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ANALYTIC EQUIVALENCE RELATIONS AND ULM-TYPE CLASSIFICATIONS

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Our main goal in this paper is to establish a Glimm-Effros type dichotomy for arbitrary analytic equivalence relations.

The original Glimm-Effros dichotomy, established by Effros [Ef], [Ef1], who generalized work of Glimm [Gl], asserts that if an F_σ equivalence relation on a Polish space X is induced by the continuous action of a Polish group G on X, then exactly one of the following alternatives holds:

(I) Elements of X can be classified up to E-equivalence by "concrete invariants" computable in a reasonably definable way, i.e., there is a Borel function $f: X \to Y$, Y a Polish space, such that $xEy \Leftrightarrow f(x) = f(y)$,
or else

(II) E contains a copy of a canonical equivalence relation which fails to have such a classification, namely the relation $xE_0y \Leftrightarrow \exists n \forall m \geq n(x(n) = y(n))$ on the Cantor space 2^ω ($\omega = \{0, 1, 2, \cdots\}$), i.e., there is a continuous embedding $g: 2^\omega \to X$ such that $xE_0y \Leftrightarrow g(x)Eg(y)$.

Moreover, alternative (II) is equivalent to:

(II)$'$ There exists an E-ergodic, nonatomic probability Borel measure on X, where E-ergodic means that every E-invariant Borel set has measure 0 or 1 and E-nonatomic means that every E-equivalence class has measure 0.

This basic classification/nonclassification dichotomy was recently shown to be true for an arbitrary Borel equivalence relation, not necessarily induced by any such group action, by Harrington, Kechris, and Louveau [HKL].

We study here the case of general analytic equivalence relations on Polish spaces. Simple examples (see §6 below) show that the above dichotomy cannot possibly hold in this context, even if in (I) we appropriately relax the requirement that f is Borel (which is clearly too strong in this case) to anything that is "reasonably definable". The problem is that finding invariants which can be taken to be members of a Polish space is not always possible in this more general situation. The clue for the correct types of invariants needed comes from a standard classification result in algebra, i.e., the Ulm classification of countable abelian p-groups up to isomorphism (which in a standard way can be viewed as an example of an analytic equivalence relation, in fact induced by a continuous action of the Polish group of all permutations on ω). Such groups are classified by their Ulm invariants

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which are countable transfinite sequences $(u_\alpha)_{\alpha<\xi}$, where $\xi < \omega_1$ (= the first uncountable ordinal) and u_α is a natural number or else the symbol ∞. Moreover, one can compute in a reasonably definable way these invariants from the group in question. It turns out that allowing for this more general type of invariants, which are clearly necessary by this example, one can establish a Glimm-Effros dichotomy for arbitrary analytic equivalence relations. More precisely, we have, letting $2^{<\omega_1}$ denote the set of countable transfinite sequences $(a_\alpha)_{\alpha<\xi}$, where $\xi < \omega_1$ and $a_\alpha \in \{0, 1\}$,

THEOREM 1. *Assume* $\forall x \in \omega^\omega$ $(x^\#$ *exists*$)$. *Then for any analytic equivalence relation* E *on a Polish space* X, *exactly one of the following holds*:

(I) *There is a map* $f : X \to 2^{<\omega_1}$ *which is* $\mathbf{\Delta}^1_2$ (*in any standard system of encoding* $2^{<\omega_1}$ *by elements of* ω^ω),

or else

(II) *There is a continuous embedding* $g : 2^\omega \to X$ *such that* $xE_0y \Leftrightarrow g(x)Eg(y)$.

Moreover, (II) *is equivalent to* (II)$'$ *as before.*

The hypothesis $\forall x \in \omega^\omega$ $(x^\#$ exists$)$ is one of the standard (and milder) large cardinal principles in set theory. It follows for example from the existence of a measurable cardinal. We do not know if one can establish in ZFC alone that either (I) or (II) above hold. (However, Sy Friedman has informed us recently that he can prove this assuming only that $\forall x \in \omega^\omega$ (there is a weakly compact cardinal in $L[x]$).) Making however one more assumption, often met in practice, namely that every E-equivalence class is Borel, allows us to prove the following:

THEOREM 2. *If* E *is an analytic equivalence relation on a Polish space* X *such that every* E*-equivalence class is Borel, then either one of* (I) *or* (II) *above holds. If, moreover, every* $\mathbf{\Sigma}^1_2$ *set is measurable, exactly one of* (I), (II) *holds and* (II) *is equivalent to* (II)$'$ *as before.*

It should be pointed out that earlier Ditzen [Di] and Foreman and Magidor (unpublished) have established, under appropriate determinacy hypotheses much stronger than $\forall x(x^\#$ exists$)$, a form of a dichotomy for $\mathbf{\Sigma}^1_1$-equivalence relations in which in alternative (I) above the invariants are members of 2^{ω_1}, i.e., ω_1-sequences of zeros and ones.

A particularly interesting case of an analytic equivalence relation having Borel equivalence classes is the one induced by the orbits of a continuous (or more generally Borel) action of a Polish group on a Polish space. This includes in particular the isomorphism relation on the set of countably infinite models of an $L_{\omega_1\omega}$ sentence (viewed always as having universe ω). In this case one can prove Theorem 1 in ZFC alone and obtain more accurate estimates for the descriptive complexity of the computation of the invariants, which if formulated in the language of effective descriptive set theory (see Theorem 2.1, below) is very close to that of the calculation of Ulm invariants for p-groups (see [BE]). In particular, in this case the function f in alternative (I) of Theorem 1 can be taken to be C-measurable, where C is the smallest σ-algebra of sets in Polish spaces containing the Borel sets and closed under the Souslin operation $\mathscr{A}$.

This result in the case of Borel actions of Polish groups was also proved independently by Becker (unpublished).

We also show that in the particular case where the Polish group is actually abelian one can improve the dichotomy by actually having in alternative (I) the invariants to be members of 2^ω (instead of $2^{<\omega_1}$) as in the case of Borel equivalence relations (and with the function f C-measurable). (After this paper was completed, Solecki in fact extended this result to all Polish groups admitting an invariant compatible metric and with f Borel.) As pointed out earlier, the Ulm classification shows that such an improvement is impossible for arbitrary Polish groups, and thus one obtains an interesting distinction in the structure of continuous actions of abelian vs. general Polish groups. This dichotomy can be also viewed as a strengthening of an earlier result of Sami [Sa1] according to which the equivalence relation induced by a Borel action of an abelian Polish group on a Polish space contains either countably many or else perfectly many orbits (i.e., the topological Vaught conjecture holds for abelian Polish groups). It should be mentioned here that, by results of Solecki [So], these equivalence relations are not necessarily Borel.

It is finally interesting to consider specifically the case of the isomorphism relation on the countable models of an $L_{\omega_1\omega}$ sentence. Call such a sentence σ *Ulm-classifiable* if alternative (I) above holds for this isomorphism relation, and *concretely classifiable* if the isomorphism relation on the models of σ is Borel and alternative (I) holds, which, by the above-stated theorem in [HKL], is the same thing as saying that one can classify countable models of σ up to isomorphism by invariants which are members of some Polish space (as opposed to $2^{<\omega_1}$) and can be computed in a Borel way. Let σ_ξ be a sentence of $L_{\omega_1\omega}$ whose countable models are exactly the countable structures (in the given language) of Scott height $\xi < \omega_1$. Then the countable models of $\sigma \wedge \sigma_\xi$ are the countable models of σ of Scott height ξ. We now have the following characterization of Ulm-classifiability.

THEOREM 3. *Let σ be an $L_{\omega_1\omega}$ sentence. Then the following are equivalent*:

(i) *σ is Ulm-classifiable.*

(ii) *For each ξ, $\sigma \wedge \sigma_\xi$ is concretely classifiable.*

(iii) *Every complete $L_{\omega_1\omega}$-theory T, containing σ and some σ_ξ, satisfies the following ω_1-compactness property*: *If every countable subset of T has a model, then T has a countable model.*

(iv) *Same as* (iii), *but with "T has a countable model" replaced by "T has a model".*

(v) *If a probability Borel measure on the space of structures of L (with universe ω) satisfies the* 0-1 *law for $L_{\omega_1\omega}$ sentences (i.e., every such sentence is true a.e. or false a.e.) and σ is true a.e., then there is a countable model $\mathcal{M}$ of σ so that the measure concentrates on the isomorphism class of $\mathcal{M}$.*

Notice that the equivalence of (iii) and (iv) is a purely model theoretic result concerning $L_{\omega_1\omega}$, which we do not know how to prove otherwise. Also, in many interesting cases σ already implies logically some σ_ξ. In these situations one can drop any reference to σ_ξ in Theorem 3. (Examples of such σ include the conjunction of the axioms for torsion free groups of finite rank, locally finite graphs, etc.)

Examples of Ulm-classifiable sentences include the (conjunction of the) axioms for an equivalence relation or a unary injective function (both of these are actually concretely classifiable), torsion abelian groups, and abelian p-groups. Non Ulm-classifiable sentences include, for example, the (conjunction of the) axioms for rank 1 torsion-free abelian groups, locally finite trees, fields, etc.

§**1. Preliminaries.** (A) We will use standard terminology and notation from descriptive set theory; see, e.g., [Mo] and [Ke]. In particular, $\mathscr{N}$ denotes the Baire space ω^ω, where $\omega = \{0, 1, 2, 3, \cdots\}$. By a *standard Borel space* we mean a Polish space with its associated σ-algebra of Borel sets; see [Ke]. For any Polish space X, $\mathscr{F}(X)$ denotes the Effros (standard) Borel space of the closed subsets of X with the σ-algebra generated by the sets of the form $\{F \in \mathscr{F}(X) : F \cap U \neq \emptyset\}$, for $U \subseteq X$ open; see again [Ke].

As usual, the Souslin operation $\mathscr{A}$ is defined by $\mathscr{A}_s P_s = \bigcup_{\alpha \in \mathscr{N}} \bigcap_{n \in \omega} P_{\alpha|n}$ for any family $\{P_s\}_{s \in \omega^{<\omega}}$ of subsets of a set X. The smallest σ-algebra of subsets of a standard Borel space X which contains the Borel sets and is closed under the operation $\mathscr{A}$ is called the class of *C-sets* in X. A function measurable with respect to this σ-algebra is called *C-measurable.*

(B) Let E, F be equivalence relations on sets X, Y, respectively. A *reduction* of E to F is a map $f : X \to Y$ such that $xEy \Leftrightarrow f(x)F f(y)$. It is an *embedding* if it is also one-to-one. If such a reduction exists which belongs to some class of functions Γ, then we write $E \leq_\Gamma F$, and in case of an embedding $E \sqsubseteq_\Gamma F$. We use the subscript "c" in the case when Γ consists of the class of continuous functions on Polish spaces.

We will often use the following result from [HKL]: Denote by E_0 the following equivalence relation on the Cantor space 2^ω:

$$xE_0y \Leftrightarrow \exists n \forall m \geq n(x(m) = y(m)).$$

Call a Borel equivalence relation E on a standard Borel space X *smooth* if it admits a *countable separating family*, i.e., a sequence (A_n) of Borel sets such that $xEy \Leftrightarrow \forall n(x \in A_n \Leftrightarrow y \in A_n)$. Equivalently, this means that $E \leq_{\Delta^1_1} \Delta(2^\omega)$, where $\Delta(S)$ denotes the equality relation on any set S. Finally, given a $\mathbf{\Sigma}^1_1$ equivalence relation E on a standard Borel space X, a probability Borel measure μ on X is *E-ergodic* if $\mu(A) = 0$ or 1 for any Borel E-invariant set A and *E-nonatomic* if $\mu([x]_E) = 0$ for any E-equivalence class $[x]_E, x \in X$. We now have

Theorem ([HKL]). *Let E be a Borel equivalence relation on a Polish space X. Then exactly one of the following holds*:

(I) *X is smooth.*

(II) *$E_0 \sqsubseteq_c X$.*

Moreover, (II) *is equivalent to*

(II)′ *There exists an E-ergodic, nonatomic probability Borel measure on X.*

Finally, this result holds effectively, i.e., if E is $\Delta^1_1(x)$ say on the space $X = \mathscr{N}$, then either (I) *E admits a $\Delta^1_1(x)$ separating family (A_n) (or equivalently $E \leq_{\Delta^1_1(x)} \Delta(2^\omega)$), or* (II) *$E_0 \sqsubseteq_c X$.*

(C) Now let G be a group, X a set and $(g, x) \mapsto g \cdot x$ an action of G on X. We denote by E_G the corresponding orbit equivalence relation: $xE_Gy \Leftrightarrow \exists g \in G(g \cdot x = y)$. If G is a Polish group and X a standard Borel space, we say that the action is *Borel* if $(g, x) \in G \times X \mapsto g \cdot x \in X$ is a Borel function.

If X is a Polish space and this function is continuous, we say that the action is *continuous*. For Borel actions, E_G is analytic but in general non-Borel, but each orbit $G \cdot x = \{g \cdot x : g \in G\}$ is Borel (see [Mi]).

Given a countable language $L = \{R_i\}_{i \in I} \cup \{f_j\}_{j \in J} \cup \{c_k\}_{k \in K}$, where R_i is an n_i-ary relation symbol, f_j an m_j-ary function symbol and each c_k a constant symbol, we can define the space X_L of countably infinite models of L with universe ω as follows:

$$X_L = \prod_{i \in I} 2^{(\omega^{n_i})} \times \prod_{j \in J} \omega^{(\omega^{m_j})} \times \omega^K.$$

Every $x \in X_L$ corresponds canonically to an L-structure $\mathscr{A}_x$ with universe ω, defined as follows:

$$\mathscr{A}_x = \langle \omega, \{R_i^{\mathscr{A}_x}\}, \{f_j^{\mathscr{A}_x}\}, \{c_k^{\mathscr{A}_x}\} \rangle,$$

where

$$R_i^{\mathscr{A}_x}(a_0, \cdots a_{n_i - 1}) \Leftrightarrow x_i^0(a_0, \cdots, a_{n_i - 1}) = 1,$$
$$f_j^{\mathscr{A}_x}(a_0, \cdots, a_{m_j - 1}) = x_j^1(a_0, \cdots, a_{m_j - 1}), \qquad c_k^{\mathscr{A}_x} = x^2(k),$$

and $x = (x^0, x^1, x^2)$, with $x^0 \in \prod_{i \in I} 2^{(\omega^{n_i})}$, $x^1 \in \prod_{j \in J} \omega^{(\omega^{m_j})}$, $x^2 \in \omega^K$. The *infinite symmetric group* S_∞ of all permutations of ω, which is a Polish group with the topology it inherits as a G_δ subspace of $\mathscr{N}$, acts continuously on X_L in the obvious way. This action is called the *logic action*. The associated equivalence relation is isomorphism $\cong$ of structures.

The invariant Borel sets under this action (or the corresponding equivalence relation $\cong$) are exactly those of the form

$$\mathrm{Mod}(\sigma) = \{x \in X_L : \mathscr{A}_x \models \sigma\},$$

for an $L_{\omega_1 \omega}$-sentence σ (López-Escobar; see, for example, [Va]).

We will often use standard results about the Scott analysis of a structure, Scott sentences, and Scott heights, for which the reader can consult for example [Ba]. We summarize the basic concepts and facts below.

Let $\mathscr{M} = \langle M, \dots \rangle$ be a structure for a language L. For $n \in \omega$ and $s \in M^n$, $s = (a_0, \dots, a_{n-1})$, let φ_s^0 be the infinitary formula $\bigwedge \{\psi(x_0, \cdots, x_{n-1}) : \psi$ is atomic or the negation of an atomic formula, and $\mathscr{M} \models \psi[a_0, \cdots, a_{n-1}]\}$. For $\alpha \in \mathrm{ORD}$, $n \in \omega$, $s \in M^n$, define φ_s^α by induction as follows:

$$\varphi_s^{\alpha+1} = \varphi_s^\alpha \wedge \bigwedge_{a \in M} \exists x_n \varphi_{s^\frown a}^\alpha \wedge \forall x_n \bigvee_{a \in M} \varphi_{s^\frown a}^\alpha,$$

and for λ limit

$$\varphi_s^\lambda = \bigwedge_{\alpha < \lambda} \varphi_s^\alpha.$$

We call φ_s^α the *α-type* of s. The *Scott height* of $\mathscr{M}$ is the least γ such that for all $\alpha \geq \gamma$, $n \in \omega$, $s \in M^n$

$$M \models \forall x_0 \cdots \forall x_{n-1} (\varphi_s^\gamma \Leftrightarrow \varphi_s^\alpha).$$

For $\alpha \in \mathrm{ORD}$, $\{\varphi_s^\alpha : s \in M^n, n \in \omega\}$ is said to be the collection of *α-types* *realized* by $\mathcal{M}$. The (*canonical*) *Scott sentence* of $\mathcal{M}$, $\sigma_{\mathcal{M}}$, is the conjunction of the sentences $\varphi_{\emptyset}^{\gamma}$, $\forall x_0 \cdots x_{n-1}[\varphi_s^{\gamma} \Rightarrow \varphi_s^{\gamma+1}]$ for all $s \in M^n$, $n \in \omega$ and $\gamma =$ Scott height of $\mathcal{M}$.

For countable $\mathcal{M}_0, \mathcal{M}_1$ we have that $\mathcal{M}_0 \cong \mathcal{M}_1$ iff $\mathcal{M}_0 \models \sigma_{\mathcal{M}_1}$. We will also make use of the following standard fact:

If $\mathcal{M}_0, \mathcal{M}_1$ are countable, have Scott height $\leq \gamma$ and realize the same γ-types, then $\mathcal{M}_0 \cong \mathcal{M}_1$. If moreover $a \in M_0$, $b \in M_1$ and $\varphi_a^{\gamma} = \varphi_b^{\gamma}$, then there is an isomorphism $j : \mathcal{M}_0 \cong \mathcal{M}_1$ with $j(a) = b$.

(D) Our notation and terminology from set theory is mostly standard, as for example in [Je]. We denote by $\omega_{\alpha}^{x_1,\dots,x_n}$ the αth admissible in $x_1, \dots, x_n$ ordinal, for $x_1, \dots, x_n \in \mathcal{N}$, and by ω_1^{CK} the first admissible ordinal. We often identify subsets of ω or ω_1 with their characteristic functions, so that, for example, $2^{<\omega_1}$ denotes interchangeably the set of bounded subsets of ω_1 or the set of α-sequences from $\{0, 1\}$ for any $\alpha < \omega_1$.

§**2. The case of group actions.** We first consider the case of an equivalence relation induced by a Polish group action. The main result is the following, proved also independently by Becker:

THEOREM 2.1. *Let G be a Polish group, $(g, x) \mapsto g \cdot x$ a Borel action of G on the Baire space $\mathcal{N}$, and E_G the associated equivalence relation. Then exactly one of the following holds*:

(I) *There is a map $U \colon \mathcal{N} \to 2^{<\omega_1}$ such that $xE_G y \Leftrightarrow U(x) = u(y)$, a parameter $z \in \mathcal{N}$ such that $U(x) \subseteq \omega_1^{x,z}$ and a formula φ of the language of set theory such that*

$$\xi \in U(x) \Leftrightarrow L_{\omega_2^{x,z}}[x, z] \models \varphi(\xi, x, z).$$

(II) $E_0 \sqsubseteq_c E_G$.

Moreover, (II) *is equivalent to*:

(II)$'$ *There exists an E_G-ergodic, nonatomic probability Borel measure.*

Let WO be the set of codes of infinite countable ordinals, i.e., $\mathrm{WO} \subseteq 2^{\omega}$ consists of all $x \in 2^{\omega}$ such that $<_x = \{(m, n) : x(\langle m, n\rangle) = 1\}$ is a wellordering of ω, where $\langle m, n\rangle$ is a fixed recursive bijection of ω^2 with ω. If $x \in \mathrm{WO}$, let $|x| =$ the ordinal isomorphic to $<_x$, and $f_x \colon \omega \to |x|$ the unique isomorphism. Any subset $a \subseteq \omega$ can be viewed as coding the subset $[(x, a)] = f_x[a] \subseteq |x|$. Thus the map $(x, a) \mapsto [(x, a)]$, where $x \in \mathrm{WO}$ and $a \subseteq \omega$, can be viewed as coding the elements of $2^{<\omega_1}$. The following is then an immediate corollary of 2.1.

COROLLARY 2.2. *Let G be a Polish group, X a standard Borel space and $(g, x) \mapsto g \cdot x$ a Borel action of G on X with associated equivalence relation E_G. Then exactly one of the following holds*:

(I) *There is a map $U \colon X \to 2^{<\omega_1}$ such that $xE_G y \Leftrightarrow U(x) = U(y)$ and U is C-measurable in the codes, i.e., there is a C-measurable map $U^* \colon X \to 2^{\omega} \times 2^{\omega}$ such that $U^*(x) \in \mathrm{WO} \times 2^{\omega}, \forall x$, and $[U^*(x)] = U(x)$.*

(II) $E_0 \sqsubseteq_c E_G$.

Again, as in 2.1, (II) *can be replaced by* (II)$'$.

We will now prove 2.1.

PROOF. First we show that (I) or (II) holds. We will use the following effective version of a result in [BK, 7.3.1].

LEMMA. *There is a transfinite sequence* (A_ξ), $\omega \leq \xi < \omega_1$, *of pairwise disjoint Borel sets with the following two properties*:

(i) $A_\xi \subseteq \mathcal{N}$, *each* A_ξ *is* E_G*-invariant, and* $\bigcup_\xi A_\xi = \mathcal{N}$.

(ii) *There is a parameter* $z \in \mathcal{N}$ *such that*:

(a) A_ξ *and* $E_G|A_\xi$ *are uniformly* $\Delta^1_1(z, w)$ *for any* $w \in \mathrm{WO}$, $|w| = \xi$,

(b) *There is a* $\Pi^1_1(z)$*-recursive partial function* $c\colon \mathcal{N} \times 2^\omega \to 2^\omega$ *such that, for each* x, $c(x, \mathcal{O}^{x,z})$ *is defined and in* WO, *where* $\mathcal{O}^{x,z}$ *is the complete* $\Pi^1_1(x,z)$ *subset of* ω, *and if* $\xi = |c(x, \mathcal{O}^{x,z})|$, *then* $x \in A_\xi$.

(The use of infinite ξ only is just a matter of technical convenience for some calculations below.)

We will assume this lemma and give its proof later. To simplify notation, we will also drop the parameter z for the rest of the proof.

So let us assume (II) fails and proceed to prove (I).

Since $E_0 \not\sqsubseteq_c E_G$, clearly $E_0 \not\sqsubseteq_c E_G|A_\xi$; thus $E_G|A_\xi$ is smooth. In fact, because of (ii)(a) and Theorem 1.4 in [HKL], we have the following: There is a Σ^1_1 set A and a Π^1_1 set B, $A, B \subseteq \omega \times \mathcal{N}^2$, such that for $w \in \mathrm{WO}$, $|w| = \xi$,

$$A(n, x, w) \Leftrightarrow B(n, x, w)$$

and if we let $S_n^w = \{x : A(n, x, w)\}$, then (S_n^w) is a separating family for $E_G|A_\xi$.

Consider now the space $(\xi)^\omega$ of all injections of ω into ξ. It is Polish, being a closed subspace of ξ^ω with the product topology. (Here $\xi \geq \omega$.) The basic neighborhoods of $(\xi)^\omega$ are of the form:

$$N_{(\xi_0,\dots,\xi_{k-1})} = \{f \in (\xi)^\omega : (\xi_0, \dots, \xi_{k-1}) \subseteq f\},$$

where $\xi_0, \dots, \xi_{k-1} < \xi$ are distinct. Put

$$S_n^{\xi,\xi_0,\dots,\xi_{k-1}} = \{x : \forall^* f \in N_{(\xi_0,\dots,\xi_{k-1})} (x \in S_n^{w_f})\},$$

where for $f \in (\xi)^\omega$, f onto, $w_f \in \mathrm{WO}$ is given by $w_f(\langle m, n\rangle) = 1 \Leftrightarrow f(m) < f(n)$, and "$\forall^*$" means "for comeager many". Note that $\forall^* f \in (\xi)^\omega (f \text{ is onto})$. We claim that $\{S_n^{\xi,\xi_0,\dots,\xi_{k-1}}\}$ is a separating family for $E_G|A_\xi$.

First, each $S_n^{\xi,\xi_0,\dots,\xi_{k-1}}$ is E_G-invariant: If $xE_G y$ and $x \in S_n^{\xi,\xi_0,\dots,\xi_{k-1}}$, then for comeager many $f \in N_{(\xi_0,\dots,\xi_{k-1})}$ we have $x \in S_n^{w_f}$, and since $S_n^{w_f}$ is E_G-invariant, the same holds for y. Next, let $x, y \in A_\xi$ and $\neg x E_G y$. Then for any $|w| = \xi$, there is n with $x \in S_n^w$ and $y \notin S_n^w$. So

$$\forall^* f \in (\xi)^\omega \exists n (x \in S_n^{w_f} \;\&\; y \notin S_n^{w_f}).$$

Since the map $f \mapsto w_f$ is Borel, the set $\{f \text{ is onto } : x \in S_n^{w_f} \;\&\; y \notin S_n^{w_f}\}$ is analytic, so by a standard category argument there are distinct $\xi_0, \dots, \xi_{k-1} < \xi$ and $n \in \omega$ such that

$$\forall^* f \in N_{(\xi_0,\dots,\xi_{k-1})} (x \in S_n^{w_f} \;\&\; y \notin S_n^{w_f}),$$

so that $x \in S_n^{\xi,\xi_0,\dots,\xi_{k-1}}$ and $y \notin S_n^{\xi,\xi_0,\dots,\xi_{k-1}}$.

Put

$$V(x) = \{(\xi, \xi_0, \ldots, \xi_{k-1}, n) : \xi_i < \xi, \xi_i \text{ distinct and } x \in S_n^{\xi, \xi_0, \ldots, \xi_{k-1}}\},$$

where ξ is such that $x \in A_\xi$. Then

$$xE_G y \Leftrightarrow V(x) = V(y).$$

Also by (ii)(b) it is clear that $\xi < \omega_2^x$ and $V(x)$ is uniformly definable from x in $L_{\omega_2^x}[x]$. For each $\xi < \omega_1$ let $\overline{\xi}$ be the cardinality of ξ in L_{ξ^+}, where ξ^+ is the least admissible ordinal $> \xi$. We can clearly encode $V(x)$ as a subset $U(x)$ of $\overline{\xi}$, where $x \in A_\xi$, so that again $U(x)$ is uniformly definable from x in $L_{\omega_2^x}[x]$, i.e., there is a formula φ of the language of set theory such that

$$\xi \in U(x) \Leftrightarrow L_{\omega_2^x}[x] \models \varphi(\xi, x).$$

So it only remains to check that $\overline{\xi} \leq \omega_1^x$. To see this it is enough to show that if $w \in \mathrm{WO}$ is such that $|w| = \omega_1^x$, then $\xi < \omega_1^w$. (Because this implies that if $\omega_1^x = \alpha$, then $\xi < \alpha^+$, so $\overline{\xi} \leq \alpha$.)

So fix $w \in \mathrm{WO}$ with $|w| = \omega_1^x$. Let $A = \{c(y, \mathscr{O}^y) : \omega_1^y \leq |w|\}$. Then $A \subseteq \mathrm{WO}$, $A \in \Sigma_1^1(w)$ (since $\omega_1^y \leq |w| \Rightarrow \mathscr{O}^y \in \Delta_1^1(y, w)$) and $c(x, \mathscr{O}^x) \in A$. By boundedness $\xi = |c(x, \mathscr{O}^x)| \leq \sup\{|v| : v \in A\} < \omega_1^w$, and we are done.

So it only remains to give the

Proof of the lemma. Let $\mathscr{F}(G)$ be the standard Borel space of closed subsets of G with the Effros Borel structure. Denote by $\mathcal{S}(G)$ the Borel subset of $\mathscr{F}(G)$ consisting of the closed subgroups of G. This is also a standard Borel space. The group G acts in a Borel way on $\mathcal{S}(G)$ by conjugation:

$$(g, F) \mapsto gFg^{-1}.$$

Note that if G_x denotes the stabilizer of x in the G-action on X, then $G_x \in \mathcal{S}(G)$ and

$$G_{g \cdot x} = gG_x g^{-1}.$$

Let

$$P = \{(x, G_x) : x \in X\} \subseteq X \times \mathcal{S}(G).$$

Then P is a $\mathbf{\Pi}_1^1$ subset of $X \times \mathcal{S}(G)$. Consider the product action of G on $X \times \mathcal{S}(G)$,

$$g \cdot (x, F) = (g \cdot x, gFg^{-1}).$$

Clearly P is invariant under this action. So by a result of Solovay (see, e.g., 34.6 in [Ke]) there is a $\mathbf{\Pi}_1^1$-rank $\varphi\colon P \to \omega_1$ which is also invariant under this action and, moreover, (i) there is a $\mathbf{\Pi}_1^1$-measurable function $g : \mathscr{N} \times \mathcal{S}(G) \to 2^\omega$ with domain P such that for $(x, G_x) \in P$ we have $g(x, G_x) \in \mathrm{WO}$ and $|g(x, G_x)| = \varphi(x, G_x)$, and (ii) the set $P^\xi = \{(x, G_x) \in P : \varphi(x, G_x) \leq \xi\}$ is uniformly $\mathbf{\Delta}_1^1$ in any code of ξ, i.e., there are $K, L \subseteq \mathcal{N} \times \mathcal{S}(G) \times 2^\omega$ in $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$ respectively, so that, for $|w| = \xi$,

$$(x, G_x) \in P^\xi \Leftrightarrow K(x, G_x, w) \Leftrightarrow L(x, G_x, w).$$

Now, let $P_\xi = \{(x, G_x) \in P : \varphi(x, G_x) = \xi\}$. Then P_ξ is Borel and, in fact, by (i) and (ii) above it is uniformly $\mathbf{\Delta}^1_1$ in any code of ξ.

Put $A_\xi = \mathrm{proj}_X(P_\xi) = \{x : (x, G_x) \in P_\xi\}$. Since P_ξ is the graph of a function, A_ξ is also Borel and clearly invariant under the G-action on X. Also $x \mapsto G_x$ is Borel on A_ξ, so by a standard fact (see, e.g., [Ke]) $E_G|A_\xi$ is Borel as well. Moreover A_ξ and $E_G|A_\xi$ are uniformly $\mathbf{\Delta}^1_1$ in any code of ξ.

Now fix a basis $\{V_n\}$ for G, and for $x \in \mathcal{N}$ let

$$a_x = \{n : V_n \cap G_x \neq \emptyset\} \in 2^\omega.$$

Clearly

$$A = \{(n, x) : n \in a_x\} \subseteq \omega \times \mathcal{N}$$

is $\mathbf{\Sigma}^1_1$. Moreover there is a Borel function $f : \mathcal{N} \times 2^\omega \to \mathcal{S}(G)$, with $f(x, a_x) = G_x$, namely

$$f(x, y) = F \Leftrightarrow \forall n(F \cap V_n \neq \emptyset \Leftrightarrow y(n) = 1).$$

Let also $B, C \subseteq \mathcal{N} \times 2^\omega$ and $R, S \subseteq \mathcal{N}^2 \times 2^\omega$ be such that $B, R \in \mathbf{\Sigma}^1_1$ and $C, S \in \mathbf{\Pi}^1_1$, and, for $w \in \mathrm{WO}$,

$$x \in A_{|w|} \Leftrightarrow B(x, w) \Leftrightarrow C(x, w),$$
$$x, y \in A_{|w|} \;\&\; xE_G y \Leftrightarrow R(x, y, w) \Leftrightarrow S(x, y, w).$$

Now choose the parameter z so that $A, B, R \in \Sigma^1_1(z)$, $C, S \in \Pi^1_1(z)$, $f \in \Delta^1_1(z)$ and the partial function $h(x, y) = g(x, f(x, y))$ is $\Pi^1_1(z)$-recursive.

Then (ii)(a) is clearly satisfied. For (ii)(b), note that a_x is uniformly recursive in $\mathcal{O}^{x,z}$ so there is a $\Pi^1_1(z)$-recursive function $c : \mathcal{N} \times 2^\omega \to 2^\omega$ such that $c(x, \mathcal{O}^{x,z}) = h(x, a_x) = g(x, f(x, a_x)) = g(x, G_x)$, and this clearly works.

Next, we show that we cannot have both (I) and (II). It is easy to check that the assertion: "For every Polish group G and every Borel action of G on $\mathcal{N}$, (I) and (II) cannot both hold" is a Π^1_3 sentence. So it is enough to prove it assuming $MA + \neg CH$; so, in particular, all $\mathbf{\Sigma}^1_2$ sets are universally measurable.

So assume (I) and (II) hold and $f : 2^\omega \to \mathcal{N}$ is an embedding verifying (II). Let μ be the usual measure on 2^ω and $\nu = f\mu$. Then ν is E_G-ergodic and nonatomic. Put

$$X_\xi = \{x \in \mathcal{N} : \xi \in U(x)\}.$$

Then X_ξ is E_G-invariant and in the class C. Moreover,

$$xE_G y \Leftrightarrow \forall \xi < \omega_1 (x \in X_\xi \Leftrightarrow y \in X_\xi).$$

Let

$$Y_\xi = \begin{cases} X_\xi & \text{if } \nu(X_\xi) = 0, \\ \sim X_\xi, & \text{if } \nu(X_\xi) = 1. \end{cases}$$

Then $\mu(Y_\xi) = 0$, $\forall \xi$, and $\bigcup_\xi Y_\xi$ is the complement of an E_G-equivalence class, so has measure 1. On the other hand, consider the prewellordering (on $\bigcup_\xi Y_\xi$)

$$x \leq y \Leftrightarrow \exists \xi (x \in Y_\xi \,\&\, \forall \eta < \xi (y \notin Y_\eta)).$$

It is easy to check that it is $\mathbf{\Sigma}^1_2$, so ν-measurable. By the usual Fubini argument (see, e.g., 17.14 in [Ke]) it follows that $\bigcup_\xi Y_\xi$ has ν-measure 0, a contradiction.

The same argument shows that one cannot have both (I) and (II)$'$, so this shows the equivalence of (II) and (II)$'$, and the proof is complete. ⊣

Let us point out that the preceding proof also shows the following fact, which provides another equivalent of (I) in Theorem 2.1. In order to state it in a succinct form, we will introduce the following terminology:

Let G be a Polish group, X a standard Borel spae, and $(g, x) \mapsto g \cdot x$ a Borel action of G on X with associated equivalence relation E_G. A transfinite sequence $(A_\xi)_{\xi<\omega_1}$ of pairwise disjoint sets $A_\xi \subseteq X$ is called *acceptable* if

(i) A_ξ is E_G-invariant, with $\bigcup_\xi A_\xi = X$.

(ii) A_ξ and $E_G|A_\xi$ are Borel, and

(iii) the prewellordering

$$x \leq y \Leftrightarrow \exists \xi (x \in A_\xi \,\&\, \forall \eta < \xi (y \notin A_\eta))$$

is in the class C.

The preceding lemma implies the existence of such acceptable sequences. We then have the following.

COROLLARY 2.3. *Let G be a Polish group, X a Polish space, and $(g, x) \mapsto g \cdot x$ a Borel action of G on X. Then for any acceptable sequence $(A_\xi)_{\xi<\omega_1}$, the following are equivalent*: (I) *of* 2.2, *and*

(I)* *For any $\xi < \omega_1$, $E_G|A_\xi$ is smooth.*

PROOF. If (I) holds, then $\neg E_0 \sqsubseteq_c E_G$; thus $\neg E_0 \sqsubseteq_c E_G|A_\xi$ for all ξ, so $E_G|A_\xi$ is smooth. Conversely, if (I) fails, then $E_0 \sqsubseteq_c E_G$. Let $f : 2^\omega \to X$ be an embedding witnessing this, μ the usual measure on 2^ω, and $\nu = f\mu$. Then ν is E_G-ergodic and nonatomic. We claim that $\nu(A_\xi) = 1$ for some ξ. Otherwise $\nu(A_\xi) = 0$ for all ξ, so, by the usual Fubini argument, $\nu(\bigcup_\xi A_\xi) = \nu(X) = 0$, a contradiction. Thus ν is $E_G|A_\xi$-ergodic and nonatomic, so $E_G|A_\xi$ is not smooth. ⊣

Consider now the special case of the logic action. Let L be a countable language and σ an $L_{\omega_1\omega}$ sentence. We call σ *Ulm-classifiable* if alternative (I) of 2.1 (or 2.2) holds for $\cong |\mathrm{Mod}(\sigma)$, i.e., the countable infinite models of σ can be classified up to isomorphism by Ulm-type invariants, i.e., to each $x \in \mathrm{Mod}(\sigma)$ we can assign in a reasonably definable way an invariant which is essentially a countable length transfinite sequence of zeros and ones. Call σ *concretely classifiable* if $\cong |\mathrm{Mod}(\sigma)$ is Borel and smooth, i.e., models of σ can be classified up to isomorphism by invariants, computed again in a reasonably definable way, which are essentially infinite sequences of zeros and ones (or equivalently members of some Polish space).

For convenience, for each $\xi < \omega_1$ let σ_ξ be an $L_{\omega_1\omega}$ sentence whose countable models are exactly the countable L-structures of Scott height ξ. Then we have the following equivalences, of which (iii) and (iv) give a purely model-theoretic way of expressing Ulm-classifiability. Note also that the equivalence of (iii) and (iv) expresses a purely model-theoretic result about $L_{\omega_1\omega}$ for which we do not know an independent proof.

THEOREM 2.4. *Let σ be an $L_{\omega_1\omega}$ sentence. Then the following are equivalent*:

(i) *σ is Ulm-classifiable.*

(ii) *For each ξ, $\sigma \wedge \sigma_\xi$ is concretely classifiable.*

(iii) *Every complete $L_{\omega_1\omega}$-theory T containing σ and some σ_ξ satisfies the following compactness property*: *If every countable subset of T has a model, then T has a countable model.*

(iv) *Same as* (iii), *but with "T has a countable model" replaced by "T has a model".*

(v) *If a probability Borel measure on the space of structures of L (with universe ω) satisfies the* 0-1 *law for $L_{\omega_1\omega}$ sentences (i.e., every such sentence is true a.e. or false a.e.) and σ is true a.e., then there is a countable model $\mathscr{M}$ of σ so that the measure concentrates on the isomorphism class of $\mathscr{M}$.*

PROOF. If A_ξ denotes the set of (countably infinite) models of σ of Scott height ξ, then (A_ξ) is acceptable (for the logic action of S_∞ on the models of σ). Noticing that $\mathrm{Mod}(\sigma \wedge \sigma_\xi) = A_\xi$, we immediately obtain the equivalence of (i), (ii), and (v) (by the argument in 2.3). Now assume that (ii) holds; we prove (iii). Fix such a $T \supseteq \{\sigma, \sigma_\xi\}$.

Let $A = \mathrm{Mod}(\sigma \wedge \sigma_\xi)$ and, for any Borel subset $B \subseteq A$ invariant under isomorphism, let σ_B be an $L_{\omega_1\omega}$ sentence with $\mathrm{Mod}(\sigma_B) = B$. This σ_B is uniquely determined up to logical equivalence. So put

$$B \in \mathscr{U} \Leftrightarrow \sigma_B \in T.$$

Assuming every countable subset of T has a model, $\mathscr{U}$ is a countably complete ultra-filter on the isomorphism-invariant Borel subsets of A. Since $\cong |A$ is smooth, it follows easily that $\mathscr{U}$ contains a single isomorphism class, i.e., for some $\mathscr{A}_x \models \sigma \wedge \sigma_\xi$ if $B = \{y : \mathscr{A}_x \cong \mathscr{A}_y\}$ then $B \in \mathscr{U}$, i.e., the Scott sentence $\sigma_{\mathscr{A}_x} \in T$, so $\mathscr{A}_x \models T$.

Conversely, assume (ii) fails and let μ be an E-ergodic, nonatomic probability measure on $A = \mathrm{Mod}(\sigma \wedge \sigma_\xi)$ for some ξ, where $E = \cong |A$. For $\tau \in L_{\omega_1\omega}$ put

$$\tau \in T \Leftrightarrow \mu(\mathrm{Mod}(\tau)) = 1.$$

Then $\{\sigma, \sigma_\xi\} \supseteq T$, T is complete (by ergodicity), and every countable subset of T has a model. On the other hand, if $\mathscr{A}_x \models T$, then the Scott sentence $\sigma_{\mathscr{A}_x} \in T$, so $\mu(\{y : \mathscr{A}_x \cong \mathscr{A}_y\}) = 1$, contradicting nonatomicity. Thus T has no countable model. So (iii) fails.

Finally we show that $\neg$(ii) implies $\neg$(iv). This is immediate from the following lemma that seems interesting in its own right.

LEMMA. *Let σ be an $L_{\omega_1\omega}$ sentence. Let μ be an ergodic nonatomic probability Borel measure for $\cong |\mathrm{Mod}(\sigma)$. Put*

$$T_\mu = \{\tau \in L_{\omega_1\omega} : \mu(\mathrm{Mod}(\tau)) = 1\}.$$

Then T_μ is a complete theory in $L_{\omega_1\omega}$ containing σ such that every countable subset of T_μ has a model but T_μ has no model.

PROOF. As before there is $\xi_0 < \omega_1$ such that $\mu(\mathrm{Mod}(\sigma \wedge \sigma_{\xi_0})) = 1$.

For any L-structure $\mathscr{M}$ and $\alpha \in \mathrm{ORD}$, let

$$\mathrm{tp}_\alpha(\mathscr{M}) = \{\varphi_s^\alpha : s \in M^n, n \in \omega\}$$

be the set of α-types realized in $\mathscr{M}$. By López-Escobar's result, for each $\alpha < \omega_1$ and $L_{\omega_1\omega}$ formula φ we can construct an $L_{\omega_1\omega}$ sentence θ_φ^α such that

$$\{x \in \mathrm{Mod}(\sigma) : \varphi \in \mathrm{tp}_\alpha(\mathscr{A}_x)\} = \{x \in \mathrm{Mod}(\sigma) : \mathscr{A}_x \models \theta_\varphi^\alpha\}$$

and moreover this equality also holds in any generic extension of the universe. It follows that for any (even uncountable) model $\mathscr{M}$ of σ we have $\varphi \in \mathrm{tp}_\alpha(\mathscr{M}) \Leftrightarrow \mathscr{M} \models \theta_\varphi^\alpha$. (Just make $\mathscr{M}$ countable in a generic extension of the universe.)

Now put, for $\alpha < \omega_1$,

$$\mathrm{tp}_\alpha(T_\mu) = \{\varphi \in L_{\omega_1\omega} : \mu(\{x \in \mathrm{Mod}(\sigma) : \varphi \in \mathrm{tp}_\alpha(\mathscr{A}_x)\}) = 1\}.$$

Claim. $\mathrm{tp}_\alpha(T_\mu)$ *is countable for all* $\alpha < \omega_1$.

Proof. Otherwise let $\varphi^\gamma \in \mathrm{tp}_\alpha(T_\mu)$, $\gamma < \omega_1$, be such that $\gamma \neq \delta \Rightarrow \varphi^\delta \neq \varphi^\gamma$. Let $V[G]$ be a ccc forcing extension of V such that $V[G] \models MA(\aleph_1)$. Since $\mu(\{x \in \mathrm{Mod}(\sigma) : \varphi^\gamma \in \mathrm{tp}(\mathscr{A}_x)\}) = 1$ also holds in $V[G]$ by absoluteness, it follows by $MA(\aleph_1)$ that $\mu(\bigcap_{\gamma<\omega_1}\{x \in \mathrm{Mod}(\sigma) : \varphi^\gamma \in \mathrm{tp}(\mathscr{A}_x)\}) = 1$ holds in $V[G]$, so there is $x \in \mathrm{Mod}(\sigma)^{V[G]}$ such that, in $V[G]$, $\mathscr{A}_x$ realizes φ^γ for all $\gamma < \omega_1$, a contradiction as $\mathscr{A}_x$ is countable.

Now fix an enumeration $\{\varphi_\alpha^{(i)} : i \in \omega\}$ of $\mathrm{tp}_\alpha(T_\mu)$, for any $\alpha < \omega_1$. We claim that, for some $\alpha < \omega_1$,

$$\mu(\{x \in \mathrm{Mod}(\sigma) : \mathrm{tp}_\alpha(\mathscr{A}_x) = \{\varphi_\alpha^{(i)} : i \in \omega\}\}) = 0.$$

Indeed, if $\alpha = \xi_0$ and $\mu(\{x \in \mathrm{Mod}(\sigma) : \mathrm{tp}_\alpha(\mathscr{A}_x) = \{\varphi_\alpha^{(i)} : i \in \omega\}\}) = 1$, then since $\mu(\mathrm{Mod}(\sigma \wedge \sigma_\alpha)) = 1$ it follows that for μ-almost all $x \in \mathrm{Mod}(\sigma \wedge \sigma_\alpha)$ we have $\mathrm{tp}_\alpha(\mathscr{A}_x) = \{\varphi_\alpha^{(i)} : i \in \omega\}$, so that, for some $x \in \mathrm{Mod}(\sigma \wedge \sigma_\alpha)$ and μ-almost all $y \in \mathrm{Mod}(\sigma \wedge \sigma_\alpha)$, $\mathscr{A}_x$ and $\mathscr{A}_y$ realize the same α-types and have Scott height α; thus they are isomorphic, i.e., $\mu(\{y : \mathscr{A}_x \cong \mathscr{A}_y\}) = 1$, a contradiction.

So let α_0 be the least α such that

$$\mu(\{x \in \mathrm{Mod}(\sigma) : \mathrm{tp}_\alpha(\mathscr{A}_x) = \{\varphi_\alpha^{(i)} : i \in \omega\}\}) = 0.$$

Suppose now $\mathscr{M} \models T_\mu$, towards a contradiction. Using the preceding observations about θ_φ^α and a simple simultaneous induction on α, we can easily get the following facts:

(i) If $\alpha < \alpha_0$, $s \in M^n$, $n \in \omega$, then $\varphi_s^\alpha \in \{\varphi_\alpha^{(i)} : i \in \omega\}$.

(ii) If $\alpha \leq \alpha_0$, $s \in M^n$, $n \in \omega$, then $\varphi_s^\alpha \in L_{\omega_1\omega}$.

By the choice of α_0 now it follows that there is $s \in M^n$, $n \in \omega$, with $\varphi_s^{\alpha_0} \notin \{\varphi_{\alpha_0}^{(i)} : i \in \omega\}$. So if $\varphi = \varphi_s^{\alpha_0}$, then

$$\mu(\{x \in \mathrm{Mod}(\sigma) : \varphi \in \mathrm{tp}_{\alpha_0}(\mathscr{A}_x)\}) = \mu(\{x \in \mathrm{Mod}(\sigma) : \mathscr{A}_x \models \theta_\varphi^{\alpha_0}\}) = 0.$$

Thus $\neg\theta_\varphi^{\alpha_0} \in T_\mu$, and so $\mathscr{M} \models \neg\theta_\varphi^{\alpha_0}$, i.e., $\varphi \notin \mathrm{tp}_{\alpha_0}(\mathscr{M})$, a contradiction. ⊣ ⊣

There are many examples of sentences σ of $L_{\omega_1\omega}$ which have the property that $\sigma \models \sigma_\xi$ for some $\xi < \omega_1$. In fact, by a result in [BK] this happens exactly when

$\cong|\mathrm{Mod}(\sigma)$ is Borel. So, for example, it holds for any σ implying the axioms for torsion free groups of finite rank or the axioms for locally finite graphs, etc. For such sentences σ, in (ii) and (iii) above one can drop any reference to σ_ξ.

Finally we will prove a stronger version of 2.2 in the case when G is abelian. (An effective version, as in 2.1, can be also formulated, but we will leave it to the reader.)

THEOREM 2.5. *Let G be an abelian Polish group and $(g,x)\mapsto g\cdot x$ a Borel action of G on a standard Borel space X with associated equivalence relation E_G. Then exactly one of the following holds*:

(I) *There is a C-measurable map $U\colon X\to 2^\omega$ such that $xE_Gy \Leftrightarrow U(x)=U(y)$.*

(II) $E_0 \sqsubseteq_c E_G$.

PROOF. By the results in [BK] we can assume that X is Polish and the action is continuous.

For $x\in X$ let $G_x=\{g : g\cdot x = x\}$ be the *stabilizer* of x. We will describe an "inductive analysis" of the stabilizer of x which can be viewed as a (somewhat loose) analog of the Scott analysis of a countable structure. This analysis works even if G is nonabelian, but commutativity is needed to establish a key invariance property.

DEFINITION 2.6. Suppose G is a Polish group acting continuously on a Polish space $X\colon (g,x)\mapsto g\cdot x$. Fix a countable basis $\mathscr{B}$ for G and a compatible complete metric d for G. For $x\in X$, put

$$\begin{aligned} G_x^0 &= \{W\in\mathscr{B} : \exists (g_i)_{i\in\omega}\subseteq W(g_i\cdot x \text{ converges to } x)\},\\ G_x^{\alpha+1} &= \{W\in\mathscr{B} : \forall\epsilon>0\exists V\in\mathscr{B}(\overline{V}\subseteq W \;\&\; \mathrm{diam}(V)<\epsilon \;\&\; V\in G_x^\alpha)\},\\ G_x^\lambda &= \bigcap_{\alpha<\lambda} G_x^\alpha, \quad \text{if } \lambda \text{ is limit.}\end{aligned}$$

It is easy to check, using the definitions, that $\alpha\le\beta\Rightarrow G_x^\alpha\supseteq G_x^\beta$, so for some least countable ordinal $\alpha(x)$ we have $G_x^{\alpha(x)}=G_x^\alpha$ $\forall\alpha\ge\alpha(x)$. A simple argument also shows the basic fact that, for $W\in\mathscr{B}$,

$$W\cap G_x\neq\emptyset \Leftrightarrow W\in G_x^{\alpha(x)}.$$

Thus, if we identify the closed subgroup G_x with $\{W\in\mathscr{B} : W\cap G_x\neq\emptyset\}$, this shows that $\{G_x^\alpha\}_{\alpha\le\alpha(x)}$ provides an "inductive analysis" of G_x.

We now have the following crucial invariance property:

LEMMA 2.7. *Let G, X be as in Definition 2.6, but assume now that moreover G is abelian. Then*

$$xE_Gy\Rightarrow\forall\alpha(G_x^\alpha=G_y^\alpha).$$

PROOF. It is clearly enough to show that $xE_Gy\Rightarrow G_x^0=G_y^0$. So fix $W\in G_x^0$ in order to show that $W\in G_y^0$. Let $\{g_i\}\subseteq W$, and $g\in G$ be such that $g_i\cdot x\to x$ and $g\cdot x=y$. By continuity, we have

$$g\cdot(g_i\cdot x)\to g\cdot x.$$

But $g\cdot(g_i\cdot x)=gg_i\cdot x=g_ig\cdot x=g_i\cdot(g\cdot x)$ by commutativity, so $g_i\cdot y\to y$; thus $W\in G_y^0$. ⊣

We can use this now to establish the following key decomposition.

LEMMA 2.8. *Let G be abelian Polish, X a Polish space, and $(g,x) \mapsto g \cdot x$ a continuous action of G on X. Then there is a C-measurable E_G-invariant function $D\colon X \to 2^\omega$ such that $D^{-1}(\{y\})$ and $E_G|D^{-1}(\{y\})$ are Borel for each $y \in 2^\omega$, and moreover there is a $\mathbf{\Delta}^1_1$-measurable function C with domain $D[X]$ such that, for $y \in D[X]$, $C(y)$ is a Borel code of $E_G|D^{-1}(\{y\})$.*

PROOF. In the notation of 2.6, let $\mathscr{B} = \{W_i\}_{i\in\omega}$ be an enumeration of the basis $\mathscr{B}$, and let $D(x)$ be a member of 2^ω that encodes the following prewellordering in some straightforward fashion: $i \leq^x j \Leftrightarrow \forall\alpha(W_i \in G_x^\alpha \Rightarrow W_j \in G_x^\alpha)$. Note that in particular $D(x)$ encodes $G_x^{\alpha(x)} = \{i : W_i \cap G_x \neq \emptyset\}$. Clearly D is E_G-invariant by 2.7. To see that it is C-measurable, note that, fixing a Borel injection $g\colon X \to 2^\omega$, we can easily check that, for some $z \in 2^\omega$ and all $x \in X$, $\alpha(x) \leq \omega_1^{g(x),z}$. Fixing $y \in D[X]$, we see again that if $D(x) = y$, then $\alpha(x) \leq$ (length of the prewellordering coded by y), from which it follows that $D^{-1}(\{y\})$ is Borel, and a Borel code of $D^{-1}(\{y\})$ can be computed uniformly in a $\mathbf{\Delta}^1_1$ way from y. Finally, from y one can compute in a uniform $\mathbf{\Delta}^1_1$ way$\{i : W_i \cap G_x \neq \emptyset\} = \{i : W_i \cap G_x^{\alpha(x)} \neq \emptyset\}$ for any x with $D(x) = y$ (this is independent of x), and therefore, again in a uniform $\mathbf{\Delta}^1_1$ way, a Borel code for a Borel transversal T_y of G_x, i.e., a Borel set meeting every (left) coset of G_x in exactly one point.

Since for $x, x' \in D^{-1}(\{y\})$

$$xE_Gx' \Leftrightarrow \exists g \in T_y(g \cdot x = x') \Leftrightarrow \exists! g \in T_y(g \cdot x = x'),$$

it follows that in a uniform $\mathbf{\Delta}^1_1$ way we can find from y a Borel code of $E_G|D^{-1}(\{y\})$, and the proof of the lemma is complete. ⊣

To finish the proof of 2.5 we can now argue as follows:

Let D, C be as in 2.8. If for some $y \in D(x)$ we have $E_0 \sqsubseteq_c E_G|D^{-1}(\{y\})$, then clearly $E_0 \sqsubseteq_c E_G$ and (II) holds. Otherwise this fails for all $y \in D(x)$, so by the result in [HKL], which holds uniformly, and using C, we can find a $\mathbf{\Delta}^1_1$-measurable function U' with domain $D[X]$ such that, for each $y \in D[X]$, $U'(y)$ is a code of a Borel function $f_y\colon X \to 2^\omega$ such that for $x, x' \in D^{-1}(\{y\})$,

$$xE_Gx' \Leftrightarrow f_y(x) = f_y(x').$$

Now for $x \in X$ we let $U(x) = \langle D(x), f_{D(x)}(x)\rangle$ (where $\langle\,\rangle$ is a Borel bijection of $2^\omega \times 2^\omega$ and 2^ω). Then U is C-measurable and $xE_Gx' \Leftrightarrow U(x) = U(x')$, so alternative (I) holds. ⊣

Theorem 2.5 has been improved first by Hjorth, who proved that (I) of 2.5 holds with U actually Borel, and then by Solecki, who showed that if G is a Polish group admitting an invariant compatible metric and $(g \cdot x) \mapsto g \cdot x$ is a continuous action of G on a Polish space X with associated equivalence relation E_G, then either (I) E_G is G_δ (so (I) of 2.5 holds with U actually Borel), or (II) $E_0 \sqsubseteq_c E_G$.

Although 2.5 has been improved on, we still consider the above proof to have independent interest. Lemmas 2.7 and 2.8 provide an analysis of the stabilizer function for abelian Polish group actions. For instance, it can be used to show that in the presence of projective determinacy, we have the topological Vaught's conjecture for abelian Polish groups even in the projective context.

THEOREM 2.9. *Assume projective determinacy. Let G be an abelian Polish group, X a standard Borel space, and $(g \cdot x) \mapsto g \cdot x$ a Borel action of G on X with E_G the associated equivalence relation. Let $Y \subseteq X$ be projective. Then $E_G|Y$ has either countably many or perfectly many equivalence classes.*

We do not give the proof of this theorem. However, it follows easily from 2.8, basic facts about determinacy as found in [Mo], and the methods of [St] or the proof of 4.5 in [Ha].

§**3. Embedding E_0 in analytic equivalence relations.** We prove here some results that are needed in the next section. Since they appear to be of independent interest, it seems best to present them separately.

The following fact, whose proof is related to that of the Effros theorem [Ef1], was noticed in [BK, 3.4.5]: Let X be a perfect Polish space and G a group acting by homeomorphisms on X with associated equivalence relation E_G. If there is a dense orbit and E_G is meager (in X^2), then $E_0 \sqsubseteq_c G$.

We note here a slight variation of this fact, for which we need the following concept.

DEFINITION 3.1. Let X be a Polish space and $E \subseteq F \subseteq X^2$, with E an equivalence relation. We write

$$E_0 \sqsubseteq_c (E, F)$$

if there is an embedding $f : 2^\omega \to X$ with

$$xE_0y \Rightarrow f(x)Ef(y), \qquad \neg xE_0y \Rightarrow \neg f(x)Ff(y).$$

(So $E_0 \sqsubseteq_c E \Leftrightarrow E_0 \sqsubseteq_c (E, E)$.)

We now have:

THEOREM 3.2. *Let X be a perfect Polish space, and G a group acting by homeomorphisms on X with associated equivalence relation E_G. Let $E_G \subseteq F \subseteq X^2$, and assume there is a dense orbit and F is meager. Then $E_0 \sqsubseteq_c (E_G, F)$.*

The proof is identical to that of Theorem 3.4.5 in [BK], so we omit it here.

This has the following corollary.

COROLLARY 3.3. *Let X be a perfect Polish space, and G a group acting by homeomorphisms on X with associated equivalence relation E_G. Let $E_G \subseteq F \subseteq X^2$, and assume there is a dense orbit and F is an equivalence relation with the Baire property such that F is not comeager. Then $E_0 \sqsubseteq_c (E_G, F)$.*

PROOF. It is enough to show that F is meager. Consider the group H of homeomorphisms of X^2 of the form $(x, y) \mapsto (g \cdot x, g' \cdot y)$, where $g, g' \in G$. If U, V are nonempty open sets in X^2, there is $h \in H$ such that $h(U) \cap V \neq \emptyset$, as follows easily from the fact that there is a dense G-orbit. By the usual topological 0-1 law (see [Ke, 8.46]) it follows that every $A \subseteq X^2$ which has the Baire property and is invariant under H is either meager or comeager. But easily F is invariant under H, being an equivalence relation containing E_G, and the proof is complete. ⊣

Now let E be an analytic equivalence relation on a Polish space X and $E \subseteq F \subseteq X^2$, where F is a coanalytic relation. Burgess [Bu] showed that there is a Borel equivalence relation E^* such that $E \subseteq E^* \subseteq F$. We show here that if $\neg E_0 \sqsubseteq_c E$, then we can also ensure that $\neg E_0 \sqsubseteq_c E^*$.

THEOREM 3.4. *Let E be a $\boldsymbol{\Sigma}^1_1$ equivalence relation on a Polish space X and $E \subseteq F \subseteq X^2$, where F is $\boldsymbol{\Pi}^1_1$. If $\neg E_0 \sqsubseteq_c E$, then there is a smooth Borel equivalence relation E^* such that $E \subseteq E^* \subseteq F$.*

PROOF. We will use the so-called second or strong reflection theorem (see [HMS] or [Ke]), which reads as follows:

Let W be a Polish space and $\Phi \subseteq \mathscr{P}(W) \times \mathscr{P}(W)$, where $\mathscr{P}(W)$ is the power set of W. Assume:

(i) Φ is hereditary, i.e., $\Phi(A, B)$ & $A' \subseteq A$ & $B' \subseteq B \Rightarrow \Phi(A', B')$;

(ii) Φ is continuous upward in the second variable, i.e., if $B_n \subseteq B_{n+1}$ and $\bigcup_n B_n = B$, then $\forall n \Phi(A, B_n) \Rightarrow \Phi(A, B)$; and

(iii) Φ is $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$, i.e., if Y, Z are Polish spaces and $A \subseteq Y \times W$, $B \subseteq Z \times W$ are $\boldsymbol{\Sigma}^1_1$, then the set $\{(y, z) \in Y \times Z : \Phi(A_y, B_z)\}$ is $\boldsymbol{\Pi}^1_1$.

Then, for any $A \subseteq W$, $A \in \boldsymbol{\Sigma}^1_1$,

$$\Phi(A, \sim A) \Rightarrow \exists B \supseteq A (B \in \boldsymbol{\Delta}^1_1 \text{ and } \Phi(B, \sim B)).$$

Now take $W = X^2$ and consider the Φ given below, where $A, B \subseteq X^2$:

$$\begin{aligned}
\Phi(A, B) \Leftrightarrow {} & \text{(i) } A \subseteq F \,\& \\
& \text{(ii) } \forall x (x, x) \notin B \,\& \\
& \text{(iii) } \forall x \forall y [(x, y) \in A \Rightarrow (y, x) \notin B] \,\& \\
& \text{(iv) } \forall x \forall y \forall z [(x, y) \in A \;\&\; (y, z) \in A \Rightarrow (x, z) \notin B] \,\& \\
& \text{(v) } \forall \text{ embedding } f\colon 2^\omega \to X [\forall^*_{2^\omega} a \forall b E_0 a \forall c E_0 a (f(b), f(c)) \in A \\
& \qquad\qquad \Rightarrow \forall^*_{2^\omega \times 2^\omega} (b, c) (f(b), f(c)) \notin B],
\end{aligned}$$

where "$\forall^*_Y y \cdots y \cdots$" means "for comeager many $y \in Y, \cdots y \cdots$".

It is clear that Φ is hereditary, and it is easy to see that Φ is continuous upward in the second variable (using the fact that $\forall n \forall^*_X x (x \in C_n) \Rightarrow \forall^*_X x \forall n (x \in C_n)$). Finally, by the standard fact that $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$ are each closed under the $\forall^*_Y$ quantifiers for Polish Y (see, e.g., 29.22 in [Ke]), it follows that Φ is $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$.

We claim now that $\Phi(E, \sim E)$ holds. This is clear for (i)–(iv). For (v), assume an embedding $f\colon 2^\omega \to X$ is a counterexample, towards a contradiction. Then there is a comeager G_δ E_0-invariant set $G \subseteq 2^\omega$ such that

$$b, c \in G \;\&\; b E_0 c \Rightarrow f(b) E f(c).$$

(To see this, notice that there is a countable group of homeomorphisms of 2^ω inducing E_0, and thus every comeager set contains a G_δ E_0-invariant set.) Put

$$E' = \{(b, c) \in G^2 : f(b) E f(c)\},$$

so that $E_0 | G \subseteq E' \subseteq G^2$ and, since $\neg \forall^*_{2^\omega \times 2^\omega} (b, c)[(f(b), f(c)) \in E]$, we have that E' is not comeager. It follows from 3.2 that $E_0 \sqsubseteq_c (E_0 | G, E')$, say via g, i.e., $g\colon 2^\omega \to G$ is an embedding and $x E_0 y \Rightarrow g(x) E_0 g(y)$, $\neg x E_0 y \Rightarrow \neg x E' y$. Let $h = f \circ g$. Then $h\colon 2^\omega \to X$ is an embedding and $x E_0 y \Leftrightarrow h(x) E h(y)$, i.e., $E_0 \sqsubseteq_c E$, a contradiction.

So by second reflection, there is a Borel $E^* \supseteq E$ such that $\Phi(E^*, \sim E^*)$, so in particular $E^* \subseteq F$, by (i), and E^* is an equivalence relation, by (ii)–(iv). We claim that E^* is smooth. Otherwise, there is an embedding $f: 2^\omega \to X$ such that $xE_0y \Leftrightarrow f(x)E^*f(y)$, and this clearly violates (v). ⊣

Burgess [Bu] derives from his reflection property that any $\boldsymbol{\Sigma}_1^1$ equivalence relation can be written canonically as the intersection of a decreasing sequence of ω_1 Borel equivalence relations. We have the analogous fact in our context.

COROLLARY 3.5. *Let X be a Polish space, E a $\boldsymbol{\Sigma}_1^1$ equivalence relation on X, and $\varphi\colon \sim E \to \omega_1$ a $\boldsymbol{\Pi}_1^1$-rank. Let $E_{(\xi)} = \sim\{(x,y)\colon \varphi(x,y) < \xi\}$, for $\xi < \omega_1$, so that $(E_{(\xi)})$ is decreasing, each $E_{(\xi)}$ is Borel and $E = \bigcap_{\xi<\omega_1} E_{(\xi)}$. Then, if $\neg E_0 \sqsubseteq_c E$, the set $\{\xi : E_{(\xi)}$ is a smooth Borel equivalence relation$\}$ is closed unbounded in ω_1.*

PROOF. Since the intersection of a sequence of smooth Borel equivalence relations is smooth, it is enough to show that the above set is unbounded. So fix $\eta < \omega_1$. Since $E \subseteq E_{(\eta)}$, by 3.3, there is a smooth Borel equivalence relation $E^{(1)}$ with $E \subseteq E^{(1)} \subseteq E_{(\eta)}$. Then $\sim E^{(1)} \subseteq \sim E$, so, by boundedness, $\sim E^{(1)} \subseteq \{(x,y) \notin E : \varphi(x,y) < \eta_1\}$ for some $\eta_1 > \eta$; thus $E^{(1)} \supseteq E_{(\eta_1)}$. By repeating this argument, we can find a sequence $\eta < \eta_1 < \eta_2 < \cdots$ and a sequence $E^{(i)}$ of smooth Borel equivalence relations such that

$$E_{(\eta)} \supseteq E^{(1)} \supseteq E_{(\eta_1)} \supseteq E^{(2)} \supseteq E_{(\eta_2)} \supseteq \cdots.$$

Let $\xi = \lim_n \eta_n$. Then $\xi > \eta$, and

$$E_{(\xi)} = \bigcap_n E_{(\eta_n)} = \bigcap_n E^{(n)}$$

is a smooth Borel equivalence relation. ⊣

§4. **$\boldsymbol{\Sigma}_1^1$ Equivalence relations with Borel classes.** For the more general case of arbitrary $\boldsymbol{\Sigma}_1^1$ equivalence relations it seems necessary to pass to a more complex type of reduction, namely $\boldsymbol{\Delta}_1 = \bigcup_{x\in\mathcal{N}} \Delta_1(x)$ (in the language of set theory). Of course these include the C-measurable functions, and a $\boldsymbol{\Delta}_1$ function with domain and range a Polish space will be $\boldsymbol{\Delta}_2^1$. We will consider in this section the case when all equivalence classes are Borel. The main result is 4.4, but before proving it we will need three useful lemmas.

LEMMA 4.1. *Let E be a Σ_1^1 equivalence relation on $\mathcal{N}$. Then $\{w \in \mathrm{WO} : E_{(|w|)}$ is a smooth equivalence relation$\}$ is Π_1^1, where $E_{(\xi)}$ is as in 3.4 with φ a Π_1^1-rank on $\sim E$, $\varphi\colon \sim E \to \omega_1$.*

PROOF. Given $w \in \mathrm{WO}$, $E_{(|w|)}$ is uniformly $\Delta_1^1(w)$, and so, by 1.4 of [HKL], if $E_{(|w|)}$ is smooth, then there is a $\Delta_1^1(w)$ function $f: \mathcal{N} \to \mathcal{N}$ such that $xE_{(|w|)}y \Leftrightarrow f(x) = f(y)$. Thus $E_{(|w|)}$ is a smooth equivalence relation iff $E_{(|w|)}$ is an equivalence relation and $\exists f \in \Delta_1^1(w) \forall x,y \in \mathcal{N}(xE_{(|w|)}y \Leftrightarrow f(x) = f(y))$, which is clearly Π_1^1. ⊣

LEMMA 4.2. *Let E be a Σ_1^1 equivalence relation on $\mathcal{N}$. There is a function $f: \mathcal{N} \times \omega_1 \to 2^{<\omega_1}$ with $f(x,\xi) \subseteq \xi$, such that if $E_{(\xi)}$ is a smooth equivalence relation and $\xi \geq \omega$, then*

$$xE_{(\xi)}y \Leftrightarrow f(x,\xi) = f(y,\xi),$$

and, moreover, if $\xi^{+(x)}$ is the least x-admissible $> \xi$, then $f(x,\xi)$ is uniformly Δ_1

definable in $L_{\xi^{+(x)}}[x]$, i.e., there are Σ_1 formulas φ, ψ such that, for ξ as above,

$$\beta \in f(x,\xi) \Leftrightarrow L_{\xi^{+(x)}}[x] \models \varphi(\beta,\xi,x) \Leftrightarrow L_{\xi^{+(x)}}[x] \models \neg\psi(\beta,\xi,x).$$

PROOF. Fix $\xi \geq \omega$ such that $E_{(\xi)}$ is a smooth equivalence relation. Then by 1.4 of [HKL] we can find for any $w \in \mathrm{WO}$, $|w| = \xi$, a family $(R_n^w)_{n\in\omega}$, separating for $E_{(\xi)}$, such that (R_n^w) is uniformly Δ_1^1 in n and w. For $\xi_0, \cdots, \xi_{k-1} < \xi$, put (in the notation of §2)

$$R_n^{\xi,\xi_0,\ldots,\xi_{k-1}} = \{x : \forall^* f \in N_{(\xi_0,\ldots,\xi_{k-1})}(x \in R_n^{w_f})\}.$$

By standard facts on admissible sets it follows that there are Σ_1 formulas φ', ψ' such that

$$\begin{aligned} x \in R_n^{\xi,\xi_0,\ldots,\xi_{k-1}} &\Leftrightarrow L_{\xi^{+(x)}}[x] \models \varphi'(\xi,(\xi_0,\ldots,\xi_{k-1},n),x) \\ &\Leftrightarrow L_{\xi^{+(x)}}[x] \models \neg\psi'(\xi,(\xi_0,\ldots,\xi_{k-1},n)x). \end{aligned}$$

Let $H : \xi \to \xi^{<\omega}$ be a map whose range consists of all $(\xi_0,\ldots,\xi_{k-1},n)$ with $\xi_0,\ldots,\xi_{k-1} < \xi$ distinct and $n \in \omega$, such that H is uniformly Δ_1 definable from ξ in L_{ξ^+}. Let

$$f(x,\xi) = \{\beta < \xi : x \in S_n^{\xi,\xi_0,\ldots,\xi_{k-1}}, \text{ where } H(\beta) = (\xi_0,\ldots,\xi_{k-1},n)\}.$$

Then

$$\begin{aligned} \beta \in f(x,\xi) &\Leftrightarrow L_{\xi^{+(x)}}[x] \models \varphi'(\xi,H(\beta),x) \\ &\Leftrightarrow L_{\xi^{+(x)}}[x] \models \psi'(\xi,H(\beta),x), \end{aligned}$$

so that $f(x,\xi)$ is uniformly Δ_1 definable in $L_{\xi^{+(x)}}[x]$. Finally,

$$xE_{(\xi)}y \Leftrightarrow f(x,\xi) = f(y,\xi)$$

by an argument similar to that in the proof of 2.1. ⊣

The following is a well-known application of the boundedness theorem for Π_1^1-ranks.

LEMMA 4.3. *Let E be a Σ_1^1 equivalence relation on $\mathcal{N}$. Suppose $[x]_E$ is Borel. Then, for some $\xi < \omega_1$, $[x]_E = [x]_{E_{(\xi)}}$.*

We now have the main theorem for this section.

THEOREM 4.4. *Let E be a Σ_1^1 equivalence relation on $\mathcal{N}$. Suppose that $\forall x([x]_E$ is Borel). Then one of the following holds:*

(I) *There is $U : \mathcal{N} \to 2^{<\omega_1}$ which is Δ_1 (in the language of set theory) and $xEy \Leftrightarrow U(x) = U(y)$.*

(II) $E_0 \sqsubseteq_c E$.

PROOF. Assume (II) fails. Let f be as in 4.2. For $x \in \mathcal{N}$, let $A_x = \{\langle \beta, \xi \rangle : E_{(\xi)}$ is a smooth equivalence relation $\& \beta < \xi \ \& \ \beta \in f(x, \xi)\}$, where $\langle \rangle$ is a Δ_1 pairing function on the ordinals mapping ω_1^2 onto ω_1. Note that by 3.5

$$xEy \Leftrightarrow A_x = A_y.$$

Below $\mathbb{P}$ denotes notions of forcing, τ denotes terms in the forcing language, and $\dot{g}$ is a name for the generic object. If $\mathbb{P} \in N$, a model of a fragment of ZF, then $\Vdash_{\mathbb{P}}^N$ denotes forcing over N.

Claim. There are η, $\theta < \omega_1$ *such that* $M = L_\eta[A_x \cap \theta] \models ZFC_N$ *(a fixed large finite fragment of ZFC) and* $\mathbb{P}$, $\tau \in M$, $p \in \mathbb{P}$ *such that*

(i) $(p, p) \Vdash_{\mathbb{P}\times\mathbb{P}}^M \tau(\dot{g}_1) E \tau(\dot{g}_2)$, *and*

(ii) *there is an* M*-generic* g *for* $\mathbb{P}$ *above* p *such that* $\tau(g)Ex$.

Granting this claim, let η_x, θ_x be least such that this claim holds for some $\mathbb{P}, \tau, p$ and then let $\mathbb{P}_x, \tau_x, p_x$ be least in the canonical wellordering of $L_{\eta_x}[A_x \cap \theta_x]$ satisfying the above.

Let $U'(x) = (\eta_x, \theta_x, A_x \cap \theta_x, \mathbb{P}_x, \tau_x, p_x)$. Then clearly $xEy \Rightarrow U'(x) = U'(y)$. Conversely, if $U'(x) = U'(y)$, then $M = L_{\eta_x}[A_x \cap \theta_x] = L_{\eta_y}[A_y \cap \theta_y]$, $\mathbb{P} = \mathbb{P}_x = \mathbb{P}_y$, $\tau = \tau_x = \tau_y$, $p = p_x = p_y$ and $(p, p) \Vdash_{\mathbb{P}\times\mathbb{P}}^M \tau(\dot{g}_1) E \tau(\dot{g}_2)$, while there exist M-generics g_1, g_2 for $\mathbb{P}$ above p such that $\tau(g_1)Ex$ and $\tau(g_2)Ey$. Let g^* be $M[g_1]$- and $M[g_2]$-generic above p, so that $\tau(g_1)E\tau(g^*)$ and $\tau(g_2)E\tau(g^*)$; thus $\tau(g_1)E\tau(g_2)$ and xEy.

We verify next that U' is a Δ_1 function. This follows from 4.1 and 4.2 together with the fact that, in the presence of (i), (ii) of the claim is a Δ_1 condition since it is equivalent to:

(ii)$'$ $\forall M$-generic g for $\mathbb{P}$ above p we have $\tau(g)Ex$.

(This is seen exactly as in the just preceding argument.)

Finally, via the canonical wellordering of $L_{\eta_x}[A_x \cap \theta_x]$, $\mathbb{P}_x, \tau_x, p_x$ can be viewed as ordinals $< \eta_x$; so, via some simple Δ_1 coding, we can view $U'(x)$ as a bounded subset of ω_1, say $U(x)$, and this completes the proof, modulo the claim.

Proof of the Claim. Let $\omega \leq \xi < \omega_1$ be such that $E_{(\xi)}$ is a smooth equivalence relation and $[x]_E = [x]_{E_{(\xi)}}$, and choose $\eta, \theta > \xi$ large enough so that $L_\eta[A_x \cap \theta] \models ZFC_N$. Let $\mathbb{P}$ be the collapse of ξ to ω and let g be M-generic for $\mathbb{P}$. We claim that there is $y \in \mathcal{N} \cap M[g]$ such that $\forall \beta < \xi$, if $H(\beta) = (\xi_0, \dots, \xi_{k-1}, n)$, then

$$y \in R_n^{\xi, \xi_0, \dots, \xi_{k-1}} \Leftrightarrow \langle \beta, \xi \rangle \in A_x.$$

This follows from the fact that if $w \in \mathrm{WO} \cap M[g]$ is such that $|w| = \xi$, then (since $\{\beta < \xi : \langle \beta, \xi \rangle \in A_x\}$ is in $M[g]$) the set of y's satisfying the above condition is Σ_1^1 in some parameter $z \in M[g]$ and has a solution (namely x) in V, so has a solution $y \in M[g]$. Now let τ be a term such that $y = \tau(g)$, and p a condition such that g meets p and

$$p \Vdash_{\mathbb{P}}^M \forall \beta < \xi (\text{if } H(\beta) = (\xi_0, \dots, \xi_{k-1}, n),$$
$$\text{then } \tau(g) \in R_n^{\xi, \xi_0, \dots, \xi_{n-1}} \Leftrightarrow \langle \beta, \xi \rangle \in A_x).$$

(Note that this is a statement in the forcing language of M by the argument in the proof of 4.2.) Thus (ii) is clearly satisfied.

Finally, we verify (i). If (g_1, g_2) are M-generic for $\mathbb{P} \times \mathbb{P}$ above (p, p), then clearly $\tau(g_1)E_{(\xi)}x$ and $\tau(g_2)E_{(\xi)}x$, so $\tau(g_1)Ex$ and $\tau(g_2)Ex$; thus $\tau(g_1)E\tau(g_2)$. ⊣

A relativized version of the preceding arguments gives as usual a result for general Polish spaces and $\mathbf{\Sigma}^1_1$ equivalence relation with Borel classes.

THEOREM 4.5. *Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space X. Suppose every equivalence class is Borel. Then one of the following holds*:

(I) *There is a map $U\colon X \to 2^{<\omega_1}$ such that $xEy \Leftrightarrow U(x) = U(y)$ and U is $\mathbf{\Delta}^1_2$ in the codes* (*in the sense of* 2.2).

(II) $E_0 \sqsubseteq_c E$.

As in the proof of 2.1 we have that *exactly one* of (I), (II) holds assuming that every $\mathbf{\Sigma}^1_2$ set is measurable (e.g., if $\aleph_1^{L[x]} < \aleph_1$, $\forall x \in \mathcal{N}$). In this case (II) is equivalent to (II)$'$as in 2.1.

§5. General $\mathbf{\Sigma}^1_1$ equivalence relations.

DEFINITION 5.1. Assume $x^\#$ exists. Then, for v a linear ordering, $\Gamma(x^\#, v)$ indicates the canonical model obtained by expanding $x^\#$ along indiscernibles $(c_i)_{i\in v}$. (A detailed discussion of this construction can be found in §30 of [Je].)

For $\alpha \in \mathrm{ORD}$, let i^x_α be the αth Silver indiscernible for $L[x]$, and let i_α be the αth Silver indiscernible for L.

LEMMA 5.2. *Suppose $x^\#$ exists, where $x \in \mathcal{N}$. Let $A \subseteq \mathrm{ORD}$ be a class of ordinals definable over $L[x]$ from the parameter x. Let v be some countable ill-founded linear ordering with limit type. Then*:

(*a*) *The Scott height of $M = (L[A]; \epsilon, A)^{\Gamma(x^\#, v)}$ is less than or equal to i^x_β, where β is the well-founded part of v*;

(*b*) *M realizes the same i^x_β-types as $(L[A]; \epsilon, A)^{\Gamma(x^\#, \beta+\omega)}$.*

A proof of this lemma can be found in 1.1 of [Hj1]; a discussion of the ideas needed for the proof can also be found in [Hj]. It follows from (b) that M will be an ω-model.

COROLLARY 5.3. *Suppose x and A are as in* 5.2 *above. Then there exist $\alpha < \beta < \omega_1$ such that*

$$(L_\alpha[x]; \epsilon, A \cap \alpha, x) \prec (L_\beta[x]; \epsilon, A \cap \beta, x) \prec (L[x]; \epsilon, A, x),$$

and there exists an ill-founded model N such that

(i) $|N| = \aleph_0$,

(ii) $\mathcal{L}(N) = (\epsilon, A)$,

(iii) *N has well-founded ordinals isomorphic to α*,

(iv) *N has Scott height less than or equal to α*,

(v) *N realizes the same α-types as $(L_\beta[A]; \epsilon, A \cap \beta)$*,

(vi) *for some countable ordering v, $N \cong (L[A]; \epsilon, A)^{\Gamma(x^\#, v)}$.*

PROOF. Let γ be any countable ordinal. Let $\alpha = i_\gamma$ and$\beta = i_{\gamma+\omega}$. The first part of the corollary is then a well-known consequence of indiscernibility.

Now for $v = \gamma + \mathbb{Q}$ we obtain (i)–(vi) by 5.2. ⊣

It is important to note that in the situation of 5.3, the set of α-types realized by N will exist in $L[A]$, since $L[A]$ can determine which α-types are realized by $L_\beta[A]$. In this sense, a *description* of N exists in $L[A]$. In fact, by Σ^1_1 absoluteness (see, for instance, Section 41 of [Je]) in any transitive model of ZFC_N in which $L_\beta[A]$ is countable, there must exist an isomorphic copy of N, since the existence of such a model is $\boldsymbol{\Sigma}^1_1$ in any code for α and the collection of α-types.

Our invariant will consist of assigning to each $x \in \mathcal{N}$ the description of some N and $a \in N$ as in 5.3. We will first obtain $A_x \subseteq \mathrm{ORD}$ so that A_x is uniformly definable, from x, over $L[x]$, and A_x depends only on $[x]_E$. Then we try to find a canonical ill-founded N, as in 5.3, which somehow captures the equivalence class of x. Assuming this all to be possible, we attempt to assign some bounded subset of the ordinals of $L[A_x]$ that canonically codes the description of N and the method by which it entraps $[x]_E$.

THEOREM 5.4. *Suppose that* $\forall x \in \mathcal{N}$ $(x^\#$ *exists*). *Let* E *be a* Σ^1_1 *equivalence relation on* $\mathcal{N}$. *Then exactly one of the following holds*:

(I) *There exists* $U\colon \mathcal{N} \to 2^{<\omega_1}$ *such that* U *is* Δ_1, *and* $\forall x, y \in \mathcal{N}(xEy \Leftrightarrow U(x) = U(y))$.

(II) $E_0 \sqsubseteq_c E$.

PROOF. We first prove that at least one of (I) and (II) must hold. Since E is Σ^1_1, we can find a recursive tree T such that E is the projection of $[T]$. We can then fix the Π^1_1-rank on $\sim E$, $\varphi\colon \sim E \to \omega_1$, where for $(x, y) \notin E$, $\varphi(x, y)$ is the rank of $T(x, y) = \{s \in \omega^{<\omega} : (s, x \upharpoonright \mathrm{lh}(s), y \upharpoonright \mathrm{lh}(s)) \in T\}$. Following the notation of §3, we then have $xE_{(\xi)}y$ iff $T(x, y)$ is ill-founded or $T(x, y)$ has rank $\geq \xi$. We view $E_{(\xi)}$ as defined by this even for $\xi \geq \omega_1$. Now suppose that $E_0 \not\sqsubseteq_c E$.

Following the argument from 4.4, we can assign to each $x \in \mathcal{N}$ a set $A_x \subseteq \omega_1$ such that $\forall y \in \mathcal{N}(xEy \Rightarrow A_x = A_y)$, and moreover, for all $\xi < \omega_1$ with $E_{(\xi)}$ a smooth equivalence relation, $A_x(\xi) =_{\mathrm{df}} \{\beta : \langle \beta, \xi\rangle \in A_x\} = A_y(\xi)$ iff $xE_{(\xi)}y$. We now pass from A_x to a new class of ordinals; it will be convenient to remove the parameter ω_1 from consideration. Let $A^*_x \subseteq \mathrm{ORD}$ be defined for each $x \in \mathcal{N}$ so that

(i) A^*_x is definable over $L[x]$ using x as the only parameter, and

(ii) $A^*_x \cap \omega_1 = A_x$.

Since $A_x(\xi)$ was defined in a uniformly $\Delta_1(x, \xi)$ fashion for any $\xi < \omega_1$, there exists a canonical choice for this class of ordinals. Since ω_1 is a uniform indiscernible, it follows that there is a unique choice of A^*_x satisfying (i) and (ii). Again by indiscernibility, but this time over $L[x, y]$, we have that $A_x = A_y$ implies $A^*_x = A^*_y$. It then follows from (i), (ii), and the previously established properties of A_x that, for all $x, y \in \mathcal{N}$,

$$xEy \Leftrightarrow A^*_x = A^*_y.$$

So the assignment $x \mapsto A^*_x$ is an invariant for the equivalence classes of E. We would be done if there existed some $x^* \in [x]_E \cap L[A^*_x]$, since we could assign to each $[x]_E$ the least such real in the canonical wellorder of $L[A^*_x]$. However, by the examples of §6, this would be overly optimistic.

In 4.4 we essentially took the approach of specifying some $p, \mathbb{P}, \tau \in L[A^*_x]$, with $p \Vdash^V_{\mathbb{P}} \tau[\dot{g}]Ex$, and then, in effect, using $(p, \mathbb{P}, \tau)$ as the invariant. While it is possible to argue that such $(p, \mathbb{P}, \tau)$ must exist in $L[A^*_x]$ even in the present circumstances, we will take an alternative route. We will assign as our invariant the *description* of some such $(p, \mathbb{P}, \tau)$.

Fix $x \in \mathcal{N}$.

Claim.

$$\forall \xi < \omega_1 \exists p, \mathbb{P}, \tau \in L[A_x^*](p \Vdash_{\mathbb{P}}^{L[x]} \tau[\dot{g}] E_{(\xi)} x).$$

Proof. This follows as in the proof of 4.4. ⊣

Thus, by indiscernibility of ω_1 over $L[x]$,

$$L[x] \models \text{“}\forall \xi \in \mathrm{ORD} \exists p, \mathbb{P}, \tau \in L[A_x^*](p \Vdash_{\mathbb{P}}^{L[x]} \tau[\dot{g}] E_{(\xi)} x)\text{”}.$$

Now applying 5.3 to x and A_x^* we obtain $\alpha < \beta < \omega_1$ such that

$$(L_\alpha[x]; \epsilon, A_x^* \cap \alpha, x) \prec (L_\beta[x]; \epsilon, A_x^* \cap \beta, x) \prec (L[x]; \epsilon, A_x^*, x),$$

and there is an ill-founded countable M, elementarily equivalent to $(L_\beta[x]; \epsilon, A_x^* \cap \beta, x)$, such that for $N = (L[A_x^*]; \epsilon, A_x^*)^M$ we have

(i) N has Scott height $\leq \alpha$, and

(ii) N realizes the same α-types as $(L_\beta[A_x^*]; \epsilon, A_x^* \cap \beta)$.

So, in particular, by the preceding claim, there exists some ill-founded $\xi^* \in (\mathrm{ORD})^M = (\mathrm{ORD})^N$ and $p, \mathbb{P}, \tau \in N$ such that $p \Vdash_{\mathbb{P}}^M \tau[\dot{g}] E_{(\xi^*)} x$. ⊣

Claim. $p \Vdash_{\mathbb{P}}^V \tau[\dot{g}] E x$.

Proof. We may assume $p \Vdash_{\mathbb{P}}^M \tau[\dot{g}] E x$ or $p \Vdash_{\mathbb{P}}^M \neg \tau[\dot{g}] E x$. Let us prove the claim in the second case; the proof in the first case is basically the same, only slightly easier.

It follows from the forcing theorem that if g is an M-generic filter on $\mathbb{P}$ below p, then in $M[g]$ there is an order-preserving map from ξ^* into $T(\tau[g], x)$ equipped with the Kleen-Brouwer ordering; it follows from M being an ω-model that $\tau[g]$ is a real and $T(\tau[g], x)$ is calculated correctly by $M[g]$. Hence $T(\tau[g], x)$ is ill-founded, and so $\tau[g] E x$. Now if g_0 is a V-generic filter on $\mathbb{P}$, then in particular it is M-generic, since all dense sets in M are also dense in V; thus, if $g_0 \subseteq \mathbb{P}$ is a V-generic filter containing p, then $\tau[g_0] E x$. Now the claim follows by the forcing theorem applied over V. ⊣

Notice, moreover, by remarks following 5.3, whenever $G \subseteq \mathrm{Coll}(\omega, (\beta^+)^{L_{\omega_1}[x]})$ is $L_{\omega_1}[x]$-generic, such an M and N exist in $L_{\omega_1}[x][G]$. The preceding calculation then gives $p \Vdash_{\mathbb{P}}^{L_{\omega_1}[x][G]} \tau[\dot{g}] E x$.

Now let γ_x be the least $\gamma \in \mathrm{ORD}$ such that there exists $\bar{\alpha} < \bar{\beta} < \gamma$ and $q, \mathbb{Q} \in L_\gamma[x]$ for which

(i) $L_\gamma[x] \models ZFC_N$, and

(ii) for any $G \subseteq \mathbb{Q}$ $L_\gamma[x]$-generic with $q \in G$, there exist $\overline{N} \in L_\gamma[x][G]$ and $(\bar{\tau}, \bar{p}, \bar{\mathbb{P}}) \in \bar{N}$ such that

  (iia) $\bar{p} \Vdash_{\bar{\mathbb{P}}}^{L_\gamma[x][G]} \bar{\tau}[\dot{g}] E x$;

  (iib) $\bar{N}$ has Scott height $\bar{\alpha}$;

  (iic) $\bar{N}$ realizes the same $\bar{\alpha}$-types as $(L_{\bar{\beta}}[A_x^*]; \epsilon, A_x^* \cap \bar{\beta})$.

Let $(\bar{\alpha}_x, \bar{\beta}_x)$ be the least such $(\bar{\alpha}, \bar{\beta})$ in some canonical wellordering of $\mathrm{ORD} \times \mathrm{ORD}$. Now let $(C_\alpha^x)_{\alpha < \bar{\beta}_x}$ be some canonical enumeration in $L_{\bar{\beta}_x \cdot \omega}[A_x^*]$ of the $\bar{\alpha}_x$-types realized in $(L_{\bar{\beta}_x}[A_x^*]; \epsilon, A_x^* \cap \bar{\beta}_x)$. Let D^x be the least $\bar{\alpha}_x$-type realized in $(L_{\bar{\beta}_x}[A_x^*]; \epsilon, A_x^* \cap \bar{\beta}_x)$ such that, for some $\mathbb{Q}_x, q_x \in L_{\gamma_x}[x]$, if $G \subseteq \mathbb{Q}_x$ is $L_{\gamma_x}[x]$-generic with $q_x \in G$ then there exists $(\bar{\tau}_x, \bar{p}_x, \bar{\mathbb{P}}_x)$ as in (iia)–(iic) above, with D^x

the $\bar{\alpha}_x$-type of $(\bar{\tau}_x, \bar{p}_x, \bar{\mathbb{P}}_x)$. Since we may assume without loss of generality that the condition $q \in \mathbb{Q}$ in (ii) above has the further property of deciding the $\bar{\alpha}$-type of $(\bar{\tau}, \bar{p}, \bar{\mathbb{P}})$, such a set D^x must exist. Now let $U(x)$ be a canonical code of $((C^x_\alpha)_{\alpha<\bar{\beta}_x}, D^x)$ inside $L_{\bar{\beta}_x\cdot\omega}[A^*_x]$.

It is immediate from the construction that $U(x)$ is uniformly Δ_1 in x. It is also immediate that $U(x)$ is a bounded subset of ω_1. It only remains to show that $U(x)$ is a complete invariant for $[x]_E$.

Claim. $xEy \Rightarrow U(x) = U(y)$.

Proof. Suppose instead that $(\bar{\alpha}_x, \bar{\beta}_x)$ is less than $(\bar{\alpha}_y, \bar{\beta}_y)$ in the canonical wellordering of ORD $\times$ ORD, but xEy. Let G_0 be $L_{\gamma_x}[x]$-generic; let $\bar{N}_x \in L_{\gamma_x}[x][G_0]$, with $\bar{N}_x$ having Scott height $\bar{\alpha}_x$, and realizing the same $\bar{\alpha}_x$-types as $(L_{\bar{\beta}_x}[A^*_x], \in, A^*_x \cap \bar{\beta}_x)$; let $(\bar{\tau}_x, \bar{p}_x, \bar{\mathbb{P}}_x) \in \bar{N}_x$ with

$$\bar{p}_x \Vdash^{L_{\gamma_x}[x][G_0]}_{\bar{\mathbb{P}}_x} \bar{\tau}_x[\dot{g}]Ex.$$

Now by the remarks following 5.3, if $G_1 \subseteq \mathrm{Coll}(\omega, \bar{\beta}_x)$ is $L_{\gamma_y}[y]$-generic, then $\exists \bar{N}_y \in L_{\gamma_y}[y][G_1]$, $(\bar{\tau}_y, \bar{p}_y, \bar{\mathbb{P}}_y) \in \bar{N}_y$, such that $\bar{N}_y$ has Scott height $\bar{\alpha}_x$, realizes the same $\bar{\alpha}_x$-types as $(L_{\bar{\beta}_x}[A^*_x], \in, A^*_x \cap \bar{\beta}_x)$, and hence $\bar{N}_x$, and such that $(\bar{\tau}_y, \bar{p}_y, \bar{\mathbb{P}}_y)$ has the same $\bar{\alpha}_x$-type as $(\bar{\tau}_x, \bar{p}_x, \bar{\mathbb{P}}_x)$. Thus there is an isomorphism $j : \bar{N}_x \cong \bar{N}_y$ with $j(\bar{\tau}_x, \bar{p}_x, \bar{\mathbb{P}}_x) = (\bar{\tau}_y, \bar{p}_y, \bar{\mathbb{P}}_y)$, and thus for any sufficiently generic $h \subseteq \bar{\mathbb{P}}$ with $\bar{p}_y \in h$ we have that $j^{-1}[h]$ is $L_{\gamma_x}[x][G_0]$-generic below $\bar{p}_x$ and hence $xE\bar{\tau}_x[j^{-1}[h]]$; but since $\bar{\tau}_x[j^{-1}[h]] = \bar{\tau}_y[h]$ and xEy, we have $yE\bar{\tau}_y[h]$ and hence $L_{\gamma_y}[y][G_1][h] \models$"$yE\bar{\tau}_y[h]$", by Σ^1_1 absoluteness for transitive models. Thus, by the forcing theorem,

$$\bar{p}_y \Vdash^{L_{\gamma_y}[y][G_1]}_{\bar{\mathbb{P}}_y} \bar{\tau}_y[\dot{h}]Ey,$$

contradicting that $(\bar{\alpha}_x, \bar{\beta}_x)$ is smaller than $(\bar{\alpha}_y, \bar{\beta}_y)$ in the canonical wellordering of ORD $\times$ ORD.

Similarly we obtain $((C^x_\alpha)_{\alpha<\bar{\beta}_x}, D^x) = ((C^y_\alpha)_{\alpha<\bar{\beta}_y}, D^y)$, and $U(x) = U(y)$, as required in order to prove the claim. ⊣

Claim. $U(x) = U(y) \Rightarrow xEy$.

Proof. Suppose $U(x) = U(y)$, $\bar{N}_x \in L_{\gamma_x}[G_0]$, $\bar{N}_y \in L_{\gamma_y}[y][G_1]$, $(\bar{\tau}_x, \bar{p}_x, \bar{\mathbb{P}}_x) \in \bar{N}_x$, $(\bar{\tau}_y, \bar{p}_y, \bar{\mathbb{P}}_y) \in \bar{N}_y$, $\bar{p}_x \Vdash^{L_{\gamma_x}[x][G_0]}_{\bar{\mathbb{P}}_x} \bar{\tau}_x[\dot{g}]Ex$, $\bar{p}_y \Vdash^{L_{\gamma_y}[y][G_1]}_{\bar{\mathbb{P}}_y} \bar{\tau}_y[\dot{g}]Ey$, and, by our supposition, that there exists $j : \bar{N}_x \cong \bar{N}_y$, $j(\bar{\tau}_x, \bar{p}_x, \bar{\mathbb{P}}_x) = (\bar{\tau}_y, \bar{p}_y, \bar{\mathbb{P}}_y)$. But now if $h \subseteq \mathbb{P}_y$ is sufficiently generic below the condition $\bar{p}_y$, then we have $\bar{\tau}_x[j^{-1}[h]] = \bar{\tau}_y[h]$, $xE\bar{\tau}_x[j^{-1}[h]]$, $yE\bar{\tau}_y[h]$, and hence xEy, as required for the claim. ⊣

To complete the proof of the theorem, we must show that (I) and (II) are mutually exclusive; however, this follows just as in 2.1, since the assumption that $\forall x \in \mathcal{N}$ ($x^\#$ exists) gives that all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable. ⊣

For the usual reasons, the preceding argument can be generalized and lifted to arbitrary Polish spaces.

THEOREM 5.5. *Let X be a Polish space, and let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on X. Suppose $\forall x \in \mathcal{N}$ ($x^\#$ exists). Then exactly one of the following holds*:

(I) *There is a function* $U: X \to 2^{<\omega_1}$, $\mathbf{\Delta}^1_2$ *in the codes, such that* $\forall x, y \in X(xEy \Leftrightarrow U(x) = U(y))$.

(II) $E_0 \sqsubseteq_c E$. ⊣

We do not know how to prove that at least one of (I) and (II) must hold in 5.4 or 5.5 without making use of the assumption of sharps; nor do we have any reason for believing that one needs anything beyond ZFC. However, it is clear that ZFC alone is insufficient to obtain an actual dichotomy. For instance, if $V = L$, then there is a Δ_1 reduction of E_0 into $\Delta(2^{<\omega_1})$, for pathological reasons.

As in the earlier case of 2.1, assuming $\forall x \in \mathcal{N}$ ($x^\#$ exists), we have that (II) is equivalent to (II)$'$.

§**6. Examples and counterexamples.** In this section we present counterexamples to some putative dichotomy theorems that might be suggested for analytic equivalence relations.

EXAMPLE 6.1. Assume $\forall x \in \mathcal{N}(\omega_1^{L[x]} < \omega_1)$. Then there is a Σ^1_1 equivalence relation E such that

(I) $E \not\leq_{\mathbf{\Delta}^1_2} \Delta(2^\omega)$;

(II) $E_0 \not\leq_{\mathbf{\Delta}^1_2} E$.

PROOF. For $x, y \in 2^\omega$, set xEy if and only if either (i) $x, y \notin \mathrm{WO}$, or (ii) $x, y \in \mathrm{WO}$ and $|x| = |y|$.

It is clear that E is Σ^1_1. If $E \leq_{\mathbf{\Delta}^1_2} \Delta(2^\omega)$ then there exists a $\mathbf{\Sigma}^1_2$ set carrying a $\mathbf{\Sigma}^1_2$ wellordering of order type ω_1, violating our hypothesis. If, on the other hand, $E_0 \leq_{\mathbf{\Delta}^1_2} E$, then let $f: 2^\omega \to 2^\omega$ be $\mathbf{\Delta}^1_2$ and such that $\forall x, y(xE_0y \Leftrightarrow f(x)Ef(y))$. Let $X_\xi = \{x \in 2^\omega : f(x) \in \mathrm{WO}$ and $|f(x)| \leq \xi$, or $f(x) \notin \mathrm{WO}\}$, for each $\xi < \omega_1$. Since each X_ξ is countable, we obtain that it has Lebesgue measure zero.

Now, recall first that we have $\mathbf{\Sigma}^1_2$ Lebesgue measurability by our assumption, and second that the induced prewellordering, $x \leq y$ iff $\forall \xi < \omega_1(y \in X_\xi \Rightarrow x \in X_\xi)$, is $\mathbf{\Delta}^1_2$. Thus, by the same Fubini argument used in 2.1, we obtain that $\bigcup_\xi X_\xi$ has Lebesgue measure zero, and therefore that 2^ω is null; this is absurd. ⊣

In the case of E induced by the Borel action of a Polish group, it is still possible to give a counterexample along the lines of 6.1, thus showing the use of transfinite ordinals to be necessary. One actually obtains a counterexample for the logic action. Before presenting the example, it will be helpful to recall some fundamental properties of Ulm invariants for torsion abelian groups. Let p be some fixed prime number.

THEOREM 6.2. *Let L be the language of groups, and let* $P \subseteq X_L$ *consist of the abelian p-groups. Then there is a* Δ_1 *function* $U: P \to (\omega \cup \{\infty\})^{<\omega_1}$ *such that*:

(I) $\forall x, y \in P(\mathscr{A}_x \cong \mathscr{A}_y \Leftrightarrow U(x) = U(y))$.

(II) *If* $\alpha < \omega_1$, *and* $f: \alpha + 1 \to \omega \cup \{\infty\}$ *is given by* $f(0) = \infty$, *and* $f(\beta) = 1$ *for* $\beta \neq 0$, $\beta \leq \alpha$, *then there exists* $x \in P$ *with* $U(x) = f$. ⊣

This theorem is a standard result in abelian group theory; see, for example [BE].

COROLLARY 6.3. *If* $\forall x \in \mathcal{N}(\omega_1^{L[x]} < \omega_1)$, *then there is a Borel action of* S_∞, *with induced equivalence relation* E, *such that*:

(I) $E \not\leq_{\mathbf{\Delta}^1_2} \Delta(2^\omega)$.

(II) $E_0 \not\leq_{\mathbf{\Delta}^1_2} E$.

PROOF. As in 6.2, let P be the set of all abelian p-groups. Then P is Δ^1_1 and there is a Δ^1_1 action $g \cdot x$ of S_∞ on P such that

$$\forall x, y \in P(\exists g \in S_\infty(g \cdot x = y) \Leftrightarrow (\mathscr{A}_x \cong \mathscr{A}_y)).$$

Claim. $E \not\leq_{\Delta^1_2} \Delta(2^\omega)$.

Proof. Suppose instead that there is a $\Delta^1_2(z)$ reduction $h: P \to 2^\omega$, for some $z \in \mathscr{N}$. Then there is a κ-sequence of functions $(f_\alpha)_{\alpha<\kappa} \in L[z]$, where $\kappa = \omega_1$, defined by

$$f_\alpha: \alpha + 1 \to \omega \cup \{\infty\}, \qquad f_\alpha(\beta) = \begin{cases} \infty, & \text{for } \beta = 0, \\ 1, & \text{otherwise.} \end{cases}$$

But then for each $x \in P$ with $U(x) = f_\alpha$ we would have $h(x) \in L[z]$. This gives a κ-sequence of reals in $L[z]$, contradicting our assumption that $\omega_1^{L[z]} < \kappa$, and hence $|\mathscr{N} \cap L[z]| = \aleph_0$. ⊣

Claim. $E_0 \not\leq_{\Delta^1_2} E$.

Proof. This follows essentially as in 2.1; the only difference is that we now use the fact that $\mathbf{\Sigma}^1_2$ Lebesgue measurability already holds in V. Starting with the assumption of $E_0 \leq_{\Delta^1_2} \Delta(2^{<\omega_1})$, we obtain the same contradiction as before. ⊣ ⊣

In fact, it is well known that one obtains, provably in ZFC, that the isomorphism relation on countable torsion abelian groups is reducible to $\Delta(2^{<\omega_1})$ by a function C-measurable in the codes. In other words, if X is the space of countable torsion abelian groups, if $G = S_\infty$ is viewed as acting on X in the usual fashion, so that E_G is the relation of isomorphism, then we are in case (I) of 2.2.

In light of these last two examples, one might want to try to obtain a dichotomy theorem along the lines of the original result from [HKL] by simply introducing a third possibility, to allow for E resembling the examples from 6.1 and 6.3 in that it has an ω_1-sequence of equivalence classes. However, even this is overly optimistic. Certain constructions of Woodin, arising in connection with the sudy of definable cardinality in the context of AD, provide a counterexample.

EXAMPLE 6.4 (Woodin). Assume $AD^{L(\mathbb{R})}$. Then there is a Σ^1_1 equivalence relation E such that

(I) $E \not\leq_{L(\mathbb{R})} \Delta(2^\omega)$;

(II) $E_0 \not\leq_{L(\mathbb{R})} E$; and

(III) there is no ω_1-sequence of equivalence classes in $L(\mathbb{R})$.

By combining some of the ideas which were used in 6.4 with certain techniques from [Sa], we will give what seems the strongest possible counterexample at the least possible cost. We will make use of the Solovay model; a discussion of the basic properties of this model can be found in §42 of [Je].

EXAMPLE 6.5. Let κ be inaccessible in L. Let $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ be L-generic, and let $\mathbb{R} = \mathbb{R}^{L[G]}$. Then in $L[G]$ there is a Σ^1_1 equivalence relation E on 2^ω such that $\forall x \in 2^\omega([x]_E \in \mathbf{\Pi}^0_3)$ and such that

(I) $E \not\leq_{L(\mathbb{R})} \Delta(2^\omega)$;

(II) $E_0 \not\leq_{L(\mathbb{R})} E$;

(III) there is no ω_1-sequence of equivalence classes in $L(\mathbb{R})$.

PROOF. The equivalence relation will concentrate on the elements $x \in 2^\omega$ such that x codes an ω-model M_x, of some fragment of set theory, with $\mathscr{L}(M_x) =$

$(\epsilon, \dot{a}, \dot{S})$ and field$(M_x) = \omega$. We will insist also that $(\omega)^{M_x} = \{2n : n \in \omega\}$ with the successor of $2n$ in M_x being $2n+2$. We will also require that

$$M_x \models \text{“}V = L[\dot{a}],\ \dot{a} \subseteq \omega\text{”}.$$

By assuming that x also codes the satisfaction relation for M_x, along the lines of §1 of [Sa], we obtain that the set of $x \in 2^\omega$ giving rise to M_x as above is a Π^0_2 set of reals; such $x \in 2^\omega$ will be called *good*.

For x as above, let β_x be the order type of the well-founded ordinals in M_x. Then let a_x be the unique element of Cantor space such that

$$n \in a_x \Leftrightarrow 2n \in (\dot{a})^{M_x}.$$

Let S_x be the collection of $z \subseteq \omega$ such that there exists some $m \in \omega$ with

$$M_x \models \text{“}m \text{ is a subset of the natural numbers”} \ \&\ m \in (\dot{S})^{M_x}$$
$$\&\ \forall n \in \omega(n \in z \Leftrightarrow M_x \models \text{“}2n \in m\text{”}) \ \&\ z \in L_{\beta_x}[a_x].$$

For x and y both good, set xEy iff $S_x = S_y$, and $a_x = a_y$, and $2^\omega \cap L_{\beta_x}[a_x] = 2^\omega \cap L_{\beta_y}[a_y]$. If only one of x and y are good, set $\neg xEy$; and if neither are good, set xEy.

Claim. $\forall x \in 2^\omega([x]_E \in \mathbf{\Pi}^0_3)$.

Proof. First of all, if x is not good, then $[x]_E \in \mathbf{\Sigma}^0_2$. Suppose instead that x is some fixed good element of 2^ω.

As we range over y, the requirement that $a_x = a_y$ is only $\mathbf{\Pi}^0_1$, and, more generally, for m, m' such that $M_x \models$ “m is a set of natural numbers” and $M_y \models$ “m' is a set of natural numbers”, the requirement that

$$\forall n \in \omega(M_y \models \text{“}2n \in m'\text{”} \Leftrightarrow M_x \models \text{“}2n \in m\text{”})$$

is $\mathbf{\Pi}^0_1$.

If $\beta_x > \omega_1^{L[a_x]}$, then the requirement that $S_x = S_y$ amounts to insisting that

$$2^\omega \cap L[a_x] \subseteq L_{\beta_y}[a_y] \ \&\ \forall z \in 2^\omega \cap L[a_x](z \in S_x \Leftrightarrow z \in S_y).$$

These conditions are both $\mathbf{\Pi}^0_3$, since $2^\omega \cap L[a_x]$ is countable. If $\beta_x < \omega_1^{L[a_x]}$ and $z_0 \in 2^\omega \cap L[a_x]$ is least such that z_0 codes a wellordering of order type β_x, then the requirement that $S_x = S_y$ and $\beta_x = \beta_y$ amounts to demanding that:

$$\forall m \in \omega(\exists n \in \omega M_y \models \text{”}2n \in m\text{”} \text{ iff } (n \notin z_0))$$
$$\&\ \forall z \in 2^\omega \cap L_{\beta_x}[a_x](z \in M_y)$$
$$\&\ \forall z \in 2^\omega \cap L_{\beta_x}[a_x](z \in S_x \text{ iff } z \in S_y).$$

Again, none of these is worse than $\mathbf{\Pi}^0_3$. ⊣

Claim. $E_0 \not\leq_{L(\mathbb{R})} \Delta(2^\omega)$.

Proof. Suppose $f : 2^\omega \to 2^\omega$ were an OD(z) reduction of E to $\Delta(2^\omega)$, where $z \in 2^\omega$. Then for each $S \in (\mathscr{P}(2^\omega))^{L[z]}$ there would exist $y_S \in 2^\omega$ such that for any x coding $(L_{\omega_1^{L[z]}}[z]; \epsilon, z, S)$ we have $f(x) = y_S$. Hence, y_S would be OD(z, S) over $L(\mathbb{R})$, and hence, by the property of the Solovay model, $y_S \in L[z]$. But since this assignment can be performed over $L[z]$, we obtain a contradiction to Cantor's theorem that $|\mathscr{P}(2^\omega)| > |2^\omega|$ inside $L[z]$. ⊣

Claim. $E_0 \not\leq_{L(\mathbb{R})} E$.

Proof. Suppose $f : 2^\omega \to 2^\omega$ is an OD(z) reduction of E_0 to E, where $z \in 2^\omega$. Applying the argument from 2.1 and using that all sets are Lebesgue measurable, we obtain that there is a measure one set $B \subseteq 2^\omega$ such that for some $a \in 2^\omega$

$$\forall y_1, y_2 \in B(f(y_1) = x_1,\ f(y_2) = x_2 \Rightarrow a_{x_1} = a_{x_2}).$$

Similarly we obtain a single $\beta < \kappa$ and $S \subseteq 2^\omega \cap L[a]$ such that on some measure one set $B_1 \subseteq B$

$$\forall y \in B_1(f(y) = x \Rightarrow \beta_x = \beta,\ S_x = S)$$

This contradicts the fact that B_1 must contain many equivalence classes. ⊣

Claim. There is no $\omega_1^{L[G]}$-sequence of E equivalence classes in $L(\mathbb{R})$.

Proof. Otherwise let $(A_\alpha)_{\alpha<\kappa}$ be such a sequence. We may assume, by thinning out the sequence, that either

$$\forall \alpha \neq \beta < \kappa(x_\alpha \in A_\alpha,\ x_\beta \in A_\beta,\ a_{x_\alpha} \neq a_{x_\beta}),$$

or

$$\forall \alpha \neq \beta < \kappa(x_\alpha \in A_\alpha,\ x_\beta \in A_\beta,\ a_{x_\alpha} = a_{x_\beta}).$$

The first possibility is out of the question, since there is no κ-sequence of reals in $L(\mathbb{R})$. So let us assume the second. There must then be some fixed $a \in 2^\omega$ such that $\forall \alpha < \kappa \forall x \in A_\alpha(S_x \subseteq 2^\omega \cap L[a])$. But since $(2^\omega)^{L[a]}$ is countable in $L(\mathbb{R})$, this again gives us a κ-sequence of reals, contradicting known properties of the Solovay model. ⊣ ⊣

Becker raised the question of whether every E_G induced by a Borel action of a Polish group must satsify either (I), (II), or (III) of 6.5. This remains open.

§7. **Open problems.** We collect here various open problems suggested by the preceding work:

(1) (Becker) In 2.1, can one choose U so that for each countable ordinal α, the set $\{x \in \mathcal{N} : \alpha \in \mathrm{dom}(U(x))\ \&\ U(x)(\alpha) = 0\}$ is Borel? This is the case for the classical Ulm invariants for abelian p-groups.

(2) Is the complexity of U in 4.4 or 5.4 best possible?

(3) Is it provable in ZFC that (I) or (II) holds in 5.4?

(4) (Becker) Is one of the alternatives of 6.5 false for the equivalence relation E_G induced by a Borel action of a Polish group?

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