any signature which includes the identity, intersection and composition operators, a representation of the *point algebra* with minimal non-zero elements $1', <, >$ interprets $<$ as a dense linear order without endpoints, necessarily infinite. On the other hand, the finite representation property is desirable, for one thing it entails the decidability of the equational theory.

There are two well-known cases where we do have the finite representation property. For the signature with identity, converse and composition only, the Cayley representation maps an algebra element a to the binary relation $\{(x, x; a) : x \in \mathcal{A}\}$ over the algebra itself. At the other extreme, for the signature consisting of Boolean operators only we may modify the standard Stone representation (which represents elements as unary relations) and represent an element a is the identity relation over the ultrafilters containing a, more generally for a signature with solely an order relation $\leq$ we may represent an element as the identity over the set of upward closed subsets of the poset containing that element. This representation of a poset has the additional property of being a completion of the poset.

An interesting case, then, is where the signature includes both composition and an order relation. The construction we consider here has aspects of the Cayley representation but also aspects of the upward closed set representation for a poset. Each element of an algebra will be represented as a set of pairs of upward closed subsets of the algebra, but in order to make the representation work for the non-Boolean operators, these subsets will be required to have certain other closure properties. As with posets, this construction yields a completion of the original algebra at the same time it provides a base for a representation of that algebra, at least for the special case of the signature consisting of a domain operator, a range operator, converse, composition and an order relation (so-called Ordered Domain Algebras). A complete, finite set of equations for this representation class was given in [1] and the finite representation property was proved in [3].

In this paper we construct a completion for ordered domain algebras (ODAs) and show that this completion forms the base of a natural representation of the algebra. We see that the completion may be viewed as an algebra of the same signature as ordered domain algebras and obeys many of the equations defining ordered domain algebras. On the other hand rather important properties, like the associativity of composition, are shown to fail in the completion.

The remainder of this paper is divided as follows. In the next section we give the basic definitions for meet-completions and standard closure operators and provide a proof of the correspondence between them. In section 3 we provide a method for extending isotone operators on a poset to a meet-completion of that poset. This provides a method for extending poset completions to completions of isotone poset expansions. We investigate some general rules governing the preservation of inequalities by completions of isotone poset expansions using this method. In section 4 we define ordered domain algebras, and in section 5 we apply the considerations of sections 2 and 3 to construct a completion for ordered domain algebras and determine which ODA equations it preserves. In the final section we show how this completion can be used as the base of a representation for that algebra.

§2. Meet-completions and closure operators. The material in this section is well known, dating back to the pioneering work of Ore [6, 5, 7]. The aim here is to provide formulations best suited for the work we undertake in later chapters.

Definition 2.1 (P^*). If P is a poset define P^* to be the complete lattice of up-sets (including $\emptyset$) of P ordered by reverse inclusion (so the order dual $P^{*\delta}$ is the lattice of up-sets ordered by inclusion with bottom element $\emptyset$).

It's easy to see the map $\iota\colon P \to P^*$ defined by $\iota(p) = p^{\uparrow}$ defines a meet-completion of P (note though that ι will not map the top element of P (if it exists) to the top element of P^*, as the top element of P^* will be $\emptyset$). This particular completion plays an important role in the theory of meet-completions.

Definition 2.2 (Closure operator). Given a poset P a closure operator on P is a map $\Gamma\colon P \to P$ such that

(1) $p \leq \Gamma(p)$ for all $p \in P$,
(2) $p \leq q \Longrightarrow \Gamma(p) \leq \Gamma(q)$ for all $p, q \in P$, and
(3) $\Gamma(\Gamma(p)) = \Gamma(p)$ for all $p \in P$.

Following [2] we say a closure operator Γ on P^* or $P^{*\delta}$ is *standard* when $\Gamma(p^{\uparrow}) = p^{\uparrow}$ for all $p \in P$.

It is well known that a meet-completion $e\colon P \to Q$ defines a standard closure operator $\Gamma_e\colon P^{*\delta} \to P^{*\delta}$ by $\Gamma_e(S) = \{p \in P : e(p) \geq \bigwedge e[S]\}$ (we take the dual of P^* as otherwise condition 1 of Definition 2.2 fails). In this case Q is isomorphic to the lattice $\Gamma_e[P^*]$ of Γ_e-closed subsets of P^* (note we are purposefully taking P^* rather than $P^{*\delta}$ here as we want to order by reverse inclusion, this is technically

an abuse of notation as Γ_e is originally defined on P^{*^δ}, but as these structures have the same carrier hopefully our meaning is clear). The isomorphism is given by the map $h_e\colon Q \to \Gamma_e[P^*]$ defined by $h_e(q) = \{p \in P : e(p) \geq q\}$. Conversely, whenever Γ is a standard closure operation on $P^{*\delta}$ it induces a meet-completion $e_\Gamma\colon P \to \Gamma[P^*]$ defined by $e_\Gamma(p) = p^\uparrow$. For $S \in P^*$ we have $\Gamma_{e_\Gamma}(S) = \{p \in P : p^\uparrow \geq \bigwedge\{p^\uparrow : p \in S\}\} = \{p : p^\uparrow \subseteq \Gamma(S)\} = \Gamma(S)$, so $\Gamma_{e_\Gamma} = \Gamma$, and, for all $p \in P$, $e_{\Gamma_e}(p) = p^\uparrow = h_e \circ e(p)$ so the diagram in the following theorem commutes:
We state the results of the preceding discussion as a theorem.

THEOREM 2.3. *If $e\colon P \to Q$ is a meet-completion then there is a unique isomorphism h_e between Q and $\Gamma_e[P^*]$ such that the following commutes:*

$$\begin{tikzcd} P \arrow[r, "e"] \arrow[d, "e_{\Gamma_e}"'] & Q \arrow[dl, leftrightarrow, "h_e"] \\ \Gamma_e[P^*] & \end{tikzcd}$$

Moreover, if $e_1\colon P \to Q_1$ and $e_2\colon P \to Q_2$ are meet-completions such that there is an isomorphism $h\colon Q_1 \to Q_2$ with $h \circ e_1 = e_2$ then $\Gamma_{e_1} = \Gamma_{e_2}$.

PROOF. The existence of the required isomorphism has been established, and uniqueness follows from Lemma 2.4 below. If $h\colon Q_1 \to Q_2$ with $h \circ e_1 = e_2$ then $\Gamma_{e_2}(S) = \{p \in P : e_2(p) \geq \bigwedge e_2[S]\} = \{p \in P : h \circ e_1(p) \geq \bigwedge h \circ e_1[S]\} = \{p \in P : h \circ e_1(p) \geq h(\bigwedge e_1[S])\} = \{p \in P : e_1(p) \geq \bigwedge e_1[S]\} = \Gamma_{e_1}(S)$. ⊣

LEMMA 2.4. *If $e_1\colon P \to Q_1$ and $e_2\colon P \to Q_2$ are meet-completions of P and $g\colon Q_1 \to Q_2$ is an isomorphism such that $g \circ e_1 = e_2$, then g is unique with this property.*

PROOF. Suppose h is another such isomorphism. Then for all $p \in P$, and for all $q \in Q$, we have $e_1(p) \geq q \iff g \circ e_1(p) \geq g(q) \iff h \circ e_1(p) \geq h(q)$, and $g \circ e_1(p) \geq g(q) \iff e_2(p) \geq g(q)$, and similarly $h \circ e_1(p) \geq h(q) \iff e_2(p) \geq h(q)$, so $\{p \in P : e_2(p) \geq g(q)\} = \{p \in P : e_2(p) \geq h(q)\}$ and thus by meet-density we are done. ⊣

LEMMA 2.5. *If P is a poset, $e\colon P \to Q$ is a meet-completion, and $n \in \omega$, then $e^n\colon P^n \to Q^n$ is a meet-completion of P^n, where we define*

$$e^n((p_1, ..., p_n)) = (e(p_1), ..., e(p_n)).$$

PROOF. Since a finite product of complete lattices is again a complete lattice it remains only to check that $e^n[P^n]$ is meet-dense in Q^n. Given $(q_1, ..., q_n) \in Q^n$ we claim that $(q_1, ..., q_n) = \bigwedge\{e^n((p_1, ..., p_n)) : e(p_i) \geq q_i \text{ for all } i \in \{1, ..., n\}\}$. Now, $(q_1, ..., q_n)$ is clearly a lower bound, so suppose $(q_1', ..., q_n')$ is another such lower bound. Then for $i \in \{1, ..., n\}$ $q_i' \leq e(p_i)$ for all $p_i \in P$ with $q_i \leq e(p_i)$, so by meet-density of $e[P]$ in Q we have $q_i' \leq q_i$, and so $(q_1', ..., q_n') \leq (q_1, ..., q_n)$ as required. ⊣

§3. Preserving inequalities in meet-completions of isotone poset expansions. If $e_1 \colon P_1 \to Q_1$ is a meet-completion, $e_2 \colon P_2 \to Q_2$ is any completion of P_2, and $f \colon P_1 \to P_2$ is an order preserving map, there is a natural method (introduced in [4]) for lifting f to an order preserving map $\hat{f} \colon Q_1 \to Q_2$, given by

$$\hat{f}(q) = \bigwedge\{e_2(f(p)) : e_1(p) \geq q\} \tag{1}$$

Given any standard closure operator $\Gamma\colon\ {P^*}^\delta \to {P^*}^\delta$ and n-ary function $f : P^n \to P$, there is a natural map $f^\bullet_\Gamma \colon \Gamma[P^*]^n \to \Gamma[P^*]$ defined by

$$f^\bullet_\Gamma(C_1, ..., C_n) = \Gamma(f[C_1 \times ... \times C_n]^\uparrow)$$

and when $\Gamma = \Gamma_e$ for some meet-completion $e\colon P \to Q$ the diagram in Figure 1 commutes (where $\hat{f}$ in this diagram is defined as in (1)). Note that if $e\colon P \to Q$ is a meet-completion then so is $e^n \colon P^n \to Q^n$, where e^n is defined by $e(p_1, ..., p_n) = (e(p_1), ..., e(p_n))$. We can use this to define lifts of isotone (order preserving) operations $P^n \to P$ to order preserving operations $\Gamma[P^*]^n \to \Gamma[P^*]$. This means that given an isotone poset expansion $\mathcal{P}$ (i.e. a structure $\mathcal{P} = (P, \leq, f_i : i \in I)$ where f_i is an n_i-ary isotone operation $P^{n_i} \to P$ for each $i \in I$, where I is some ordinal), such as an ODA, we can use Γ to define a completion of $\mathcal{P}$ with the corresponding signature of operations. We note that frequently inequalities that hold with respect to the operations of $\mathcal{P}$ will fail in this completion. The remainder of this section is devoted to an examination of some conditions which guarantee inequality preservation.

$$\begin{tikzcd}
P^n \arrow[r, "e^n"'] \arrow[rr, bend left, "(e_{\Gamma_e})^n"] \arrow[d, "f"'] & Q^n \arrow[r, leftrightarrow, "\cong"'] \arrow[d, "\hat{f}"] & \Gamma_e[P^*]^n \arrow[d, "f^\bullet_{\Gamma_e}"] \\
P \arrow[r, "e"] \arrow[rr, bend right, "e_{\Gamma_e}"'] & Q \arrow[r, leftrightarrow, "\cong"] & \Gamma_e[P^*]
\end{tikzcd}$$

FIGURE 1. Lifting operations in terms of closure operators

DEFINITION 3.1 (Γ_ι). Define Γ_ι to be the identity on ${P^*}^\delta$

LEMMA 3.2. *Let $\mathcal{P} = (P, \leq, f_i : i \in I)$ be an isotone poset expansion, let $x_1, ..., x_n$ be distinct variables, and let $\phi(x_1, ..., x_n)$ be a term in the language of $\mathcal{P}$ such that x_i does not appear more than once in ϕ for all $i \in \{1, ..., n\}$. Define $\phi^\bullet_{\Gamma_\iota}(x_1, ..., x_n)$ by replacing each occurrence of f_i in ϕ with $f^\bullet_{i\Gamma_\iota}$. Then, for all $(C_1, ..., C_n) \in {P^*}^n$, $\phi^\bullet_{\Gamma_\iota}(C_1, ..., C_n) = \phi[C_1 \times ... \times C_n]^\uparrow$, where $\phi[C_1 \times ... \times C_n]^\uparrow$ is defined to be $\{\phi(x_1, ..., x_n) : x_i \in C_i \text{ for all } i \in \{1, ..., n\}\}^\uparrow$.*

PROOF. Straight forward induction on the construction of ϕ. ⊣

Note that the condition that no variable occurs more than once in ϕ is required in Lemma 3.2, as otherwise even the base case fails. For example if $\phi(x) = f(x,x)$ for binary operation f then $f[C \times C]^\uparrow \neq \{f(x,x) : x \in C\}^\uparrow$ in general.

DEFINITION 3.3. Given poset expansion $\mathcal{P} = (P, \leq, f_i : i \in I)$ define $\mathcal{P}^* = (P^*, \supseteq, f^\bullet_{i\Gamma_\iota} : i \in I)$.

PROPOSITION 3.4. *Let $\mathcal{P} = (P, \leq, f_i : i \in I)$ be an isotone poset expansion, and let $\phi(x_1, ..., x_n)$ and $\psi(x_1, ..., x_n)$ be terms in the language of $\mathcal{P}$ such that $x_i = x_j \Longrightarrow i = j$ for all $i, j \in \{1, ..., n\}$. Define $\phi^\bullet_{\Gamma_\iota}(x_1, ..., x_n)$ and $\psi^\bullet_{\Gamma_\iota}(x_1, ..., x_n)$ as in Lemma 3.2. Then*

$$\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n) \iff \mathcal{P}^* \models \phi^\bullet_{\Gamma_\iota}(x_1, ..., x_n) \leq \psi^\bullet_{\Gamma_\iota}(x_1, ..., x_n).$$

PROOF. Let $(C_1, ..., C_n) \in P^{*n}$ and suppose $\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n)$. Then

$$\begin{aligned} p \in \psi^\bullet_{\Gamma_\iota}(C_1, ..., C_n) &\iff p \geq \psi(x_1, ..., x_n) \text{ for some } (x_1, ..., x_n) \in C_1 \times ... \times C_n \\ &\Longrightarrow p \geq \phi(x_1, ..., x_n) \\ &\iff p \in \phi^\bullet_{\Gamma_\iota}(C_1, ..., C_n) \end{aligned}$$

So $\psi^\bullet_{\Gamma_\iota}(C_1, ..., C_n) \subseteq \phi^\bullet_{\Gamma_\iota}(C_1, ..., C_n)$, and thus $\mathcal{P}^* \models \phi^\bullet_{\Gamma_\iota}(x_1, ..., x_n) \leq \psi^\bullet_{\Gamma_\iota}(x_1, ..., x_n)$ as required. Conversely, if $\mathcal{P}^* \models \phi^\bullet_{\Gamma_\iota}(x_1, ..., x_n) \leq \psi^\bullet_{\Gamma_\iota}(x_1, ..., x_n)$ then in particular, for all $(p_1, ..., p_n) \in P^n$, $\phi[p_1^\uparrow \times ... \times p_n^\uparrow]^\uparrow \supseteq \psi[p_1^\uparrow \times ... \times p_n^\uparrow]^\uparrow$, and this can happen only when $\phi(p_1, ..., p_n) \geq \psi(p_1, ..., p_n)$, so $\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n)$. ⊣

COROLLARY 3.5. *Let $\mathcal{P}$, $\phi(x_1, ..., x_n)$, and $\psi(x_1, ..., x_n)$, be as in Proposition 3.4 and suppose $\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n)$. Let Γ be a standard closure operator on $P^{*\delta}$ and define $\Gamma[\mathcal{P}] = (\Gamma[P^*], \supseteq, f^\bullet_{i\Gamma} : i \in I)$. Define $\phi^\bullet_\Gamma(x_1, ..., x_n)$ and $\psi^\bullet_\Gamma(x_1, ..., x_n)$ in a similar manner to Lemma 3.2, and suppose for all $(C_1, ..., C_n) \in \Gamma[P^*]^n$ we have $\psi^\bullet_\Gamma(C_1, ..., C_n) = \Gamma(\psi[C_1 \times ... \times C_n]^\uparrow])$. Then*

$$\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n) \Longrightarrow \Gamma[\mathcal{P}] \models \phi^\bullet_\Gamma(x_1, ..., x_n) \leq \psi^\bullet_\Gamma(x_1, ..., x_n).$$

PROOF. By Proposition 3.4 we have $\mathcal{P}^* \models \phi^\bullet_{\Gamma_\iota}(x_1, ..., x_n) \leq \psi^\bullet_{\Gamma_\iota}(x_1, ..., x_n)$, so in particular $\psi[C_1 \times ... \times C_n]^\uparrow \subseteq \phi[C_1 \times ... \times C_n]^\uparrow$ for all $(C_1, ..., C_n) \in \Gamma[P^*]^n$. We must always have $\Gamma(\phi[C_1 \times ... \times C_n]^\uparrow) \subseteq \phi^\bullet_\Gamma(C_1, ..., C_n)$, and similar for ψ, so $\Gamma(\psi[C_1 \times ... \times C_n]^\uparrow) \subseteq \Gamma(\phi[C_1 \times ... \times C_n]^\uparrow) \subseteq \phi^\bullet_\Gamma(C_1, ..., C_n)$. If $\psi^\bullet_\Gamma(C_1, ..., C_n) = \Gamma(\psi[C_1 \times ... \times C_n]^\uparrow])$ then $\psi^\bullet_\Gamma(C_1, ..., C_n) \subseteq \phi^\bullet_\Gamma(C_1, ..., C_n)$, and thus $\Gamma[\mathcal{P}] \models \phi^\bullet_\Gamma(x_1, ..., x_n) \leq \psi^\bullet_\Gamma(x_1, ..., x_n)$ as required. ⊣

COROLLARY 3.6. *With all notation as in Corollary 3.5, suppose $\mathcal{P} \models \phi(x_1, ..., x_n) = \psi(x_1, ..., x_n)$. Then if*

(1) $\psi^\bullet_\Gamma(C_1, ..., C_n) = \Gamma(\psi[C_1 \times ... \times C_n]^\uparrow])$, *and*
(2) $\phi^\bullet_\Gamma(C_1, ..., C_n) = \Gamma(\phi[C_1 \times ... \times C_n]^\uparrow])$

for all $(C_1, ..., C_n) \in \Gamma[P^*]^n$, then

$$\Gamma[\mathcal{P}] \models \phi^\bullet_\Gamma(x_1, ..., x_n) = \psi^\bullet_\Gamma(x_1, ..., x_n)$$

PROOF. It is always true that $\psi^\bullet_\Gamma(C_1, ..., C_n) \supseteq \Gamma(\psi[C_1 \times ... \times C_n]^\uparrow) = \Gamma(\phi[C_1 \times ... \times C_n]^\uparrow) \subseteq \phi^\bullet_\Gamma(C_1, ..., C_n)$ and the result follows. ⊣

§4. Ordered domain algebras.

DEFINITION 4.1. The class $\mathbf{R}(;,\mathbf{dom},\mathbf{ran},\smile,0,\mathbf{id},\leq)$ is defined as the isomorphs of $\mathcal{A}=(A,;,\mathbf{dom})$ where $A\subseteq\wp(U\times U)$ for some base set U and

$$\begin{aligned}x\,;y&=\{(u,v)\in U\times U:(u,w)\in x\text{ and }(w,v)\in y\text{ for some }w\in U\}\\ \mathbf{dom}(x)&=\{(u,u)\in U\times U:(u,v)\in x\text{ for some }v\in U\}\\ \mathbf{ran}(x)&=\{(v,v)\in U\times U:(u,v)\in x\text{ for some }u\in U\}\\ x^\smile&=\{(v,u)\in U\times U:(u,v)\in x\}\\ \mathbf{id}&=\{(u,v)\in U\times U:u=v\}\end{aligned}$$

for every $x,y\in A$ ($\leq$ is interpreted as $\subseteq$, and 0 is interpreted as the empty set).

Let **Ax** denote the following formulas:

Partial order: $\leq$ is reflexive, transitive and antisymmetric, with lower bound 0.

isotonicity and normality: the operators $\smile,;,\mathbf{dom},\mathbf{ran}$ are isotonic, e.g. $a\leq b\to a\,;c\leq b\,;c$ etc. and normal $0^\smile=0\,;a=a\,;0=\mathbf{dom}(0)=\mathbf{ran}(0)=0$.

Involuted monoid: ; is associative, **id** is left and right identity for ;, $\mathbf{id}^\smile=\mathbf{id}$ and $\smile$ is an involution: $(a^\smile)^\smile=a$, $(a\,;b)^\smile=b^\smile\,;a^\smile$.

Domain/range axioms:

(D1) $\mathbf{dom}(a)=(\mathbf{dom}(a))^\smile\leq\mathbf{id}=\mathbf{dom}(\mathbf{id})$

(D2) $\mathbf{dom}(a)\leq a\,;a^\smile$

(D3) $\mathbf{dom}(a^\smile)=\mathbf{ran}(a)$

(D4) $\mathbf{dom}(\mathbf{dom}(a))=\mathbf{dom}(a)=\mathbf{ran}(\mathbf{dom}(a))$

(D5) $\mathbf{dom}(a)\,;a=a$

(D6) $\mathbf{dom}(a\,;b)=\mathbf{dom}(a\,;\mathbf{dom}(b))$

(D7) $\mathbf{dom}(\mathbf{dom}(a)\,;\mathbf{dom}(b))=\mathbf{dom}(a)\,;\mathbf{dom}(b)=\mathbf{dom}(b)\,;\mathbf{dom}(a)$

Two consequences of these axioms (use (D6), (D7) for the first, use (D4), (D5) for the second) are

(D8) $\mathbf{dom}(\mathbf{dom}(a)\,;b)=\mathbf{dom}(a)\,;\mathbf{dom}(b)$

(D9) $\mathbf{dom}(a)\,;\mathbf{dom}(a)=\mathbf{dom}(a)$

A model of these axioms is called an *ordered domain algebra*.

Each of the axioms (D1)–(D8) has a dual axiom, obtained by swapping domain and range and reversing the order of compositions, and we denote the dual axiom by a ∂ superscript, thus for example, $(D6)^\partial$ is $\mathbf{ran}(b\,;a)=\mathbf{ran}(\mathbf{ran}(b)\,;a)$. The dual axioms can be obtained from the axioms above, using the involution axioms and (D3).

Another consequence of the ODA axioms is the following lemma, which we shall use later.

LEMMA 4.2. *Let $\mathcal{B}$ be any ODA and let $b,c\in\mathcal{B}$. Then*

$$\mathbf{dom}(b\,;c)\,;b\geq b\,;\mathbf{dom}(c)$$

and

$$b\,;\mathbf{ran}(c\,;b)\geq\mathbf{ran}(c);b$$

PROOF.

$$\begin{aligned}\mathbf{dom}(b\,;c)\,;b &= \mathbf{dom}(b\,;\mathbf{dom}(c))\,;b && \text{by (D6)}\\ &\geq \mathbf{dom}(b\,;\mathbf{dom}(c))\,;b\,;\mathbf{dom}(c) && \text{(D1)}\\ &= b\,;\mathbf{dom}(c) && \text{(D5)}\end{aligned}$$

The other part is similar. ⊣

§5. A completion.

DEFINITION 5.1 (Γ_D). Given an ODA A with underlying poset P, define $\Gamma_D\colon\ {P^*}^\delta \to {P^*}^\delta$ by defining the closed sets of P^* to be those $X \in P^*$ such that $\{\mathbf{dom}(x)\,;y\,;\mathbf{ran}(z) : x,y,z \in X\}^\uparrow = X$.

LEMMA 5.2. *Γ_D is a standard closure operator on ${P^*}^\delta$.*

PROOF. Routine. ⊣

LEMMA 5.3. *Given $X \in P^*$, if we define $X_0 = X$, and $X_{n+1} = \{\mathbf{dom}(x)\,;y\,;\mathbf{ran}(z) : x,y,z \in X_n\}^\uparrow$ for all $n \in \omega$, then $\Gamma_D(X) = \bigcup_\omega X_n$.*

PROOF. It's easy to show that $X_n \subseteq X_{n+1}$ for all $n \in \omega$, so given $x,y,z \in \bigcup_\omega X_n$ there is $k \in \omega$ with $x,y,z \in X_k$. Thus $\mathbf{dom}(x)\,;y\,;\mathbf{ran}(z)^\uparrow \subseteq X_{k+1} \subseteq \bigcup_\omega X_n$. Clearly any closed set containing X must contain $\bigcup_\omega X_n$, so we must have $\Gamma_D(X) = \bigcup_\omega X_n$ as required. ⊣

We can use the theory on lifting maps to lift the ODA operations to operations on the completion induced by Γ_D.

DEFINITION 5.4 ($\Gamma_D[\mathcal{A}]$). Given an ODA $\mathcal{A}$ with underlying poset P, we define $\Gamma_D[\mathcal{A}] = (\Gamma[P^*], \supseteq, f^\bullet_{\Gamma_D} : f \in \{;, \mathbf{dom}, \mathbf{ran}, ^\smile, 0, \mathbf{id}\})$.

Henceforth we shall denote $f^\bullet_{\Gamma_D}$ by $f^\bullet$.

LEMMA 5.5. *Given an ODA $\mathcal{A}$ with underlying poset P and the closure operator Γ_D. Then for all $f \in \{\mathbf{dom}, \mathbf{ran}, ^\smile, 0, \mathbf{id}\}$, $f^\bullet(C_1, ..., C_n) = f[C_1 \times ... \times C_n]^\uparrow$.*

PROOF. First note that $0^\bullet$ and $\mathbf{id}^\bullet$ are just $0^\uparrow$ and $\mathbf{id}^\uparrow$. For $\mathbf{dom}$ let $C \in \Gamma_D[P^*]$ and let $x,y,x \in C$. Then $\mathbf{dom}(\mathbf{dom}(x))\,;\mathbf{dom}(y)\,;\mathbf{ran}(\mathbf{dom}(z)) = \mathbf{dom}(x)\,;\mathbf{dom}(y)\,;\mathbf{dom}(z) = \mathbf{dom}(\mathbf{dom}(x)\,;y)\,;\mathbf{dom}(z)$ by ODA axioms (D4), (D7), and (D8). As C is Γ_D-closed we must have $\mathbf{dom}(x)\,;y \in C$, so we have something of form $\mathbf{dom}(x')\,;\mathbf{dom}(z)$ for $x', z \in C$. Another application of (D8) gives $\mathbf{dom}(x')\,;\mathbf{dom}(z) = \mathbf{dom}(\mathbf{dom}(x')\,;z)$, and thus as C is closed we have something of form $\mathbf{dom}(y')$ for $y' \in C$, which is in $\mathbf{dom}[C]$. The $\mathbf{ran}$ case is similar, and the $^\smile$ case follows from axiom (D3) and the fact that $^\smile$ is an involution. ⊣

NOTATION 5.6. *Given $S \subseteq P$ we define $S^\smile = \{s^\smile : s \in S\}^\uparrow$, and we define $\mathbf{dom}(S)$ and $\mathbf{ran}(S)$ similarly. Given $S, T \subseteq P$ we define $S\,;T = \{s\,;t : s \in S \text{ and } t \in T\}^\uparrow$. By Lemma 5.5, when S is Γ_D-closed the unary operations on S defined in this way will coincide with their interpretation in $\Gamma_D[\mathcal{A}]$, e.g. $\mathbf{dom}(C) = \mathbf{dom}[C]^\uparrow = \Gamma_D(\mathbf{dom}[C]^\uparrow) = \mathbf{dom}^\bullet(C)$ for all Γ_D-closed sets C, though this is not the case for ;.*

We ask how close $\Gamma_D[\mathcal{A}]$ is to being an ODA. Most of the axioms (D1)-(D8) hold (Proposition 5.7), with the exceptions being (D2) and (D6) (Examples 5.11 and 5.12), the operations on $\Gamma_D[\mathcal{A}]$ remain isotone and normal, $\mathbf{id}^\bullet$ remains a left and right identity for composition and $^{\smile\bullet}$ is still an involution (Lemma 5.9). The dramatic deviation is that $;^\bullet$ is not necessarily associative (Example 5.13). The remainder of this section will be taken up with proving the claims in this paragraph.

Proposition 5.7. *Given ODA $\mathcal{A}$, axioms (D1), (D3), (D4), (D5), and (D7) hold in* $\Gamma_D[\mathcal{A}]$.

Proof. That $\Gamma_D[\mathcal{A}] \models \{(D1), (D3), (D4)\}$ follows easily from Corollary 3.5 and Lemma 5.5. Since $\mathbf{dom}^\bullet(C_1) ;^\bullet \mathbf{dom}^\bullet(C_2) = \Gamma_D(\{\mathbf{dom}(C_1) ; \mathbf{dom}(C_2))$ for all $C_1, C_2 \in \Gamma_D[\mathcal{A}]$, by Corollary 3.5 it is a necessary and sufficient condition for $\Gamma_D[\mathcal{A}] \models (D7)$ that $\Gamma_D(\mathbf{dom}(\mathbf{dom}(C_1);\mathbf{dom}(C_2)) = \mathbf{dom}[\Gamma_D(\mathbf{dom}(C_1);\mathbf{dom}(C_2))]^\uparrow$ for all $C_1, C_2 \in \Gamma_D[\mathcal{A}]$. We shall show that $\mathbf{dom}(C_1);\mathbf{dom}(C_2)$ is Γ_D-closed, as in that case the required equality follows from Lemma 5.5: Let $x_1, x_2, x_3 \in C_1$, and let $y_1, y_2, y_3 \in C_2$. Then

$$\begin{aligned}&\mathbf{dom}(\mathbf{dom}(x_1) ; \mathbf{dom}(y_1)) ; \mathbf{dom}(x_2) ; \mathbf{dom}(y_2) ; \mathbf{ran}(\mathbf{dom}(x_3) ; \mathbf{dom}(y_3))\\ =\,&\mathbf{dom}(x_1) ; \mathbf{dom}(x_2) ; \mathbf{dom}(x_3) ; \mathbf{dom}(y_1) ; \mathbf{dom}(y_2) ; \mathbf{dom}(y_3)\end{aligned}$$

by axioms (D4) and (D7). Since $\mathbf{dom}[C_1]^\uparrow$ and $\mathbf{dom}[C_2]^\uparrow$ are closed by Lemma 5.5 it's easy to show that $\mathbf{dom}(x_1) ; \mathbf{dom}(x_2) ; \mathbf{dom}(x_3) \in \mathbf{dom}[C_1]^\uparrow$ and $\mathbf{dom}(y_1) ; \mathbf{dom}(y_2) ; \mathbf{dom}(y_3) \in \mathbf{dom}[C_2]^\uparrow$ and thus $\Gamma_D[\mathcal{A}] \models (D7)$ as required. That $\Gamma_D[\mathcal{A}] \models (D5)$ follows easily from Lemma 5.5. $\dashv$

Lemma 5.8. *For all $S \in P^*$, $\Gamma_D(S)^\smile = \Gamma_D(S^\smile)$.*

Proof. Since $S^\smile \subseteq \Gamma_D(S)^\smile$ and $\Gamma_D(S)^\smile$ is Γ_D-closed by Lemma 5.5, $\supseteq$ follows from properties of closure operators. Define $X_0 = S$ and X_n as in Lemma 5.3 for all $n \in \omega$. Then $X_0^\smile = S^\smile \subseteq \Gamma_D(S^\smile)$, and for all $k \in \omega$ and every $a \in X_k$ we have $a \geq b = \mathbf{dom}(b_1) ; b_2 ; \mathbf{ran}(b_3)$ for some $b_1, b_2, b_3 \in X_{k-1}$, so $b^\smile = \mathbf{dom}(b_3^\smile) ; b_2^\smile ; \mathbf{ran}(b_1^\smile)$ by involution and axioms (D1) and (D4), and so if $X_{k-1}^\smile \subseteq \Gamma_D(S)^\smile \Longrightarrow X_k^\smile \subseteq \Gamma_D(S)^\smile$. Since $\Gamma_D(S)^\smile = \bigcup_{n\in\omega} X_n^\smile$ we are done. $\dashv$

Lemma 5.9. *For all $f \in \{;, \mathbf{dom}, \mathbf{ran}, ^\smile, 0, \mathbf{id}\}$ the extension $f^\bullet$ is isotone and normal, moreover*

(1) *$\mathbf{id}^\bullet$ is a left and right identity for $;^\bullet$, and*
(2) *$^{\smile\bullet}$ is an involution.*

Proof. Isotonicity of the operations is automatic from the lifting process, and normality follows from the fact that $0^\bullet = 0^\uparrow$. That $\mathbf{id}^\bullet$ is a left and right identity for $;^\bullet$ follows easily from the definition of $;^\bullet$ and the fact that $\mathbf{id}^\bullet = \mathbf{id}^\uparrow$. To see that $(a ; b)^\smile \leq b^\smile ; a^\smile$ holds in $\Gamma_D[\mathcal{A}]$ define $\phi = (a ; b)^\smile$ and $\psi = b^\smile ; a^\smile$. Then using Lemma 5.5 it's easy to see that $\psi^\bullet(C, D) = \Gamma_D(\psi[C \times D]^\uparrow)$ for all $C, D \in \Gamma_D[\mathcal{A}]$, and that $\phi^\bullet(C, D) = \Gamma_D(\phi[C \times D]^\uparrow)$ follows from Lemma 5.8. $\dashv$

Lemma 5.10. *Let $X, Y \in \Gamma_D[\mathcal{A}]$. If $\mathbf{dom}(X) = \mathbf{dom}(Y)$ and $\mathbf{ran}(X) = \mathbf{ran}(Y)$ then $X \cup Y \in \Gamma_D[\mathcal{A}]$.*

PROOF. Let $z_1, z_2, z_3 \in X \cup Y$. We are required to prove that $\mathbf{dom}(z_1); z_2; \mathbf{ran}(z_3) \in X \cup Y$. Without loss, let $z_2 \in X$. Since $\mathbf{dom}(X) = \mathbf{dom}(Y)$ we know that $\mathbf{dom}(z_1) \in \mathbf{dom}(X)$ and similarly $\mathbf{ran}(z_3) \in \mathbf{ran}(X)$, hence $\mathbf{dom}(z_1); z_2; \mathbf{ran}(z_3) \in X$, by the closure of X. ⊣

EXAMPLE 5.11. *To show that (D2) can fail in* $\Gamma_D[\mathcal{A}]$. Let $\mathcal{A}$ be the full proper ODA over a base of four elements $\{a, b, c, d\}$. Define $x, y \in \mathcal{A}$ by $x = \{(a, b), (c, d)\}$, and $y = \{(a, d), (c, b)\}$. Then $\mathbf{dom}(x) = \mathbf{dom}(y)$ and $\mathbf{ran}(x) = \mathbf{ran}(y)$, and consequently $C = \{x, y\}^{\uparrow}$ is Γ_D-closed. We aim to show that $\Gamma_D(C\,;C^{\smile}) \not\subseteq \mathbf{dom}(C)$. Now, in particular $x\,;y^{\smile} \in \Gamma_D(C\,;C^{\smile})$, and $x\,;y^{\smile} = \{(a, c), (c, a)\}$, and $\mathbf{dom}(C) = \mathbf{dom}(x)^{\uparrow} = \mathbf{dom}(y)^{\uparrow} = \{(a, a), (c, c)\}^{\uparrow}$, so $x\,;y^{\smile} \notin \mathbf{dom}(C)$, and thus $\Gamma_D(C\,;C^{\smile}) \not\subseteq \mathbf{dom}(C)$, and $\Gamma_D[\mathcal{A}] \not\models (D2)$.

EXAMPLE 5.12. *To show that (D6) can fail in* $\Gamma_D[\mathcal{A}]$. Let $\mathcal{A}$ be the full proper ODA over the two element base $\{a, b\}$. Define $x = \{(a, b), (b, a)\}$ and let $\mathbf{id} = \{(a, a), (b, b)\}$ be the identity as normal. Let $C = \{x, \mathbf{id}\}^{\uparrow}$. Then, as $\mathbf{dom}(x) = \mathbf{dom}(\mathbf{id})$ and $\mathbf{ran}(x) = \mathbf{ran}(\mathbf{id})$, C is Γ_D-closed. Define $B = \{(b, b)\}^{\uparrow}$. Then

$$\begin{aligned}
\mathbf{dom}(\Gamma_D(C\,;B)) &= \mathbf{dom}(\Gamma_D(\{x\,;\{(b,b)\},\ \ \mathbf{id}\,;\{(b,b)\}\}^{\uparrow}))\\
&= \mathbf{dom}(\Gamma_D(\{\{(a,b),(b,a)\}\,;\{(b,b)\},\ \ \{(b,b)\}\}^{\uparrow}))\\
&= \mathbf{dom}(\Gamma_D(\{\{(a,b)\},\{(b,b)\}\}^{\uparrow}))\\
&= \mathbf{dom}(\emptyset^{\uparrow})\\
&= \emptyset^{\uparrow}
\end{aligned}$$

However, $\mathbf{dom}(C) = \{\{(a, a), (b, b)\}\}^{\uparrow} = \mathbf{id}$, and so $\mathbf{dom}(C)\,; B = B = \mathbf{dom}(B) \neq \emptyset^{\uparrow}$, and thus $\Gamma_D[\mathcal{A}] \not\models ((D6))$.

EXAMPLE 5.13. *To show that associativity can fail in* $\Gamma_D[\mathcal{A}]$. Let $\mathcal{A}$ be the full proper ODA over a base of five elements $\{a, b, c, d, e\}$, let $x = \{(a, a)\}$, let $y = \{(a, b), (c, d)\}$, let $z = \{(a, d), (c, b)\}$, and let $u = \{(b, e), (d, e)\}$. Define $A = x^{\uparrow}$, $B = \{y, z\}^{\uparrow}$, and $C = u^{\uparrow}$. Then A and C are principal and hence Γ_D-closed, and $\mathbf{dom}(z) = \mathbf{dom}(y)$ and $\mathbf{ran}(z) = \mathbf{ran}(y)$ so C is also Γ_D-closed. Now, $\Gamma_D(A\,;B) = \Gamma_D(\{x\,;y, x\,;z\}^{\uparrow}) = \Gamma_D(\{\{(a, b)\}, \{(a, d)\}\}^{\uparrow}) = \emptyset^{\uparrow}$, as $\{a, b\}\,;\mathbf{ran}(\{a, d\}) = \emptyset$, so $\Gamma_D(\Gamma_D(A\,;B)\,;C) = \emptyset^{\uparrow}$. However, $B\,;C = \{y\,;u, z\,;u\}^{\uparrow} = \{(a, e), (c, e)\}^{\uparrow}$, which is principal and hence Γ_D-closed. Thus $\Gamma_D(A\,;\Gamma_D(B\,;C)) = \Gamma_D(A\,;B\,;C) = \Gamma_D(\{x\,;y\,;u, x\,;z\,;u\}^{\uparrow}) = \Gamma_D(\{(a, e)\}^{\uparrow}) = \{(a, e)\}^{\uparrow} \neq \emptyset^{\uparrow}$, and so $\Gamma_D(\Gamma_D(A\,;B)\,;C) \neq \Gamma_D(A\,;\Gamma_D(B\,;C))$.

§6. The representation theorem.

THEOREM 6.1. *Let $\mathcal{A}$ be an ordered domain algebra. The map $h : \mathcal{A} \to \wp(\Gamma_D[\mathcal{A}] \times \Gamma_D[\mathcal{A}])$ defined by*

$$(X, Y) \in h(a) \iff X\,;^{\bullet} a^{\uparrow} \subseteq Y \text{ and } Y\,;^{\bullet}(a^{\smile})^{\uparrow} \subseteq X$$

is a representation of $\mathcal{A}$ over the base $\Gamma_D[\mathcal{A}]$.

First, some preparatory lemmas.

LEMMA 6.2. *Let $\mathcal{A}$ be an ODA.*

(1) *If $a \in \mathcal{A}$, $X \in \Gamma_D[\mathcal{A}]$ and $\mathbf{dom}(a) \in \mathbf{ran}(X)$ then $\mathbf{ran}(X; a^{\uparrow}) \in \Gamma_D[\mathcal{A}]$.*

(2) *If* $a \in \mathcal{A}$, $X \in \Gamma_D[\mathcal{A}]$, $\delta \in \mathbf{dom}(\Gamma_D[\mathcal{A}])$ *and* $\mathbf{ran}(X) \supseteq \mathbf{dom}(a^{\uparrow};\delta)$ *then* $X;a^{\uparrow};\delta \in \Gamma_D[\mathcal{A}]$.

Proof. For the first part, let $x_i \in X$ (for $i = 1,2,3$). We know that $\mathbf{ran}(x_i;a) \in \mathbf{ran}(X;a)$ and we are required to prove that

$$\mathbf{dom}(\mathbf{ran}(x_1;a));\mathbf{ran}(x_2;a);\mathbf{ran}(\mathbf{ran}(x_3;a)) \in \mathbf{ran}(X;a)$$

Well,

$$\begin{aligned}
&\mathbf{dom}(\mathbf{ran}(x_1;a));\mathbf{ran}(x_2;a);\mathbf{ran}(\mathbf{ran}(x_3;a)) \\
&= \mathbf{ran}(x_1;a);\mathbf{ran}(x_2;a);\mathbf{ran}(x_3;a) && \text{by (D4)} \\
&\geq \mathbf{ran}(\underbrace{x_1;\mathbf{ran}(x_2);\mathbf{ran}(x_3)}_{\in X};a) && \text{by (D1), (D6)} \\
&\in \mathbf{ran}(X;a) && \text{since } X \in \Gamma_D[\mathcal{A}]
\end{aligned}$$

For the second part, let $x_i \in X$ and $d_i \in \delta$ (for $i = 1,2,3$), we are required to prove that

$$\mathbf{dom}(x_1;a;d_1);(x_2;a;d_2);\mathbf{ran}(x_3;a;d_3) \in X;a;\delta$$

For this,

$$\begin{aligned}
&\mathbf{dom}(x_1;a;d_1);(x_2;a;d_2);\mathbf{ran}(x_3;a;d_3) \\
&= \mathbf{dom}(x_1;\mathbf{dom}(a;d_1));x_2;a;d_2;\mathbf{ran}(\mathbf{ran}(x_3;a);d_3) && \text{by (D6)} \\
&= \underbrace{\mathbf{dom}(x_1;\mathbf{dom}(a;d_1));x_2}_{=x_2' \in X};a;d_2;\mathbf{ran}(x_3;a);d_3 && \text{see (†) below} \\
&= x_2';a;\mathbf{ran}(x_3;a);d_2;d_3 && \text{by (D7)} \\
&\geq x_2';\mathbf{ran}(x_3);a;d_2;d_3 && \text{Lemma 4.2} \\
&\in X;a;\delta && \text{since } X \in \Gamma_D[\mathcal{A}],\ \delta \in \mathbf{dom}(\Gamma_D[\mathcal{A}])
\end{aligned}$$

(†) this follows from $(D7)$ and the facts that $X \in \Gamma_D[\mathcal{A}]$, $\mathbf{ran}(X) \supseteq \mathbf{dom}(a;\delta)$. ⊣

Lemma 6.3. *h is 1-1.*

Proof. Let $a \not\leq b \in \mathcal{A}$. By isotonicity, (D5), (D2), and Lemma 5.5, $(\mathbf{dom}(a))^{\uparrow};a^{\uparrow} \subseteq a^{\uparrow}$ and $a^{\uparrow};(a^{\smile})^{\uparrow} \subseteq (\mathbf{dom}(a))^{\uparrow}$, so $((\mathbf{dom}(a))^{\uparrow}, a^{\uparrow}) \in h(a)$. Also, we cannot have $\mathbf{dom}(a);b \geq a$, by transitivity, isotonicity and (D1), since $a \not\leq b$. Thus $((\mathbf{dom}(a))^{\uparrow}, a^{\uparrow}) \notin h(b)$, and we are done. ⊣

Lemma 6.4. $\{^{\smile}, 0, \mathbf{id}, \leq\}$ *are correctly represented*

Proof. $h(0) = \emptyset$, by normality and the partial order axioms, and $\leq$ is correctly represented by the partial order axioms and isotonicity. We have $h(\mathbf{id}) = \{(X,X) : X \in Cl(\mathcal{A})\}$ by the involuted monoid axioms, and $^{\smile}$ is correctly represented by the involution axioms. ⊣

LEMMA 6.5. *Let $a, b \in \mathcal{A}$, $X, Z \in \Gamma_D[\mathcal{A}]$ and suppose $X; a; b \subseteq Z$, and $Z; b^\smile; a^\smile \subseteq X$. Then the sets*

$$\begin{aligned}\alpha =& X\,; a^\uparrow\,; \mathbf{ran}(Z\,;(b^\smile)^\uparrow),\\ \beta =& Z\,;(b^\smile)^\uparrow\,; \mathbf{ran}(X\,; a^\uparrow), \text{ and}\\ & \alpha \cup \beta\end{aligned}$$

are closed.

PROOF. Consider $\alpha = X\,; a^\uparrow\,; \mathbf{ran}(Z\,;(b^\smile)^\uparrow)$ first. If $z \in Z$ then

$$\begin{aligned}\mathbf{dom}(a; \mathbf{ran}(z; b^\smile)) &= \mathbf{dom}(a; \mathbf{dom}(b; \mathbf{ran}(z)) && \text{by } (D3), (D6)\\ &= \mathbf{dom}(a; b; \mathbf{ran}(z)) && (D6)\\ &= \mathbf{ran}(\mathbf{ran}(z); b^\smile; a^\smile) && (D3)\\ &= \mathbf{ran}(z; b^\smile; a^\smile) && (D6)\\ &\in \mathbf{ran}(X) && Z; b^\smile; a^\smile \subseteq X\end{aligned}$$

hence $\mathbf{dom}(a^\uparrow; \mathbf{ran}(Z; b^{\smile\uparrow})) \subseteq \mathbf{ran}(X)$ and by Lemma 6.2(2) (with $\delta = \mathbf{ran}(Z\,; b^{\smile\uparrow})$) α is closed. Similarly β is closed. Note that $\mathbf{dom}(\alpha) = \mathbf{dom}(\beta)$ and $\mathbf{ran}(\alpha) = \mathbf{ran}(\beta)$. By Lemma 5.10, $\alpha \cup \beta$ is also closed. ⊣

LEMMA 6.6. ; *is correctly represented.*

PROOF. If $(X, Y) \in h(a)$ and $(Y, Z) \in h(b)$, then

$$\begin{aligned}&X\,;^\bullet a^\uparrow \subseteq Y,\\ &Y\,;^\bullet (a^\smile)^\uparrow \subseteq X,\\ &Y\,;^\bullet b^\uparrow \subseteq Z, \text{ and}\\ &Z\,;^\bullet (b^\smile)^\uparrow \subseteq Y.\end{aligned}$$

Hence $X\,;^\bullet (a\,; b)^\uparrow \subseteq Z$ and $Z\,;^\bullet ((a\,; b)^\smile)^\uparrow = Z\,;^\bullet (b^\smile\,; a^\smile)^\uparrow \subseteq X$ by associativity and the involution axioms. So $(X, Z) \in h(a\,; b)$.

Conversely, assume that $(X, Z) \in h(a\,; b)$, i.e. that

$$\begin{aligned}&X;^\bullet (a; b)^\uparrow \subseteq Z, \text{ and}\\ &Z;^\bullet (b^\smile; a^\smile)^\uparrow \subseteq X.\end{aligned}$$

Let $Y = \alpha \cup \beta = X; a^\uparrow; \mathbf{ran}(Z; b^{\smile\uparrow}) \ \cup\ Z; b^{\smile\uparrow}; \mathbf{ran}(X; a^\uparrow)$. Then Y is closed by Lemma 6.5. We claim that $(X, Y) \in h(a)$, and $(Y, Z) \in h(b)$. To prove the claim we must show that $X;^\bullet a^\uparrow \subseteq Y$ and $Y;^\bullet a^{\smile\uparrow} \subseteq X$. For the first inclusion, we have $X; a \subseteq \alpha \subseteq Y$. For the other inclusion, let $y \in Y$, we have to prove that $y;^\bullet a^{\smile\uparrow} \in X$. Since $y \in Y = (X; a; \mathbf{ran}(Z; b^\smile)) \cup (Z; b^\smile; \mathbf{ran}(X; a))$ there are $x \in X$, $z \in Z$ and either $y \geq x; a; \mathbf{ran}(z; b^\smile)$ or $y \geq z; b^\smile; \mathbf{ran}(x; a)$. In the former case,

$$\begin{aligned}y; a^\smile &\geq x; a; \mathbf{ran}(z; b^\smile); a^\smile\\ &\geq x; \mathbf{dom}(a; \mathbf{ran}(z; b^\smile)) && \text{by (D2)}\\ &\geq x; \mathbf{dom}(a; b; z^\smile) && (D3), (D6)\\ &\in X && Z; b^\smile; a^\smile \subseteq X, X \text{ closed}\end{aligned}$$

while in the latter case

$$\begin{aligned}
y; a^{\smile} &= z; b^{\smile}; \mathbf{ran}(x; a); a^{\smile} && \\
&\geq z; b^{\smile}; \mathbf{dom}(a^{\smile}; \mathbf{ran}(x)); a^{\smile} && \text{by (D3), (D6)} \\
&\geq z; b^{\smile}; a^{\smile}; \mathbf{dom}(x^{\smile}) && \text{Lemma 4.2} \\
&\in X; \mathbf{ran}(X) = X && X \text{ is closed}
\end{aligned}$$

⊣

Lemma 6.7. **dom** *and* **ran** *are correctly represented.*

Proof. If $(X, Y) \in h(\mathbf{dom}(a))$, then $X\,;(\mathbf{dom}(a))^{\uparrow} \subseteq Y$. Since $\mathbf{dom}(a) \leq \mathbf{id}$ by (D1), we have that, for every $x \in X$, there is $y \in Y$ such that $x \geq x\,; \mathbf{dom}(a) \geq y$. Since Y is (upwards) closed, we get $X \subseteq Y$. Similarly, we get $Y \subseteq X$ by $Y\,;((\mathbf{dom}(a))^{\smile})^{\uparrow} \subseteq Y\,;(\mathbf{dom}(a))^{\uparrow} \subseteq X$ (using (D1)). Hence $X = Y$, i.e., $(X, X) \in h(\mathbf{dom}(a))$. Note also that $\mathbf{dom}(a) \in \mathbf{ran}(X)$, since $\mathbf{dom}(a) \in \mathbf{ran}(Y\,;(\mathbf{dom}(a)^{\uparrow}) \subseteq \mathbf{ran}(x)$.

Define the closed element $Z = X\,; a^{\uparrow}$. Then $(X, Z) \in h(a)$, since $X\,; a^{\uparrow} \subseteq Z$ by definition, and

$$X\,; a^{\uparrow}\,;(a^{\smile})^{\uparrow} \subseteq X\,;(\mathbf{dom}(a))^{\uparrow} \subseteq X$$

by (D2), and $\mathbf{dom}(a) \in \mathbf{ran}(X)$. Conversely, suppose $(X, Z) \in h(a)$ (for some Z). Then $X\,; a^{\uparrow} \subseteq Z$ and $Z\,;(a^{\smile})^{\uparrow} \subseteq X$. Since $Z\,;(a^{\smile})^{\uparrow} \subseteq X$, we have $\mathbf{dom}(a) = \mathbf{ran}(a^{\smile}) \in \mathbf{ran}(Z\,;(a^{\smile})^{\uparrow}) \subseteq \mathbf{ran}(X)$, whence $X\,;(\mathbf{dom}(a))^{\uparrow} \subseteq X$, i.e. $(X, X) \in h(\mathbf{dom}(a))$. So **dom** is correctly represented. Showing that **ran** is properly represented is similar. ⊣

We have shown that h yields a representation of $\mathcal{A}$, and clearly when $\mathcal{A}$ is finite the base of this representation is also finite. This concludes the proof of Theorem 6.1.

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