Positive Fragments of Relevance Logic and Algebras of Binary Relations

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Abstract

We prove that algebras of binary relations whose similarity type includes intersection, union, and one of the residuals of relation composition form a non-finitely axiomatizable quasivariety and that the equational theory is not finitely based. We apply this result to the problem of the completeness of the positive fragment of relevance logic with respect to binary relations.

Keywords: substructural logics, relevance logic, dynamic semantics, completeness, finite axiomatizability, relation algebras

1 Introduction

Recently there has been a renewed interest in relation-algebraic semantics for substructural and, in particular, relevance logics, see, e.g., [BDM09, Ko07, Ma07, Ma10, Mi09, Mi10]. The idea is roughly the following: formulas are interpreted as binary relations and valid formulas correspond to valid algebraic equations of binary relations. In this paper, we look at the positive fragment $\mathbf{R}_+$ of relevance logic and the associated semantics by binary relations, and prove a non-finite axiomatizability result, Theorem 2.2.

Probably the simplest way of providing an algebraic semantics to (substructural) logics is to consider the Lindenbaum–Tarski algebra of the logic. For instance, the Lambek calculus (LC) is complete w.r.t. ordered residuated semigroups. Similar completeness result holds for relevance logic $\mathbf{R}^{\mathbf{t}}$ w.r.t. De Morgan monoids, cf. [Du66, AB75, RM73]. This type of semantics is rather abstract: the semantics is given by a class of algebras usually defined by a set of axioms.

A more concrete, Kripke-style semantics can be given by evaluating formulas over possible worlds, considering certain connectives as modalities and interpreting them by accessibility relations over the set of possible worlds. For instance, fusion ∘ is treated as a binary modality with a corresponding ternary accessibility relation R:

$$a \models \varphi \circ \psi \iff \text{ there are } b, c \text{ such that } Rabc, \ b \models \varphi \text{ and } c \models \psi \tag{1}$$

cf. [RM72]. Hence we can interpret certain logics via classes of frames of the form (S, R_i), where S is the set of possible worlds and the R_is are accessibility relations over S interpreting the intensional connectives such as fusion or various modalities (in the above example of fusion, $R \subseteq S \times S \times S$). Of course, we may have to restrict the class of frames, e.g., commutativity of ∘ requires: for all a, b, c,

$$R(a, b, c) \Rightarrow R(a, c, b) \tag{2}$$

and idempotency of $\circ$ requires: for every a,

$$R(a, a, a) \tag{3}$$

Examples of this type of semantics include the Routley–Meyer semantics for relevance logic $\mathbf{R}$ and its positive fragment $\mathbf{R}_+$ [ABD92, RM73] and relational semantics for LC [Do92, Du93]. In passing we note that the above algebraic and Kripke-style semantics are not independent: certain axiomatically given classes of algebras can be characterized as (subalgebras of) complex algebras of classes of frames. We mention the classical result of [JT52] that every relation algebra (see Definition 2.1 below) is a subalgebra of the complex algebra of a frame (or, using the terminology of algebraic logic, of an atom structure). Similar connections via canonical extensions can be established for substructural logics, see, e.g., [DGP05].

Even more concrete semantics can be provided by requiring that the possible worlds have certain properties and the accessibility relations are determined by the nature of the possible worlds. For instance, we may require that possible worlds are ordered pairs, hence formulas are interpreted as binary relations, and the accessibility relation interpreting fusion is

$$R(a_0, a_1)(b_0, b_1)(c_0, c_1) \Longleftrightarrow a_0 = b_0, \ a_1 = c_1 \text{ and } b_1 = c_0 \tag{4}$$

i.e., fusion is interpreted as relation composition. This bonds well with the intuition behind dynamic semantics, see, e.g., [vB96]. In this setting, possible worlds can be thought of as transitions (or information channels) and fusion as dynamic conjunction (the conjuncts are executed/stated sequentially). Implication can be interpreted as the residual(s)[1] of composition

$$(a_0, a_1) \models \varphi \to \psi \Longleftrightarrow \text{for every } x, \ (x, a_0) \models \varphi \text{ implies } (x, a_1) \models \psi \tag{5}$$

and relevant (or De Morgan) negation as converse-complement

$$(a_0, a_1) \models\, \sim \varphi \Longleftrightarrow (a_1, a_0) \not\models \varphi \tag{6}$$

Finite axiomatizability w.r.t. this type of semantics includes the completeness of the relevance logic $\mathbf{RM}$ [Ma10] and the LC [AM94], and the representation of De Morgan lattices (distributive lattices with de Morgan negation) [Du82].

Looking at the relevance logic $\mathbf{R}$, propositions are interpreted over a family of commuting and dense binary relations, and the interpretation of the logical connectives is as follows: conjunction $\wedge$ is intersection, disjunction $\vee$ is union, fusion $\circ$ is relation composition, implication $\to$ is interpreted as the residual of relation composition, and relevant negation $\sim$ as converse-complement. While formulas are evaluated at sets of transitions, truth is restricted to states (or situations), i.e., to ordered pairs of the form (a, a) — see the precise definition of this semantics in Section 2.3. Recently [Ma07] showed that relevance logic $\mathbf{R}$ indeed can be soundly interpreted over commutative and dense families of relations, and posed the problem of completeness of $\mathbf{R}$ w.r.t. this semantics. It turned out that completeness is impossible in this case, cf. [Mi09], although completeness can be achieved by adding the mingle axiom to $\mathbf{R}$ and the corresponding requirement that all relations are transitive to the semantics, cf. [Ma10].

As [RM73] points out the Routley–Meyer semantics satisfies the *variable sharing*, or *relevance principle*: if φ and ψ do not share atomic propositions, then $\varphi \to \psi$ is not valid. Interpreting implication $\to$ as the residual of fusion avoids some arguably counterintuitive features of classical implication as well. Note that $(\varphi_1 \wedge \varphi_2) \to \psi \models (\varphi_1 \to \psi) \vee (\varphi_2 \to \psi)$ is classically valid but does not hold when $\to$ is interpreted as the residual of fusion. Let φ_i stand for "you answer question i" and ψ stand for "you pass the exam", and assume that the rule is that a student who answers both questions passes the exam. We expect that a teacher would have difficulties explaining to two students, one of whom answered question 1 and the other answered question 2, why neither of them passed the exam.[2]

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[1]In the special case where composition is commutative, the right and left residuals of composition coincide, see below.

[2]We heard a similar example from I. Hodkinson, who attributes it to D. Gabbay.

Although our results are motivated by substructural logics, they may also be of interest to those studying pure algebraic logic. Recall that a relation algebra $\mathfrak{A} = (A, 0, 1, \cdot, +, -, ;, \smile, 1')$ is a Boolean algebra with operators, and it is representable, an RRA, if it is isomorphic to an algebra of binary relations — see the precise definition in Section 2.2. It is known that RRA is a variety [Ta55], but it cannot be defined by any finite set of first-order formulas [Mo64], hence the equational theory is not finitely based. There has been some interest in identifying subsignatures τ of the full relation algebra signature such that the corresponding representation class $\mathsf{Q}(\tau)$ (defined formally below) is defined by a finite set of axioms, or where the set of equations valid over $\mathsf{Q}(\tau)$ follows from a finite set of equations, see [Sc91, Mi04, AM10] for surveys. It is known that $\mathsf{Q}(\tau)$ is a finitely axiomatizable quasivariety if $\tau \subseteq \{0, 1, +, \cdot, -, 1', \smile\}$. But the general rule is that $\mathsf{Q}(\tau)$ is not finitely axiomatizable whenever $; \in \tau$. A notable exception is $\tau = \{\cdot, ;\}$, see [BS78], but $\mathsf{Q}(\tau)$ is known to be not finitely axiomatizable when τ contains $\{\cdot, \smile, ;\}$ or $\{+, \cdot, ;\}$, see [HM00] and [An91], respectively. Also $\mathsf{Q}(\tau)$ is not finitely axiomatizable if τ equals $\{+, ;\}$, $\{\cdot, 1', ;\}$, or $\{\leq, 1', ;\}$, cf. [An88], [HM07], [Hi05] for respective proofs. If we restrict τ to positive subsignatures of RRA, and we look at the variety $\mathsf{V}(\tau)$ generated by $\mathsf{Q}(\tau)$, then the picture is more promising: $\mathsf{V}(\tau)$ is finitely based if and only if not all of $;$, $\cdot$ and $\smile$ are contained in τ, see [AM10] for this result and for precise references to specific cases.

In this paper we extend these results by looking at generalized subsignatures τ of RRA, where operations definable in RRA can be included in τ. We show that whenever τ includes $\cdot, +$ and either the left $/$ or the right residual $\backslash$ of composition, then the quasivariety $\mathsf{Q}(\tau)$ cannot be defined by finitely many formulas, nor can the variety $\mathsf{V}(\tau)$ generated by $\mathsf{Q}(\tau)$ be finitely axiomatized, cf. Theorem 2.3 below. Our proof works with the additional requirement of the algebras being dense and commutative, a sufficient condition for soundly interpreting relevance logic $\mathbf{R}$, see below. Hence we can apply these results to the positive fragment of the relevance logic $\mathbf{R}$: the relevance logic $\mathbf{R}_+$ of the language $\{\wedge, \vee, \rightarrow\}$ is not complete w.r.t. algebras of binary relations even if we include finitely many additional axioms to the standard axiomatization of $\mathbf{R}_+$, Theorem 2.2.

The rest of the paper is organized as follows. In the next section, we recall the basics of relevance logic and relation algebras, and state our main results. Section 3 is devoted to the proofs of the key lemmas of the algebraic result. We conclude with some open problems.

2 Relevance logic and relation algebras

We continue with recalling the basics of relevance logic.

2.1 Relevance logic

We recall that the logic $\mathbf{R}$ of relevant implication is a finite Hilbert-style derivation system in the language $\mathcal{L}(\mathbf{R}) = \{\rightarrow, \wedge, \sim\}$ see, e.g., [RM73, AB75, ABD92]. In $\mathbf{R}$ the connectives $\vee$ and $\circ$ are defined as

$$\varphi \vee \psi = \sim (\sim \varphi \wedge \sim \psi) \text{ and } \varphi \circ \psi = \sim (\varphi \rightarrow \sim \psi)$$

In [AB75], **R** is defined by the following axioms

$$A1 \quad \varphi \to \varphi$$
$$A2 \quad \varphi \to ((\varphi \to \psi) \to \psi)$$
$$A3 \quad (\varphi \to \psi) \to ((\psi \to \eta) \to (\varphi \to \eta))$$
$$A4 \quad (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$$
$$A5 \quad (\varphi \wedge \psi) \to \varphi$$
$$A6 \quad (\varphi \wedge \psi) \to \psi$$
$$A7 \quad ((\varphi \to \psi) \wedge (\varphi \to \eta)) \to (\varphi \to (\psi \wedge \eta))$$
$$A8 \quad \varphi \to (\varphi \vee \psi)$$
$$A9 \quad \psi \to (\varphi \vee \psi)$$
$$A10 \quad ((\varphi \to \eta) \wedge (\psi \to \eta)) \to ((\varphi \vee \psi) \to \eta))$$
$$A11 \quad (\varphi \wedge (\psi \vee \eta)) \to ((\varphi \wedge \psi) \vee \eta)$$
$$A12 \quad (\varphi \to\sim \psi) \to (\psi \to\sim \varphi)$$
$$A13 \quad \sim\sim \varphi \to \varphi$$

and inference rules

$$R1 \quad \varphi, \varphi \to \psi \vdash \psi$$
$$R2 \quad \varphi, \psi \vdash \varphi \wedge \psi$$

We will write $\mathbf{R} \vdash \varphi$ if φ is derivable from the axioms, i.e., if φ is a theorem of $\mathbf{R}$.

The derivation system **RM** has the additional *mingle* axiom

$$\varphi \to (\varphi \to \varphi)$$

The logic $\mathbf{R^t}$ is defined by expanding the language of $\mathbf{R}$ by a logical constant $\mathbf{t}$: $\mathcal{L}(\mathbf{R^t}) = \mathcal{L}(\mathbf{R}) \cup \{\mathbf{t}\}$, and adding the following two axioms to those of $\mathbf{R}$:

$$\mathbf{t} \quad \text{and} \quad \mathbf{t} \to (\varphi \to \varphi)$$

The positive fragment $\mathbf{R_+}$ of $\mathbf{R}$ is defined by the axioms $A1 - A11$ and the derivation rules $R1$ and $R2$. For more details on relevance logic we refer the reader to the monographs [AB75, ABD92].

[Ma07] observes that a sound semantics is provided for relevance logic by algebras of binary relations. Next we recall the definition of representable relation algebras and then describe how to interpret relevance logic.

2.2 Relation algebras

Let us recall the basic definitions about relation algebras, cf. [HH02, Ma06].

Definition 2.1 *1. A* relation algebra, *an* RA, *is an algebra*

$$\mathfrak{A} = (A, 0, 1, \cdot, +, -, ;, \breve{}, 1')$$

such that $(A, 0, 1, \cdot, +, -)$ *is a Boolean algebra, and the following equations hold, for every* $x, y, z \in A$:

$$(R1) \quad x \,;\, (y \,;\, z) = (x \,;\, y) \,;\, z$$
$$(R2) \quad (x + y) \,;\, z = (x \,;\, z) + (y \,;\, z)$$
$$(R3) \quad x \,;\, 1' = x$$
$$(R4) \quad x^{\breve{}\breve{}} = x$$
$$(R5) \quad (x + y)^{\breve{}} = x^{\breve{}} + y^{\breve{}}$$
$$(R6) \quad (x \,;\, y)^{\breve{}} = y^{\breve{}} \,;\, x^{\breve{}}$$
$$(R7) \quad x^{\breve{}} \,;\, (-(x \,;\, y)) \leq -y$$

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where $x \leq y$ abbreviates $x + y = y$. We denote the class of all relation algebras by RA. We call a relation algebra $\mathfrak{A}$ integral if $1'$ is an atom (minimal non-zero element) in $\mathfrak{A}$, and symmetric if every element is self converse ($x = x^{\smile}$).

2. By a proper relation algebra, a PRA, we mean an algebra $\mathfrak{A} = (A, 0, 1, \cdot, +, -, ;, ^{\smile}, 1')$ such that $A \subseteq \mathcal{P}(W)$ (the power set of W) for some equivalence relation W, $0 = \emptyset$, $1 = W$, $\cdot$ is intersection, $+$ is union, $-$ is complement w.r.t. W, ; is relation composition, $^{\smile}$ is relation converse, and $1'$ is the identity relation Id_W restricted to W. More formally, for all elements $x, y \in A$,

$$x\,;y = \{(u, v) \in W : (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w\}$$
$$x^{\smile} = \{(u, v) \in W : (v, u) \in x\}$$
$$1' = \{(u, v) \in W : u = v\}$$

We denote the class of proper relation algebras by PRA. Given an $\mathfrak{A} \in$ PRA and W as above, we call W the unit of $\mathfrak{A}$. By a relation set algebra, an Rs, we mean a proper relation algebra $\mathfrak{A}$ with a square unit: $W = U \times U$ for some set U, the base of $\mathfrak{A}$. We denote the class of relation set algebras by Rs.

The class RRA of representable relation algebras is defined as

$$\mathsf{RRA} = \mathbf{I}\mathsf{PRA}$$

i.e., we close the class PRA under isomorphic copies.

Note that the (quasi)equational theories of Rs, PRA and RRA coincide. In fact we have

$$\mathsf{RRA} = \mathbf{I}\mathsf{PRA} = \mathbf{SP}\mathsf{Rs}$$

i.e., representable algebras are given by the closure of the class Rs under products and (isomorphic copies of) subalgebras. Indeed, if we take an arbitrary $\mathfrak{A} \subseteq \prod_{i \in I} \mathfrak{A}_i$ where each $\mathfrak{A}_i \in$ Rs, then $\mathfrak{A}$ may be embedded into the full algebra on $\mathcal{P}(W)$, where W is the disjoint union of the units W_i of $\mathfrak{A}_i$, i.e., $\mathfrak{A} \in$ RRA. Conversely, an $\mathfrak{A} \in$ PRA with an equivalence relation unit W is a subalgebra of the product of the restrictions $\mathfrak{A}_i$ of $\mathfrak{A}$ to the equivalence classes W_i of W, whence $\mathfrak{A} \in \mathbf{SP}$Rs, see [HH02, Lemma 3.7].

We may introduce additional operations in addition to, or instead of, the operations for RA as follows. An RA-definable operation is a term $t(\bar{x})$ using only the constants and the operations in $\{0, 1, +, \cdot, -, 1', ^{\smile}, ;\}$ and the variables in $\bar{x}$ (a sequence of variables). The main additional operations considered in this paper are the two residuals of composition, $\backslash$ and $/$

$$x \backslash y = -(x^{\smile}\,;-y) \text{ and } x / y = -(-x\,;y^{\smile})$$

It is easy to check that in any relation algebra $\mathfrak{A}$, and for any $x, y, z \in A$,

$$y \leq x \backslash z \text{ iff } x\,;y \leq z \text{ iff } x \leq z / y \tag{7}$$

We note that for any $x_1, x_2, y \in A$,

$$(x_1 \backslash y) \cdot (x_2 \backslash y) = (x_1 + x_2) \backslash y$$
$$(y \backslash x_1) \cdot (y \backslash x_2) = y \backslash (x_1 \cdot x_2) \tag{8}$$

with similar equations holding for the left residual. Thus each residual is monotonic in one argument and reverse monotonic (or antitonic, or downward monotonic) in the other argument. The interpretation of the residuals in a PRA with unit W is given by

$$x \backslash y = \{(u, v) \in W : \text{for every } w, (w, u) \in x \text{ implies } (w, v) \in y\}$$
$$x / y = \{(u, v) \in W : \text{for every } w, (v, w) \in y \text{ implies } (u, w) \in x\}$$
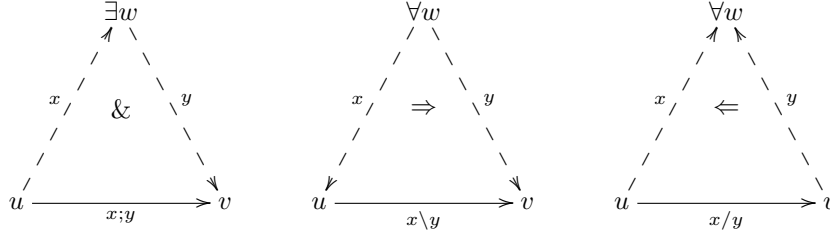
Figure 1: Composition and its residuals

see Figure 1. We also note that converse-complement $\sim x$ is defined as $\sim x = -(x^{\smile})$ in RA, and it is interpreted in a PRA with unit W as

$$\sim x = \{(u,v) \in W : (v,u) \notin x\}$$

Let τ be a subset of the definable operations in RA. We will denote the class of subalgebras of the τ-reducts, the τ-subreducts, of the elements of Rs by $\mathsf{R}(\tau)$.

Next we define subclasses of $\mathsf{R}(\tau)$ satisfying some extra conditions. We call an algebra $\mathfrak{A}$ in $\mathsf{R}(\tau)$ commutative if, for every $x, y \in A$ and u, v, w,

$$(u, w) \in x \text{ and } (w, v) \in y \text{ imply } (u, w') \in y \text{ and } (w', v) \in x \text{ for some } w' \tag{9}$$

and dense if, for every $x \in A$ and u, v,

$$(u, v) \in x \text{ implies } (u, w) \in x \text{ and } (w, v) \in x \text{ for some } w \tag{10}$$

cf. the similar "frame conditions" (2) and (3) on Routley–Meyer frames [RM73]. Let $\mathsf{R}^{cd}(\tau)$ denote the subclass of $\mathsf{R}(\tau)$ where each algebra is commutative and dense. If composition ; is definable in $\mathsf{R}(\tau)$, then commutativity and density are expressible by the equations $x\,;\,y = y\,;\,x$ and $x \leq x\,;\,x$, respectively. It is easy to check that $x \setminus y$ and $y \,/\, x$ coincide in commutative algebras, hence in $\mathsf{R}^{cd}(+, \cdot, \setminus)$.

Let $\mathsf{Q}(\tau)$ and $\mathsf{Q}^{cd}(\tau)$ denote the quasivarieties generated by $\mathsf{R}(\tau)$ and $\mathsf{R}^{cd}(\tau)$, respectively, and let $\mathsf{V}(\tau)$ and $\mathsf{V}^{cd}(\tau)$ denote the corresponding varieties. In general, $\mathsf{Q}(\tau)$ and $\mathsf{Q}^{cd}(\tau)$ are not closed under homomorphic images, whence the generated varieties $\mathsf{V}(\tau)$ and $\mathsf{V}^{cd}(\tau)$ might be strictly bigger than these quasivarieties (in contrast to the case of the full RA signature). In passing we note that $\mathsf{R}(\tau)$ is closed under ultraproducts (but may not be closed under products), whence $\mathsf{Q}(\tau) = \mathbf{SP}\mathsf{R}(\tau)$, i.e., elements of the generated quasivariety are representable as algebras of binary relations just like in the case of the full RA signature.

2.3 Main results

Relevance logic $\mathbf{R}$ can be interpreted into $\mathsf{R}^{cd}(\cdot, \sim, \setminus)$ as follows. Let $\mathfrak{A} \in \mathsf{R}^{cd}(\cdot, \sim, \setminus)$ with unit W. A valuation is a map v from the set of propositional atoms into A that is extended to compound formulas by interpreting the logical connectives $\wedge$, $\rightarrow$ and $\sim$ by the corresponding algebraic operations $\cdot$, $\setminus$ and $\sim$, respectively. We define

$$\mathsf{Id}_W = \{(u, v) \in W : u = v\}$$

Note that $a \setminus a \supseteq \mathsf{Id}_W$ for any $a \in A$, but Id_W may not be an element of $\mathfrak{A}$. Nevertheless, it can be used to define the semantics of $\mathbf{R}$ in $\mathfrak{A}$ as follows:

$$\mathfrak{A} \models \varphi \text{ iff } \mathsf{Id}_W \subseteq v(\varphi) \text{ for every valuation } v.$$

The formula φ is valid in $\mathsf{R}^{cd}(\cdot, \sim, \backslash)$, in symbols $\mathsf{R}^{cd}(\cdot, \sim, \backslash) \models \varphi$, iff $\mathfrak{A} \models \varphi$ for every $\mathfrak{A} \in \mathsf{R}^{cd}(\cdot, \sim, \backslash)$. As [Ma07] shows this is a sound semantics for $\mathbf{R}$:

$$\mathbf{R} \vdash \varphi \text{ implies } \mathsf{R}^{cd}(\cdot, \sim, \backslash) \models \varphi$$

In [Mi09] it is shown that completeness does not hold. In passing we note that $\mathbf{RM}$ is complete and sound w.r.t. $\mathsf{R}^{cdt}(\cdot, \sim, \backslash)$, the subclass of $\mathsf{R}^{cd}(\tau)$ defined by the additional requirement of transitivity: $x \, ; x \leq x$, [Ma10].

The proof in [Mi09] heavily uses the existence of the connective $\sim$. Thus it leaves the problem whether $\mathbf{R}_+$ is complete w.r.t. $\mathsf{R}^{cd}(\cdot, +, \backslash)$ open (of course, $\vee$ is interpreted as $+$). A related question is whether we can finitely axiomatize the logic of $\mathsf{R}^{cd}(\cdot, +, \backslash, ;)$ (where ; is the interpretation of fusion $\circ$). Here we show that no completeness is possible in these cases.

Theorem 2.2 *Relevance logic $\mathbf{R}_+$ is not complete w.r.t. to the semantics $\mathsf{R}^{cd}(\cdot, +, \backslash)$. In fact, the logic of $\mathsf{R}^{cd}(\cdot, +, \backslash)$ is not axiomatizable by a finite set of axioms and the derivation rules $R1$ and $R2$. The same non-finite axiomatizability holds for $\mathsf{R}^{cd}(\tau)$ and $\mathsf{R}(\tau)$, where $\{\cdot, +, \backslash\} \subseteq \tau$ and the elements of τ are definable using $\cdot$, $+$, $\backslash$, $/$, $\sim$ and $1'$.*

Proof: Observe that an equation $\sigma \leq \tau$ is valid in $\mathfrak{A} \in \mathsf{R}^{cd}(\cdot, +, \backslash)$ if and only if $\mathsf{Id}_W \subseteq \sigma \setminus \tau$ in $\mathfrak{A}$. Thus the theorem is an immediate consequence of Theorem 2.3 below. ∎

We formulate the main algebraic result of the paper.

Theorem 2.3 *Let τ be a similarity type, definable by the operations of relation algebra, such that $+, \cdot, \backslash$ are definable by τ. Then the equational and quasiequational theories of $\mathsf{R}(\tau)$ and $\mathsf{R}^{cd}(\tau)$ are not finitely based.*

Symmetrically, we may prove the same results if τ defines $+, \cdot, /$.

Proof: For every $n \in \omega$ we will define commutative and dense $\mathfrak{A}_n \in \mathsf{RA}$ and show that

1. for every $2 \leq n \in \omega$ and $\tau \supseteq \{\cdot, +, \backslash\}$, the τ-reduct $\mathfrak{D}_n$ of $\mathfrak{A}_n$ is not in the variety $\mathsf{V}(\tau)$ generated by $\mathsf{R}(\tau)$, since there is an equation e_n in the language $\{\cdot, +, \backslash\}$ such that e_n fails in $\mathfrak{D}_n$, but e_n is valid in $\mathsf{R}(\tau)$, hence in $\mathsf{V}(\tau)$, Lemma 3.1.

2. any non-trivial ultraproduct of $\mathfrak{A}_n$ is representable (is in RRA), whence the ultraproduct of $\mathfrak{D}_n$ is in $\mathsf{Q}^{cd}(\tau) \subseteq \mathsf{V}^{cd}(\tau)$, Lemma 3.10.

Hence the complements of the classes $\mathsf{Q}(\tau)$, $\mathsf{V}(\tau)$, $\mathsf{Q}^{cd}(\tau)$ and $\mathsf{V}^{cd}(\tau)$ are not closed under ultraproducts, and the result follows by Łoś theorem. ∎

3 Non-finite axiomatizability

The rest of the paper is devoted to make the proof of Theorem 2.3 complete.

3.1 Rainbow algebras

It is, in general, very difficult to know whether a given relation algebra is representable, indeed for finite algebras this question is undecidable [HH01b]. Nevertheless, special types of relation algebras can be defined where it is relatively easy to determine if the algebra is representable or not. The idea is to define relation algebras so that it is not too hard to work out how many rounds the second player can survive in a game to test representability (see Definition 3.2 below). A group of such constructions goes under the name of *rainbow algebra*, mainly because each non-identity atom of these algebras has a colour — green, yellow, white, black or red. Rainbow algebras have been used to solve many combinatorial problems for relation algebras and their fragments

[Hi95, HH97, Ho97, HM00, HH00, HH01a] and there is a version for cylindric algebras [HH09]. It is typically the case with these algebras that the first player ($\forall$) can use green and yellow atoms to force the existence of a large set of points where each distinct pair of points from this set is forced (by the consistency rules) to be labelled by a red atom. In [HH02, Definition 16.10] these sets are called *red cliques*. The second player ($\exists$) is allowed to choose, within the constraints of consistency, which red atom to use. We will see that the constraints in the finite algebras $\mathfrak{A}_n$ defined below restrict the moves of $\exists$ so that she eventually loses the representability game, whence $\mathfrak{A}_n$ is not representable. But as n grows, she can survive more and more rounds in the game played on $\mathfrak{A}_n$, and in fact $\exists$ can win the game played on the ultraproduct $\mathfrak{A}$ of $\mathfrak{A}_n$ ($n \in \omega$), whence $\mathfrak{A}$ is representable.

Rainbow algebras are integral, but commutativity, symmetry and density may fail. It is not hard to make rainbow algebras symmetric (hence commutative) and a symmetric relation algebra with a rule like (20) below was defined in [HM00]. In the definition below we modify the construction so that the algebras are also dense, this type of construction appeared in [Mi09]. In these modified rainbow algebras, it is possible for player $\forall$ to make a series of moves using green and yellow atoms to force a large set of points where nearly all pairs of distinct points from the set are labelled by a red atom, but some edges may be labelled by a yellow atom. We call these sets 'red clusters', and the strategy to deal with them is new. We will explain this more carefully in the remainder of this section.

We define relation algebras $\mathfrak{A}_n$ for $n \in \omega$. Let n be any natural number $n = \{0, 1, \ldots, n-1\}$. We define $\mathfrak{A}_n$ to be the finite relation algebra (in RA) with the following atoms $\mathsf{At}(\mathfrak{A}_n)$:

- identity: $1'$

- greens: g_i for $i \in n+1$

- yellows: y_i for $i \in n$

- blacks: b_i for $i \in n$

- whites: w_S for $S \subseteq (n+1) \times n$, $|S| \leq 2$

- reds: r_i for $i \in n$

All the atoms are self converse. Given this, a triple (x, y, z) of atoms is said to be an *inconsistent (or forbidden) triangle* if $x \cdot (y \,;\, z) = y \cdot (z \,;\, x) = z \cdot (x \,;\, y) = 0$. Using additivity, composition is determined by specifying that the set of inconsistent triangles are precisely the permutations of the following:

$$(1', x, y) \qquad \text{unless } x = y \tag{11}$$

$$(\mathsf{g}_i, \mathsf{g}_j, \mathsf{y}_k) \qquad \text{unless } |i - j| = 1 \tag{12}$$

$$(\mathsf{g}_i, \mathsf{g}_j, \mathsf{g}_k) \qquad \text{unless } i = j = k \tag{13}$$

$$(\mathsf{g}_i, \mathsf{g}_j, \mathsf{w}_S) \tag{14}$$

$$(\mathsf{g}_i, \mathsf{g}_j, \mathsf{r}_k) \qquad \text{unless } k = |i - j| \leq 5 \text{ or } 5 < |i - j| \equiv_5 k \tag{15}$$

$$(\mathsf{y}_i, \mathsf{y}_j, \mathsf{b}_k) \tag{16}$$

$$(\mathsf{y}_i, \mathsf{y}_j, \mathsf{y}_k) \qquad \text{unless } i = j = k \tag{17}$$

$$(\mathsf{y}_i, \mathsf{r}_j, \mathsf{b}_k) \qquad \text{unless } i = k \tag{18}$$

$$(\mathsf{g}_i, \mathsf{y}_j, \mathsf{w}_S) \qquad \text{unless } (i, j) \in S \tag{19}$$

$$(\mathsf{r}_i, \mathsf{r}_j, \mathsf{r}_k) \qquad \text{unless } i = j = k \text{ or } i + j = k \text{ or } i + k = j \text{ or } j + k = i \tag{20}$$

where $\equiv_5$ denotes equivalence modulo 5. For any three integers i, j, k we define $T(i, j, k)$ to mean that either $i = j = k$, $i + j = k$, $i + k = j$ or $j + k = i$. So (20) says that $(\mathsf{r}_i, \mathsf{r}_j, \mathsf{r}_k)$ is forbidden unless $T(i, j, k)$. We might also use the phrase 'the indices add up' in case $T(i, j, k)$. The set of consistent triangles of atoms is denoted by C:

$$C = \{(x, y, z) : x, y, z \in \mathsf{At}(\mathfrak{A}_n), (x, y, z) \text{ is consistent}\}$$

Note that $\mathfrak{A}_n$ is a dense $(x \leq x \,;\, x)$, commutative $(x \,;\, y = y \,;\, x)$, integral $(x \,;\, y = 0$ implies $x = 0$ or $y = 0$, hence $1'$ is an atom) and symmetric $(x^{\smile} = x)$ algebra. Also, for any natural numbers i, j, k, if $i \leq j \leq k$, then $|j - i| + |k - j| = |k - i|$, hence for arbitrary i, j, k if $|i - j|, |j - k|, |i - k| < n$, then the triangle

$$\left(\mathsf{r}_{|j-i|}, \mathsf{r}_{|k-j|}, \mathsf{r}_{|k-i|}\right) \tag{21}$$

is consistent.

It is not difficult to check that $\mathfrak{A}_n$ is indeed a relation algebra, for $n \geq 5$. All the axioms but $(R1)$ are straightforward to check. Below we sketch why $(R1)$ holds. It suffices to show that whenever we have atoms a, x, y, z such that $a \leq (x \,;\, y) \,;\, z$, then there is an atom b such that $a \leq x \,;\, b$ and $b \leq y \,;\, z$. If $x = a$ and $y = z$, then we can choose $b = 1'$. Otherwise we can try to use a white atom or a black atom for b. The only case when this might not work is when both x and a are green and both y and z are yellow (or the other way round), say $x = \mathsf{g}_i$, $a = \mathsf{g}_j$ and $y = \mathsf{y}_k$, $z = \mathsf{y}_l$. In this case, we can choose $b = \mathsf{r}_{|i-j|(\text{mod } 5)}$.

3.2 Non-representability

Next we show that the non-representability of $\mathfrak{D}_n$, the $\{\cdot, +, \backslash\}$-reduct of $\mathfrak{A}_n$, is witnessed by an equation e_n.

Lemma 3.1 *For every $2 \leq n \in \omega$, there is an equation e_n in the language $\{\cdot, +, \backslash\}$ such that*

1. *e_n fails in the $\{\cdot, +, \backslash\}$-reduct $\mathfrak{D}_n$ of $\mathfrak{A}_n$,*

2. *e_n is valid over Rs, hence over $\mathsf{R}(\cdot, +, \backslash)$.*

Hence $\mathfrak{D}_n \notin \mathsf{V}(\cdot, +, \backslash)$.

Proof: For each atom $\alpha \in \mathsf{At}(\mathfrak{A}_n)$, let v_α be a variable symbol, and let $V = \{v_\alpha : \alpha \in \mathsf{At}(\mathfrak{A}_n)\}$ denote the collection of the above variables. Note that V is finite since $\mathfrak{A}_n$ is finite. For each atom $\alpha \in \mathsf{At}(\mathfrak{A}_n)$ let $\overline{v_\alpha}$ stand for $\sum\{v_\beta : \beta \in \mathsf{At}(\mathfrak{A}_n) \setminus \{\alpha\}\}$. Let $\Omega = \sum\{v_\alpha \cdot v_\beta : \alpha, \beta \in \mathsf{At}(\mathfrak{A}_n),\ \alpha \neq \beta\}$.

For each atom $\alpha \in \mathsf{At}(\mathfrak{A}_n)$, we define terms $\sigma_n(\alpha)$ by induction.

$$\sigma_0(\alpha) = v_\alpha \cdot (\Omega \setminus \Omega)$$

$$\sigma_{m+1}(\alpha) = \sigma_m(\alpha) \cdot \prod_{\beta \in \mathsf{At}(\mathfrak{A}_n)} \left(v_\beta \setminus \sum_{(\alpha, \beta, \gamma) \in C} \sigma_m(\gamma)\right)$$

Terms τ_m are also defined by induction:

$$\tau_0 = \overline{v_{\mathsf{g}_0}}$$

$$\tau_{m+1} = \overline{v_{\mathsf{g}_{m+1}}} + (\sigma_m(\mathsf{y}_m) \setminus \tau_m)$$

We can now define the equation e_n as

$$\sigma_n(\mathsf{g}_n) \leq \tau_n$$

For item 1, let ι be the evaluation of the variables that maps v_α to α, for $\alpha \in \mathsf{At}(\mathfrak{A}_n)$. Note that $\iota(\Omega) = 0$ and $\iota(\overline{v_\alpha}) = -\iota(v_\alpha) = -\alpha$, for any $\alpha \in \mathsf{At}(\mathfrak{A}_n)$. We will show that $\iota(\sigma_n(\mathsf{g}_n)) = \mathsf{g}_n$ and $\iota(\tau_n) = -\mathsf{g}_n$, whence e_n fails in $\mathfrak{D}_n$.

For any $\alpha \in \mathsf{At}(\mathfrak{A}_n)$, we have $\iota(\sigma_0(\alpha)) = \iota(v_\alpha \cdot \Omega \setminus \Omega) = \alpha \cdot 0 \setminus 0 = \alpha$, and inductively

$$\iota(\sigma_{m+1}(\alpha)) = \alpha \cdot \prod_{\beta \in \mathsf{At}(\mathfrak{A}_n)} \left(\beta \setminus \sum_{(\alpha, \beta, \gamma) \in C} \gamma\right) = \alpha$$

since $\alpha \,;\, \beta = \beta \,;\, \alpha \leq \sum_{(\alpha, \beta, \gamma) \in C} \gamma$, whence $\alpha \leq \beta \setminus \sum_{(\alpha, \beta, \gamma) \in C} \gamma$ by equivalence (7). Hence $\iota(\sigma_n(\alpha)) = \alpha$, and in particular $\iota(\sigma_n(\mathsf{g}_n)) = \mathsf{g}_n$.

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Next we show that $\iota(\tau_m) = -\mathsf{g}_m$, for $m \leq n$, by induction over m. We have $\iota(\tau_0) = \iota(\overline{v_{g_0}}) = -\mathsf{g}_0$. By the previous paragraph, $\iota(\sigma_m(\mathsf{y}_m)) = \mathsf{y}_m$. Then, inductively,

$$\iota(\tau_{m+1}) = -\mathsf{g}_{m+1} + (\mathsf{y}_m \setminus -\mathsf{g}_m) = -\mathsf{g}_{m+1}$$

since $\mathsf{g}_{m+1} \not\leq \mathsf{y}_m \setminus -\mathsf{g}_m$ by $(\mathsf{y}_m, \mathsf{g}_{m+1}, \mathsf{g}_m) \in C$.

For item 2, assume that we have a representable algebra $\mathfrak{D} \in \mathsf{R}(\cdot, +, \setminus)$. We have to show that e_n is valid in $\mathfrak{D}$. We will establish this by reductio ad absurdum. To this end, assume that we have a valuation $\iota : V \to D$ and a pair (u_n, v) such that $(u_n, v) \in \iota(\sigma_n(\mathsf{g}_n))$ but $(u_n, v) \notin \iota(\tau_n)$. See Figure 2 for the argument below.
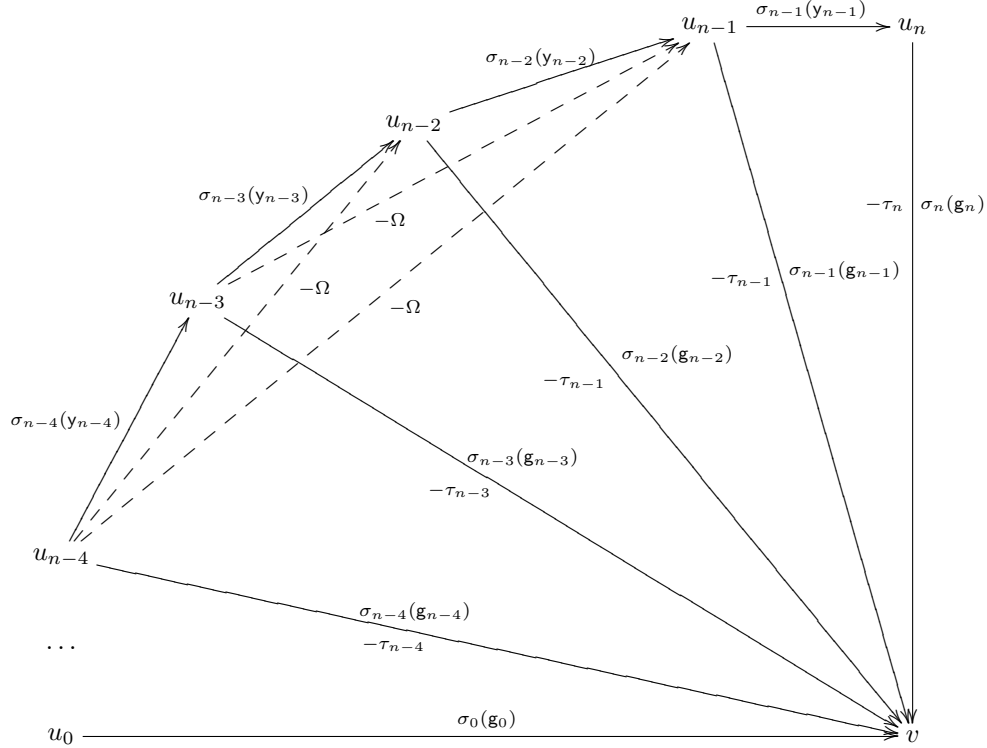

Figure 2: The sequence of points

Since $(u_n, v) \notin \iota(\tau_n)$, we have $(u_n, v) \notin \iota(\sigma_{n-1}(\mathsf{y}_{n-1})) \setminus \iota(\tau_{n-1})$, so there is u_{n-1} with $(u_{n-1}, u_n) \in \iota(\sigma_{n-1}(\mathsf{y}_{n-1}))$ and $(u_{n-1}, v) \notin \iota(\tau_{n-1})$. Continuing in the same way, there are $u_0, u_1, \ldots, u_n$ such that

$$(u_i, u_{i+1}) \in \iota(\sigma_i(\mathsf{y}_i)) \text{ and } (u_i, v) \notin \iota(\tau_i)$$

In particular, $(u_i, u_{i+1}) \in \iota(v_{\mathsf{y}_i}) = \mathsf{y}_i$, since $\iota(v_{\mathsf{y}_i}) \supseteq \iota(\sigma_i(\mathsf{y}_i))$, and $(u_i, v) \notin \iota(\overline{v_{\mathsf{g}_i}})$, since $\iota(\tau_i) \supseteq \iota(\overline{v_{\mathsf{g}_i}})$.

We noted that $(u_i, v) \notin \iota(\overline{v_{\mathsf{g}_i}})$. It follows that $(u_i, v) \notin \iota(v_\alpha)$ for any $\alpha \in \mathsf{At}(\mathfrak{A}_n)$ such that $\alpha \neq \mathsf{g}_i$, whence $(u_i, v) \notin \iota(\Omega)$ by the definition of Ω.

We claim that

$$(u_i, v) \in \iota(\sigma_i(\mathsf{g}_i))$$

for every $i \in n + 1$. We know that $(u_n, v) \in \iota(\sigma_n(\mathsf{g}_n))$ by assumption. This implies $(u_n, v) \in \prod_{\alpha \in \mathsf{At}(\mathfrak{A}_n)} (v_\alpha \setminus \sum_{(\alpha, \mathsf{g}_n, \gamma) \in C} \iota(\sigma_{n-1}(\gamma)))$. Since $(u_{n-1}, u_n) \in \iota(v_{\mathsf{y}_{n-1}})$, by considering $\alpha = \mathsf{y}_{n-1}$ and the interpretation of $\setminus$ in representable algebras, we deduce that $(u_{n-1}, v) \in \iota(\sigma_{n-1}(\gamma))$, for some $\gamma \in \mathsf{At}(\mathfrak{A}_n)$ such that $(\mathsf{y}_{n-1}, \mathsf{g}_n, \gamma) \in C$. We know that $(u_{n-1}, v) \notin \iota(\overline{v_{\mathsf{g}_{n-1}}})$, hence $\gamma = \mathsf{g}_{n-1}$ and $(u_{n-1}, v) \in \iota(\sigma_{n-1}(\mathsf{g}_{n-1}))$. The claim follows inductively.

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Our next claim is that

$$(u_i, u_j) \notin \iota(\Omega)$$

for $i < j \leq n$. In passing we note that from this claim it follows that all the points u_i are distinct. We have already seen that $(u_i, v) \notin \iota(\Omega)$. Suppose for contradiction that $(u_i, u_j) \in \iota(\Omega)$, for some $i < j \leq n$. If $j < n$, then $(u_j, u_{j+1}) \in \iota(\sigma_j(\mathsf{y}_j)) \subseteq \iota(\Omega \setminus \Omega)$. Since $(u_i, u_j) \in \iota(\Omega)$ by the indirect assumption, we get $(u_i, u_{j+1}) \in \Omega$. Repeating this, we deduce that $(u_i, u_n) \in \iota(\Omega)$. Repeating one extra time, since $(u_n, v) \in \iota(\Omega \setminus \Omega)$, we get $(u_i, v) \in \iota(\Omega)$, a contradiction.

Next we claim that, for each $i < j \leq n$, there is a unique atom $\alpha_{ij} \in \mathsf{At}(\mathfrak{A}_n)$ such that

$$(u_i, u_j) \in \iota(\sigma_i(\alpha_{ij})) \subseteq \iota(v_{\alpha_{ij}})$$

and for any $i < k < j \leq n$,

$$(\alpha_{ik}, \alpha_{kj}, \alpha_{ij}) \in C \tag{22}$$

First recall that $(u_i, u_j) \notin \iota(\Omega)$, hence such an atom α_{ij}, if it exists at all, is unique. We have already seen that $(u_i, u_{i+1}) \in \iota(\sigma_i(\mathsf{y}_i)) \subseteq \iota(v_{\mathsf{y}_i})$, so

$$\alpha_{i,i+1} = \mathsf{y}_i \tag{23}$$

for $i < n$. Now assume that $|j - i| \geq 2$ and let $i < k < j$. Then, inductively, $(u_k, u_j) \in \iota(\sigma_k(\alpha_{kj})) \subseteq \iota(\prod_{\alpha \in \mathsf{At}(\mathfrak{A}_n)}(v_\alpha \setminus \sum_{(\alpha, \alpha_{kj}, \gamma) \in C} \sigma_{k-1}(\gamma))) \subseteq \iota(v_{\alpha_{ik}} \setminus \sum_{(\alpha_{ik}, a_{kj}, \gamma) \in C} \sigma_{k-1}(\gamma))$. By the inductive hypothesis, we have $(u_i, u_k) \in \iota(v_{\alpha_{ik}})$. Then by the definition of $\setminus$ in representable algebras, $(u_i, u_j) \in \iota(\sigma_{k-1}(\gamma))$ for some $\gamma = \alpha_{ij}$ such that the triple $(\alpha_{ik}, \alpha_{kj}, \alpha_{ij}) \in C$. Since $i \leq k - 1$, we get $(u_i, u_j) \in \iota(\sigma_{k-1}(\gamma)) \subseteq \iota(\sigma_i(\gamma)) \subseteq \iota(v_\gamma) = \gamma$.

We claim that

$$(\alpha_{ij}, \mathsf{g}_j, \mathsf{g}_i) \in C \tag{24}$$

for $i < j \leq n$ as well. Since $(u_j, v) \in \iota(\sigma_j(\mathsf{g}_j))$, we have $(u_j, v) \in \iota(v_{\mathsf{g}_j} \cdot (v_{\alpha_{ij}} \setminus \sum_{(\alpha_{ij}, \mathsf{g}_j, \gamma) \in C} v_\gamma))$. Recall that $(u_i, u_j) \in \iota(v_{\alpha_{ij}})$. Hence $(u_i, v) \in \iota(v_\gamma) \in \mathsf{At}(\mathfrak{A}_n)$ such that $(\alpha_{ij}, \mathsf{g}_j, \gamma) \in C$. But we have seen that $(u_i, v) \notin \iota(\overline{v_{\mathsf{g}_i}})$, hence $\gamma = \mathsf{g}_i$ and $(\alpha_{ij}, \mathsf{g}_j, \mathsf{g}_i) \in C$ as desired.

Next we claim that for $i + 2 \leq j \leq n$,

$$\alpha_{ij} = \mathsf{r}_{j-i}$$

The case $i = 0$, $j = n$ will then provide a contradiction, since there is no atom r_n. See Figure 3 (where we use the notation $\iota(v_\alpha) = \alpha$ for any variable v_α with $\alpha \in \mathsf{At}(\mathfrak{A}_n)$) for the argument below. We will use that certain triples of elements must satisfy the consistency conditions on atomic triangles on $\mathfrak{A}_n$, cf. the claims (22) and (24) above.

We prove the claim by induction over $j - i$, the base case is $j - i = 2$. We know that $\alpha_{i,i+1} = \mathsf{y}_i$ and $\alpha_{i+1,i+2} = \mathsf{y}_{i+1}$ by (23), so $(\mathsf{y}_i, \mathsf{y}_{i+1}, \alpha_{i,i+2}) \in C$ by (22). Thus $\alpha_{i,i+2}$ cannot be the identity, yellow or black by (11), (17) and (16). We also know that $(\alpha_{i,i+2}, \mathsf{g}_i, \mathsf{g}_{i+2}) \in C$ by (24). So $\alpha_{i,i+2}$ cannot be green or white by (13) and (14) Hence it must be red. By (15), the only possibility is $\alpha_{i,i+2} = \mathsf{r}_2$, proving the base case.

Next we suppose $i < j \leq n$ and $j = i + 3 \leq n$, and we consider the atom $\alpha_{i,i+3}$. We already have $\alpha_{i,i+2} = \alpha_{i+1,i+3} = \mathsf{r}_2$ and $\alpha_{i,i+1} = \mathsf{y}_i$ and $\alpha_{i+2,i+3} = \mathsf{y}_{i+2}$. So $\alpha_{i,i+3}$ must satisfy

$$(\mathsf{y}_i, \mathsf{r}_2, \alpha_{i,i+3}), (\mathsf{r}_2, \mathsf{y}_{i+2}, \alpha_{i,i+3}), (\alpha_{i,i+3}, \mathsf{g}_i, \mathsf{g}_{i+3}) \in C$$

and hence $\alpha_{i,i+3} = \mathsf{r}_3$, by (13), (12), (14), and (15). Similarly, $\alpha_{ij} = \mathsf{r}_{j-i}$ whenever $i < j \leq n$ and $j - i \leq 5$.

Now let $i < j \leq n$ and $5 < j - i$. Assume, as an induction hypothesis, that $\alpha_{i',j'} = \mathsf{r}_{j'-i'}$ whenever $i' < j' \leq n$ and $j' - i' < j - i$. As before, α_{ij} cannot be the identity, black, yellow, green or white, so $\alpha_{ij} = \mathsf{r}_m$, for some $m < n$. By (24), $(\mathsf{r}_m, \mathsf{g}_i, \mathsf{g}_j) \in C$, and by (15) it follows that $m \equiv_5 j - i$. Also, by (22), $(\mathsf{r}_2, \mathsf{r}_{j-i-2}, \mathsf{r}_m) \in C$, and hence by (20) the indices must add up: $S(2, j-i-2, m)$. That is, either (i) $m = 2 = j - i - 2$, or (ii) $m = (j - i - 2) + 2 (= j - i)$, or (iii) $j - i - 2 = 2 + m$ (so $j - i - 4 = m$), or (iv) $2 = m + (j - i - 2)$. The first alternative is impossible, since
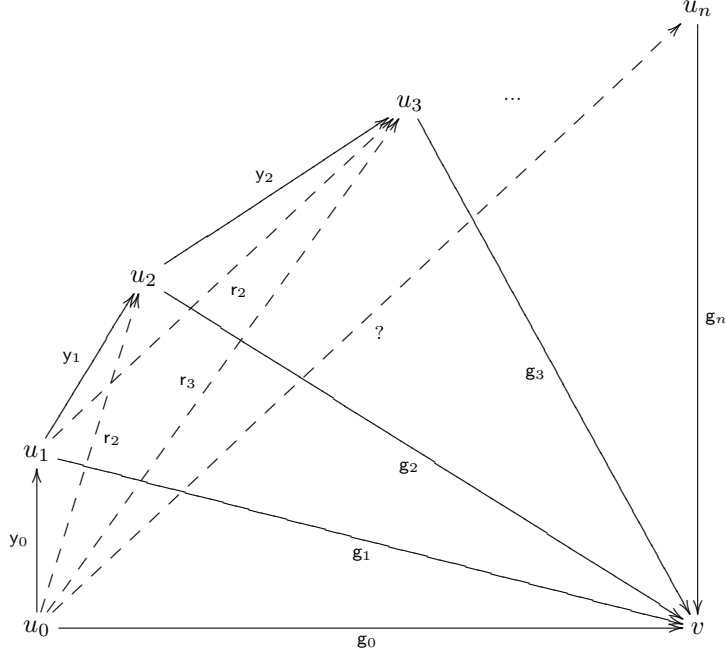
Figure 3: Red arrows

$j - i - 2 \geq 3 > 2$. The fourth alternative is impossible, since $m \geq 0$ and $j - i - 2 > 5 - 2 = 3 > 2$. The third alternative is also impossible, since $m \equiv_5 j - i$ but $j - i - 4 \not\equiv_5 j - i$. Hence it is the second alternative that must hold, and thus $\alpha_{ij} = r_{j-i}$. This completes the induction step.

In particular, we would have $(u_0, u_n) \in \rho$ for a red atom ρ with index n, but r_n does not exist. We have arrived at a contradiction, hence e_n holds in $\mathfrak{D}$. ∎

To complete the proof of Theorem 2.3 it remains to show that an ultraproduct of $\mathfrak{A}_n$ is representable. To this end we recall from [HH97] a (variant of the) game connected to representability.

3.3 The game

Definition 3.2 *Let $\mathfrak{A}$ be a relation-type algebra.*

1. *A* network *is a complete, directed, finite graph with edges labeled by atoms of $\mathfrak{A}$: i.e., $N = (N_1, N_2)$ for some set N_1 (the set of nodes) and some map $N_2 : N_1 \times N_1 \to \mathsf{At}(\mathfrak{A})$ that satisfies, for every $x, y, z \in N_1$,*

   (a) *$N_2(x, y) \leq 1'$ iff $x = y$,*

   (b) *$N_2(x, y) ; N_2(y, z) \geq N_2(x, z)$.*

   Given two networks $N = (N_1, N_2)$ and $N' = (N_1', N_2')$, we write $N \subseteq N'$ if $N_1 \subseteq N_1'$ and for all $x, y \in N_1$, $N_2'(x, y) = N_2(x, y)$.

   If no confusion is likely, we will henceforth omit the subscripts and let N denote the network, the set of nodes and the labelling function, distinguishing cases by context. Sometimes, when confusion is possible, we may write $\mathsf{nodes}(N)$ for the set of nodes N_1 of the network.

2. *Let $t \in \omega$. We define a game $\mathcal{G}_t(\mathfrak{A})$ between two players, $\forall$ (male), and $\exists$ (female). During a match of the game, they build a finite chain $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t$ of networks in the following way. In the initial round $\forall$ picks any atom α and $\exists$ plays a network N_0 containing nodes m_0, n_0 such that $N(m_0, n_0) = \alpha$ (if $\alpha \leq 1'$, then $m_0 = n_0$, else $m_0 \neq n_0$). If $m_0 \neq n_0$,*

the edges (m_0, n_0) and (n_0, m_0) belong to $\forall$. In a subsequent round i $(i < t)$, after N_{i-1} has been played,

- $\forall$ chooses an edge (m, n) from N_{i-1} and atoms $x, y \in \mathsf{At}(A)$ such that $x;y \geq N_{i-1}(m, n)$,
- $\exists$ responds with a network $N_i \supseteq N_{i-1}$ containing a node l such that $N_i(m, l) = x$ and $N_i(l, n) = y$.

For irreflexive edges of N_i where both nodes belong to N_{i-1}, the owner of the edge in N_i is the same as it was in N_{i-1}. The edges $(m, l), (l, n)$ and their converses belong to $\forall$. All other irreflexive edges in N_i belong to $\exists$. $\exists$ wins a match of the game $\mathcal{G}_t(\mathfrak{A})$ if she can respond to every move of $\forall$ by a network N_i $(i \leq t)$. We say that $\exists$ has a winning strategy if she can win all matches.

The following proposition [HH97, Proposition 15] provides us with a sufficient condition for representability of atomic relation algebras.

Proposition 3.3 Let $\mathfrak{A}$ be an atomic relation algebra. Then $\exists$ has a winning strategy in $\mathcal{G}_t(\mathfrak{A})$ for all $t \in \omega$ iff $\mathfrak{A}$ is elementarily equivalent to a completely representable relation algebra.[3] Hence, because RRA is elementary, if $\exists$ has a winning strategy in $\mathcal{G}_t(\mathfrak{A})$ for all $t \in \omega$, then $\mathfrak{A}$ is representable.

Definition 3.4 Let N be a network over $\mathfrak{A}_n$ and let $x \in N$. Define

$$\mathsf{G}_N(x) = \{y \in N : N(x, y) \text{ is green}\}$$
$$\mathsf{Y}_N(x) = \{y \in N : N(x, y) \text{ is yellow}\}$$

If $\mathsf{G}_N(x)$ is non-empty, define an equivalence relation $\sim_x$ over $\mathsf{G}_N(x)$ by letting $y \sim_x z$ iff there is a yellow path $(\rho_0, \ldots, \rho_k)$ contained in $\mathsf{G}_N(x)$, where $y = \rho_0$, $z = \rho_k$ and $N(\rho_i, \rho_{i+1})$ is yellow for every $i < k$.

Certain subsets of $X \subseteq \mathsf{nodes}(N)$ are called red clusters. There are two types, see Figure 4. The set X is a red cluster if $|X| \geq 2$ and either

Type 1: there are distinct nodes $x, y \in N$ such that $N(x, y)$ is not green or white (the case where $N(x, y)$ is green is covered by the other type of red cluster, the case where $N(x, y)$ is white is not a red cluster and is handled separately) and

$$X = \mathsf{G}_N(x) \cap \mathsf{Y}_N(y)$$

Type 2: there is $x \in N$ and X is a $\sim_x$-equivalence class of $\mathsf{G}_N(x)$.

If X is a red cluster and $X = \mathsf{G}_N(x) \cap \mathsf{Y}_N(y)$ or X is a $\sim_x$-equivalence class of $G_N(x)$ (some x, y), then x is called a green root of the red cluster X. If $X = \mathsf{G}_N(x) \cap \mathsf{Y}_N(y)$ is a red cluster of the second type, the node y is called a yellow root of X. By definition, every red cluster has a green root, but we do not assume that it has a unique green root.

Let $s \in \omega$ and let $x \in N$. If $X \subseteq \mathsf{G}_N(x)$, then an s-good map f over x is a map $f : X \to (n+1)$ such that for $y, z \in X$ if $N(x, y) = \mathsf{g}_i$ and $N(x, z) = \mathsf{g}_j$ (some $i, j \leq n$), then (i) $i \leq j \iff f(y) \leq f(z)$, (ii) $f(y) - f(z) \equiv_5 j - i$, (iii) if $|i - j| < 6 \cdot 2^s$, then $f(y) - f(z) = i - j$ and (iv) if $|i - j| \geq 6 \cdot 2^s$, then $|f(y) - f(z)| \geq 6 \cdot 2^s$.

Observe by the definition of red cluster above, that the red clusters of the network N are completely determined by the sets of green, yellow and white edges of N, in other words if N, N' are two networks with the same set of nodes and for all $u, v \in \mathsf{nodes}(N)$, $N(u, v)$ is green (yellow, white) if and only if $N'(u, v)$ is green (respectively yellow, white), then for any $X \subseteq \mathsf{nodes}(N)$, X is a red cluster of N iff X is a red cluster of N', and x is a green (or yellow) root of the red cluster X in N iff it is a green (or yellow) root of the same red cluster in N'.

The following proposition can be shown by a standard binary chop argument, cf. [HM00].

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[3] A complete representation of a relation algebra $\mathfrak{B}$ is an isomorphism from $\mathfrak{B}$ to a representable relation algebra that preserves arbitrary meets and joins whenever they exist in $\mathfrak{B}$. The algebras considered here are finite, and for finite algebras representability is the same as complete representability.
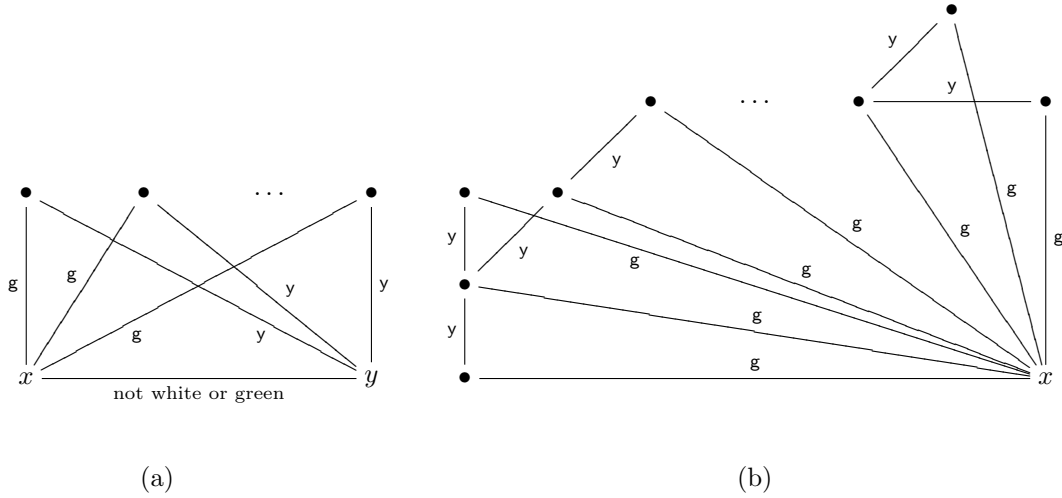
Figure 4: Type 1 and 2 red clusters

Proposition 3.5 *Let $x \in N$, $X \subseteq \mathsf{G}_N(x)$, $f : X \to (n+1)$ be an s-good map over x and $l \in \mathsf{G}_N(x) \smallsetminus X$. Then there is an $(s-1)$-good map $f^+ : X \cup \{l\} \to (n+1)$ over x extending f. Furthermore, if $n \geq 6 \cdot 2^t$ and $|X| = 2$, then there is a t-good map $f : X \to (n+1)$.*

3.4   The ultraproduct

We will now show that an ultraproduct of the $\mathfrak{A}_n$ ($n \in \omega$) is representable. First we describe a winning strategy for $\exists$ in the game $\mathcal{G}_t(\mathfrak{A}_n)$ with $6 \cdot 2^t \leq n$.

Suppose after $s < t$ rounds of $\mathcal{G}_t(\mathfrak{A}_n)$ the network N_s has just been played. In the next round $\forall$ picks nodes $m, n \in N_s$ and atoms α, β such that $\alpha \, ; \beta \geq N(m,n)$. We assume the following.

Inductive hypothesis: For any red cluster X and any green root p of X, there is an associated $(t-s)$-good function $f = f_{N,X,p} : X \to (n+1)$ over p, and for all distinct $y, z \in X$ either $N(y,z)$ is yellow or $N(y,z) = \mathsf{r}_{|f(y)-f(z)|}$.

$\exists$'s strategy is to follow the first applicable of the instructions below.

1. If there is a node $l \in N_s$ such that $N_s(m,l) = \alpha$ and $N_s(l,n) = \beta$, then she does not extend N_s.

Suppose there is no such node in N_s. In each of the following cases, $\exists$ will extend N_s to N_{s+1} containing exactly one extra node l such that $N_{s+1}(m,l) = \alpha$ and $N_{s+1}(l,n) = \beta$. To define N_{s+1} she has to define each label $N_{s+1}(l,p)$ for $p \in N_s \smallsetminus \{m,n\}$ (she lets $N_{s+1}(p,l) = N_{s+1}(l,p)$, since the algebra is symmetric). She only ever uses white, black or red for her labels.

2. For each $p \in N_s$, if either $N_s(p,m)$ or x are not green and either $N_s(p,n)$ or y are not green, then $\exists$ lets $N_{s+1}(p,l) = \mathsf{w}_S$ where

$$S = \big\{ (i,j) : \{N(p,m), \alpha\} = \{\mathsf{g}_i, \mathsf{y}_j\} \text{ or } \{N(p,n), \beta\} = \{\mathsf{g}_i, \mathsf{y}_j\} \big\}$$

Observe that $|S| \leq 2$.

In the remaining cases, $\exists$ only uses black or red atoms to label edges, see parts 3, 4 and 5 of $\exists$'s strategy below. Hence the red clusters of N_{s+1} are already determined, as we remarked. It is

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therefore safe to refer to a red cluster of N_{s+1}, although we have not yet completed the definition of N_{s+1}.

The following Proposition is a consequence of the first two parts of $\exists$'s strategy.

Proposition 3.6 *Let N be a network played in a match of $G_t(\mathfrak{A}_n)$ in which $\exists$ never uses green or yellow to label edges and she always uses parts 1 and 2 of her strategy whenever they apply (other edges may be labelled by arbitrary black or red atoms). Let $X \subseteq \mathsf{nodes}(N)$ be a set of at least four nodes. In the play of the match so far, let $x \in X$ be the node which was last to be added (we may refer to x as the most recently added node of X). Then x is incident with at most two edges in X labelled by $\forall$. Suppose there are $u, v \in X$ such that $(u, x), (v, x)$ were labelled by $\forall$. For all $y \in X \smallsetminus \{x, u, v\}$, (i) it is not the case that $N(x, u) = N(u, y)$ and $N(x, v) = N(v, y)$ and (ii) either $N(y, u)$ and $N(u, x)$ are both green or $N(y, v)$ and $N(v, x)$ are both green or $N(x, y) = \mathsf{w}_S$ where*

$$S = \big\{(i, j) : \{\mathsf{g}_i, \mathsf{y}_j\} = \{N(x, u), N(u, y)\} \text{ or } \{\mathsf{g}_i, \mathsf{y}_j\} = \{N(x, v), N(v, y)\}\big\}$$

Lemma 3.7 *Let N occur in a play of $\mathcal{G}_t(\mathfrak{A}_n)$ in which $\exists$ never chooses green or yellow to label edges, and she always uses parts 1 and 2 of her strategy whenever they apply. If X and Y are distinct red clusters of N, and u, v are distinct nodes in $X \cap Y$, then $N(u, v)$ is yellow.*

Proof: Both red clusters X and Y can be of type one or type two. First we consider two red clusters which are both of the first type, $X = \mathsf{G}_N(x) \cap \mathsf{Y}_N(y)$ and $Y = \mathsf{G}_N(x') \cap \mathsf{Y}_N(y')$, where $x, y, x', y' \in N$. Since X and Y are clusters of the first type, we know that $N(x, y)$ and $N(x', y')$ are not white or green. In this case there cannot be two distinct nodes in the intersection of the two clusters because of the following. Suppose for contradiction that $u, v \in (\mathsf{G}_N(x) \cap \mathsf{Y}_N(y)) \cap (\mathsf{G}_N(x') \cap \mathsf{Y}_N(y'))$ where $u \neq v$. Consider the set of nodes $Z = \{x, y, x', y', u, v\}$. If x is the most recently added node from this set then, by Proposition 3.6 and since $(x, u), (x, v)$ belong to $\forall$, $\exists$ chose $N(x, y)$ and she chose a white atom $N(x, y) = \mathsf{w}_S$, for some S, contrary to our assumptions. Similarly, y, x', y' cannot be the most recently added node, hence it must be u or v. Now for all $w \in \{x, y, x', y'\}, N(u, w)$ and $N(v, w)$ are green or yellow, hence (u, w) and (v, w) belong to $\forall$. Since either u or v is incident with at most two $\forall$-edges in Z, we must have $|\{x, y, x', y'\}| \leq 2$. Hence $\{x, y\} = \{x', y'\}$ as $x \neq y$ and $x' \neq y'$. Note that $x = y'$ is impossible, since $N(x, u)$ is green and $N(y', u)$ is yellow. Similarly $y = x'$ is impossible, hence $x = x'$ and $y = y'$. This contradicts our assumption that $\mathsf{G}_N(x) \cap \mathsf{Y}_N(x)$, $\mathsf{G}_N(x') \cap \mathsf{Y}_N(y')$ were distinct red clusters.

Now consider a red cluster $\mathsf{G}_N(x) \cap \mathsf{Y}_N(y)$ of the first type and a red cluster Y of the second type, say Y is a $\sim_{x'}$-equivalence class of $\mathsf{G}_N(x')$, for some $x' \in N$. For contradiction suppose $u \neq v$, $u, v \in (\mathsf{G}_N(x) \cap \mathsf{Y}_N(y)) \cap Y$ and $N(u, v)$ is not yellow. Let $(\rho_0, \ldots, \rho_k)$ be a yellow path from u to v in $\mathsf{G}_N(x')$. By our assumption that $N(u, v)$ is not yellow we know that $k \geq 2$. Consider the set $Z = \{x, y, x', \rho_i : i \leq k\}$. As before, the most recently added node from this set cannot be $x, y, x', \rho_1, \ldots, \rho_{k-1}$, either because the node is incident with three $\forall$-edges in Z or, in the cases of x and y, because it would imply, by Proposition 3.6, that $N(x, y)$ was white, contrary to assumption. Hence the most recently added node is either u or v. But u is incident with the $\forall$-edges $(u, x), (u, y), (u, x')$ and (u, ρ_1), while v is incident with $(v, x), (v, y), (v, x')$ and (v, ρ_{k-1}). Since either u or v is incident with at most two $\forall$-edges in Z, either $\{x', \rho_1\} \subseteq \{x, y\}$ or $\{x', \rho_{k-1}\} \subseteq \{x, y\}$. Now, $x' = y$ is impossible as $N(y, u)$ is yellow but $N(x', u)$ is green, hence $x' = x$. Since $\rho_i = x$ is impossible ($N(x', \rho_i)$ is not the identity), either $\rho_1 = y$ or $\rho_{k-1} = y$. Either alternative, $\rho_1 = y$ or $\rho_{k-1} = y$, is impossible as $N(x', \rho_i)$ is green but $N(x, y)$ is not green.

Finally, consider two red clusters X and Y of the second type, say X is a $\sim_x$-equivalence class over $\mathsf{G}_N(x)$ and Y is a $\sim_y$-equivalence class over $\mathsf{G}_N(y)$, for some $x, y \in N$. Let u, v be distinct nodes in $X \cap Y$. Let $(\rho_0, \ldots, \rho_k)$ be a yellow path from u to v in $\mathsf{G}_N(x)$ and let $(\rho'_0, \ldots, \rho'_m)$ be a yellow path from u to v in $\mathsf{G}_N(y)$. Let $Z = \{x, y, \rho_i, \rho'_j : i \leq k, \ j \leq m\}$. Suppose, for contradiction, that $N(u, v)$ is not yellow. Then each of the two yellow paths from u to v must have length at least two. But then x, y and each of the internal nodes of the two yellow paths are incident with at least three $\forall$-edges in Z. That leaves the two external nodes of the paths, namely u and v, as the only candidates for most recently added edge. But u is incident with the $\forall$-edges
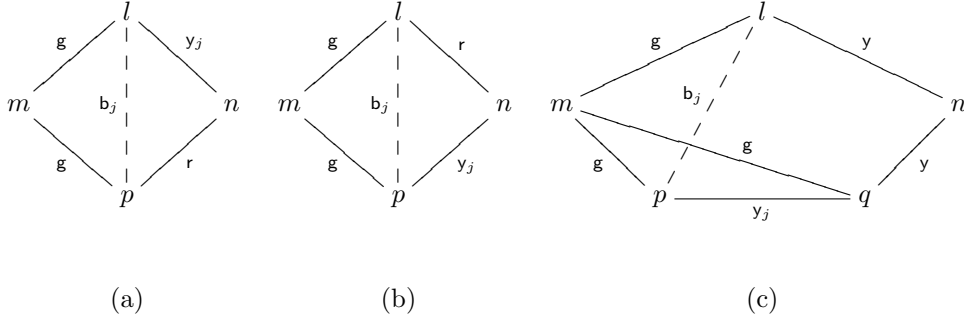
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Figure 5: Choosing a black atom to label (l, p)

$(x, u), (y, u), (u, \rho_1)$ and (u, ρ_1') in Z, while v is incident with the $\forall$-edges $(x, v), (y, v), (\rho_{k-1}, v)$ and (ρ_{m-1}', v). By Proposition 3.6, either u or v is incident with at most two $\forall$-edges in Z. Hence $x = y$ and either $\rho_1 = \rho_1'$ or $\rho_{k-1} = \rho_{m-1}'$. But then X, Y are $\sim_x$-equivalence classes of $\mathsf{G}_N(x)$ with a non-empty intersection, whence identical, contradicting our assumption that X, Y are distinct. ∎

Recall that the red clusters of N_{s+1} are already determined.

Lemma 3.8 *If X is a red cluster of N_{s+1} and $l \in N_{s+1} \smallsetminus N_s$, then either*

- *$|X| = 2$ or*

- *$X \smallsetminus \{l\}$ is a red cluster of N.*

Proof: Let X be a red cluster of N_{s+1} and suppose $|X| > 2$. The new node l cannot be a green or yellow root of X since it is incident with at most two green edges and at most two yellow edges (if l is a green (yellow) root of X, then each edge (l, x) for $x \in X$ is green (yellow)). Let $x \in N_s$ be a green root of X. Since N_{s+1} is an extension of N, the set $X \smallsetminus \{l\}$ is a red cluster of N with green root x. ∎

Hence, by Proposition 3.5, we can associate a $(t - (s+1))$-good map with each red cluster of N_{s+1} in such a way that if $X \smallsetminus \{l\}$ is a red cluster of N_s then the $(t - (s + 1))$-good map associated with X is an extension of the $(t - s)$-good map associated with $X \smallsetminus \{l\}$ in N.

We continue with $\exists$'s strategy. From now on, we will assume that α and $N_s(m, p)$ are both green. The case where β and $N_s(n, p)$ are both green is entirely symmetric.

3. Suppose that $N_s(p, m)$ and α are green. Suppose p and l do not belong to the same red cluster of N_{s+1}. Suppose also it is not the case that β and $N_s(n, p)$ are both yellow. Then $\exists$ lets $N_{s+1}(l, p) = \mathsf{b}_j$ for some $j < n$ determined as follows (see Figure 5).

   (a) If $\beta = \mathsf{y}_j$ and $N_s(n, p)$ is red, then $\exists$ lets $N_{s+1}(p, l) = \mathsf{b}_j$.

   (b) If β is red and $N_s(n, p) = \mathsf{y}_j$, then $\exists$ lets $N_{s+1}(p, l) = \mathsf{b}_j$.

   (c) If β is yellow and there is $q \in N_s$ such that $N_s(m, q)$ is green, $N_s(q, n)$ is yellow and $N_s(p, q) = \mathsf{y}_j$, then $\exists$ lets $N_{s+1}(p, l) = \mathsf{b}_j$.

   (d) Otherwise (none of the previous subcases apply) $\exists$ lets $N_{s+1}(p, l) = \mathsf{b}_0$.

We will check that this part of the strategy is well defined later. In the remaining cases, α and $N_s(m, p)$ are both green and either β and $N_s(n, p)$ are both yellow, or l and p belong to the same red cluster of N_{s+1} (both conditions might apply). In these cases $\exists$ lets $N_{s+1}(l, p)$ be a red atom.
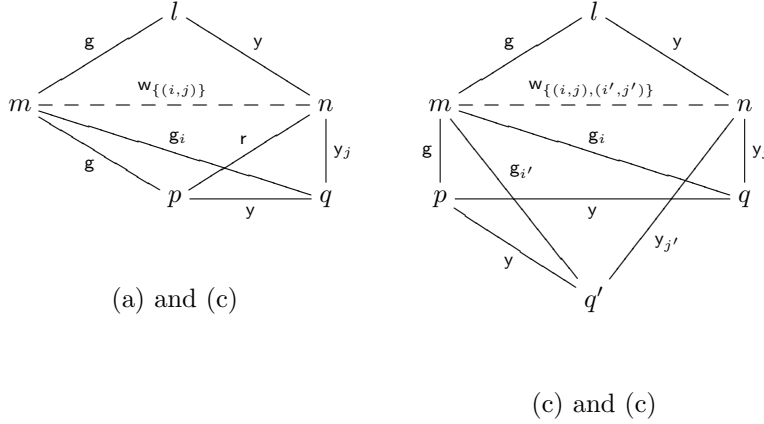
(a) and (c)

(c) and (c)

Figure 6: If 3 applied in two ways

4. If $\alpha = g_i$ and $N_s(m,p) = g_j$ are both green, β and $N_s(n,p)$ are both yellow and $N_s(m,n) = w_S$ (some S), then $\exists$ lets $N_{s+1}(l,p) = r_{|i-j|(\text{mod } 5)}$.

5. Suppose that l and p belong to the same red cluster X of N_{s+1}. The edge $N_{s+1}(l,p)$ will be red (not yellow) so, by Lemma 3.7, this red cluster is unique. Let f^+ be the $(t-(s+1))$-good map associated with X. $\exists$ lets $N_{s+1}(l,p) = r_{|f^+(l)-f^+(p)|}$.

It remains to prove that this strategy is well defined and that N_{s+1} is in fact a network.

We must show that $\exists$'s strategy is well defined in case 3, where she chooses black atoms. We claim that when labelling the edge (l,p) (some $p \in N_s$), it is impossible for two different parts from 3a–3c to apply, and if there are nodes $q, q' \in N_s$ such that 3c applies using q and also using q', then $q = q'$. To prove this claim it is clear first of all that if 3b applies (so α and β are green and red, respectively), then none of the other parts can apply (for parts 3a and 3c, α and β are green and yellow, respectively).

Suppose, for contradiction, that 3a and 3c both apply, as in the first part of Figure 6. By 3a, $N_s(m,p)$ is green, $N_s(n,p)$ is red, and by 3c there is $q \in N_s$ such that $N_s(p,q)$ is yellow, $N_s(n,q) = y_j$ (some $j < n$) and $N_s(m,q) = g_i$ (some $i \le n$). We also know by the conditions for part 3 of $\exists$'s strategy, that l and p do not belong to the same red cluster of N_{s+1}. We claim that under these conditions

$$N_s(m,n) = w_{\{(i,j)\}}$$

To prove this claim, consider the nodes $\{m,n,p,q\}$. It is easily seen, by our assumptions, that the four nodes are distinct. One of these four nodes was the most recently added node from the set. By Proposition 3.6, this node is incident with at most two edges belonging to $\forall$ within that set. This most recently added node cannot be q, since it is incident with green/yellow edges $(q,n), (q,m)$ and (q,p). If the most recent node is p, then by Proposition 3.6 either $N_s(m,n)$ is green or $N_s(p,n)$ is white. The latter contradicts our assumption that $N_s(p,n)$ is red. If $N_s(m,n)$ is green, then l and p belong to the same red cluster of N_{s+1}, since there is a yellow path from l to p in $G_{N_{s+1}}(m)$, namely (l,n,q,p) — again this is contrary to our assumptions. Hence the most recently added node from $\{m,n,p,q\}$ must be m or n. If it is m, then by Proposition 3.6, $\exists$ would have let $N_s(m,n) = w_{\{(i,j)\}}$. If n is the most recently added node from $\{m,n,p,q\}$, then we have to consider two possibilities. It could be that $N_s(p,n)$ was labelled by $\forall$, but then n is incident with two $\forall$-edges, namely (p,n) and (n,q), and then by Proposition 3.6 we have $N_s(m,n) = w_{\{(i,j)\}}$, as before. The other possibility is that, in the round when n was first played, the two edges labelled by $\forall$ were (q,n) and (u,n), for some $u \in N \setminus \{m,n,p,q\}$. Now consider $\{m,n,p,q,u\}$. Recall that n is the most recently added node. Since $N_s(p,n)$ is red and labelled by $\exists$, it must be that $N_s(p,u)$ and $N_s(u,n)$ are both green, by Proposition 3.6. But then, by Proposition 3.6 again, $N_s(m,n) = w_{\{(i,j)\}}$, as before. This proves the claim.
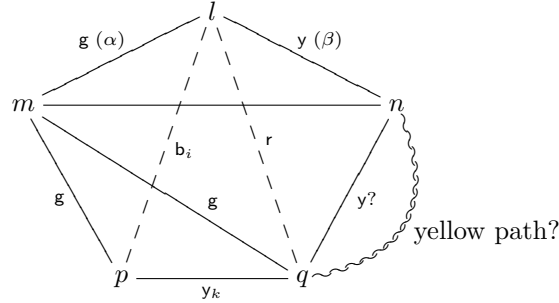
Figure 7: Checking consistency

But then, since the two atoms α and β used in $\forall$'s current move have to satisfy $\alpha\,;\beta \geq N_s(m,n)$ and we are assuming that α is green and β is yellow, it must be that $\alpha = \mathsf{g}_i$ and $\beta = \mathsf{y}_j$. Then $\exists$ would have used part 1 of her strategy, since there is a witness node $q \in N_s$, contrary to assumption. This proves that 3a and 3c cannot simultaneously apply.

Finally we check that 3c cannot apply simultaneously because of distinct nodes $q, q' \in N_s$, see the second part of Figure 6. For contradiction suppose it did. Then $N_s(m,p)$, $N_s(m,q)$ and $N_s(m,q')$ are all green and $N_s(p,q)$, $N_s(q,n)$, $N_s(p,q')$ and $N_s(q',n)$ are all yellow. Consider the nodes $\{m,n,p,q,q'\}$. It is easy to check that there are at least four nodes in this set (the only possible equality is $q = q'$). By Proposition 3.6, the most recently added node is n, as all the other nodes are incident with at least three $\forall$-edges from this set. Proposition 3.6 also implies that $N_s(m,n) = \mathsf{w}_{\{(i,j),(i',j')\}}$ where $N_s(m,q) = \mathsf{g}_i$, $N_s(q,n) = \mathsf{y}_j$, $N_s(m,q') = \mathsf{g}_{i'}$ and $N_s(q',n) = \mathsf{y}_{j'}$. But then the current move (m,n,x,y) would have been covered by part 1 of her strategy (using either q for $\alpha = \mathsf{g}_i$ and $\beta = \mathsf{y}_j$ or q' for $\alpha = \mathsf{g}_{i'}$ and $\beta = \mathsf{y}_{j'}$ as a witness), contrary to assumption.

Now we check that her strategy is well defined when she uses parts 4 and 5 and chooses a red atom. If part 5 applies, then by Definition 3.4, whether the red cluster is of type one or type two, $N_s(m,n)$ cannot be white, hence part 4 cannot apply. If part 5 applies, then it is well defined, since the red cluster containing l and p is unique, by Lemma 3.7.

Note that the use of part 5 of $\exists$'s strategy ensures that our inductive condition remains true in N_{s+1}. Having shown that $\exists$'s strategy is well defined, we are now ready to prove that she can win the game.

Lemma 3.9 *If* $6 \cdot 2^t \leq n$, *then* $\exists$ *has a winning strategy in* $\mathcal{G}_t(\mathfrak{A}_n)$.

Proof: It suffices to prove that N_{s+1} is indeed a network. Triangles not involving the new node l are consistent, since N_s is a network. The triangle (m,n,l) must be consistent, since $\forall$ is required to pick α and β with $\alpha\,;\beta \geq N_s(m,n)$. Primary triangles (l,m,p) and (l,n,p) for $p \in N_s \smallsetminus \{m,n\}$ can easily be checked to be consistent, by inspecting each case of $\exists$'s strategy. It remains to check the secondary triangles (l,p,q), where $p,q \in N_s \smallsetminus \{m,n\}$. From her strategy, $N_{s+1}(l,p)$ and $N_{s+1}(l,q)$ are white, black or red. The only forbidden triples involving two atoms of these colours are (18) and (20).

To check forbidden triple (18), suppose that $N_{s+1}(l,p) = \mathsf{b}_i$, $N_{s+1}(l,q) = \mathsf{r}_j$ (some $i,j < n$) and $N_s(p,q) = \mathsf{y}_k$ (some $k < n$). Then α, $N_s(m,p)$ and $N_s(m,q)$ are all green (or symmetrically β, $N_s(n,p)$ and $N_s(n,q)$ are all green), otherwise $\exists$ would have chosen white instead of red and black. See Figure 7. Since $N_{s+1}(l,p)$ is black, $\exists$ must have used part 3 of her strategy when labelling (l,p). There are two possibilities which account for $N_{s+1}(l,q)$ being red, viz. parts 4 and 5. So either β and $N(n,q)$ are both yellow, or there is a yellow path from l to q within $\mathsf{G}_{N_{s+1}}(m)$. Either way β is yellow, as shown in the figure. First suppose that $N_s(n,q)$ is yellow and $N_s(m,n)$ is not green (this covers the cases where (l,q) was labelled using part 4 of $\exists$'s strategy, and also part 5 when l and q belong to a red cluster of the first type). In this case, part 3c applies when labelling (l,p), so $\exists$ must have selected $N_{s+1}(l,p) = \mathsf{b}_k$, i.e., $k = i$, so the triangle is consistent.

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Next suppose that there is a yellow path ρ from l to q within $\mathsf{G}_{N_{s+1}}(m)$. Then, by appending p to ρ, we see that there is a yellow path from l to p within $\mathsf{G}_{N_{s+1}}(m)$, so l and p belong to the same red cluster of N_{s+1}. This is contrary to our assumption that part 3 applied when labelling (l, p).

To check forbidden triple (20), suppose $N_{s+1}(l, p)$, $N_{s+1}(l, q)$ and $N(p, q)$ are all red. Since $N_{s+1}(l, p)$ is red, it must have been selected by either part 4 or 5 of $\exists$'s strategy. In the former case, $N_s(m, n) = \mathsf{w}_S$, α and $N_s(m, p)$ are green, β and $N_s(n, p)$ are yellow. Since $N_s(m, n)$ is white, (l, q) must also have been labelled using part 4 of $\exists$'s strategy (part 5 requires that $N_s(m, n)$ is not white), so $N_s(m, q)$ is also green and $N_s(n, q)$ is yellow. Consider the set $\{m, n, p, q\}$. By Proposition 3.6 there are two possibilities for the most recently added node. If it is either m or n, then $\exists$ chose $N_s(m, n) = \mathsf{w}_{\{(i', j'), (i^*, j^*)\}}$ where $N_s(m, p) = \mathsf{g}_{i'}$, $N_s(p, n) = \mathsf{y}_{j'}$, $N_s(m, q) = \mathsf{g}_{i^*}$ and $N_s(q, n) = \mathsf{y}_{j^*}$. If $\alpha = \mathsf{g}_i$ and $\beta = \mathsf{y}_j$, then, since $\alpha \; ; \; \beta \geq N_s(m, n) = \mathsf{w}_{\{(i', j'), (i^*, j^*)\}}$, either $(i, j) = (i', j')$ or $(i, j) = (i^*, j^*)$. Either way, $\exists$ would have responded to the current move using part 1 of her strategy, using either p or q as a witness, contrary to assumption. The other possibility is that the most recently added node of $\{m, n, p, q\}$ is either p or q, but then $\exists$ would have chosen a white atom to label $N_s(p, q)$, by Proposition 3.6, contrary to our assumption that $N_s(p, q)$ is red.

So we may suppose that both (l, p) and (l, q) were labelled using part 5 of $\exists$'s strategy. Thus l and p belong to a red cluster of N_{s+1}, and l and q belong to a red cluster of N_{s+1}. We claim that l, p and q all belong to the same red cluster of N_{s+1}. To prove this claim, note that each node in a red cluster is incident with at least one green edge and at least one yellow edge, so α must be green and β must be yellow (or the other way round). If the red cluster containing l and p is of type 1, then $N_s(m, n)$ is neither white nor green, so the red cluster containing l and q is also of type 1. In this case, α, $N_s(m, p)$ and $N_s(m, q)$ are green, and β, $N_s(n, p)$ and $N_s(n, q)$ are yellow, so $l, p, q \in \mathsf{G}_{N_{s+1}}(m) \cap \mathsf{Y}_{N_{s+1}}(n)$, i.e., all three points belong to the same red cluster. If l, p belong to a red cluster of type 2, then $N_s(m, n)$ must be green and there is a yellow path from l to p in $\mathsf{G}_{N_{s+1}}(m)$. Then the red cluster containing l and q must also be of type 2, and there is a yellow path from l to q in $\mathsf{G}_{N_{s+1}}(m)$. Hence, l, p and q all belong to the same $\sim_m$-equivalence class of $\mathsf{G}_{N_{s+1}}(m)$, so again all three points belong to the same red cluster, as claimed.

But then l, p and q belong to a red cluster X of N_{s+1} with associated $(t - (s+1))$-good map $f^+ : X \to (n+1)$, and the edges of the triangle (l, p, q) have labels $\mathsf{r}_{|f(l) - f(p)|}$, $\mathsf{r}_{|f(p) - f(q)|}$ and $\mathsf{r}_{|f(l) - f(q)|}$, whence the triangle is consistent by 21. $\blacksquare$

Lemma 3.10 *Any non-principal ultraproduct $\mathfrak{A}$ of $(\mathfrak{A}_n : n \in \omega)$ over ω is in* RRA. *Hence the ultraproduct $\mathfrak{D}$ of the $\{\cdot, +, \backslash\}$-reducts $\mathfrak{D}_n$ of $\mathfrak{A}_n$ is representable as well: $\mathfrak{D} \in \mathsf{Q}^{cd}(\cdot, +, \backslash)$.*

Proof: We have seen that $\exists$ has a winning strategy in $G_t(\mathfrak{A}_n)$, provided $6 \cdot 2^t \leq n$. A standard argument now shows that $\exists$ has a winning strategy in a countably long game over a non-principal ultraproduct $\mathfrak{A}$ of the $\mathfrak{A}_n$s. The rough idea of $\exists$'s winning strategy in $G_\omega(\mathfrak{A})$ is the following. For each n, let $t(n) = \lfloor \log_2(\frac{n}{6}) \rfloor$. Let U be the non-principal ultrafilter used to define the ultraproduct. In the initial round of $G_\omega(\mathfrak{A})$, $\forall$ picks an atom $[a]$ of $\mathfrak{A}$. Here $a = (a_0, a_1, \ldots)$ for some sequence of elements $a_i \in \mathfrak{A}_i$, and $[a]$ denotes the equivalence class of a (where equivalence is 'identical over a set of indices belonging to U'). For a 'large' set S_0 of indices n (i.e., $S_0 \in U$) the element a_n must be an atom of $\mathfrak{A}_n$. There are only finitely many values of n for which $t(n) = 0$, hence $S_1 = \{n \in S_0 : t(n) > 0\} \in U$. For each $n \in S_1$, $\exists$ supposes that $\forall$ plays the atom a_n in the initial round of $G_{t(n)}(\mathfrak{A}_n)$, and she responds using her winning strategy for $G_{t(n)}(\mathfrak{A}_n)$ with the $\mathfrak{A}_n$-network N_0^n. We may suppose that each of these networks has at most two nodes for a 'large' subset $S_2 \subseteq S_1$, and that they all have the same set of nodes, i.e., $\{N_0^n : n \in S_2\}$ all have the same nodes, X say. For indices n not in S_2 we may (re-)define N_0^n as an arbitrary network with nodes X. We may define the 'ultraproduct' N_0 of the N_0^ns as the network with nodes X and for $x, y \in X$ we have $N_0(x, y) = [(N_0^n(x, y) : n \in \omega)]$. N_0 is the initial network in the match of $G_\omega(\mathfrak{A})$. At the start of round $r > 0$, suppose inductively that we have $\mathfrak{A}_n$-networks N_{r-1}^n and that N_{r-1} is the ultraproduct of the N_{r-1}^ns in the above sense. We also suppose inductively that for 'many' indices n, the network N_{r-1}^n occurs in a match of $G_{t(n)}(\mathfrak{A}_n)$ in which $\exists$ has been using her winning strategy. For his move, $\forall$ picks an edge (x, y) of N_{r-1} and atoms $[a], [b]$ of $\mathfrak{A}$ such that

Figure 8: Reducts and ultraproducts

$[a]; [b] \geq N_{r-1}(x, y)$. Let $a = (a_0, a_1, \ldots)$ and $b = (b_0, b_1, \ldots)$. For each $n > 6 \cdot 2^r$, $\exists$ supposes that $\forall$ plays the edge (x, y) and $a_n, b_n \in \mathsf{At}(\mathfrak{A}_n)$ in round r of $G_{t(n)}(\mathfrak{A}_n)$. For 'many' values of n, this will be a legitimate move for $\forall$ in that game. For all of these games, provided $n > 6 \cdot 2^r$, $\exists$ plays the network N_r^n according to her winning strategy in that game — note that only finitely many strategies will have expired. For values of n outside this 'large' set, we can let N_r^n be an arbitrary network — this will not have any effect on the ultraproduct network N_r. In this way, $\exists$ maintains the inductive hypotheses. Since N_r is the ultraproduct of a sequence of networks N_r^n, it must be indeed a network of $\mathfrak{A}$, hence $\exists$ will not lose in round r. The details of this argument may be found in [HH97, Proposition 15], or for a more general argument see [HH02, Section 10.5.1].

Trivially, since $\exists$ has a winning strategy in $G_\omega(\mathfrak{A})$, she has a winning strategy in $G_n(\mathfrak{A})$ for all $n \in \omega$. By Proposition 3.3, the ultraproduct $\mathfrak{A} \in \mathsf{RRA} = \mathbf{SP}\mathsf{Rs}$. Hence there are $\mathfrak{B}_i \in \mathsf{Rs}$ for $i \in I$ such that $\mathfrak{A} \subseteq \prod_{i \in I} \mathfrak{B}_i$. Note that $\mathfrak{A}$ is commutative and dense, since each $\mathfrak{A}_n$ $(n \in \omega)$ has these properties. Since every $\mathfrak{B}_i$ $(i \in I)$ is a homomorphic image of $\mathfrak{A}$, see [HH02, Lemma 3.7], each $\mathfrak{B}_i$ is commutative and dense as well. Let $\mathfrak{D}_n$ be the $\{\cdot, +, \backslash\}$-reduct of $\mathfrak{A}_n$, let $\mathfrak{D}$ be the ultraproduct of $\mathfrak{D}_n$ $(n \in \omega)$ and $\mathfrak{C}_i$ be the $\{\cdot, +, \backslash\}$-reduct of $\mathfrak{B}_i$ for $i \in I$, i.e., $\mathfrak{C}_i \in \mathsf{R}^{cd}(\cdot, +, \backslash)$, see Figure 8. Then $\mathfrak{D}$ is the $\{+, \cdot, \backslash\}$-reduct of $\mathfrak{A}$, and hence $\mathfrak{D} \subseteq \prod_{i \in I} \mathfrak{C}_i$. That is, $\mathfrak{D} \in \mathbf{SP}\mathsf{R}^{cd}(\cdot, +, \backslash) \subseteq \mathsf{Q}^{cd}(\cdot, +, \backslash)$ as desired. ∎

4 Conclusion

The variety and quasivariety generated by $\mathsf{R}(\cdot, ;, \backslash, /)$ coincide, and it is a finitely axiomatizable class, [AM94]. By adding the axioms of commutativity and density, we get finite axiomatization for the (quasi)variety $\mathsf{Q}^{cd}(\cdot, ;, \backslash)$ generated by $\mathsf{R}^{cd}(\cdot, ;, \backslash)$. We have seen above that we lose finite axiomatizability (with or without commutativity and/or density) if we include $+$.

Problem 4.1 Are the (quasi)equational theories of (commutative and/or dense subclasses of) $\mathsf{R}(+, \backslash, /)$ and $\mathsf{R}(+, \backslash, /, ;)$ finitely based?

We showed in Theorem 2.3 that both the equational and quasiequational theories of $\mathsf{R}(\cdot, +, \backslash, /)$ are not finitely axiomatizable, but we left the question whether $\mathsf{Q}(\cdot, +, \backslash, /)$ and $\mathsf{V}(\cdot, +, \backslash, /)$ coincide open. We can formulate the problem in a more general setting.

Problem 4.2 For which similarity types τ of residuated algebras $\mathsf{Q}(\tau) = \mathsf{V}(\tau)$?

In connection with this problem we note that $\mathsf{Q}(\cdot, ;, \backslash, /) = \mathsf{V}(\cdot, ;, \backslash, /)$, cf. [Pr90, Mi10], and it would be interesting to see a more general method for translating quasiequations to equivalent equations using the residuals.

A challenging line of research is to find finite quasiequational axiomatizations for non-finitely axiomatizable varieties.

Problem 4.3 For a non-finitely axiomatizable variety V, find a finitely axiomatizable quasivariety Q such that V is the variety generated by Q.

Translating such a result to logic would provide us with a weakly sound and complete Hilbert-style calculus. We note that the axiomatization given in [Ko94a] for Kleene algebras is such an example.

Finally we note that Theorem 2.3 can be strengthened so that τ includes the reflexive-transitive closure * operation as well. Indeed, in $\mathfrak{A}_n$, $a\,;a\,;a = 1$ for every non-identity atom a. Thus we can define

$$x^* = \begin{cases} 1' & x \in \{0, 1'\} \\ 1 & \text{otherwise} \end{cases}$$

for every $x \in A_n$. Thus x^* can be defined as $(x\,;x\,;x) + 1'$ in $\mathfrak{A}_n$, hence in the ultraproduct $\mathfrak{A}$. As before, the τ-reduct of $\mathfrak{A}_n$ is not representable. By the definition of * above, we have that the ultraproduct is representable even if $^* \in \tau$. In particular, we have non-finite axiomatizability for representable residuated Kleene lattices, a.k.a. action lattices [Ko94b].

Corollary 4.4 *The quasivariety and variety generated by* $\mathsf{R}(\cdot, +, ;, \backslash, /, {}^*, 0, 1')$ *are not finitely axiomatizable.*

Acknowledgements   Thanks are due to the two anonymous referees for their useful comments and suggestions.

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