The peculiarities of Stoic propositional logic

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Aristotle, the founder of logic, nowhere defines the concepts of argument and of validity. He simply uses them in his definition of a syllogism as "an argument in which, certain things being posited, something other than those things laid down results of necessity through the things laid down" (*Topics* I.1.100a25-27; cf. *Sophistical Refutations* 164b27-165a2, *Prior Analytics* I.1.24b18-20). In reconstructing Aristotle's early theory of syllogisms, John Woods (2001) uses Aristotle's reticence to interpret the basic concepts of argument and validity very liberally: arguments may have any number of premisses, even zero, and validity is the absence of a counter-model, constrained only by a requirement that premiss(es) and conclusion belong to the same discipline. Thus Woods finds in Aristotle's earliest logical writings considerable resources for his ongoing sophisticated defence of classical validity against contemporary relevantist objections. The properties of Aristotelian syllogisms which relevantists find so congenial–exclusion of redundant premisses, non-identity of the conclusion with any premiss, multiplicity of premisses–turn out to be constraints over and above those imposed by the requirement that a syllogism be a valid argument.

Between Aristotle, writing in the fourth century BCE, and Boole (1847), writing more than two millennia later, only one logician published a system of logic. That was Chrysippus (c. 280-207 BCE), the third head of the Stoic school. Chrysippus' system of propositional logic was dominant for 400 years, until bits of it were eventually absorbed into a confused amalgamation with Aristotle's categorical logic, a bowdlerization nicely described by Speca (2001). For centuries the system from which these surviving bits were extracted was forgotten. Only the careful work of such scholars as Mates (1953), Frede (1974), Hülser (1987-1988) and Bobzien (1996, 1999) has allowed us to appreciate once again the achievement of Chrysippus.

Despite its rigour and soundness, the system is oddly incomplete. One can show that a conjunction follows from its conjuncts, but not that either conjunct follows from the conjunction. One can detach the antecedent from a conditional, but not put it back on; in other words, there is no deduction theorem, no rule of "conditional proof" or "if introduction". One can show what follows from an exclusive disjunction and the affirmation or denial of one of its disjuncts, but not what the exclusive disjunction follows from. Further, there is no evidence that anybody ever tried to extend the system.

Why was Stoic propositional logic so incomplete? I shall argue that many of its peculiarities can be explained by the rather restrictive accounts of argument and of validity which Chrysippus adopted as the foundation of his system. The omissions from the system were not accidental oversights, or not just accidental oversights, but were dictated by the requirement that everything demonstrable in the system be a valid argument. With much more complex and restrictive accounts of argument and of validity than those adopted by Woods in his reconstruction of Aristotle's earlier logic, Chrysippus was forced into a much more restrictive formal system than contemporary classical propositional logic.

1. The system

1.1 Its language: The language of the system is a punctuation-free regimented Greek, whose syntax supposedly corresponds to the structure of the incorporeal propositions (*axiômata*) signified by its sentences. In contemporary symbolism:

1. If p is a proposition, then so is $\neg p$. (Read: *not p*. Cf. Diocles 7.69, A.L. 2.88-90.)

2. If p and q are propositions, then so is $\rightarrow p \rightarrow q$. (Read: *if p then q*. Cf. Diocles 7.71, A.L. 2.109-111, Gellius 16.8.9 = FDS 953.)

3. If $p_1, ..., p_n$ (n > 1) are propositions, then so are $\& p_1 \& ... \& p_n$ (read: *both p_1 and .. and p_n*; cf. Diocles 7.72) and $\veebar p_1 \veebar ... \veebar p_n$ (read: *either p_1 or .. or p_n.*; cf. Diocles 7.72).

Only basic propositions (not defined here) and propositions formed by a finite number of applications of the above three rules are propositions in the system.

Every Stoic proposition is either true or false (Diocles 7.65, Cicero *On Fate* 38). Negation and conjunction are classically truth-functional: the negation of a true proposition is false, and of a false proposition true (A.L. 2.103); a conjunction is true if all its conjuncts are true and false if a conjunct is false (A.L. 2.125, Gellius 16.8.11 = FDS 967). The conditional connective *if* indicates that the consequent follows from the antecedent (Chrysippus, *Dialectical Definitions*, apud Diocles 7.71); hence a conditional is true if the contradictory of the consequent "conflicts with" (*machetai*) the antecedent, and false otherwise (Diocles 7.73, P.H. 2.111). Disjunction is exclusive; according to the quasi-connexionist version of its truth-conditions,[1] a disjunction is true if and only if one of its disjuncts is true and each disjunct conflicts with each other disjunct (P.H. 2.191).

An argument is "a system <composed> of premisses and a conclusion" (D.L. 7.45). The plural of "premisses" is quite intentional: Chrysippus denied that there are one-premissed arguments (A.L. 2.443), for unknown reasons. An argument is valid if and only if the contradictory of its conclusion conflicts with the conjunction of its premisses (Diocles 7.77). Thus validity is the same as truth of the conditional whose antecedent is the conjunction of the argument's premisses and whose consequent is the argument's conclusion (P.H. 2.137, A.L.

[1] There are also truth-functional (Diocles 7.72) and fully connexionist (Gellius 16.8.13 = FDS 976) versions.

2.415-417; cf. A.L. 2.112).

The concept of conflict is thus basic to the truth-conditions for conditionals and disjunctions, and to the concept of argument validity. Unfortunately our sources do not preserve a complete account of this concept. Conflicting propositions cannot be simultaneously true (I.L. 4.2, Apollonius Dyscolus 218 = FDS 926, Gellius 16.8.14 = FDS 976). The incompatibility need not be logical; we are told[2] (Diocles 7.73) that *Not it is light* conflicts with *It is day*. Some pairs of conflicting propositions can be simultaneously false, as in the example, e.g. at night by lamplight. A proposition cannot conflict with itself (P.H. 2.111). Conflict implies some sort of connection (*sunartêsis*, cf. P.H. 2.111); hence the mere fact that a proposition is always false, or even necessarily false, is not sufficient for it to conflict with any arbitrarily chosen proposition. Any proposition conflicts with its contradictory (Apollonius Dyscolus 218 = FDS 926); hence it cannot be a requirement that each conflicting proposition is at some time true, or even possibly true. To sum up, one proposition conflicts with another only if (1) they are distinct, (2) they cannot both be simultaneously true, and (3) this impossibility is not due to the necessary falsity of one of them. These necessary conditions may not be jointly sufficient; the difficulty is to specify when an always false, or necessarily false, proposition conflicts with another proposition.

1.2 Its primitives: The primitives of Stoic propositional logic are "undemonstrated[3] arguments"

[2]Falsely. North of the Arctic Circle, human beings distinguish day and night even in the middle of winter, when it is dark for weeks at a time. My thanks to Rani Lill Anjum for the counter-example. In what follows, I shall occasionally pretend that it does not exist.

[3]The usual translation is *indemonstrable*. But the suffix *-tos* is ambiguous, corresponding either to the English suffix *-ed* or to the English suffix *-able*/*-ible*. We are told that the Stoics' arguments are *anapodeiktoi* because they do not need demonstration (Diocles 7.79; cf. A.L. 2.223), an explanation which makes far more sense if we translate *anapodeiktoi* as *undemonstrated* than if we translate it as *indemonstrable*. Further, the latter translation implies

(*anapodeiktoi logoi*) of five "moods" (*tropoi*):

A1. $\to p \to q, p \vdash q$ (*modus ponendo ponens*; cf. Diocles 7.80, A.L. 2.224, P.H. 2.157, I.L. 6.6)

A2. $\to p \to q, \neg q \vdash \neg p$ (*modus tollendo tollens*; cf. Diocles 7.80,. A.L. 2.225, P.H. 2.157, I.L. 6.6)

A3. $\neg\& p \& q, p \vdash \neg q$ (*modus ponendo tollens*; cf. A.L. 2.226, Diocles 7.80, P.H. 2.158, I.L. 14.4)

A4. $\veebar p \veebar q, p \vdash \neg q$ (cf. Diocles 7.81; cf. P.H. 2.158, I.L. 6.6)

A5. $\veebar p \veebar q, \neg p \vdash q$ (*modus tollendo ponens*; cf. D.L. 7.81, P.H. 2.158, ps.-Galen 15, 608,1-2 = FDS 1129)

For present purposes, I ignore extended descriptions of third, fourth and fifth undemonstrated arguments to cover conjunctions with multiple conjuncts and disjunctions with multiple disjuncts (Cicero, *Topics* 54 = FDS 1138; Philoponus, *In an. pr.* 245,19-246,14 = FDS 1133).

1.3 Its rules of inference: There are rules, called *themata*, for generating new valid arguments from arguments already known to be valid, apparently four in number.[4]

The first *thema* is a contraposition rule for arguments, of the same sort as we find already

(falsely) that the validity of arguments of these types cannot be demonstrated, whereas the former translation merely indicates (correctly) that no demonstration of these arguments is given; such arguments, as Sextus Empiricus reports, "have no need of demonstration because of its being at once conspicuous in their case that they are valid" (A.L. 2.223).

[4]Galen refers to the Stoics' first, second, third and fourth *themata* (*On the Doctrines of Hippocrates and Plato* 2.3.18-19 = FDS 1160) in a polemical context in which it suits his purposes to mention all their *themata*. Further, we have references elsewhere to the first (ps.-Apuleius, *De interpretatione* 191,5-10 = FDS 11621), second (Alexander, *In an. pr.* 164,30-31; 284,15), third (Alexander, *In an. pr.* 278,6-7,11-14; 284,15; Simplicius, *In de caelo* 236,33-237,4) and fourth (Alexander, *In an. pr.* 284,15) *themata*, but to no other.

in Aristotle's *Topics* (8.14.163a32-36) and *Sophistical Refutations* (33.182b37-183a2):[5]

> T1. If from two [or more][6] a third follows, from either one [or all but one] of them together with the contradictory of the conclusion there follows the contradictory of the remaining one. (Ps.-Apuleius, *De Interpretatione* 191,5-11 = FDS 1161)

The term *contradictory* is more general than the term *negation*. A proposition and its negation are said to be *contradictories* (*antikeimena*; cf. Diocles 7.73, A.L. 2.89); hence the contradictory of a negated proposition may fall short of it by a negative. The first *thema* thus allows us to demonstrate variants of the moods in which we have such a contradictory rather than a negation, or in which the other conjunct or disjunct of the "leading premiss" (*hêgemonikon*) occurs in the "added premiss" (*proslêgon*).[7] It also allows us to "reduce" (*anagein*) second undemonstrated arguments to corresponding first undemonstrated arguments, and conjunction introductions to third undemonstrated arguments, as in the following example:

$$\cfrac{\cfrac{}{\neg \& \text{ F } \& \text{ G, F} \vdash \neg\text{G}}\ \scriptstyle(\text{A3})}{\text{F, G} \vdash \& \text{ F } \& \text{ G}}\ \scriptstyle(\text{T1})$$

The second, third and fourth *themata* are taken by Alexander of Aphrodisias (*In an. pr.*

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[5]Woods (2001), following the entire western logical tradition, calls this rule *argument conversion*. The label is misleading, since conversion of a proposition involves changing two components (subject and predicate, antecedent and consequent) without changing their sign. The label *argument contraposition* signals the analogy to contraposition of propositions, which involves changing the sign of two components which are interchanged.

[6]Galen (I.L. 6.5) mentions this variant formulation for multi-premissed syllogisms.

[7]The descriptions of the undemonstrated arguments, which were apparently more authoritative, allow for these variations. The first *thema* makes it unnecessary to complicate the moods to provide for them.

284,13-15 = FDS 1165) to be equivalent to the following "synthetic theorem" of the Peripatetics:

> Whenever from some something follows, and that which follows along with one or more yields something, then those from which it follows, along with the one or more with which it yields that something, will themselves also yield the same thing. (Alexander *In an. pr.* 274,21-24 = FDS 1166; 278,8-11 = FDS 1167; cf. 283,15-17 = FDS 1165)

Symbolically, if $p_1, ..., p_n \vdash q_i$ (n > 1) and $q_1, ..., q_i, ... , q_m \vdash r$, (m > 1), then $p_1, ..., p_n, q_1, ..., q_{i-1}, q_{i+1}, ... , q_n \vdash r$ ($1 \leq i \leq m$). The synthetic theorem is thus a rule for chaining arguments together, or in contemporary language a cut rule (Gentzen 1969). It is perfectly general, except that it requires both the subordinate argument (i.e. the one whose conclusion q_i is a premiss of the other argument) and the superordinate argument (i.e. the one whose premiss q_i is a conclusion of the other argument) to have at least two premisses, in accordance with Aristotle's definition of a syllogism.

The third *thema* has survived in two versions, one reported by Alexander (fl. c. 200), the other by Simplicius (writing after 532). Following Bobzien (1996, 1999), I shall use Simplicius' version:[8]

> T3. If from two a third follows, and that which follows along with another [or others][9] from outside yields something, then also from the first two and the one

[8]Bobzien's choice is unusual; it rests on the fact that existing reconstructions using Alexander's version are difficult, if not impossible, to apply in practice. In forthcoming work, I propose a reconstruction on the basis of Alexander's version which is practically workable.

[9]The extension is justified by the parallel extension to the first *thema* and by the fact that Alexander's version allows a second input argument (in his case, the subordinate argument) with more than two premisses.

[or ones] assumed in addition from outside there will follow the same one.

(Simplicius, *In de caelo* 237,2-4 = FDS 1168)

Symbolically, if $p_1, p_2 \vdash q_i$ and $q_1, ..., q_i, ..., q_n \vdash r$, then $p_1, p_2, q_1, ..., q_{i-1}, q_{i+1}, ..., q_n \vdash r$ ($1 < n$, $1 \leq i \leq n$; $p_1 \neq q_j$ and $p_2 \neq q_j$ for each j such that $1 \leq j \leq n$). The third *thema* differs from the synthetic theorem in two respects. First, the subordinate argument has exactly two premisses rather than any number greater than one; this limitation is not substantive, since all chains of reasoning must begin with undemonstrated arguments with two premisses. Second, the superordinate argument cannot have a premiss of the subordinate argument as its premiss, as the words "from outside" indicate; this latter difference is substantive, and provides the key to the reconstruction of the second and fourth *themata*, which are not extant. The following proof illustrates the use of the third *thema*:

$$\dfrac{\dfrac{}{\to D \to L, D \vdash L}\,(A1) \qquad \dfrac{}{\neg\& L \& N, L \vdash \neg N}\,(A3)}{\neg\& L \& N, \to D \to L, D \vdash \neg N}\,(T3)$$ [10]

Bobzien (1996, 151) has proposed as the second *thema* the special case where the superordinate argument uses no external premisses:[11]

T2. If from two a third follows, and that which follows along with one or both from which it follows yields something, then also from the first two there will

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[10]The abbreviations are mnemonic for the simple propositions in an argument (*Not both it is light and it is night; and if it is day it is light; but it is day; therefore not it is night*) which Sextus Empiricus analyses (A.L. 2.234-238) using a so-called "dialectical theorem".

[11]In this respect she follows Ierodiakonou (1990, 72-74). Hitchcock (1982) makes a similar proposal. I have changed Bobzien's wording to conform to my translation of Simplicius.

follow the same one.

Symbolically, if $p_1, p_2 \vdash q$ and either $q, p_i \vdash r$ (i = 1 or i = 2) or $q, p_1, p_2 \vdash r$, then $p_1, p_2 \vdash r$. The following proof illustrates the use of the second *thema*:

$$\dfrac{\dfrac{\dfrac{}{\to \mathrm{K} \to \mathrm{D}, \mathrm{K} \vdash \mathrm{D}}\,(\mathrm{A1}) \qquad \dfrac{\dfrac{}{\to \mathrm{K} \to \neg\mathrm{D}, \mathrm{K} \vdash \neg\mathrm{D}}\,(\mathrm{A1})}{\mathrm{D}, \mathrm{K} \vdash \neg \to \mathrm{K} \to \neg\mathrm{D}}\,(\mathrm{T1})}{\to \mathrm{K} \to \mathrm{D}, \mathrm{K} \vdash \neg \to \mathrm{K} \to \neg\mathrm{D}}\,(\mathrm{T2})}{\to \mathrm{K} \to \mathrm{D}, \to \mathrm{K} \to \neg\mathrm{D} \vdash \neg\mathrm{K}}\,(\mathrm{T1})$$ [12]

This proof illustrates how the combination of the first *thema* and one of the cut rules (in this case, the second *thema*) gives the effect of a *reductio ad absurdum* rule. The two first undemonstrated arguments have conclusions which contradict each other; from them, by a series of manoeuvres, one derives an argument whose conclusion is the contradictory of one premiss of the two undemonstrated arguments and whose premisses are their remaining premisses. Because the first *thema* is formulated in terms of contradictories rather than negations, the system has both a negation-introduction rule (illustrated here) and a negation-elimination rule of the sort popularized by Fitch (1952).

The fourth *thema* (Bobzien 1996, 151) provides for the case where the superordinate argument includes both one or two premisses of the subordinate argument and one or more

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[12]The capital letters are mnemonic for the constituents of the "argument through two moodals" reported by Origen (*Contra Celsum* 7.15 = FDS 1181): *If you know that you are dead, you are dead; if you know that you are dead, you are not dead; therefore, you do not know that you are dead*. Galen reports (*On the Doctrines of Hippocrates and Plato* 2.3.18 = FDS 1160) that such syllogisms were analysed using the first and second *themata*. A "moodal" (*tropikon*) is a type of non-simple proposition which can be the leading premiss of a mood (*tropos*) of undemonstrated argument, i.e. either a conditional or a negated conjunction or a disjunction.

premisses external to it:

> T4. If from two a third follows, and that which follows along with one or both from which it follows and one or more from outside yields something, then also from the first two and that or those from outside there will follow the same one.

Symbolically, if $p_1, p_2 \vdash q_j$ and either $q_1, \ldots, q_j, \ldots, q_n, p_i \vdash r$ (i = 1 or i = 2) or $q_1, \ldots, q_j, \ldots, q_n, p_1, p_2 \vdash r$, then $p_1, p_2, q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n \vdash r$ ($1 < n$, $1 \leq j \leq n$; $p_1 \neq q_k$, $p_2 \neq q_k$ for $1 \leq k \leq n$). The following proof of constructive dilemma illustrates the use of the fourth *thema*:

$$\dfrac{\dfrac{\dfrac{}{\to p \to r, \neg r \vdash \neg p}\,{\scriptstyle(A2)} \qquad \dfrac{\dfrac{\dfrac{}{\to q \to r, \neg r \vdash \neg q}\,{\scriptstyle(A2)} \qquad \dfrac{\dfrac{}{\veebar p \veebar q, \neg p \vdash q}\,{\scriptstyle(A5)}}{\neg q, \neg p \vdash \neg \veebar p \veebar q}\,{\scriptstyle(T1)}}{\to q \to r, \neg r, \neg p \vdash \neg \veebar p \veebar q}\,{\scriptstyle(T3)}}{\veebar p \veebar q, \neg r, \neg p \vdash \neg \to q \to r}\,{\scriptstyle(T1)}}{\veebar p \veebar q, \to p \to r, \neg r \vdash \neg \to q \to r}\,{\scriptstyle(T4)}}{\veebar p \veebar q, \to p \to r, \to q \to r \vdash r}\,{\scriptstyle(T1)}$$

Thus the system had a derived rule of disjunction elimination using conditionals.

1.4 Metatheorems: THEOREM 1 (*cut*): The second, third and fourth *themata* have the same demonstrative power in the system as an unrestricted cut rule.

PROOF: The *themata* cover all cases in which the subordinate argument has two premisses. Hence it is sufficient to prove that it is not necessary to have a cut rule for subordinate arguments with other than two premisses. Since each undemonstrated argument has two premisses and no *thema* permits one to derive an argument with fewer than two premisses, there is no need for a cut rule for subordinate arguments with fewer than two premisses. In a straightforward chaining of several undemonstrated arguments together, with no application of the first *thema*, a reduction which generates a subordinate argument with more than two premisses can be transformed into one which generates only two-premiss subordinate arguments, simply by changing the order in which the cut rule is applied; the trick is to always find two premisses of the analysandum from which a proposition not in the analysandum follows. Applications of the first *thema*, which are sometimes required (as in the reduction above using the fourth *thema*) to produce two premisses from which something follows, do not affect this basic strategy. QED

THEOREM 2 (*negation introduction and negation elimination*): If $p_1, ..., p_n \vdash q$ (n > 1) and $p_{n+1}, ..., p_{n+m} \vdash \neg q$ (m > 1), then $p_1, ..., p_{i-1}, p_{i+1}, ..., p_n \vdash$ ctr(p_i) ($1 \leq$ i $\leq$ m), where ctr(p_i) is r if p_i is $\neg r$ for some proposition r, and otherwise ctr(p_i) is $\neg p_i$, and any premiss which occurs both in $p_1, ..., p_n \vdash q$ and $p_{n+1}, ..., p_{n+m} \vdash \neg q$ occurs in $p_1, ..., p_{i-1}, p_{i+1}, ..., p_n$ as many times as it occurs in that one of the two arguments in which it occurs more often.

PROOF: Apply the first *thema* to whichever argument has p_i as a premiss, putting ctr(p_i) as the resulting argument's conclusion. By theorem 1, $p_1, ..., p_{i-1}, p_{i+1}, ..., p_n \vdash$ ctr(p_i). QED

In addition to a standard conditional elimination rule (A1), the system has conjunction introduction (from A3, by T1–see the example above of the use of the first *thema*), disjunction elimination using conditionals (from A2 twice and A5–see the example above of the use of the fourth *thema*), and the following qualified version of disjunction elimination using arguments: For any propositions $p, q, r, p_1, ..., p_m, ..., p_n$ ($1 \leq$ m, m < n), if $p_1, ..., p_m, p \vdash r$ and $p_{m+1}, ..., p_n, q \vdash r$, then $\veebar p \veebar q, p_1, ..., p_n \vdash r$ (where any premiss which occurs both in $p_1, ..., p_m, p \vdash r$ and $p_{m+1}, ..., p_n, q \vdash r$ occurs in $p_1, ..., p_n$ as many times as it occurs in that one of the two arguments in which it occurs more often; proved by adapting the demonstration of disjunction elimination using conditionals).

2. Peculiarities

2.1 Absence of theorems: THEOREM 3: For any proposition p, $\nvdash p$.

PROOF: Only arguments with at least two premisses are demonstrable in the system. QED

EXPLANATION: Anything demonstrated in the system is a syllogism, which is a species of valid argument (Diocles 7.78). Every argument has at least two premisses. Hence there are no theorems in the system. In particular, although the Stoics held that an argument is valid if and only if the conditional whose antecedent is the conjunction of its premisses and whose consequent is its conclusion is true (P.H. 2.137, A.L. 2.415-417; cf. A.L. 2.112), Chrysippus did not incorporate this principle in his system; even if $p_1, ..., p_n \vdash q$ (n > 1), $\nvdash \rightarrow \& p_1 \& ... \& p_n \rightarrow q$. Nor does the system permit proofs of several moods of logical truths which are expressible in the system, e.g. $\neg\& p \& \neg p$ (law of non-contradiction, known to Aristotle), $\veebar p \veebar \neg p$ (law of excluded middle, known to Aristotle, and the syntactic counterpart of the Stoics' principle of bivalence), $\rightarrow p \rightarrow p$ (law of identity).

2.2 Absence of one-premissed syllogisms: THEOREM 4: For any propositions p and q, $p \nvdash q$.

PROOF: Every undemonstrated argument has two premisses. No *thema* permits one to derive an argument with fewer than two premisses. QED

EXPLANATION: Since they are arguments, syllogisms must have more than one premiss. In particular, for any proposition p, $p \nvdash p$ (irreflexivity), even though any proposition p follows from itself and there is no general ban on syllogisms with a conclusion identical with a premiss.[13]

[13]The Stoics recognized some "non-differently concluding" (*adiaphorôs perainontes*) syllogisms, e.g. *Either it is day or it is light; but it is day; therefore it is day* (Alexander *In Top.* 10,11-12), which can be reduced to a fourth and a fifth undemonstrated argument using the

Also, for any propositions p and q, & p & $q \nvdash p$ and & p & $q \nvdash q$ (counter-conjunction-elimination), even though any conjunct follows from a conjunction. Also, there are no propositions p, q and r such that $p \vdash r$, $q \vdash r$ and $\veebar p \veebar q \vdash r$ (absence of simple disjunction elimination using arguments), even though, if a proposition follows from each disjunct in a disjunction, then it follows from the whole disjunction.

The rejection of one-premissed arguments shows that the premisses of a Stoic argument did not constitute a set, contrary to Peter Milne's suggestion (1995, 41). If they did, Stoic propositional logic would easily generate one-premissed arguments, a fact Chrysippus could hardly fail to have noticed. For example, the application of the first *thema* to the first undemonstrated argument *If not it is day, it is day; not it is day; therefore it is day* produces the argument *Not it is day; not it is day; therefore if not it is day, it is day*. If the two premisses of the latter argument are a set, it is a set with one member, so that the argument would have one premiss. Thus repetitions of premisses can occur. Furthermore, there is no general rule for eliminating them, i.e. no rule of simplification; if there were, the system would easily generate one-premissed arguments. The fact that repetitions can occur and that there is no rule permitting their elimination explains why Chrysippus needed three cut rules; he had to rule out repetitions of a premiss which occurred in both arguments being chained together.

2.3 Non-monotonicity: LEMMA: If $p_1, ..., p_n \vdash q$ and some basic proposition r occurs in the propositions $p_1, ..., p_n$ and q, then r occurs in at least two of those propositions.

PROOF: Each basic proposition in an undemonstrated argument occurs either in both premisses or in both a premiss and the conclusion. If the first *thema* is applied to an argument in which a

second *thema*.

basic proposition occurs in at least two of its components, then that basic proposition occurs in at least two components of the resulting argument. If a basic proposition occurs in at least two components of either of two arguments chained together using the second, third or fourth *thema*, then it occurs in at least two components of the resulting argument. (The proof of this last proposition requires a rather complex enumeration of cases, in order to show that any basic proposition in the conclusion of the subordinate argument must occur both in a premiss of the subordinate argument and in a component of the superordinate argument other than the premiss which drops out.) QED

THEOREM 5: For some propositions $p_1, ..., p_n, q$ and $r, p_1, ..., p_n \vdash q$ but $r, p_1, ..., p_n \nvdash q$ (n > 1).

PROOF: $\rightarrow \mathrm{F} \rightarrow \mathrm{G}, \neg\mathrm{G} \vdash \neg\mathrm{F}$ but, by the preceding lemma, $\rightarrow \mathrm{F} \rightarrow \mathrm{G}, \neg\mathrm{G}, \mathrm{H} \nvdash \neg\mathrm{F}$. QED

EXPLANATION: The concept of conflict was interpreted in such a way that an argument could be invalid not only through obvious disconnection (*diartêsis*), bad form, or deficiency of a premiss, but also through redundancy (A.L. 2.429-432). Redundancy is the introduction of a superfluous premiss into an argument which would otherwise be valid, as in the argument *If it is day, it is light; but it is day; and virtue benefits; therefore it is day*. The explanation of redundancy as a source of invalidity[14] is perhaps that, according to the Stoics' first *thema*, if this

[14]The system permits demonstration of some arguments with a redundant premiss, for example *If there exists a sign, there exists a sign; if not there exists a sign, there exists a sign; but either not there exists a sign or there exists a sign; therefore, there exists a sign* (A.L. 2.281), where the third-mentioned premiss is redundant. But this argument has the same form as the argument *If it is day, it is light; if it is night by lamplight, it is light; either it is day or it is night by lamplight; therefore it is light*, which is valid and has no redundant premiss. Similarly for all other demonstrable arguments with a redundant premiss. Hence, to preserve the soundness of the system, one needs to create an exception for arguments with the same form as a valid argument without a redundant premiss. Such an exception, unlike that proposed by Bobzien (1996, 180), covers not only syllogisms but also other types of valid arguments.

argument is valid, then so is its contrapositive: *If it is day, it is light; but it is day; and not it is day; therefore not virtue benefits.* But this argument is clearly invalid on the connexive criterion of validity; the contradictory of the conclusion has nothing to do with the conjunction of the premisses. Chrysippean propositional logic is close to being counter-monotonic; the only exceptions to counter-monotonicity which come readily to mind involve adding as a premiss a "duplicated"[15] conditional whose antecedent and consequent are identical to the conclusion, as in the argument $\rightarrow \neg F \rightarrow \neg F, \rightarrow F \rightarrow G, \neg G \vdash \neg F$.

2.4 Absence of a deduction theorem: THEOREM 6: For some propositions $p_1, ..., p_n$, q and r ($1 \leq n$), $p_1, ..., p_n, q \vdash r$ but $p_1, ..., p_n \nvdash \rightarrow q \rightarrow r$.

PROOF: If n=1, $p_1, ..., p_n \nvdash \rightarrow q \rightarrow r$, since only arguments with at least two premisses are demonstrable in the system. More generally, the system contains no general procedure for demonstrating the validity of an argument with a conditional conclusion. For such a demonstration, the conditional must be embedded in a more complex premiss, either a disjunction or a more complex conditional. QED

EXPLANATION: The deduction theorem is unsound on the connexionist account of the conditional. That is, there are cases where a proposition q follows from a combination of some proposition p with one or more other premisses but where it does not follow from those other premisses that q follows from p. For example, *grass is green* follows from the combination of *snow is white* and *not both snow is white and not grass is green*, but it does not follow from *not*

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[15] Duplicated propositions are propositions in which the same proposition occurs twice (Diocles 7.69, A.L. 2.109). The Stoics recognized as valid "duplicated arguments" in which such propositions occur as premisses (Alexander, *In an. pr.* 164, 28-31 = FDS 1169; *In an. pr.* 18,17-18; 20,10-12 = FDS 1171; *In Top.* 10,8-10 = FDS 1170).

both snow is white and grass is not green that *grass is green* follows from *snow is white*. In other words, *Not both snow is white and not grass is green*; *but snow is white*; *therefore, grass is green* is valid, but *if snow is white, then grass is green* does not follow from *not both snow is white and not grass is green*. In general, in fact, *if p, then q* does not follow from *not both p and not q*.

2.5 Absence of hypothetical syllogism, dilemma and other apparently valid moods:

THEOREM 7: (a) $\rightarrow p \rightarrow q, \rightarrow q \rightarrow r \nvdash \rightarrow p \rightarrow r$ (no hypothetical syllogism), (b) $p, \neg q \nvdash \veebar p \veebar q$, (c) $\neg\veebar p \veebar q, p \nvdash q$, and (d) $\rightarrow p \rightarrow r, \rightarrow q \rightarrow s, \veebar p \veebar q \nvdash \veebar r \veebar s$ (no dilemma).

PROOF AND EXPLANATION: The system is sound, as can be shown by applying the above truth-conditions for negations, conjunctions, conditionals and disjunctions and the definition of the validity of an argument to the five types of undemonstrated argument and by working out that each of the *themata* preserve validity. But each of the above moods has counter-examples. (a) Put *you know that you are dead* for p, *you are dead* for q, *not you know that you are dead* for r. (b) Put *grass is green* for p and *snow is pink* for q.[16] (c) Put *grass is green* for p and *snow is pink* for q. (d) Put *it is day* for p, *it is light* for r, *it is night* for q, *not the sun is shining* for s. All four counter-examples exploit the fact that the conclusion is true only if there is a conflict between two propositions. Non-syllogisticity of the above moods can also be shown by re-interpreting the connectives as truth-functions in a semantics of multiple truth-values, in such a way that undemonstrated arguments have no counter-model, the *themata* preserve absence of a counter-model, but each mood has a counter-model; cf. Milne (1995).

2.6 Absence of some valid moods: THEOREM 8: (a) $\rightarrow p \rightarrow q, \neg p \rightarrow \neg q \nvdash \veebar p \veebar \neg q$, (b) $\veebar p \veebar q, p \nvdash \rightarrow p \rightarrow \neg q$, and (c) $\& p \& q, \rightarrow p \rightarrow r \nvdash r$.

[16] Milne's complaint (1995, 52-53) that this mood is indemonstrable is thus misplaced.

PROOF: Non-syllogisticity of these moods can be shown by reinterpreting the connectives as truth-functions in a semantics of multiple truth-values. For (a), cf. Milne (1995).

EXPLANATION: These moods are valid. Their indemonstrability must be regarded as an unintended consequence of the way Chrysippus constructed his system. He evidently included as undemonstrated moods all two-premiss valid argument schemes with a complex leading premiss with two propositional variables, an added premiss which was one of the propositional variables or its contradictory, and a conclusion which was the other propositional variable or its contradictory. To these primitives he added the rule for contraposing arguments which Aristotle had explicitly stated in his early logical writings, as well as a three-part cut rule. Special cases like the above are inadvertently not provided for, even though they are valid two-premiss arguments. To add more primitives to the system to accommodate such special cases is likely to be a never-ending task, because there is no single principle which accommodates them all.

3. Conclusion

3.1 The requirement of at least two premisses: Chrysippus assumed that an argument must have at least two premisses. Hence each primitive of his system has two premisses, and the rules for demonstrating arguments on the basis of primitives do not permit reduction of the number of premisses below the minimum of the input argument(s). Hence the system has no logical truths as theorems, nor can it demonstrate any one-premiss arguments, even when one proposition follows in Chrysippus' sense from another. In particular, it cannot demonstrate conjunction elimination or simple disjunction elimination using arguments. It also has no repetition rule, even though on Chrysippus' criterion every proposition follows from itself.

3.2 Acceptance of argument contraposition: Chrysippus used Aristotle's rule of argument contraposition as his first *thema*. In order to avoid generation of one-premiss arguments using this *thema*, he had to regard an argument with one premiss repeated twice as a two-premiss argument. Thus repetitions count; the premisses of a Stoic syllogism are not a set. Further, on pain of violating the two-premiss requirement, there is no rule for eliminating a repeated premiss. Hence, in order to avoid generating unwanted repetitions in chaining two arguments together, Chrysippus needed three versions of the cut rule, to cover cases where the premisses of the superordinate argument (other than the conclusion of the subordinate argument) come entirely from the subordinate argument, entirely from outside the subordinate argument, or partly from it and partly from outside it.

3.3 Connexionism: Chrysippus was a connexionist about true conditionals, about true disjunctions, and about valid arguments: in a true conditional there had to be a connection between antecedent and consequent, in a true disjunction an incompatibility between each pair of disjuncts, and in a valid argument a connection between premisses and conclusion. The connexionist criterion for a true conditional rules out the usual deduction theorem (conditional proof), as well as proofs of a conditional from its consequent or from the contradictory of its antecedent. It also rules out hypothetical syllogism, a mood discovered by Theophrastus which is invalid on the connexionist account. Similarly, the connexionist criterion for a true disjunction rules out proofs of a disjunction from one disjunct and the contradictory of the other, as well as dilemma. The connexionist criterion for a valid argument rules out *ex falso quodlibet* and *e quolibet verum*. Apparently because of Chrysippus' acceptance of argument contraposition, it rules out redundant premisses, except in cases where the argument has a mood without redundant premisses. Hence the system is non-monotonic, and close to being counter-monotonic.

3.4 Inadvertent omissions: Most of the apparent peculiarities of the system are thus necessary consequences of Chrysippus' assumptions about the concepts of argument and of validity. The Chrysippean-valid multi-premiss arguments which are not demonstrable in his system appear to have been overlooked as a consequence of the way Chrysippus selected his five types of primitives, each with one complex premiss and one simpler premiss and simpler conclusion. It is rather surprising, in fact, given Chrysippus' numerous pre-systematic restrictions on the concept of a valid argument and the apparent absence of any effort to ensure completeness, that his system allows one to prove the validity of practically all the formally valid moods expressible in the system which we use in real reasoning and argument. From this point of view, the Chrysippean restrictions on valid arguments may not be as unmotivated as we might suppose.

Abbreviations

A.L. = Sextus Empiricus, *Against the Logicians* (= *Against the Professors* 7-8)

Alexander *In An. pr.* = Alexander of Aphrodisias, *In Aristotelis Analyticorum Priorum Librum I Commentarium*

Alexander *In Top*. = Alexander of Aphrodisias, *In Aristotelis Topicorum Libros Octo Commentaria*

D.L. = Diogenes Laertius

FDS = Hülser, *Die Fragmente zur Dialektik der Stoiker*

I.L. = Galen, *Institutio Logica* (original Greek title *Eisagôgê Dialektikê*)

P.H. = Sextus Empiricus, *Outlines of Pyrrhonism* (*Pyrrhôneiôn Hupotupôseôn*)

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