Expressing Validity: Towards a Self-Sufficient Inferentialism

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Abstract: For semantic inferentialists, the basic semantic concept is validity. An inferentialist theory of meaning should offer an account of the meaning of "valid." If one tries to add a validity predicate to one's object language, however, one runs into problems like the v-Curry paradox. In previous work, I presented a validity predicate for a non-transitive logic that can adequately capture its own meta-inferences. Unfortunately, in that system, one cannot show of any inference that it is invalid. Here I extend the system so that it can capture invalidities.

Keywords: inferentialism, naive validity, v-Curry paradox, non-transitive consequence, substructural logic

1 Introduction

We want to understand what it means for our sentences to have a certain content. If our theory of meaning is to be general, it will have to give an account of the contents of the sentences in which it is formulated. Let's call a theory of meaning that does that "self-sufficient." Such a theory applies to a formulation of itself.

For representationalist theories of meaning, self-sufficiency requires a representationalist account of expressions like "is true" or "refers to." In Tarski's (1944, p. 348) terms, the representationalist needs a "semantically closed language." Hence, the liar paradox is a problem that a representationalist who claims to offer a self-sufficient account of meaning must address. For otherwise the representationalist's theory will be built on a concept of which it offers no account.

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The inferentialist, however, should not point to this splinter in her rival's eye, as there may be a beam in her own eye. For there is an analogue of the liar paradox for inferentialist theories of meaning. That is the topic of this paper. According to inferentialism, what it means for a sentence to have a certain content is that the sentence plays a particular inferential role (Brandom, 1994, 2008; Peregrin, 2014; Sellars, 1953, 1974). We can think of the inferential role of a sentence, $S,$ as two sets of pairs. The first set contains all and only the pairs $\langle X, Y \rangle$ such that the inference from X and S to Y is valid, which I will write as $X, S \vdash Y$. And the second set contains all and only the pairs $\langle X, Y \rangle$ such that $X \vdash S, Y$.[2] So, the inferentialist explains meaning in terms of validity, which I denote by "$\vdash$."[3] Hence, a self-sufficient inferentialist theory of meaning must give an account of expressions like "is valid" or "$\vdash$."

Unfortunately, adding a validity predicate to an object language can easily lead to triviality. My goal in this paper is to contribute to a self-sufficient inferentialism by making some progress on how we can save object language expressions for validity from triviality. In particular, I will present a logic with a validity predicate that captures the meta-inferences of that logic while also allowing us to prove that many arguments are invalid.

The paper is structured as follows. I start with a recap of the debate in Section 2. In Section 3, I present a logic with a validity predicate that captures this logic's meta-inferences and proves for all invalid inferences that don't contain the validity predicate that they are invalid. Section 4 concludes.

2 The Story So Far

In this section, I recapitulate some problems with validity predicates that arise if we allow for self-reference. As self-reference is hard to avoid if one wants to construct a self-sufficient semantic theory (after all, the theory must apply to itself), I will assume that restricting self-reference is off the table. I use "$\overline{A}$" as a canonical name for the sentence A, and "$\overline{\Gamma}$" as a canonical

[2] I ignore Language Entry and Departure Transitions here (Sellars, 1974, pp. 423–424).

[3] Note that since the inferentialist's notion of validity is wider than logical validity, it will be no help below to say that there is no problem with logical validity (Field, 2017; Ketland, 2012). To see that the relevant notion of validity is broader, notice that the consequence relation that the inferentialist uses is not closed under substitution. It is, e.g., an important part of the meaning of the atomic sentence "*a* is pink" that the inference from "*a* is pink" and "*b* is crimson" to "*b* is darker than *a*" is valid. You cannot, however, substitute "*a* is cramine" for "*a* is pink" *salva consequentia* here.

name for the set Γ, etc. Thus, I assume self-reference by fiat, even without Gödel-coding.

2.1 The v-Curry Paradox

We want to introduce sentences like $Val(\overline{\Gamma}, \overline{\Delta})$ that express that the inference from Γ to Δ is valid, i.e., that $\Gamma \vdash \Delta$, where Γ and Δ may be sets or sentences. What does it take for a predicate to express validity? Beall and Murzi (2013) suggest that, intuitively, Val must obey so-called validity detachment (VD) and validity proof (VP):

$$\frac{}{\Gamma,\ Val(\overline{\Gamma}, \overline{\Delta}) \Rightarrow \Delta}\ \text{VD} \qquad \frac{\Gamma \Rightarrow \Delta}{\Rightarrow\ Val(\overline{\Gamma}, \overline{\Delta})}\ \text{VP}$$

Unfortunately, this yields triviality if we accept contraction and cut. To see this, let κ be (inter-substitutable with) $Val(\overline{\kappa}, \overline{\bot})$, and call substituting κ for $Val(\overline{\kappa}, \overline{\bot})$ or vice versa "κ-substitution." We can now reason thus:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\kappa,\ Val(\overline{\kappa}, \overline{\bot}) \Rightarrow \bot}\ \text{VD}}{\kappa, \kappa \Rightarrow \bot}\ \kappa\text{-substitution}}{\kappa \Rightarrow \bot}\ \text{contraction}}{\Rightarrow\ Val(\overline{\kappa}, \overline{\bot})}\ \text{VP}}{\Rightarrow \kappa}\ \kappa\text{-substitution} \qquad \dfrac{\dfrac{\dfrac{}{\kappa,\ Val(\overline{\kappa}, \overline{\bot}) \Rightarrow \bot}\ \text{VD}}{\kappa, \kappa \Rightarrow \bot}\ \kappa\text{-substitution}}{\kappa \Rightarrow \bot}\ \text{contraction}}{\Rightarrow \bot}\ \text{cut}$$

Non-transitive theorists, like Ripley (2013), respond to such problems by rejecting cut. The most popular non-transitive logic is ST (Cobreros, Egré, Ripley, & van Rooij, 2012, 2013).

2.2 Faithfulness

Barrio, Rosenblatt, and Tajer (2017) have criticized the non-transitive approach by arguing that, whether or not obeying VD and VP is necessary for expressing validity, it is not sufficient. They suggest a further necessary condition, which I call "faithfulness."

Definition 1 (Faithfulness) *A validity predicate,* Val*, is faithful just in case* $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \vdash Val(\overline{\Theta}, \overline{\Lambda})$ *is provable iff* $\Theta \vdash \Lambda$ *follows from* $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$ *via a valid meta-inference.*

Barrio et al. (2017) show that if we add a validity predicate to ST in the most obvious way, then the validity predicate is not faithful. I have defended the non-transitive approach against this criticism in earlier work (Hlobil, 2018b). In this paper, I want to expand on my earlier response and address a remaining issue.

Faithfulness can be understood in different ways, depending on what we mean by "valid meta-inference." We could take a meta-inference to be valid (a) if it is an instance of an admissible meta-rule, or (b) if it is an instance of a derivable meta-rule.[4] A meta-rule is admissible in a logic $\mathcal{L}$ iff, for all instances, the conclusion-sequent holds in $\mathcal{L}$ if all the premise-sequents hold in $\mathcal{L}$. A meta-rule is derivable in a sequent calculus iff, for all instances, there is a proof-tree with the conclusion-sequent as its root and all the leaves being either premise-sequents or axioms of the sequent calculus.

We should adopt option (b) and reject (a). That is because if we adopt (a), faithfulness yields triviality if the conditional that we use to define "admissible rule" obeys modus ponens and contraction (in the sense that "if A, then (if A, then B)" is equivalent to "if A, then B").[5] However, modus ponens and contraction are plausible, and the conditional of ST obeys them. Hence, advocates of the non-transitive approach should reject (a).

Someone might be tempted to reject the idea that a predicate that expresses validity must be faithful. That, however, is a bad idea for inferentialists. In formulating and using her semantic theory, the inferentialist is constantly reasoning from premises about validity to conclusion about validity. If the inferentialist rejects the left-to-right direction of faithfulness, then she admits that the inferential role—i.e., the meaning—of "valid" is such that it underwrites inferences that are incorrect, by the standard of what really follows from what. If the inferentialist rejects the right-to-left direction of faithfulness, then she holds that "x is valid; therefore, y is valid" may be correct, by the standard of what actually follows from what, and it may still be ruled invalid, by the meaning of the word "valid," as explained by the inferentialist. Either way, the inferentialist admits that when she describes

[4]Dicher and Paoli (2019) suggest a notion of local validity of meta-inferences. This notion is tied to the semantic idea of valuations. I am ignoring it here because of my inferentialist motivation.

[5]I showed this in (Hlobil, 2018a). Fjellstad (ms) has developed the following nice presentation of the point: Let κ be the sentence $Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)})$, where Γ and Δ are arbitrary. Assume faithfulness for reductio. (i) By faithfulness, $Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}) \vdash Val(\overline{\Gamma}, \overline{\Delta})$ is provable if and only if it is the case that if $Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}) \vdash Val(\overline{\Gamma}, \overline{\Delta})$, then $\Gamma \vdash \Delta$. (ii) By contraction of the conditional, if $Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}) \vdash Val(\overline{\Gamma}, \overline{\Delta})$, then $\Gamma \vdash \Delta$. But this is the right-hand-side of (i) above. (iii) So, by modus ponens, $Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}) \vdash Val(\overline{\Gamma}, \overline{\Delta})$. (iv) Since this is the antecedent of (ii), by modus ponens, $\Gamma \vdash \Delta$.

the inferential roles of sentences by using "valid," she is not—by the meaning of this word (as she explains it)—beholden to what actually follows from what. But that amounts to admitting that she is not talking about the actual inferential roles of these sentences. Hence, the inferentialist cannot reject faithfulness.

2.3 Using Contraction Against VD

Unfortunately, another problem arises: contraction, faithfulness, and VD jointly yield triviality. To see this, let κ be $Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)})$. Now:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\kappa, Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}) \Rightarrow Val(\overline{\Gamma}, \overline{\Delta})}\text{VD}}{Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}), Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}) \Rightarrow Val(\overline{\Gamma}, \overline{\Delta})}\kappa\text{-substitution}}{Val(\overline{\kappa}, \overline{Val(\Gamma, \Delta)}) \Rightarrow Val(\overline{\Gamma}, \overline{\Delta})}\text{contraction}}{\kappa \Rightarrow Val(\overline{\Gamma}, \overline{\Delta})}\kappa\text{-substitution}$$

By faithfulness, the fact that the third line of this tree is provable tells us that, by the application of a derivable meta-rule, we can infer $\Gamma \Rightarrow \Delta$ from $\kappa \Rightarrow Val(\overline{\Gamma}, \overline{\Delta})$. But we have just proven the latter. Hence, there is a way to continue our proof-tree to reach $\Gamma \Rightarrow \Delta$.

We are thus forced to choose between VD and contraction. The option of rejecting contraction is well explored in the literature. I want to look at rejecting VD. But before we worry about VD, we should ask whether we can, in fact, get a faithful validity predicate if we reject VD. Fortunately, the answer is "yes."

2.4 Assuming Sequents

I showed in (Hlobil, 2018b) how to add a faithful validity predicate to ST by rejecting VD. I call this extension of ST "NG," and Fig. 1 gives a variant of it. I use lower case Latin letters and the subscript "0" on sets (e.g., in Ax1) for atomic sentences and sets thereof respectively.

If we dropped the VLR rule from NG, we would have a formulation of ST. And if we also dropped the rules for the truth-predicate, we would have classical propositional logic (with conjunction and the conditional defined in the usual way). Notice that weakening is absorbed into the axioms and that we assume contraction and permutation, i.e., we work with sets on the left and the right.

Axioms of NG

$$\text{Ax1:} \quad \Gamma_0, p \Rightarrow p, \Delta_0$$

Rules of NG

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\ \text{LN} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ \text{RN}$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \text{L}\vee \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \text{R}\vee$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, Tr(\overline{A}) \Rightarrow \Delta}\ \text{LT} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow Tr(\overline{A}), \Delta}\ \text{RT}$$

$$\cfrac{\cfrac{\cfrac{{}^{1:}\ \Gamma_1 \Rightarrow \Delta_1}{\vdots} \quad \ldots \quad \cfrac{{}^{m:}\ \Gamma_m \Rightarrow \Delta_m}{\vdots}}{\Theta \Rightarrow \Lambda}\ \text{applications of rules of NG}}{Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_m}, \overline{\Delta_m}) \Rightarrow Val(\overline{\Theta}, \overline{\Lambda})}\ \text{VLR, } [1, \ldots, m]$$

Figure 1: System NG

What is special about NG is that we can assume and discharge sequents by using the VLR-rule. The superscripts that number the assumed sequents are not part of those sequents; they merely help us to keep track of our assumptions. We allow empty discharges.

3 Adding Invalidities to NG

The problem with NG that I want to address in this paper is that it cannot prove of any inference that it is invalid. This is a problem because if the inferentialist cannot show of any inference that it is invalid, then, for all we know, the inferential roles of our expressions include any inference you care to name. And while the meanings of our words may be opaque to us in some respects, it is implausible that this opacity is so enormous.

3.1 Formulating STV

To address this problem, I want to see how we might extend NG so that we can prove some invalidities. Let's call this extension STV. Ideally the proofs of invalidity in STV should mirror our meta-theoretic knowledge of invalidities in STV. After all, we want to give a treatment, in the object language, of our use of "$\vdash$" in the meta-language. So we should begin by asking ourselves how we usually know about invalidities. For standard sequent calculi, we usually know about invalidities by observing that a root-first proof search fails. My strategy here is to mimic such proof-searches within STV.

At the level of atomic sequents, we perform a root-first proof search simply by seeing whether the sequent has the form of an axiom. To mirror this within STV, let's add the following axioms:

Ax2 If $\Gamma_0 \cap \Delta_0 = \emptyset$ and neither *Val* nor *Tr* occur in $\Gamma_0 \cup \Delta_0$, then $Val(\overline{\Gamma_0}, \overline{\Delta_0}) \Rightarrow$ is an axiom.

With these axioms we can prove of any Val-free and Tr-free atomic sequent that is not an axiom of ST that it is invalid. After all, for any atomic sequent, $\Gamma_0 \Rightarrow \Delta_0$, that is not of the form of Ax1, Ax2 gives us $Val(\overline{\Gamma_0}, \overline{\Delta_0}) \Rightarrow$, and by RN we get $\Rightarrow \neg\, Val(\overline{\Gamma_0}, \overline{\Delta_0})$.

Next we observe that the notion of a valid meta-inference used in faithfulness builds in transitivity. We reason transitively in our sequent calculus, and the notion of a valid meta-inference must respect this transitivity of our meta-theoretic reasoning. It does that by relying on the notion of a derivable meta-rule, which cares only about the leaves and the root of a proof-tree and not about the lemmas that must be established in the proof-tree along the way. Let's make this feature of our meta-theoretical reasoning explicit at the object-level of STV by adding the following restricted cut rule.

If the principal operator of every sentence in $\Gamma \cup \Delta \cup \Lambda \cup \Theta$ is Val and there are no open assumptions, we can apply the rule:

$$\frac{\Gamma \Rightarrow \Delta,\, Val(\overline{\Phi}, \overline{\Psi}) \qquad Val(\overline{\Phi}, \overline{\Psi}), \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta}\ \text{Val-Cut}$$

We say that a proof-tree of STV is closed iff all undischarged sequents are axioms of STV. A sequent, $\Gamma \Rightarrow \Delta$, is provable in STV (for which we write $\Gamma \vdash_{\mathsf{STV}} \Delta$) iff there is a closed proof-tree of NG that has the sequent as its root.

3.2 Validity and Invalidity in STV

The validity predicate of STV captures all the validities and invalidities of ST. For the validities, it is easy to see that STV captures all of its own validities, which include all ST validities.

Proposition 1 *If* $\Gamma \vdash_{\mathsf{STV}} \Delta$, *then* $\vdash_{\mathsf{STV}} \mathit{Val}(\overline{\Gamma}, \overline{\Delta})$.

Proof. Suppose that $\Gamma \vdash_{\mathsf{STV}} \Delta$ and, hence, $\Gamma \Rightarrow_{\mathsf{STV}} \Delta$ is the root of a closed proof-tree. By VLR, $\Rightarrow_{\mathsf{STV}} \mathit{Val}(\overline{\Gamma}, \overline{\Delta})$ can be proven in a closed proof-tree. Therefore, $\vdash_{\mathsf{STV}} \mathit{Val}(\overline{\Gamma}, \overline{\Delta})$. □

For the invalidities, note that I take the language of ST, $\mathfrak{L}_{\mathsf{ST}}$, to not include the validity predicate. The validity predicate of STV captures all ST invalidities in $\mathfrak{L}_{\mathsf{ST}}$.

Proposition 2 *If* $\Gamma \nvdash_{\mathsf{ST}} \Delta$ *and* $\Gamma \cup \Delta \in \mathfrak{L}_{\mathsf{ST}}$, *then* $\vdash_{\mathsf{STV}} \neg \mathit{Val}(\overline{\Gamma}, \overline{\Delta})$.

Proof. Suppose that $\Gamma \nvdash_{\mathsf{ST}} \Delta$ and *Val* does not occur in $\Gamma \cup \Delta$. So there is no proof-tree in ST with $\Gamma \Rightarrow_{\mathsf{ST}} \Delta$ as its root. Hence, a root-first proof search (in which we forbid loops and bottom-to-top rule applications) for $\Gamma \Rightarrow_{\mathsf{ST}} \Delta$ will result in a tree with at least one leaf $\Theta_0 \Rightarrow_{\mathsf{ST}} \Lambda_0$ such that $\Theta_0 \cap \Lambda_0 = \emptyset$ (and hence $\Theta_0 \nvdash_{\mathsf{ST}} \Lambda_0$). Since $\Theta_0 \cap \Lambda_0 = \emptyset$ and *Val* does not occur in $\Theta_0 \cup \Lambda_0$, we have $\mathit{Val}(\overline{\Theta_0}, \overline{\Lambda_0}) \Rightarrow_{\mathsf{STV}}$ by Ax2. Since all the rules of ST are invertible, the inverse of the path from $\Theta_0 \Rightarrow_{\mathsf{ST}} \Lambda_0$ to $\Gamma \Rightarrow_{\mathsf{ST}} \Delta$ is a derivable rule application in ST and, hence, in STV. Call this derivable rule DERIV. We can now reason thus:

$$\dfrac{\dfrac{\dfrac{^{1:}\ \Gamma \Rightarrow \Delta}{\Theta_0 \Rightarrow \Lambda_0}\,\text{DERIV}}{\mathit{Val}(\overline{\Gamma}, \overline{\Delta}) \Rightarrow \mathit{Val}(\overline{\Theta_0}, \overline{\Lambda_0})}\,\text{VLR,[1]} \qquad \mathit{Val}(\overline{\Theta_0}, \overline{\Lambda_0}) \Rightarrow}{\dfrac{\mathit{Val}(\overline{\Gamma}, \overline{\Delta}) \Rightarrow}{\Rightarrow \neg \mathit{Val}(\overline{\Gamma}, \overline{\Delta})}\,\text{RN}}\,\text{Val-Cut}$$

This proof-tree does not have any open assumptions. Therefore, $\vdash_{\mathsf{STV}} \neg \mathit{Val}(\overline{\Gamma}, \overline{\Delta})$. □

I will show below that the converses of both propositions also hold and, hence, that the validity predicate of STV (applied to sentences of $\mathfrak{L}_{\mathsf{ST}}$) strongly represents ST-validity.[6]

[6]To be explicit, $R(\overline{x}, \overline{y})$ strongly represents relation $\mathcal{R}(x, y)$ in STV iff (i) [$\vdash_{\mathsf{STV}} R(\overline{x}, \overline{y})$ iff $\mathcal{R}(x, y)$] and (ii) [$\vdash_{\mathsf{STV}} \neg R(\overline{x}, \overline{y})$ iff not $\mathcal{R}(x, y)$].

Before we turn to that, however, notice that if STV doesn't prove the empty sequent, there must be "validity-gaps," i.e., there must be inferences such that STV proves neither that they are valid nor that they are invalid.

Proposition 3 *If $\emptyset \nvdash_{\mathsf{STV}} \emptyset$, then there are some sets, Γ, Δ, such that neither $\vdash_{\mathsf{STV}} Val(\overline{\Gamma}, \overline{\Delta})$ nor $\vdash_{\mathsf{STV}} \neg Val(\overline{\Gamma}, \overline{\Delta})$.*

Proof. By example. Let $\kappa = Val(\overline{\kappa}, \overline{\emptyset})$. Suppose we had $\vdash_{\mathsf{STV}} \neg\kappa$ and hence $\kappa \Rightarrow_{\mathsf{STV}}$. By VLR, we would get $\Rightarrow_{\mathsf{STV}} \kappa$. And Val-Cut would then yield the empty sequent. Similarly, if we had $\Rightarrow_{\mathsf{STV}} \kappa$, then faithfulness (which I will prove below) would give us $\kappa \Rightarrow_{\mathsf{STV}}$, and this would yield the empty sequent. □

I will now show that STV doesn't prove the empty sequent. We will first need to establish the following lemma, which will also make it easy to see, along the way, that the validity predicate of STV is faithful.

Lemma 1 *If $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \Rightarrow Val(\overline{\Theta_1}, \overline{\Lambda_1}), \ldots, Val(\overline{\Theta_m}, \overline{\Lambda_m})$ is provable in STV, then, for some $1 \leq i \leq m$, the sequent $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \Rightarrow Val(\overline{\Theta_i}, \overline{\Lambda_i})$ is provable in a proof-tree where the root comes by VLR.*

Proof. We argue by induction on proof-height. Suppose the lemma holds for sequents provable in trees strictly lower than *k*. A proof of height *k* of a target sequent must come by VLR, Ax1, a bottom-to-top rule, or Val-Cut.

If it comes by VLR, we are done. If it comes by Ax1, then the left and the right share a sentence. So, for some $1 \leq l \leq n$, $\Gamma_l = \Theta_i$ and $\Delta_l = \Lambda_i$. We assume $\Theta_i \Rightarrow \Lambda_i$ and immediately use VLR, discharging, for all $1 \leq r \leq n$ (vacuously except for $r = l$), the assumptions $\Theta_r \Rightarrow \Lambda_r$. We thus prove $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \Rightarrow Val(\overline{\Theta_i}, \overline{\Lambda_i})$ by VLR.

If the target sequent comes by a bottom-to-top rule application, the premise sequent must include a sentence, A, whose principal connective is not *Val*. The premise of that must come by a top-to-bottom application of the same rule or by Val-Cut. If it comes by the top-to-bottom rule application, we apply our hypothesis to the premise of that step and we are done. If it comes by Val-Cut, the same reasoning applies to the premise of Val-Cut that contains A. The premise comes either by introducing A or by Val-Cut. If the first, we can eliminate this introduction and subsequent elimination. If the second, the same reasoning applies again. The chain of Val-Cut applications must end because the leaves of proof-trees contain only finitely many sentences.

If the root comes by Val-Cut, our two premise-sequents are $Val(\overline{\Gamma_1}, \overline{\Delta_1})$, $\ldots$, $Val(\overline{\Gamma_x}, \overline{\Delta_x}) \Rightarrow Val(\overline{\Theta_1}, \overline{\Lambda_1}), \ldots, Val(\overline{\Theta_y}, \overline{\Lambda_y}), Val(\overline{\Xi}, \overline{\Pi})$ and $Val(\overline{\Xi}, \overline{\Pi}), Val(\overline{\Gamma_{x+1}}, \overline{\Delta_{x+1}}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \Rightarrow Val(\overline{\Theta_{y+1}}, \overline{\Lambda_{y+1}}), \ldots, Val(\overline{\Theta_m}, \overline{\Lambda_m})$, for some renumbering (and perhaps doubling) of the $\overline{\Gamma}$s, $\overline{\Delta}$s, $\overline{\Theta}$s, and $\overline{\Lambda}$s in the target sequent.

By our induction hypothesis, for both sequents, a sequent like it with all but one sentence on the right deleted can be proven via VLR. Now, if for the first sequent the remaining sentence on the right is not $Val(\overline{\Xi}, \overline{\Pi})$, then we get the desired result by adding some empty discharges. So let's assume that we can prove $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_x}, \overline{\Delta_x}) \Rightarrow Val(\overline{\Xi}, \overline{\Pi})$ via VLR. Then there is a proof-tree starting with (some of) the sequents that correspond to the sentences on the left and with a root that corresponds to the sentence on the right. Moreover, from the second premise of our Val-Cut application, we know that, for some s, $(y + 1) \leq s \leq m$, and $Val(\overline{\Xi}, \overline{\Pi}), Val(\overline{\Gamma_{x+1}}, \overline{\Delta_{x+1}}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \Rightarrow Val(\overline{\Theta_s}, \overline{\Lambda_s})$ is derivable via VLR. Thus, there is a corresponding proof-tree with the sequents on the left as assumed leaves. We can combine the two proof-trees to get one proof-tree in which $\Theta_s \Rightarrow \Lambda_s$ is derived from the assumptions $\Gamma_1 \Rightarrow \Delta_1, \ldots,$ $\Gamma_n \Rightarrow \Delta_n$, where $\Xi \Rightarrow \Pi$ is the root of a subproof with leaves $\Gamma_1 \Rightarrow \Delta_1, \ldots,$ $\Gamma_x \Rightarrow \Delta_x$. □

With this lemma in hand, it is easy to show that the validity predicate of STV is faithful.

Theorem 1 *The validity predicate of* STV *is faithful, i.e.,* $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \Rightarrow Val(\overline{\Theta}, \overline{\Lambda})$ *is provable in* STV *iff there is a proof-tree in* STV *with* $\Theta \Rightarrow \Lambda$ *as its root and (some of)* $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ *as its leaves, i.e., this is an application of a derivable rule.*

Proof. The left-to-right direction is immediate from VLR. The right-to-left direction is immediate from Lemma 1. □

We can now establish that STV does not prove the empty sequent. And as we have already seen above, this implies that STV does not prove all of its own invalidities.

Lemma 2 STV *does not prove the empty sequent, i.e.,* $\emptyset \nvdash_{\mathsf{STV}} \emptyset$.

Proof. The empty sequent can come only by Val-Cut. The premise-sequents are $\Rightarrow Val(\overline{\Gamma}, \overline{\Delta})$ and $Val(\overline{\Gamma}, \overline{\Delta}) \Rightarrow$. By Lemma 1, the first premise is provable via VLR. Hence, $\Gamma \Rightarrow \Delta$ is provable. However, $Val(\overline{\Gamma}, \overline{\Delta}) \Rightarrow$ is provable only if a root-first proof search for $\Gamma \Rightarrow \Delta$ fails. Thus, there would have be atomic sets Γ_0 and Δ_0 such that $\Gamma_0 \Rightarrow \Delta_0$ and $Val(\overline{\Gamma_0}, \overline{\Delta_0}) \Rightarrow$. The first has to come by Ax1 and the second by Ax2. Because of the former, $\Gamma_0 \cap \Delta_0 \neq \emptyset$. But that means that $Val(\overline{\Gamma_0}, \overline{\Delta_0}) \Rightarrow$ cannot come by Ax2. $\square$

It follows that there is no inference such that STV proves that the inference is valid and also proves that it is invalid.

Corollary 1 *There are no sets* Γ *and* Δ*, such that both* $\Rightarrow Val(\overline{\Gamma}, \overline{\Delta})$ *and* $Val(\overline{\Gamma}, \overline{\Delta}) \Rightarrow$.

As already intimated, however, Lemma 2 together with Proposition 3 also implies:

Proposition 4 *There are some sets,* Γ *and* Δ*, such that neither* $\vdash_{\mathsf{STV}} Val(\overline{\Gamma}, \overline{\Delta})$ *nor* $\vdash_{\mathsf{STV}} \neg Val(\overline{\Gamma}, \overline{\Delta})$.

In other words, STV doesn't decide all questions about validity. In particular, as we have seen in the proof of Proposition 3, STV shows neither that what the v-Curry sentence says is true, namely that the empty set follows validly from it, nor that it is false.

We have seen above that STV proves all the validities and invalidities of ST. We can now show the converse as well.

Lemma 3 *If* $\vdash_{\mathsf{STV}} Val(\overline{\Gamma}, \overline{\Delta})$ *and* $\Gamma \cup \Delta$ *is* Val*-free, then* $\Gamma \vdash_{\mathsf{ST}} \Delta$.

Proof. Suppose that $\Rightarrow_{\mathsf{STV}} Val(\overline{\Gamma}, \overline{\Delta})$ and $\Gamma \cup \Delta$ is *Val*-free. Our root must have come by VLR. So, $\Gamma \Rightarrow_{\mathsf{STV}} \Delta$ is provable. Since this does not contain *Val*, it cannot come by VLR or Val-Cut. So it must come by one of the ST-rules. This applies to all the premise sequents and, repeating the reasoning, to all sequents in the proof-tree. Hence, $\Gamma \vdash_{\mathsf{ST}} \Delta$. $\square$

Theorem 2 *The* STV *validity predicate, applied to* ST *sentences, strongly represents* ST *validity, i.e., for all* $\Gamma \cup \Delta \subseteq \mathfrak{L}_{\mathsf{ST}}$*, we have, first, that* $\vdash_{\mathsf{STV}} Val(\overline{\Gamma}, \overline{\Delta})$ *iff* $\Gamma \vdash_{\mathsf{ST}} \Delta$*; and, second, that* $\vdash_{\mathsf{STV}} \neg Val(\overline{\Gamma}, \overline{\Delta})$ *iff* $\Gamma \nvdash_{\mathsf{ST}} \Delta$.

Proof. The right-to-left directions of the two biconditionals are Propositions 1 and 2 above. The left-to-right direction of the first biconditional is Proposition 3. For the left-to-right direction of the second biconditional, we prove the contrapositive. Suppose that $\Gamma \vdash_{\mathsf{ST}} \Delta$. We know that this implies $\vdash_{\mathsf{STV}} Val(\overline{\Gamma}, \overline{\Delta})$. Assume for reductio that $\vdash_{\mathsf{STV}} \neg Val(\overline{\Gamma}, \overline{\Delta})$. That would mean that $Val(\overline{\Gamma}, \overline{\Delta}) \vdash_{\mathsf{STV}}$. But this is ruled out by Corollary 1. $\square$

Let's take stock. The validity predicate of STV is faithful, and it captures the validities and invalidities of ST (over a *Val*-free language) perfectly. Moreover, STV doesn't prove of any inference that it is valid and also that it is invalid. However, there are some inferences for which STV proves neither that they are valid nor that they are invalid.

3.3 STV Embeds LP in ST

There is another perspective on the results above that is worth mentioning. It is well-known that LP is the external logic of ST (Barrio, Pailos, & Szmuc, 2019; Barrio, Rosenblatt, & Tajer, 2015; Dicher & Paoli, 2019; Pynko, 2010). That is (Barrio et al., 2015, p. 557):

Fact 1 *Let* $\mathtt{t}$ *be a translation function such that* $\mathtt{t}(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \supset \bigvee \Delta$ *and* $\mathtt{t}(\Rightarrow \Delta) = \bigvee \Delta$*. Then an* ST*-meta-inference is admissible iff its* $\mathtt{t}$*-translation is* LP*-valid, i.e.,* $\Theta \vdash_{\mathsf{ST}} \Lambda$ *whenever* $\Gamma_1 \vdash_{\mathsf{ST}} \Delta_1, \ldots, \Gamma_n \vdash_{\mathsf{ST}} \Delta_n$ *holds if and only if* $\mathtt{t}(\Gamma_1 \Rightarrow \Delta_1), \ldots, \mathtt{t}(\Gamma_n \Rightarrow \Delta_n) \vdash_{\mathsf{LP}} \mathtt{t}(\Theta \Rightarrow \Lambda)$*.*

So, from an ST-perspective, LP tells us how to reason with claims about validity, i.e., with sequents. The validity predicate of STV allows us to codify this reasoning not at the meta-inferential level of STV but at the inferential level. In particular, for any inference that is LP-valid, we can translate the sentences that occur in it (using the inverse of $\mathtt{t}$) into sequents of ST and, then, into *Val*-sentences of STV. Either the resulting inference is STV-valid or STV proves the negation of one of the premises, i.e.:

Proposition 5 $\mathtt{t}(\Gamma_1 \Rightarrow \Delta_1), \ldots, \mathtt{t}(\Gamma_n \Rightarrow \Delta_n) \vdash_{\mathsf{LP}} \mathtt{t}(\Theta \Rightarrow \Lambda)$ *if and only if either* $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \vdash_{\mathsf{STV}} Val(\overline{\Theta}, \overline{\Lambda})$ *or, for some* $1 \leq i \leq n$*,* $\vdash_{\mathsf{STV}} \neg Val(\overline{\Gamma_i}, \overline{\Delta_i})$*, where* $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_n, \Theta$*, and* Λ *are* *Val-free.*

Proof. Left-to-right: Suppose that $\mathtt{t}(\Gamma_1 \Rightarrow \Delta_1), \ldots, \mathtt{t}(\Gamma_n \Rightarrow \Delta_n) \vdash_{\mathsf{LP}} \mathtt{t}(\Theta \Rightarrow \Lambda)$. By Fact 1, either one of $\Gamma_1 \vdash_{\mathsf{ST}} \Delta_1, \ldots, \Gamma_n \vdash_{\mathsf{ST}} \Delta_n$ fails, or $\Theta \vdash_{\mathsf{ST}} \Lambda$ holds. If the latter, then $\vdash_{\mathsf{STV}} Val(\overline{\Theta}, \overline{\Lambda})$ and we can prove this via VLR. Adding vacuous discharges we can hence also prove: $Val(\overline{\Gamma_1}, \overline{\Delta_1})$, $\ldots$, $Val(\overline{\Gamma_n}, \overline{\Delta_n}) \vdash_{\mathsf{STV}} Val(\overline{\Theta}, \overline{\Lambda})$. If the former, i.e., $\Gamma_i \nvdash_{\mathsf{ST}} \Delta_i$, then by Proposition 2, $\vdash_{\mathsf{STV}} \neg Val(\overline{\Gamma_i}, \overline{\Delta_i})$.

Right-to-left: If, for some $1 \leq i \leq n$, $\vdash_{\mathsf{STV}} \neg Val(\overline{\Gamma_i}, \overline{\Delta_i})$, then, by Theorem 2, $\Gamma_i \nvdash_{\mathsf{ST}} \Delta_i$. Hence, $\Theta \vdash_{\mathsf{ST}} \Lambda$ whenever $\Gamma_1 \vdash_{\mathsf{ST}} \Delta_1, \ldots, \Gamma_n \vdash_{\mathsf{ST}} \Delta_n$. So, by Fact 1, $\mathtt{t}(\Gamma_1 \Rightarrow \Delta_1), \ldots, \mathtt{t}(\Gamma_n \Rightarrow \Delta_n) \vdash_{\mathsf{LP}} \mathtt{t}(\Theta \Rightarrow \Lambda)$. For the other disjunct, suppose that $Val(\overline{\Gamma_1}, \overline{\Delta_1}), \ldots, Val(\overline{\Gamma_n}, \overline{\Delta_n}) \vdash_{\mathsf{STV}} Val(\overline{\Theta}, \overline{\Lambda})$.

By faithfulness, $\Theta \Rightarrow \Lambda$ follows in ST via a derivable meta-rule from $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$. So, whenever we have $\Gamma_1 \vdash_{\mathsf{ST}} \Delta_1, \ldots, \Gamma_n \vdash_{\mathsf{ST}} \Delta_n$, we also have $\Theta \vdash_{\mathsf{ST}} \Lambda$. Therefore, by Fact 1, $\mathtt{t}(\Gamma_1 \Rightarrow \Delta_1), \ldots, \mathtt{t}(\Gamma_n \Rightarrow \Delta_n) \vdash_{\mathsf{LP}} \mathtt{t}(\Theta \Rightarrow \Lambda)$. $\square$

This allows us to view LP as a theory that describes how we should reason with sentences about validity. Given that MP fails in LP, the failure of VD in STV can no longer surprise us. The advocate of STV can see LP as merely a disguised formulation of the logic of validity claims.

4 Conclusion

I have presented a way to add to ST a validity predicate that captures the derivable meta-inference of the resulting logic and strongly represents ST-validity over the *Val*-free fragment of the language. Where does that leave us with respect to a self-sufficient inferentialism?

No doubt STV falls short of providing a fully satisfactory account of the meaning of "valid" as this term is used by the inferentialist. It has at least three serious limitations. First, STV is propositional. And since the usual sequent rules for the quantifiers are not invertible, it is not obvious how the method for proving invalidities presented here could be extended to first-order invalidities. Second, STV deems some inferences invalid without proving, in the object language, that they are invalid, e.g., the inference from the v-Curry sentence to an absurd conclusion. This is an aspect of the inferential role of "valid," as used by the inferentialist who puts forward STV, about which her inferentialist account of "*Val*" is silent. Third, in proving things about STV, the inferentialist makes use of powerful mathematical machinery, e.g., in proofs by induction. But there is no analogue of that machinery at the object level of STV.

On the other hand, STV provides a model of what are arguably the two basic ways of knowing about validities and invalidities respectively, given an inferentialist perspective in a sequent calculus setting. We know about validities via proofs, and we know about invalidities via failed root-first proof-searches. STV offers an account of the use of "valid" in these activities. That brings the inferentialist a step closer to an account of her use of "valid" in stating her theory. Further steps will have to wait for another occasion.

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