Individual-Actualism and Three-Valued Modal Logics, Part 2: Natural-Deduction Formalizations

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INDIVIDUAL-ACTUALISM AND THREE-VALUED MODAL LOGICS, PART 2: NATURAL-DEDUCTION FORMALIZATIONS

5. FORMALIZING POSSIBILISTIC LOGICS BASED ON K

A sequent calculus $\underline{\mathbf{X}}$ may be viewed as a class-function assigning each appropriate language $L = L_y$ to a simultaneous inductive definition of two sets of sequents in L:

$$Th\ \underline{\mathbf{X}}(L) = \text{the theorems of } \underline{\mathbf{X}}(L);$$

$$WkTh\ \underline{\mathbf{X}}(L) = \text{the weak theorems of } \underline{\mathbf{X}}(L).$$

The base-clauses of this definition shall be called axioms; the inductive clauses shall be rules. $\underline{\mathbf{X}}$ will be sound relative to a given logic **X** iff for any appropriate L:

all members of $Th\ \underline{\mathbf{X}}(L)$ are **X**-valid;

all members of $WkTh\ \underline{\mathbf{X}}(L)$ are weakly **X**-valid.

$\underline{\mathbf{X}}$ will be complete relative to **X** iff for any appropriate L:

all **X**-valid sequents of L belong to $Th\ \underline{\mathbf{X}}(L)$;

all weakly **X**-valid sequents of L belong to $Wk\ Th\ \underline{\mathbf{X}}(L)$.

Where $\underline{\mathbf{X}}(L)$ is fixed, use these abbreviations:

$$\Gamma, \Delta \vdash \phi : (\Gamma, \Delta, \phi) \in Th\ \underline{\mathbf{X}}(L);$$

$$\Gamma, \Delta \vdash^{w} \phi : (\Gamma, \Delta, \phi) \in Wk\ Th\ \underline{\mathbf{X}}(L).$$

For $\Gamma \subseteq \Delta \subseteq fml(L)$, (Γ, Δ) is $\underline{\mathbf{X}}(L)$-inconsistent iff $\Gamma, \Delta \vdash \bot$; otherwise (Γ, Δ) is $\underline{\mathbf{X}}(L)$-consistent. Where $\underline{\mathbf{X}}(L)$ is fixed, we'll just write "consistent" or "inconsistent". Notation: where $\Phi \subseteq fml(L)$, let:

$$\Box\Phi = \{\Box\phi : \phi \in \Phi\};\ \Box^{-1}\Phi = \{\phi : \Box\phi \in \Phi\};$$

define $\underline{\Box}\Phi$ and $\underline{\Box}^{-1}\Phi$ similarly.

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The sequent calculus $\underline{\mathbf{K}}^p$ is formed as follows. For any $L = L_y$ the axioms of $\underline{\mathbf{K}}^p(L)$ are those among the following which are sequents in L:

(1) $\{\phi\}, \{\phi\} \vdash \phi$, for a $\phi \in fml(L)$;

(2) $\{\ \}, \{\phi\} \vdash^w \phi$, for ϕ as above;

(3) $\{\ \}, \{\bot\} \vdash \bot$;

(4) $\{\mathbf{u}\}, \{\mathbf{u}\} \vdash \bot$; this is unnecessary if 'T', '**u**' $\in \mathrm{lex}_y$;

(5) $\{\neg\mathbf{u}\}, \{\neg\mathbf{u}\} \vdash \bot$;

(6) $\{(\tau_0 \not\approx \tau_1)\}, \{(\tau_0 \not\approx \tau_1)\} \vdash (\tau_i \approx \tau_i)$ for $i < 2$;

(6_s) $\{(\tau_0 \not\approx_s \tau_1)\}, \{(\tau_0 \not\approx_s \tau_1), (\tau_i \not\approx_s \tau_i)\} \vdash (\tau_{1-i} \approx_s \tau_{1-i})$ for $i < 2$;

(7) $\{(\tau_0 \approx \tau_0), (\tau_1 \approx \tau_1)\}, \{(\tau_0 \approx \tau_0), (\tau_1 \approx \tau_1), (\tau_0 \approx \tau_1)\} \vdash (\tau_0 \approx \tau_1)$;

(7_s) $\{(\tau_i \approx_s \tau_i)\}, \{(\tau_i \approx_s \tau_i), (\tau_0 \approx_s \tau_1)\} \vdash (\tau_0 \approx_s \tau_1)$, where $i < 2$;

(8) $\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1)\} \vdash \Box(\tau_0 \approx \tau_1)$;

(8_s) as above with '$\approx_s$' replacing '$\approx$';

(8*) as in (8) with '$\underline{\Box}$' replacing '$\Box$';

(8_s^*) as in (8_s) with '$\underline{\Box}$' replacing '$\Box$'.

(9) $\{\ \}, \{(\tau_0 \not\approx \tau_1)\} \vdash^w \Box(\tau_0 \not\approx \tau_1)$;

(9_s) as above with '$\approx_s$' replacing '$\approx$';

(9*) as in (9) with '$\underline{\Box}$' replacing '$\Box$';

(9_s^*) as in (9_s) with '$\underline{\Box}$' replacing '$\Box$';

(10*) $\{\neg\underline{\Box}\phi\}, \{\neg\underline{\Box}\phi\} \vdash \underline{\Box}(\phi \supseteq \phi)$, for every $\phi \in fml(L)$;

(11*) $\{\underline{\Box}(\phi \supseteq \phi)\}, \{\underline{\Box}(\phi \supseteq \phi), \underline{\Box}\phi\} \vdash \underline{\Box}\phi$, for every $\phi \in fml(L)$.

The rules of $\underline{\mathbf{K}}^p(L)$ include all those presented in §2 and §5, of [2], namely the following:

Structural rules

(Thinning) if $\Gamma \subseteq \Gamma'$; $\Delta \subseteq \Delta'$ and $\Gamma' \subseteq \Delta'$,

$$\frac{\Gamma, \Delta \vdash \phi}{\Gamma', \Delta' \vdash \phi}; \quad \frac{\Gamma, \Delta \vdash^{w} \phi}{\Gamma', \Delta' \vdash \phi};$$

(Weakening) $\dfrac{\Gamma, \Delta \vdash \phi}{\Gamma, \Delta \vdash^{w} \phi}$; (⊥ Strengthening) $\dfrac{\Gamma, \Delta \vdash^{w} \bot}{\Gamma, \Delta \vdash \bot}$

Introduction and Elimination rules:

(Strong Indirect Proof) $\dfrac{\Gamma, \Delta \cup \{\neg\phi\} \vdash \bot}{\Gamma, \Delta \vdash \phi}$;

(Weak Indirect Proof) $\dfrac{\Gamma \cup \{\neg\phi\}, \Delta \cup \{\neg\phi\} \vdash \bot}{\Gamma, \Delta \vdash^{w} \phi}$;

('⊇' Bivalence) $\dfrac{\Gamma, \Delta \vdash (\phi \supseteq \psi)}{\Gamma, \Delta \cup \{\phi\} \vdash \phi}$; $\dfrac{\Gamma, \Delta \vdash (\phi \supseteq \psi)}{\Gamma, \Delta \cup \{\psi\} \vdash \psi}$;

(Strong '⊇' Elimination) $\dfrac{\Gamma, \Delta \vdash (\phi \supseteq \psi) \quad \Gamma, \Delta \vdash^{w} \phi}{\Gamma, \Delta \vdash \psi}$;

(Strong '⊃' Elimination) As above with '⊃' in place of '⊇';

(Weak '⊇' Elimination) $\dfrac{\Gamma, \Delta \vdash^{w} (\phi \supseteq \psi) \quad \Gamma, \Delta \vdash \phi}{\Gamma, \Delta \vdash^{w} \psi}$;

(Weak '⊃' Elimination) As above with '⊃' in place of '⊇';

(Strong '⊇' Introduction) $\dfrac{\Gamma, \Delta \cup \{\phi\} \vdash \phi \quad \Gamma, \Delta \cup \{\psi\} \vdash \psi \quad \Gamma, \Delta \cup \{\phi\} \vdash \psi}{\Gamma, \Delta \vdash \{\phi \supseteq \psi)}$;

(Strong '$\supset$' Introduction) $$\frac{\Gamma, \Delta \cup \{\phi\} \vdash \psi}{\Gamma, \Delta \vdash (\phi \supset \psi)};$$

(Weak '$\supseteq$' Introduction) $$\frac{\Gamma, \cup \{\phi\}, \Delta \cup \{\phi\} \vdash^{w} \psi}{\Gamma, \Delta \vdash^{w} (\phi \supseteq \psi)};$$

(Weak '$\supset$' Introduction) As above with '$\supset$' in place of '$\supseteq$'.

('T' Introduction) $$\frac{\Gamma, \Delta \vdash \phi}{\Gamma, \Delta \vdash T\phi};$$

('T' Elimination) $$\frac{\Gamma, \Delta \vdash^{w} T\phi}{\Gamma, \Delta \vdash \phi}.$$

In what follows, we always suppose that τ is substitutable for v in ϕ.

('$\underline{\exists}$' Bivalence): $$\frac{\Gamma, \Delta \vdash (\underline{\exists} v)\phi}{\Gamma \cup \{E(v)\}, \Delta \cup \{E(v), \phi\} \vdash \phi};$$

(Strong '$\underline{\exists}$' Elimination): $$\frac{\Gamma, \Delta \vdash (\underline{\exists} v)\phi \qquad \Gamma \cup \{E(v), \phi\}, \Delta \cup \{E(v), \phi\} \vdash \psi}{\Gamma, \Delta \vdash \psi};$$

where v is not free in ψ or in any member of Δ;

(Strong '$\exists$' Elimination): as above with '$\exists$' replacing '$\underline{\exists}$';

(Weak '$\underline{\exists}$' Elimination): $$\frac{\Gamma, \Delta \vdash^{w} (\underline{\exists} v)\phi \qquad \Gamma \cup \{E(v)\}, \Delta \cup \{E(v), \phi\} \vdash^{w} \psi}{\Gamma, \Delta \vdash^{w} \psi},$$

with v as above;

(Weak '$\exists$' Elimination): as above with '$\exists$' replacing '$\underline{\exists}$'.

(Strong '$\underline{\exists}$' Introduction): $$\frac{\Gamma, \Delta \vdash \phi(v/\tau) \qquad \Gamma, \Delta \vdash E(\tau) \qquad \Gamma \cup \{E(v)\}, \Delta \cup \{E(v), \phi\} \vdash \phi}{\Gamma, \Delta \vdash (\underline{\exists} v)\phi},$$

where v is not free in any member of Δ;

(Strong '∃' Introduction):
$$\begin{array}{l}\Gamma, \Delta \vdash \phi(v/\tau) \\ \Gamma, \Delta \vdash E(\tau) \\ \hline \Gamma, \Delta \vdash (\exists v)\phi\,;\end{array}$$

(Weak '$\underline{\exists}$' Introduction):
$$\begin{array}{l}\Gamma, \Delta \vdash^{w} \phi(v/\tau) \\ \Gamma, \Delta \vdash E(\tau) \\ \hline \Gamma, \Delta \vdash^{w} (\underline{\exists} v)\phi\,;\end{array}$$

(Weak '∃' Introduction: as above with '∃' replacing '$\underline{\exists}$';

(Strong Congruence):
$$\begin{array}{l}\Gamma, \Delta \vdash \phi(v/\tau_0) \\ \Gamma, \Delta \vdash (\tau_0 \approx \tau_1) \\ \hline \Gamma, \Delta \vdash \phi(v/\tau_1)\,;\end{array}$$

(Weak Congruence):
$$\begin{array}{l}\Gamma, \Delta \vdash^{w} \phi(v/\tau_0) \\ \Gamma, \Delta \vdash (\tau_0 \approx \tau_1) \\ \hline \Gamma, \Delta \vdash^{w} \phi(v/\tau_1)\ .\end{array}$$

Form rules (. . . '∃' - - -$_s$) and (. . . '$\underline{\exists}$' - - -$_s$) from (. . . '∃' - - -) and (. . . '$\underline{\exists}$' - - -) by replacing 'E' by 'E_s'; form (. . . Congruence$_s$) from (. . . Congruence) by replacing '$\approx$' by '$\approx_s$'.

In addition, we adopt these rules:

(Strong '$\Box$' Introduction):
$$\frac{\Gamma, \Delta \vdash \phi}{\Box\Gamma, \Box\Delta, \vdash \Box\phi\,;}$$

(Weak '$\Box$' Introduction): as above with '$\vdash^{w}$' replacing '$\vdash$';

(Strong '$\underline{\Box}$' Introduction):
$$\frac{\Gamma, \Delta \vdash \phi}{\underline{\Box}\Gamma, \underline{\Box}\Delta, \vdash \underline{\Box}\phi\,,}$$

where all members of $\Delta - \Gamma$ are of the form $(\tau_0 \not\approx \tau_1)$ or $(\tau_0 \not\approx_s \tau_1)$;

(Weak '$\underline{\Box}$' Introduction): as above with '$\vdash^{w}$' replacing '$\vdash$'.

Consider the following additional axioms:

(nn) $\{\ \}, \{\ \} \vdash (\exists v)E(v)$;

(nn_s) as above with 'E_s' replacing 'E';

(nn^*) as in (nn), with '$\underline{\exists}$' replacing '$\exists$';

(nn_s^*) as in (nn_s) with '$\underline{\exists}$' replacing '$\exists$';

(ea) $\{\phi\}, \{\phi\} \vdash E(\tau_i)$ where ϕ is either $\mathbf{P}(\tau_0, \ldots, \tau_{n-1})$ or $\neg\mathbf{P}(\tau_0, \ldots, \tau_{n-1})$, for $n \geqslant 1$, $i < n$;

(ea_s) as above with 'E_s' replacing 'E';

(eat) $\{E(\tau_0), \ldots, E(\tau_{n-1})\}, \{E(\tau_0), \ldots, E(\tau_{n-1}), \mathbf{P}(\tau_0, \ldots, \tau_{n-1})\} \vdash \mathbf{P}(\tau_0, \ldots, \tau_{n-1})$, where $n \geqslant 1$.

(eat_s) as above with 'E_s' replacing 'E'.

Form $\underline{\mathbf{K}}^p_{nn}(L)$ by adding (nn) or an appropriate variant to the axioms of $\underline{\mathbf{K}}^p(L)$; form $K^p_{ea}(L)$ similarly; form $L^p_{eat}(L)$ by adding (ea) and (eat), or appropriate variants, to the axioms of $\underline{\mathbf{K}}^p(L)$. Until further notice, 'x' is replaceable by the empty symbol, 'nn', 'ea' and 'eat'.

THEOREM p. 1. $\underline{\mathbf{K}}^p_x$ is sound and complete with respect to $\mathbf{K}^p_x$. Soundness follows by the usual induction on the length of derivations. The only rules which deserve comment govern '$\underline{\Box}$'. Where Frame($\mathfrak{A}$) $= (W, R)$ and α is an $\mathfrak{A}$-assignment, suppose:

$(\mathfrak{A}, w) \vDash_p \underline{\Box}\Gamma[\alpha]$; $(\mathfrak{A}, w) \vDash^w_p \underline{\Box}\Delta[\alpha]$;

all members of $\Delta - \Gamma$ are of the form $(\tau_0 \not\approx \tau_1)$;

(Γ, Δ, ϕ) is $\mathbf{K}^p_x$-valid.

If there is no u so that wRu, $(\mathfrak{A}, w) \vDash_p \underline{\Box}\phi[\alpha]$; suppose that there is such a u. For any such u, $(\mathfrak{A}, u) \vDash_p \Gamma[\alpha]$. Suppose $(\tau_0 \not\approx \tau_1) \in \Delta - \Gamma$; since $(\mathfrak{A}, w) \vDash^w_p \underline{\Box}(\tau_0 \not\approx \tau_1)[\alpha]$, either $(\mathfrak{A}, w) \vDash_p \underline{\Box}(\tau_0 \not\approx \tau_1)[\alpha]$, in which case $(\mathfrak{A}, u) \vDash_p (\tau_0 \not\approx \tau_1)[\alpha]$, or else $(\mathfrak{A}, w) \mid_p \underline{\Box}(\tau_0 \approx \tau_1)[\alpha]$; in the latter case for some v with wRv, $(\mathfrak{A}, v) \mid_p (\tau_0 \approx \tau_1)[\alpha]$; so den$(\Gamma, \alpha, \tau_i)\uparrow$ for some $i < 2$; so $(\mathfrak{A}, u) \vDash^w_p (\tau_0 \not\approx \tau_1)[\alpha]$. So $(\mathfrak{A}, u) \vDash^w_p \Delta[\alpha]$; so $(\mathfrak{A}, u) \vDash_p \phi[\alpha]$. We've shown that $(\mathfrak{A}, w) \vDash_p \underline{\Box}\phi[\alpha]$. A parallel argument applies when '$\approx_s$' $\in$ lex$_y$. A parallel argument shows that (W'$\underline{\Box}$'I) is sound.

It should be noticed that the restriction on members of $\Delta - \Gamma$ is essential; for example:

$(\{(\neg F\mathbf{P}) \supseteq \mathbf{Q}\}, \{(\neg F\mathbf{P}) \supseteq \mathbf{Q}, \mathbf{P}\}, \mathbf{Q})$ is $\mathbf{K}^p_x$-valid,
$(\{\underline{\Box}((\neg F\mathbf{P}) \supseteq \mathbf{Q})\}, \{\underline{\Box}((\neg F\mathbf{P}) \supseteq \mathbf{Q}), \underline{\Box}\mathbf{P}\}, \underline{\Box}\mathbf{Q})$ is not $\mathbf{K}^p_x$-valid.

Where $y = 0, \ldots,$ something like axioms (10*) and (11*) are needed. Unfortunately, I can see no way to replace them by an introduction or elimination rule, or by simpler axioms.

Notice that we have not introduced calculi $\underline{\mathbf{K}}^p_a$ or $\underline{\mathbf{K}}^p_{at}$ to handle the logics $\mathbf{K}^p_a$ and $\mathbf{K}^p_{at}$ respectively. Finding such calculi looks like a tricky matter. For example, (S'$\Box$'I) doesn't preserve $\mathbf{K}^p_a$-validity: for $i < 2$,

$(\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1)\}, E(\tau_i))$ is $\mathbf{K}^p_a$-valid;

$(\{\Box(\tau_0 \approx \tau_1)\}, \{\Box(\tau_0 \approx \tau_1)\}, \Box E(\tau_i))$ is not $\mathbf{K}^p_a$-valid.

The source of this anomaly is that $(\mathfrak{A}, w)$ may be denotation-wise actualistic and α may be a $(\mathfrak{A}, w)$-assignment while either $(\mathfrak{A}, u)$ is not denotation-wise actualistic or α is not an $(\mathfrak{A}, u)$-assignment. The following observation replaces a direct formalization of $\underline{\mathbf{K}}^p_x$, where '$x$' is replaced by '$a$', '$at$', '$a$ & nn' or 'at & nn'. Where '$\approx$' $\in \text{lex}^m_y$, let:

$$\Delta' = \{E(\tau)\text{: for some } \phi \in \Delta, \tau \in \text{Param}(\phi)\}.$$

Where '$\approx_s$' $\in \text{lex}_y$ and $y = 1, \ldots,$ let:

$$\Delta' = \{(\tau \approx_s \tau) \supset E_x(\tau)\text{: for some } \phi \in \Delta, \tau \in \text{Param}(\phi)\};$$

where $y = 0, \ldots,$ replace '$\supset$' by '$\supseteq$' in the preceding equation. In what follows, replace 'x' by 'a', 'at', 'a & nn', or 'at & nn' and replace 'x*' by 'ea', 'eat', 'ea & nn' or 'eat & nn' respectively;

For $\Gamma \subseteq \Delta \subseteq fml(L)$:

(Γ, Δ, ϕ) is $\mathbf{K}^p_x$-valid iff $(\Gamma, \Delta \cup \Delta', \phi)$ is $\mathbf{K}^p_{x^*}$-valid;

(Γ, Δ, ϕ) is weakly $\mathbf{K}^p_x$-valid iff $(\Gamma, \Delta \cup \Delta', \phi)$ is weakly $\mathbf{K}^p_{x^*}$-valid.

We'll now address the completeness of $\underline{\mathbf{K}}^{\rho}_{x}$. We'll make much use of the following notation. Where A is any set:

$A^{<\omega} = \{\langle x_0, \ldots, x_{n-1}\rangle : n < \omega, x_0, \ldots, x_{n-1} \in A\}$;

$|\langle x_0, \ldots, x_{n-1}\rangle| = n$; $\langle x_0, \ldots, x_n\rangle^- = \langle x_0, \ldots, x_{n-1}\rangle$;

$\langle x_0, \ldots, x_{n-1}\rangle^* x = \langle x_0, \ldots, x_{n-1}, x\rangle$, where $0 < n < \omega$;

where $w = \langle x_0, \ldots, x_{n-1}\rangle$, $u \subseteq w$ iff $u = \langle x_0, \ldots, x_i\rangle$ for some $i < n$;

for $u, w \in W$, u and v are incompatible iff there is no $w \in W$ so that $u, v \subseteq w$;

for $w_0, w_1 \in A^{<\omega}$, $w_0 \cap w_1 =$ the longest u so that $u \subseteq w_0, w_1$;
W is a tree on A iff $W \subseteq A^{<\omega}$ and for all $u \subseteq w \in W$, $u \in W$.

Fix a tree W on A. A name-array on W is a function $\mathscr{C}$ on W such that for each $w \in W$ $\mathscr{C}(w)$ is a set of individual constants such that:

card$(\mathscr{C}(w)) =$ card(A);

if $|w| \geqslant 1$ then $\mathscr{C}(w^-) \subseteq \mathscr{C}(w)$ and card$(\mathscr{C}(w) - \mathscr{C}(w^-)) =$ card(A);

if $v = u \cap w$ then $\mathscr{C}(v) = \mathscr{C}(u) \cap \mathscr{C}(w)$.

Let $\bar{\mathscr{C}} = \cup\{\mathscr{C}(w) : w \in W\}$; for $\mathbf{c} \in \bar{\mathscr{C}}$ let $u(\mathbf{c})$ be the $\mathbf{c}$-minimal w so that $\mathbf{c} \in \mathscr{C}(w)$; let $u(v) = \langle\ \rangle$ for $v \in \mathbf{Var}$. Given $L = L_y(\mathbf{Pred}, \mathbf{C})$, let $\kappa = \text{card}(\mathit{fml}(L)) = \max(\omega, \text{card}(\mathbf{Pred} \cup \mathbf{C}))$. (We take cardinals to be initial ordinals; card$(\kappa) = \kappa$.) Fix a name array on a tree W on κ so that $\mathbf{C} \subseteq \mathscr{C}(\langle\ \rangle)$ and card$(\mathscr{C}(\langle\ \rangle) - \mathbf{C}) = \kappa$. For $w \in W$ let $L^w = L_y(\mathbf{Pred}, \mathscr{C}(w))$, $\bar{L} = L_y(\mathbf{Pred}, \bar{\mathscr{C}})$. A diagram on W for L shall be an ordered pair (D_0, D_1), where D_0 and D_1 are functions on W with $D_0(w) \subseteq D_1(w) \subseteq \mathit{fml}(L')$, for some $L' = L_y(\mathbf{Pred}, \mathbf{C} \cup \mathbf{C}')$ and some $\mathbf{C}'$.

Where the diagram (D_0, D_1) is fixed, we'll let $\Gamma_w = D_0(w)$, $\Delta_w = D_1(w)$, for all $w \in W$. We adopt these definitions:

D is $\mathscr{C}$-strict iff for all $w \in W$, $\Delta_w \subseteq \mathit{fml}(L^w)$.

D is safe iff for any $w^*x \in W$: if $\phi \in \Delta_w$ then $\phi \in \Delta_{w*x}$, where ϕ is $(\tau_0 \not\approx_s \tau_1)$ or $(\tau_0 \not\approx \tau_1)$.

D is $\underline{\mathbf{K}}^p_x$-consistent iff for each $w \in W$ (Γ_w, Δ_w) is $\underline{\mathbf{K}}^p_x$-consistent;

D is $\Box$-normal iff for all $w^*x \in W$:
$\Box^{-1}\Gamma_w \subseteq \Gamma_{w*x}$, $\Box^{-1}\Delta_w \subseteq \Delta_{w*x}$;

D is $\underline{\Box}$-normal iff for all $w^*x \in W$:
$\underline{\Box}^{-1}\Gamma_w \subseteq \Gamma_{w*x}$,
if $\underline{\Box}\psi \in \Delta_w - \Gamma_w$ then for some $w^*z \in W$, ψ, $\psi \in \Delta_{w*z}$.

D is normal iff either $y = 1, \ldots$ and D is $\Box$-normal or $y = 0, \ldots$ and D is $\underline{\Box}$-normal.

Where $\Gamma \subseteq \Delta \subseteq fml(L)$ and $L = L_y(\mathbf{Pred}, \mathbf{C})$ we adopt these definitions:

(Γ, Δ) is $\neg$-complete for L iff for every $\phi \in fml(L)$ either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$ or $\phi, \neg\phi \in \Delta$;

(Γ, Δ) is $\exists$-complete for L iff for every $\phi \in fml(L)$:
if $(\exists v)\phi \in \Gamma$ then for some $\mathbf{c} \in \mathbf{C}$, $\phi(v/\mathbf{c}) \in \Gamma$,
if $(\exists v)\phi \in \Delta$ then for some $\mathbf{c} \in \mathbf{C}$, $\phi(v/\mathbf{c}) \in \Delta$;

(Γ, Δ) is $\underline{\exists}$-complete for L iff for every $\phi \in fml(L)$:
if $(\underline{\exists} v)\phi \in \Gamma$ then for some $\mathbf{c} \in \mathbf{C}$, $\phi(v/\mathbf{c}) \in \Gamma$, and for every $\tau \in \mathbf{Var} \cup \mathbf{C}$ either $\phi(v/\tau)$ or $\phi(v/\tau) \in \Gamma$;
if $(\underline{\exists} v)\phi \in \Delta$ then for some $\mathbf{c} \in \mathbf{C}$, $\phi(v/\mathbf{c}) \in \Delta$.

Where D is a diagram on W:

D is $\neg$-complete for $\mathscr{C}$ iff for each $w \in W$, (Γ_w, Δ_w) is $\neg$-complete for L^w;

D is $\exists$-complete for $\mathscr{C}$ iff for each $w \in W$, (Γ_w, Δ_w) is $\exists$-complete for L^w;

D is $\underline{\exists}$-complete for $\mathscr{C}$ iff for each $w \in W$, (Γ_w, Δ_w) is $\underline{\exists}$-complete for L^w;

D is $\Diamond$-complete iff for every $w \in W$:
if $\neg\Box\phi \in \Gamma_w$ then for some $w^*x \in W$, $\neg\phi \in \Gamma_{w*x}$;
if $\neg\Box\phi \in \Delta_w$ then for some $w^*x \in W$, $\neg\phi \in \Delta_{w*x}$;

D is $\underline{\Diamond}$-complete iff for every $w \in W$:
if $\neg\underline{\Box}\phi \in \Gamma_w$ then for some $w{*}x \in W$, $\neg\phi \in \Gamma_{w*x}$,
and for every $w{*}y \in W$ either ϕ or $\neg\phi \in \Gamma_{w*y}$;
if $\neg\underline{\Box}\phi \in \Delta_w$ then for some $w{*}x \in W$, $\neg\phi \in \Delta_{w*x}$;

D is complete for $\mathscr{C}$ iff D is $\neg$-complete for $\mathscr{C}$ and:
if $y = 1, \ldots$ then D is $\exists$-complete for $\mathscr{C}$ and $\Diamond$-complete;
if $y = 0, \ldots$ then D is $\underline{\exists}$-complete for $\mathscr{C}$ and $\underline{\Diamond}$-complete.

The following facts are easy to see and shall be used without further notice in what follows.

If D is $\mathscr{C}$-strict, consistent and $\neg$-complete for $\mathscr{C}$, $w \in W$ and $\phi \in fml(L^w)$ then:

if $\Gamma_w, \Delta_w \vdash \phi$ then $\phi \in \Gamma_w$;

if $\Gamma_w, \Delta_w \vdash^w \phi$ then $\phi \in \Delta_w$.

Where $y = 1, \ldots$ and D is $\mathscr{C}$-strict, $\underline{\mathbf{K}}^p_x$-consistent, $\Box$-normal and $\neg$-complete for $\mathscr{C}$, D is safe; this follows using axioms (9) or (9_s) and the fact that $\Box^{-1}\Delta_w \subseteq \Delta_{w*x}$ for any $w{*}x \in W$. Notice that this argument doesn't carry over to $y = 0, \ldots$, since $\underline{\Box}$-normality doesn't imply that $\underline{\Box}^{-1}\Delta_w \subseteq \Delta_{w*x}$.

For what follows, suppose that D is a $\mathscr{C}$-strict, safe, normal, consistent complete for $\mathscr{C}$ diagram on W for L.

LEMMA *p*. 1. *Where* '$\approx$' $\in$ *lex*$_y$: *if for* $w \in W$, $(\tau_0 \approx \tau_1) \in \Gamma_w$ *then for any* $u \in W$ *with* $\tau_0, \tau_1 \in \boldsymbol{Var} \cup \mathscr{C}(u)$, $(\tau_0 \approx \tau_1) \in \Gamma_u$; *where* '$\approx_s$' $\in$ *lex*$_y$, *replace* '$\approx$' *by* '$\approx_s$' *in the preceding.*

Proof. Suppose '$\approx$' $\in$ lex$_y$. Let $u_0 = u(\tau_0) \cap u(\tau_1)$. By axioms of group (8) and the normality of D, it suffices to show that if $(\tau_0 \approx \tau_1) \in \Gamma_w$ then $(\tau_0 \approx \tau_1) \in \Gamma_{u_0}$. If $(\tau_0 \not\approx \tau_1) \in \Delta_{u_0}$, since D is safe and $u_0 \subseteq w$, $(\tau_0 \not\approx \tau_1) \in \Delta_w$; so (Γ_w, Δ_w) is inconsistent, a contradiction. Where '$\approx_s$' $\in$ lex$_y$ a similar argument applies. QED

LEMMA *p*. 2. *Where* '$\approx$' $\in$ *lex*$_y$:

(1) $\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1)\} \vdash (\tau_0 \approx \tau_0)$

(2) $\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1)\} \vdash (\tau_1 \approx \tau_0)$

(3) $\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1)\} \vdash (\tau_1 \approx \tau_1)$

(4) $\{(\tau_0 \approx \tau_1), (\tau_1 \approx \tau_2)\}, \{(\tau_0 \approx \tau_1), (\tau_1 \approx \tau_2)\} \vdash (\tau_0 \approx \tau_2)$

Where '$\approx_s$' $\in$ lex$_y$, replace '$\approx$' by '$\approx_s$' in the preceding.

Proof. Select $v \in$ **Var** distinct from τ_0 and τ_1. We have:

$$\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1), (\tau_0 \not\approx \tau_0)\} \vdash^w (\tau_0 \not\approx v)(v/\tau_0);$$

by (WC):

$$\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1), (\tau_0 \not\approx \tau_0)\} \vdash^w (\tau_0 \not\approx v)(v/\tau_1);$$

but cut and indirect proof, (1) as follows. So

$$\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1)\} \vdash (v \approx \tau_0)(v/\tau_0);$$

by (SC):

$$\{(\tau_0 \approx \tau_1)\}, \{(\tau_0 \approx \tau_1)\} \vdash (v \approx \tau_0)(v/\tau_1),$$

which is (2). Using (1) and (2), (3) follows. (4) also follows using (SC) and writing $(\tau_0 \approx \tau_1)$ as $(\tau_0 \approx v)(v/\tau_1)$. Where '$\approx_s$' replaces '$\approx$', the preceding arguments still apply. QED

Let U_0 = **Var** $\cup\ \mathscr{C}$. For $\tau_0, \tau_1 \in U_0$, let $\tau_0 \sim \tau_1$ iff:

if '$\approx$' $\in$ lex$_y^m$ then for some $w \in W$, $(\tau_0 \approx \tau_1) \in \Gamma_w$;

if '$\approx_s$' $\in$ lex$_y^m$ then for some $w \in W$, $(\tau_0 \approx_s \tau_1) \in \Gamma_w$.

Let $U_1 = Fld(\sim)$; by Lemma *p*. 2, $U_1 = \{\tau\colon \tau \sim \tau\}$ and $\sim$ is a symmetric. Notice that $\sim$ need not be transitive; where $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2 \in \mathscr{C}$ so that $u(\mathbf{c}_1) \subseteq u(\mathbf{c}_0), u(\mathbf{c}_2)$, but $u(\mathbf{c}_0)$ and $u(\mathbf{c}_2)$ are incompatible, we may have $\mathbf{c}_0 \sim \mathbf{c}_1$ and $\mathbf{c}_1 \sim \mathbf{c}_2$ but not $\mathbf{c}_0 \sim \mathbf{c}_2$, because there is no w so that $\mathbf{c}_0, \mathbf{c}_2 \in \mathscr{C}(w)$.

LEMMA *p*. 3. *Let* $\tau_0, \ldots, \tau_n$ *be a* $\sim$*-chain and* $u(\tau_0), u(\tau_n) \subseteq w$. *Where* '$\approx$' $\in$ *lex*$_y^m$, $(\tau_0 \approx \tau_n) \in \Gamma_w$. *Where* '$\approx_s$' $\in$ *lex*$_y$, $(\tau_0 \approx_s \tau_n) \in \Gamma_w$.
Proof is by induction on $n \geqslant 1$. If $n = 1$, this is Lemma *p*. 1. Suppose $n > 1$ and for all $m < n$ the lemma holds. For each $i < n$, either $u(\tau_i) \subseteq u(\tau_{i+1})$ or $u(\tau_{i+1}) \subseteq u(\tau_i)$. So for some $k \leqslant n$: for all $i \leqslant n$ $u(\tau_k) \subseteq u(\tau_i)$; call such a k a root for $\{\tau_0, \ldots, \tau_n\}$. By choice of w either $u(\tau_0) \subseteq u(\tau_n)$ or $u(\tau_n) \subseteq u(\tau_0)$.

CASE 1. $u(\tau_0) \subseteq u(\tau_n)$. Fix a root $k < n$ for $\{\tau_0, \ldots, \tau_n\}$. If $0 < k$, applying the induction hypothesis to $\tau_0, \ldots, \tau_k$ and to $\tau_k, \ldots, \tau_n$, we have $(\tau_0 \approx \tau_k)$, $(\tau_k \approx \tau_n) \in \Gamma_w$; by Lemma *p*. 2, $(\tau_0 \approx \tau_n) \in \Gamma_w$. Suppose there is no root $k > 0$ for $\{\tau_0, \ldots, \tau_n\}$; so $k = 0$ is the only such root. Let k' be a root for $\{\tau_1, \ldots, \tau_n\}$. Without loss of generality, suppose that $u(\tau_0) \subseteq u(\tau_{k'}) \subseteq u(\tau_n)$. Suppose $k' < n$; applying the induction hypothesis to $\tau_0, \ldots, \tau_{k'}$, and to $\tau_{k'}, \ldots, \tau_n$, we get $(\tau_0 \approx \tau_{k'})$, $(\tau_{k'} \approx \tau_n) \in \Gamma_w$; so $(\tau_0 \approx \tau_n) \in \Gamma_w$. If $k' = n$ the result is obvious.

CASE 2. $u(\tau_n) \subseteq u(\tau_0)$; then there is a root $k > 0$ for $\{\tau_0, \ldots, \tau_n\}$; if $k < n$ we proceed as in the previous case when $0 < k$; if $k = n$, let k' be a root for $\{\tau_0, \ldots, \tau_{n-1}\}$; we proceed as in the previous case when $k = 0$. If '$\approx_s$' $\in \text{lex}_y^m$, the above argument applies after replacements. QED

Let $\sim^*$ be the transitive closure of $\sim$. Clearly $\sim^*$ is an equivalence relation on U_1. For $\tau \in U_1$ let $[\tau]$ = the equivalence class of τ under $\sim^*$, i.e. $\{\tau' : \tau \sim^* \tau'\}$. Let $U = U_1/\sim^* = \{[\tau] : \tau \in U_1\}$. Where '$\approx$' $\in \text{lex}_y$, let $\bar{U}(w) = \{[\tau] : E(\tau) \in \Gamma_w\}$ for $w \in W$; where '$\approx_s$' $\in \text{lex}_y$, replace 'E' by 'E_s'. By (SC):

$$\{(\tau_0 \approx \tau_1), E(\tau_i)\}, \{(\tau_0 \approx \tau_1), E(\tau_i)\} \vdash E(\tau_{1-i}),$$

for $i < 2$ and '$\approx$' $\in \text{lex}_y$; so by Lemma *p*. 2, $\bar{U}(w)$ is well-defined; a similar argument works if '$\approx_s$' $\in \text{lex}_y$. We are now ready to convert D into a model $\mathfrak{M}$ for L and an assignment α. Let:

$$R = \{(w^-, w) : w \in W, |w| \geqslant 1\};$$

$$\mathscr{E}(\mathbf{P})(w, [\tau_0], \ldots, [\tau_{n-1}]) \simeq \begin{cases} 1 & \text{if } \mathbf{P}(\tau_0, \ldots, \tau_{n-1}) \in \Gamma_w; \\ 0 & \text{if } \neg\mathbf{P}(\tau_0, \ldots, \tau_{n-1}) \in \Gamma_w; \end{cases}$$

$$\mathscr{N}(\mathbf{c}) \simeq [\mathbf{c}] \text{ for } \mathbf{c} \in \mathscr{C};$$

$$\alpha(v) = [v] \text{ for } v \in \mathbf{Var};$$

here if $\mathbf{c} \notin U_1$, $\mathscr{N}(\mathbf{c})\uparrow$; similarly for $\alpha_D(v)$. Let $\mathfrak{A} = (W, R, U, \bar{U}, \mathscr{E}, \mathscr{N})$, $\mathfrak{M} = (\mathfrak{A}, \langle\ \rangle)$. It's easy to see that $\mathfrak{M}$ is $\underline{\mathbf{K}}_x^p$-model for L.

LEMMA *p*. 4. *Let $v_0, \ldots, v_{n-1} \in$ **Var** be distinct, $\phi \in fml(L^w)$ for $w \in W$, $\tau_0, \ldots, \tau_{n-1} \in$ **Var** $\cup\ \mathscr{C}(w)$ so that $\tau_0, \ldots, \tau_{n-1}$ are substitutable* *for* $v_0, \ldots, v_{n-1}$ *in* ϕ; *let* $\phi' = \phi(v_0, \ldots, v_{n-1}/\tau_0, \ldots, \tau_{n-1})$ *and*

$$\alpha' = \alpha^{v_0, \ldots, v_{n-1}}_{[\tau_0], \ldots, [\tau_{n-1}]}; \text{ here if } \tau_i \notin U_1 \text{ then } \alpha'(v_i)\uparrow.$$

Then:

(i) for $\tau \notin \{v_0, \ldots, v_{n-1}\}$, $\mathrm{den}(\mathfrak{A}_D, \alpha', \tau) \simeq [\tau]$;

(ii) $\mathrm{den}(\mathfrak{A}_D, \alpha', v_i) \simeq [\tau_i]$ for $i < n$;

(iii) $\phi' \in \Gamma_w$ iff $(\mathfrak{A}, w) \vDash_p \phi[\alpha']$;

(iv) $\neg\phi' \in \Gamma_w$ if $(\mathfrak{A}, w)\ {}_p\!\dashv \phi[\alpha']$;

(v) $\phi', \neg\phi' \in \Delta_w$ iff $(\mathfrak{A}, w) \mid_p \phi[\alpha']$.

(i) and (ii) are obvious; the rest is proved by induction on the depth of ϕ.

Suppose ϕ is $(\tau \approx \sigma)$ and ϕ' is $(\tau' \approx \sigma')$. If $\phi' \in \Gamma_w$ then $\tau' \sim \sigma'$; by (i) and (ii) $\mathrm{den}(\mathfrak{A}, \alpha', \tau) = [\tau'] = [\sigma'] = \mathrm{den}(\mathfrak{A}, \alpha', \sigma)$; so $(\mathfrak{A}, w) \vDash_p \phi[\alpha']$. If $(\mathfrak{A}, w) \vDash_p \phi[\alpha']$ then $\mathrm{den}(\mathfrak{A}, \alpha', \tau) = [\tau'] = [\sigma'] = \mathrm{den}(\mathfrak{A}_D, \alpha', \sigma)$, all being defined; since $\tau' \sim^* \sigma'$, by Lemma *p*. 3, $\phi' \in \Gamma_w$. If $\neg\phi' \in \Gamma_w$, by Axiom (6) $\tau', \sigma' \in U_1$, so $\mathrm{den}(\mathfrak{A}, \alpha', \tau) = [\tau']$ and $\mathrm{den}(\mathfrak{A}, \alpha', \sigma) = [\sigma']$ all these being defined; by Lemma *p*. 3 and the consistency of D, $\tau' \nsim^* \sigma'$; so $[\tau'] \neq [\sigma']$; so $(\mathfrak{A}, w)\ {}_p\!\dashv \phi[\alpha']$. If $(\mathfrak{A}, w)\ {}_p\!\dashv \phi[\alpha']$ then $\mathrm{den}(\mathfrak{A}, \alpha', \tau) = [\tau'] \neq [\sigma'] = \mathrm{den}(\mathfrak{A}, \alpha', \sigma)$, all being defined. Since $\tau', \sigma' \in U_1$ and $\sim$ is reflexive, $(\tau' \approx \tau')$, $(\sigma' \approx \sigma') \in \Gamma_w$; by axiom (7) either ϕ' or $\neg\phi' \in \Gamma_w$; in the former case, $\tau' \sim \sigma'$, a contradiction; so $\neg\phi' \in \Gamma_w$. Where ϕ is $(\tau \approx_s \sigma)$ a similar argument applies. For all other atomic ϕ the arguments are straightforward.

Where ϕ is $(\phi_0 \supset \phi_1)$, $(\phi_0 \supseteq \phi_1)$, $(\exists v)\psi$ or $(\underline{\exists} v)\psi$ the induction steps are straightforward applications of the $\neg$-completeness, $\exists$-completeness or $\underline{\exists}$-completeness of (Γ_w, Δ_w) and the rules governing '$\supset$', '$\supseteq$', '$\exists$' and '$\underline{\exists}$'. Where ϕ is $\Box\psi$, the induction step is a straightforward application of the $\Diamond$-completeness and $\Box$-normality of D. Suppose ϕ is $\underline{\Box}\psi$. If $\phi' \in \Gamma_w$ then for all $w^*x \in W$, $\psi' \in \Gamma_{w^*x}$; so $(\mathfrak{A}, w^*x) \vDash_p \psi[\alpha']$; so $(\mathfrak{A}, w) \vDash \phi[\alpha']$. If $\neg\phi' \in \Gamma_w$ by Axiom (10*) $\underline{\Box}(\psi' \supseteq \psi') \in \Gamma_w$; so for any $w^*x \in W$ $(\psi' \supseteq \psi') \in \Gamma_{w^*x}$; we want to have $(\mathfrak{A}, w^*x) \vDash_p (\psi \supseteq \psi)[\alpha']$. This will hold if we've defined the depth of formulae correctly; so that:

$$\text{depth}(\theta_0 \supseteq \theta_1) = 1 + \max\{\text{depth}(\theta_0), \text{depth}(\theta_1)\};$$

$$\text{depth}(\underline{\Box}\theta) = 2 + \text{depth}(\theta).$$

Therefore either $(\mathfrak{A}, w^*x) \models_p \psi[\alpha']$ or $(\mathfrak{A}, w^*x)\ {}_p\!\dashv\ \psi[\alpha']$. By $\underline{\Diamond}$-completeness, for some $w^*y \in W$, $\neg\psi' \in \Gamma_{w*y}$; so $(\mathfrak{A}, w^*y)\ {}_p\!\dashv\ \psi[\alpha']$; thus $(\mathfrak{A}, w)\ {}_p\!\dashv\ \phi[\alpha']$. If $\phi' \in \Delta_w - \Gamma_w$, $\underline{\Box}$-normality yields a $w^*x \in W$ so that $\psi', \neg\psi' \in \Delta_{w*x}$; so $(\mathfrak{A}, w^*x) \mid_p \psi[\alpha']$; so $(\mathfrak{A}, w) \models_p^w \phi[\alpha']$. If $\phi' \in \Delta_w$, by $\underline{\Diamond}$-completeness there is a $w^*x \in W$ with $\psi' \in \Delta_{w*x}$; so $(\mathfrak{A}, w^*x) \not\models_p \psi[\alpha']$; therefore $(\mathfrak{A}, w) \models_p^w \phi[\alpha']$. From these (iii), (iv) and thus (v) all hold for ϕ.

Theorem *p*. 1 now follows as usual from the following model-existence theorem.

THEOREM *p*. 2. *Where* $\Gamma \subseteq \Delta \subseteq fml(L)$ *and* (Γ, Δ) *is* $\underline{\mathbf{K}}_x^p$-*consistent, there is a* $\mathbf{K}_x$-*model* $\mathfrak{M} = (\mathfrak{A}, w)$ *and an* $\mathfrak{A}$-*assignment* α *so that* $\mathfrak{M} \models_p \Gamma[\alpha]$ and $\mathfrak{M} \models_p^w \Delta[\alpha]$.

Proof. Assume the antecedent. Fix a name-array $\mathscr{C}'$ on $\kappa^{<\omega}$. We'll construct a tree W on κ and a diagram D on W, $\mathscr{C} = \mathscr{C}' \mid W$ for L so that D is strict, safe, $\underline{\mathbf{K}}_x^p$-consistent, normal, complete, and $\Gamma \subseteq \Gamma_{\langle\,\rangle}$, $\Delta \subseteq \Delta_{\langle\,\rangle}$. Then the reduct of $\mathfrak{M}$ to L and α are as required.

CASE 1. $y = 1, \ldots$ We inflate (Γ, Δ) to a consistent, $\neg$-complete and $\exists$-complete pair $(\Gamma_{\langle\,\rangle}, \Delta_{\langle\,\rangle})$, $\Gamma_{\langle\,\rangle} \subseteq \Delta_{\langle\,\rangle} \subseteq fml(L^{\langle\,\rangle})$, using the usual Henkin-style induction on a well-ordering of $fml(L^{\langle\,\rangle})$. Letting $W_0 = \{\langle\,\rangle\}$, suppose we have constructed $W_n \subseteq \kappa^{<\omega}$ so that for all $w \in W_n$, $|w| \leqslant n$ and (Γ_w, Δ_w) has been defined and is consistent. Where $w \in W_n$, $|w| = n$ and for some $\phi \in fml(L^w)$ $\neg\Box\phi \in \Delta_w$, we'll define Γ_{w*x} and Δ_{w*x} for all $x \in \kappa$. Let $\langle\psi_x : x < \kappa\rangle$ be a listing, perhaps with repetitions, of all ψ such that $\neg\Box\psi \in \Delta_w$. Let:

$$\Delta^0_{w*x} = \Box^{-1}\Delta_w \cup \{\neg\psi_x\};$$

$$\Gamma^0_{w*x} = \begin{cases} \Box^{-1}\Gamma_w & \text{if } \neg\Box\psi_x \notin \Gamma_w; \\ \Box^{-1}\Gamma_w \cup \{\neg\psi_x\} & \text{otherwise.} \end{cases}$$

If $(\Gamma^0_{\omega\bullet x}, \Delta^0_{w\bullet x})$ were inconsistent then (Γ_w, Δ_w) would be inconsistent; so $(\Gamma^0_{w\bullet x}, \Delta^0_{w\bullet x})$ is consistent. By the usual Henkin construction, we obtain $\Gamma_{w\bullet x} \subseteq \Delta_{w\bullet x}$, with $\Gamma^0_{w\bullet x} \subseteq \Gamma_{w\bullet x}$ and $\Delta^0_{w\bullet x} \subseteq \Delta_{w\bullet x}$ so that $(\Gamma_{w\bullet x}, \Delta_{w\bullet x})$ is consistent, $\neg$-complete and $\exists$-complete for $L_{w\bullet x}$. After ω-steps we've constructed the desired diagram.

CASE 2. $y = 0, \ldots$ Construct $\Gamma_{\langle\,\rangle}$ and $\Delta_{\langle\,\rangle}$ to be $\underline{\mathbf{K}}^p_x$-consistent, $\neg$-complete for $L_{\langle\,\rangle}$, and $\underline{\exists}$-complete for $L_{\langle\,\rangle}$. Again, this involves the usual Henkin construction; the additional requirement when $(\underline{\exists} v)\phi \in \Gamma_{\langle\,\rangle}$ can be guaranteed using the distinctive features of the '$\underline{\exists}$'-rules. Suppose W_n has been constructed as above. For $w \in W_n$, $|w| = n$, if for some ψ either $\neg\underline{\Box}\psi \in \Delta_w$ or $\underline{\Box}\psi \in \Delta_w - \Gamma_w$, we must construct $\Gamma_{w\bullet x} \subseteq \Delta_{w\bullet x}$ for all $x \in \kappa$; let $\langle \psi_x : x < \kappa\rangle$ be a κ-listing of all such ψ. Where '$\approx$' $\in$ lex$_y$, if $\neg\underline{\Box}\psi_x \in \Delta_w$ and $\underline{\Box}\psi_x \notin \Delta_x - \Gamma_w$, let:

$$\Delta^0_{w\bullet x} = \underline{\Box}^{-1}\Gamma_w \cup \{\neg\psi_x\} \cup \{(\tau_0 \not\approx \tau_1) : (\tau_0 \not\approx \tau_1 \in \Delta_w\};$$

$$\Gamma^0_{w\bullet x} = \begin{cases} \underline{\Box}^{-1}\Gamma_w & \text{if } \neg\underline{\Box}\psi_x \in \Gamma_w; \\ \underline{\Box}^{-1}\Gamma_w \cup \{\neg\psi_x\} & \text{otherwise.} \end{cases}$$

CLAIM. $(\Gamma^0_{w\bullet x}, \Delta^0_{w\bullet x})$ is consistent. Suppose not.
If $\neg\underline{\Box}\psi_x \in \Gamma_w$ then:

$$\underline{\Box}^{-1}\Gamma_w, \underline{\Box}^{-1}\Gamma_w \cup \{(\tau_0 \not\approx \tau_1) : (\tau_0 \not\approx \tau_1) \in \Delta_w\} \vdash \psi_x;$$

By $(S\text{`}\Box\text{'}I)$,

$$\Gamma_w, \Gamma_w \cup \{\underline{\Box}(\tau_0 \not\approx \tau_1) : (\tau_0 \not\approx \tau_1) \in \Delta_w\} \vdash \underline{\Box}\psi_x;$$

using (9*) we have: $\Gamma_w, \Delta_w \vdash \underline{\Box}\psi_x$, a contradiction. If $\neg\underline{\Box}\psi_x \notin \Gamma_w$ then:

$$\underline{\Box}^{-1}\Gamma_w, \underline{\Box}^{-1}\Gamma_w \cup \{(\tau_0 \not\approx \tau_1) : (\tau_0 \not\approx \tau_1) \in \Delta_w\} \vdash^w \underline{\Box}\psi_x.$$

By $(W\Box I)$ and (9*) we have $\Gamma_w, \Delta_w \vdash^w \underline{\Box}\psi_x$; so $\underline{\Box}\psi_x \in \Delta_w - \Gamma_w$, contrary to assumption.

If $\underline{\Box}\psi_x \in \Delta_w - \Gamma_w$, let

$$\Delta^0_{w\bullet x} = \underline{\Box}^{-1}\Gamma_w \cup \{\psi_x, \neg\psi_x\} \cup \{(\tau_0 \not\approx \tau_1) : (\tau_0 \not\approx \tau_1) \in \Delta_w\};$$

$$\Gamma^0_{w\bullet x} = \underline{\Box}^{-1}\Gamma_w.$$

CLAIM. $(\Gamma^0_{w\bullet x}, \Delta^0_{w\bullet x})$ is consistent. For suppose we have: $\Gamma^0_{w\bullet x}, \Delta^0_{w\bullet x} \vdash \bot$. Then:

$$\underline{\Box}^{-1}\Gamma_w, \underline{\Box}^{-1}\Gamma_w \cup \{(\tau_0 \not\approx \tau_1) : (\tau_0 \not\approx \tau_1) \in \Delta_w\} \vdash (\psi_x \supseteq \psi_x);$$

by $(S`\Box\text{'}I)$,

$$\Gamma_w, \Gamma_w \cup \{\underline{\Box}(\tau_0 \not\approx \tau_1) : (\tau_0 \not\approx \tau_1) \in \Delta_w\} \vdash \underline{\Box}(\psi_x \supseteq \psi_x).$$

By (9*) and (11*): $\Gamma_w, \Gamma_w \vdash \underline{\Box}\psi_x$; so $\underline{\Box}\psi_x \in \Gamma_w$, contrary to supposition. Where '$\approx_s$' $\in \text{lem}_y$, replace '$\approx$' by '$\approx_s$' throughout the preceding discussion. Now $(\Gamma^0_{w\bullet x}, \Delta^0_{w\bullet x})$ may be inflated to a $(\Gamma_{w\bullet x}, \Delta_{w\bullet x})$ which is consistent, $\neg$-complete, and $\underline{\exists}$-complete. After ω steps, we've constructed the desired diagram D. To see that D is $\underline{\Diamond}$-complete, notice that Axiom (10*) or (10^*_s) insures that if $\neg\underline{\Box}\psi \in \Gamma_w$ then for any $x \in \kappa$ $(\psi \supseteq \psi) \in \Gamma_{w\bullet x}$; so either ψ or $\neg\psi \in \Gamma_{w\bullet x}$.

6. FORMALIZING SEMI-POSSIBILISTIC AND ACTUALISTIC LOGICS BASED ON K

Formalizing logics of the form $\mathbf{K}^{sp}_x$ shall be a useful stepping-stone from §5 to formalizing logics of the form $\mathbf{K}^a_x$. Formalizing $\mathbf{K}^{sp}_x$ involves a peculiar difficulty for the case in which $y = 0, \ldots$ So we shall define $\underline{\mathbf{K}}^{sp}_x(L_y)$ where $y = 1, \ldots$; then we'll consider the case in which y is 0, T or 0, T, **u**. As the axioms of $\underline{\mathbf{K}}^{sp}_x(L_y)$ we take:

(1)–(5) above;

(11) $\{\ \}, \{\ \} \vdash^w (\tau \approx \tau)$;

(11_s) $(\ \}, \{\ \} \vdash^w (\tau \approx_s \tau)$;

(12) $\{\phi\}, \{\phi\} \vdash E(\tau_i)$, where ϕ is either $(\tau_0 \approx \tau_1)$ or $(\tau_0 \not\approx \tau_1)$ and $i < 2$;

(12_s) as above with '$\approx_s$' and 'E_s' replacing '$\approx$' and 'E';

(13) $\{E(\tau_0), E(\tau_1)\}, \{E(\tau_0), E(\tau_1), (\tau_0 \approx \tau_1)\} \vdash (\tau_0 \approx \tau_1)$;

(13_s) $\{E_s(\tau_i), E_s(\tau_i), (\tau_0 \approx_s \tau_1)\} \vdash (\tau_0 \approx_s \tau_1)$, for $i < 2$.

We take over all rules from $\underline{\mathbf{K}}_x^p$ appropriate to L_y where $y = 1, \ldots,$ except for (SC), (SC$_s$), (WC) and (WC$_s$); these are replaced by:

(Extended Strong Congruence):
$$\frac{\Gamma, \Delta \vdash \phi(v/\tau_0) \qquad \Gamma, \Delta \vdash \neg\Box^n(\tau_0 \not\approx \tau_1)}{\Gamma, \Delta \vdash \phi(v/\tau_1)} \quad ;$$

(Extended Weak Congruence):
$$\frac{\Gamma, \Delta \vdash^w \phi(v/\tau_0) \qquad \Gamma, \Delta \vdash \neg\Box^n(\tau_0 \not\approx \tau_1)}{\Gamma, \Delta \vdash^w \phi(v/\tau_1)} \quad ;$$

(Extended Strong Congruence$_s$): as in (ESC) with '$\approx_s$' replacing '$\approx$';

(Extended Weak Congruence$_s$): as in (EWC) with '$\approx_s$' replacing '$\approx$'.

It's worth noticing that axioms of groups (6) and (7) are $\mathbf{K}_x^{sp}$-valid, while those of groups (8) and (9) are not; Axioms (11) are (11$_s$), (13) and (13$_s$) are $\mathbf{K}_x^p$-valid, while (12) and (12$_s$) are not.

THEOREM *sp*. 1. Restricted to $L = L_y$ for $y = 1, \ldots,$ $\underline{\mathbf{K}}_x^{sp}$ is sound and complete with respect to $\mathbf{K}_x^{sp}$. Soundness follows easily. We'll now prove completeness by indicating the needed modifications of the proof of Theorem *p*. 1. In order to save space, proofs of the following lemmas have been omitted, at the suggestion of *the Journal of Philosophical Logic*.

Suppose we're given L, a tree W on $\kappa^{<\omega}$, a name-array $\mathscr{C}$ on W and a $\mathscr{C}$-strict, $\Box$-normal $\underline{\mathbf{K}}_x^{sp}$-consistent complete diagram D on W for L.

LEMMA *sp*. 1. *Where* '$\approx$' $\in lex_y$: *if* $(\tau_0 \approx \tau_1) \in \Gamma_w$ *then* (i) *for every* $u \in W$ *with* $\tau_0, \tau_1 \in \mathbf{Var} \cup \mathscr{C}(u)$, $(\tau_0 \approx \tau_1) \in \Delta_u$; *and* (ii) $E(\tau_0) \in \Gamma_u$ *iff* $E(\tau_1) \in \Gamma_u$. *Where* '$\approx_s$' $\in lex_y$, *replace* '$\approx$' by '$\approx_s$'.

Proof uses (*ESC*), Axiom (11), (S'$\Box$'I) and $\Box$-normality. Note: it is with this lemma that we run into trouble if $y = 0, \ldots$; the role here played by $\Box$-normality could not be played by $\underline{\Box}$-normality.

LEMMA *sp*. 2. *As in Lemma p. 2, with* $\underline{\mathbf{K}}_x^{sp}(L)$ *replacing* $\underline{\mathbf{K}}_x^{p}(L)$. Proof uses Axiom (11).

Define U_0, $\sim$ and U_1 as before; again $U_1 = \{\tau: \tau \sim \tau\}$ and $\sim$ is symmetric.

LEMMA *sp*. 3. *Let* $\tau_0, \ldots, \tau_n$ *be a* $\sim$*-chain and* $u(\tau_0), u(\tau_n) \subseteq w$. *Where* '$\approx$' $\in lex_y$, *if* $E(\tau_0) \in \Gamma_w$ *then* $(\tau_0 \approx_s \tau_n) \in \Gamma_w$; *where* '$\approx_s \in lex_y$, *replace* '$\approx$' *and* '$E$' *by* '$\approx_s$' *and* '$E_s$'. Proof is by induction on $n \geqslant 1$. If $n = 1$, use Lemma *sp*. 1 and Axiom (13). For $n > 1$, use Axiom (12) and Lemma *sp*. 2, and the ideas used for Lemma *p*. 3.

Define $\sim^*$, U, $\bar{U}(w)$ for $w \in W$, R, $\mathscr{E}$, $\mathscr{N}$, $\mathfrak{A}$, $\mathfrak{M}$, and α as before; again $\mathfrak{M}$ is a $\mathbf{K}_x$-model for $\bar{L}$.

LEMMA *sp*. 4. *For* $v_0, \ldots, v_{n-1}, \phi, w, \tau_0, \ldots, \tau_{n-1}, \phi'$ and α' as in Lemma *p*. 4:

(i) and (ii) of Lemma *p*. 4 hold;

(iii) $\phi' \in \Gamma_w$ iff $(\mathfrak{A}, w) \vDash_{sp} \phi[\alpha']$;

(iv) $\neg\phi' \in \Gamma_w$ iff $(\mathfrak{A}, w)\ {}_{sp}\!\Dashv \phi[\alpha']$;

(v) $\phi', \neg\phi' \in \Delta_w$ iff $(\mathfrak{A}, w) \mid_{sp} \phi[\alpha']$.

Proof. To handle identities, use Axioms (12) and (13), and Lemma *sp*. 3. All other cases are straightforward.

Theorem *sp*. 1 will now follow from the following model-existence theorem:

THEOREM *sp*. 2. Where $y = 1, \ldots, \Gamma \subseteq \Delta \subseteq fml(L_y)$ and (Γ, Δ) is $\underline{\mathbf{K}}_x^{sp}$-consistent, there is a $\mathbf{K}_x$-model $\mathfrak{M} = (\mathfrak{A}, w)$ for L_y and an $\mathfrak{A}$-assignment α so that $M \vDash_{sp} \Gamma[\alpha]$ and $M \vDash_{sp}^{w} \Delta[\alpha]$.

Proof. Just like the proof of Theorem *p*. 2 for $y = 1, \ldots$.

Note. If $y = 0$, T or 0, T, $\mathbf{u}$, we can define $\underline{\mathbf{K}}_x^{sp}(L_y)$ by adding Axioms (10*) and (11*) to the previous sequent calculus, replacing (ESC), (EWC), (S'□'I) and (W'□'I) by:

$$\text{(ESC*)}\quad \frac{\begin{array}{l}\Gamma, \Delta \vdash \phi(v/\tau_0)\\ \Gamma, \Delta \vdash \neg\underline{\Box}^n T(\tau_0 \not\approx \tau_1)\end{array}}{\Gamma, \Delta \vdash \phi(v/\tau_0)}\quad ;$$

(WSC*) $\dfrac{\Gamma, \Delta \vdash^{w} \phi(v/\tau_0) \qquad \Gamma, \Delta \vdash \neg\underline{\Box}^{n} T(\tau_0 \not\approx \tau_1)}{\Gamma, \Delta \vdash^{w} \phi(v/\tau_1)}$;

(S'$\underline{\Box}$'I*) $\dfrac{\Gamma, \Gamma \vdash \phi}{\underline{\Box}\Gamma, \underline{\Box}\Gamma \vdash \underline{\Box}\phi}$;

($W\underline{\Box}I$*) $\dfrac{\Gamma, \Gamma \vdash^{w} \phi}{\underline{\Box}\Gamma, \underline{\Box}\Gamma \vdash^{w} \underline{\Box}\phi}$.

(Notice that (S'$\underline{\Box}$'I) and (W'$\Box$'I) would not preserve $\mathbf{K}^{sp}_{x}$-validity.) Soundness and completeness of $\underline{\mathbf{K}}^{sp}_{x}(L_y)$ in this case follow much as before. Lemma *sp*. 1 must be revised to take this form:

> if $(\tau_0 \approx \tau_1) \in \Gamma_w$ then (i) for every $u \in W$ with $\tau_0, \tau_1 \in$ **Var** $\cup\, \mathscr{C}(u)$, $\neg T(\tau_0 \not\approx \tau_1) \in \Gamma_u$, and (ii) $E(\tau_0) \in \Gamma_u$ iff $E(\tau_1) \in \Gamma_u$.

The rest of the argument combines features of the previous argument, together with features of the proof of Theorem *p*. 2 for the case of $y = 0, \ldots$ Problem: Formalize the logic $\mathbf{K}^{sp}_{x}$ when y is either 0 or 0, **u** or 0, s or 0, **u**, s.

We now consider formalization of the actualistic based on **K**. Let's first consider $L = L_y$ for $y = 1, \ldots$ We form $\underline{\mathbf{K}}^{a}_{a}(L)$ from $\underline{\mathbf{K}}^{sp}_{ea}(L)$ by adding these additional axioms:

(14) $\{(\tau_0 \not\approx \tau_1)\}, \{(\tau_0 \not\approx \tau_1)\} \vdash^{w} \Box(\tau_0 \not\approx \tau_1)$;

(14_s) as above with '$\approx_s$' replacing '$\approx$';

(actualism): $\{\ \}, \{\neg E(\tau)\} \vdash^{w} \Box\neg E(\tau)$;

(actualism$_s$): $\{\neg E_s(\tau)\}, \{\neg E_s(\tau)\} \vdash \neg\ \Box\neg E_s(\tau)$.

We also gain elegance without loss of strength by replacing (ESC) and (EWC) by (SC) and (WC) from $\underline{\mathbf{K}}^{p}_{x}$. We form $\underline{\mathbf{K}}^{a}_{nn}(L)$ by adding (*nn*) or (*nn$_s$*) to $\underline{\mathbf{K}}^{a}_{a}(L)$; we form $\underline{\mathbf{K}}^{a}_{at}(L)$ by adding (*eat*) or (*eat$_s$*) to $\underline{\mathbf{K}}^{a}_{a}(L)$; form $\underline{\mathbf{K}}^{a}_{at\&nn}(L)$ by adding all these axioms. Hereafter, 'x' is replaceable by 'a', '*nn*', '*at*' or '*at & nn*'.

THEOREM *a*. 1. Restricted to $L = L_y$ for $y = 1, \ldots, \underline{\mathbf{K}}^{a}_{x}$ is sound and complete with respect to $\mathbf{K}^{a}_{x}$. Soundness follows easily. To prove completeness, we'll indicate the needed modifications of the previous argument.

Suppose we're given L, W, $\mathscr{C}$ and a $\mathscr{C}$-strict, $\Box$-normal, $\underline{\mathbf{K}}^{a}_{x}$-consistent complete diagram D on W for L. Where '$\approx$' $\in$ lex$_y$, let D be settled iff for all $w{*}x \in W$: iff $\neg E(\tau) \in \Delta_w$ then $\neg E(\tau) \in \Delta_{w*x}$; where '$\approx_s$' $\in$ lex$_y$, replace 'E' by 'E_s'. Let D be proper iff for all $w{*}x \in W$, if $(\tau_0 \approx \tau_1) \in \Gamma_w$ then: $E(\tau_0) \in \Gamma_{w*x}$ iff $E(\tau_1) \in \Gamma_{w*x}$. Suppose that D is settled and proper.

LEMMA *a*. 1. *Where* '$\approx$' $\in$ *lex*$_y$: *if* $(\tau_0 \approx \tau_1) \in \Gamma_w$ *then for every* $u \in W$ *with* $\tau_0, \tau_1 \in \mathbf{Var} \cup \mathscr{C}(u)$: (i) *if* $i < 2$ *and* $E(\tau_i)$ *then* $(\tau_0 \approx \tau_1) \in \Gamma_u$; *and* (ii) $E(\tau_0) \in \Gamma_u$ *iff* $E(\tau_1) \in \Gamma_u$. *Where* '$\approx_s$' $\in$ *lex*$_y$, *replace* '$\approx$' *and* 'E' *by* '$\approx_s$' *and* 'E_s'.

Proof uses Axioms (11), (12), (13) and (14), rule (WC), and the assumption that D is settled, $\Box$-normal and proper.

LEMMA *a*. 2. *just like Lemma sp*. 2, *for* $\underline{\mathbf{K}}^{a}_{x}(L)$ *instead of* $\underline{\mathbf{K}}^{sp}_{x}(L)$. Define U_0, $\sim$ and U_1 as before.

LEMMA *a*. 3. *just like Lemma sp*. 3; *the proof also carries over.*

Define $\sim^*$, U, $\bar{U}(w)$ for $w \in W$, R, $\mathscr{E}$, $\mathscr{N}$, $\mathfrak{A}$, $\mathfrak{M}$ and α as before. Notice that $\mathfrak{A}$ is a settled structure, since D is settled; clearly $\mathfrak{M}^w$ is actualistic and α^w is an M^w-assignment for each $w \in W$.

LEMMA *a*. 4. *For* $v_0, \ldots, v_{n-1}$, ϕ, w, $\tau_0, \ldots, \tau_{n-1}$, ϕ' *and* α' *as in Lemma p*. 4:

(i) for $\tau \notin \{v_0, \ldots, v_{n-1}\}$, den$(\mathfrak{A}^w, \alpha'^w, \tau) = [\tau]$ iff $[\tau] \in \bar{U}(w)$; otherwise den$(\mathfrak{A}^w, \alpha'^w, \tau)\uparrow$;

(ii) den$(\mathfrak{A}^w, \alpha'^w, v_i) = [\tau_i]$ if $[\tau_i] \in \bar{U}(w)$; otherwise den$(\mathfrak{A}^w, \alpha'^w, v_i)\uparrow$;

(iii) $\phi' \in \Gamma_w$ iff $(\mathfrak{A}^w, w) \vDash \phi[\alpha'^w]$;

(iv) $\neg\phi' \in \Gamma_w$ iff $(\mathfrak{A}^w, w) \dashv \phi[\alpha'^w]$;

(v) $\phi', \neg\phi' \in \Delta_w$ iff $(\mathfrak{A}^w, w) \mid \phi[\alpha'^w]$.

The proof of this is a straightforward imitation of the proof of Lemma *sp*. 4.

Theorem *a*. 1 will now follow from the following model-existence theorem.

THEOREM *a*. 2. Where $y = 1, \ldots, \Gamma \subseteq \Delta \subseteq fml(L_y)$ and (Γ, Δ) is $\underline{\mathbf{K}}^a_x$-consistent, there is a $\mathbf{K}^a_x$-model $\mathfrak{M}$ and an $\mathfrak{M}$-assignment α so that $\mathfrak{M} \models \Gamma[\alpha]$ and $\mathfrak{M} \models^w \Delta[\alpha]$.

Suppose (Γ, Δ) are as required; fix a name-array $\mathscr{C}'$ on W. We must construct a tree W on κ and a diagram D for W, $\mathscr{C} = \mathscr{C}' \mid W$ so that D is strict, proper, $\Box$-normal, complete, $\underline{\mathbf{K}}^a_x$-consistent, and $\Gamma \subseteq \Gamma_{\langle\ \rangle}$, $\Delta \subseteq \Delta_{\langle\ \rangle}$. It will follow that D is settled, using (actualism) or (actualism$_s$) and $\Box$-normality.

The reduct of $(\mathfrak{A}^{\langle\ \rangle}, \langle\ \rangle)$ to L and $\alpha^{\langle\ \rangle}$ will be as required.

The only novelty in this construction comes from the need to make D proper.

CASE 1. '$\approx$' $\in$ lex$_y$. Suppose we have constructed $W_n \subseteq \kappa^{<\omega}$ so that for all $w \in W_n$, $|w| \leqslant n$ and (Γ_w, Δ_w) has been defined and is consistent. If for some $\phi \in fml(L^w)$ $\neg\Box\phi \in \Delta_w$ then we'll define Γ_{w*x} and Δ_{w*x} for all $x \in \kappa$. Let $\langle \psi_x : x < \kappa \rangle$ be a listing, repetitions allowed, of all such ψ. Let:

$$\Delta^0_{w*x} = \Box^{-1}\Delta_w \cup \{\neg\psi_x\};$$

$$\Gamma^0_{w*x} = \begin{cases} \Box^{-1}\Gamma_w & \text{if } \neg\Box\psi_x \notin \Gamma_w; \\ \Box^{-1}\Gamma_w \cup \{\neg\psi_x\} & \text{otherwise.} \end{cases}$$

$(\Gamma^0_{w*x}, \Delta^0_{w*x})$ is consistent, since otherwise, using $(S\text{`}\Box\text{'}I)$ or $(W\text{`}\Box\text{'}I)$ (Γ_w, Δ_w) would be inconsistent. We'll construct $\Gamma^1_{w*x} \subseteq \Delta^1_{w*x}$ so that $(\Gamma^1_{w*x}, \Delta^1_{w*x})$ is consistent and:

$$\text{for } \tau \in \mathbf{Var} \cup \mathscr{C}(w) \text{ either } E(\tau) \in \Gamma^1_{w*x} \text{ or } \neg E(\tau) \in \Delta^1_{w*x};$$

$$\text{if } (\tau \approx \tau') \in \Gamma_w \text{ then: } E(\tau) \in \Gamma^1_{w*x} \text{ iff } E(\tau') \in \Gamma^1_{w*x}.$$

Let $\langle \tau_y \rangle_{y<\kappa}$ be a κ-listing of $\mathbf{Var} \cup \mathscr{C}(w)$, repetitions allowed. Our strategy is this:

$$\text{if } \neg\Box\psi_x \notin \Gamma_w \text{ we try to put each } \neg E(\tau_y) \text{ into } \Delta^1_{w*x};$$

$$\text{if } \neg\Box\psi_x \in \Gamma_w \text{ we try to put each } E(\tau_y) \text{ into } \Gamma^1_{w*x}.$$

Suppose $\neg\Box\psi_x \notin \Gamma_w$. Let $\Delta^{0,0}_{w*x} = \Delta^0_{w*x}$. Suppose for $y < \kappa$ that $\Delta^{0,y}_{w*x}$ has been defined and $(\Gamma^0_{w*x}, \Delta^{0,y}_{w*x})$ is consistent. Let:

$$\Delta^{0,y+1}_{w*x} = \begin{cases} \Delta^{0,y}_{w*x} \cup \{\neg E(\tau_y)\} & \text{if } (\Gamma^0_{w*x}, \Delta^{0,y}_{w*x} \cup \{\neg E(\tau_y)\}) \text{ is consistent;} \\ \Delta^0_{w*x} & \text{otherwise.} \end{cases}$$

Where y is a limit ordinal let $\Delta^{0,y}_{w*x} = \bigcup_{z<y} \Delta^{0,z}_{w*x}$; so $(\Gamma^0_{w*x}, \Delta^{0,\kappa}_{w*x})$ is consistent; and if $\neg E(\tau_y) \notin \Delta^{0,\kappa}_{w*x}$ then: $\Gamma^0_{w*x},\ \Delta^{0,\kappa}_{w*x} \vdash E(\tau_y)$. Let:

$$\Gamma^1_{w*x} = \Gamma^0_{w*x} \cup \{E(\tau_y) : \neg E(\tau_y) \notin \Delta^{0,\kappa}_{w*x}\}$$

$$\Delta^1_{w*x} = \Delta^{0,\kappa}_{w*x} \cup \{E(\tau_y) : \neg E(\tau_y) \notin \Delta^{0,\kappa}_{w*x}\};$$

then $(\Gamma^1_{w*x}, \Delta^1_{w*x})$ is consistent. Suppose that $(\tau \approx \tau') \in \Gamma_w$, $\neg E(\tau') \in \Delta^{0,\kappa}_{w*x}$ and $\neg E(\tau) \notin \Delta^{0,\kappa}_{w*x}$. Where τ is τ_y: $\Gamma^0_{w*x}, \Delta^{0,y}_{w*x} \vdash E(\tau)$. So for some $\sigma_0, \ldots, \sigma_{n-1}$ with $\neg E(\sigma_0), \ldots, \neg E(\sigma_{n-1}) \in \Delta^{0,y}_{w*x} - \Delta^0_{w*x}$:

$$\Box^{-1}\Gamma_w, \Box^{-1}\Delta_w \cup \{\neg\psi_x, \neg E(\sigma_0), \ldots, \neg E(\sigma_{n-1})\} \vdash E(\tau);$$

Therefore:

$$\Gamma_w, \Delta_w \vdash \Box((\neg\psi_x \,\&\, \neg E(\sigma_0) \,\&\, \ldots \,\&\, \neg E(\sigma_{n-1})) \supset E(\tau));$$

using (SC):

$$\Gamma_w, \Delta_w \vdash \Box((\neg\psi_x \,\&\, \neg E(\sigma_0) \,\&\, \ldots \,\&\, \neg E(\sigma_{n-1})) \supset E(\tau'));$$

so

$$(\neg\psi_x \,\&\, \neg E(\sigma_0) \,\&\, \ldots \,\&\, \neg E(\sigma_{n-1})) \supset E(\tau') \in \Gamma^0_{w*x}.$$

Thus $\Gamma^0_{w*x}, \Delta^0_{w*x} \vdash E(\tau')$; but since $\neg E(\tau') \in \Delta^{0,\kappa}_{w*x}$, this violates the consistency of $(\Gamma^0_{w*x}, \Delta^{0,\kappa}_{w*x})$. Thus for $(\tau \approx \tau') \in \Gamma_w$: if $\neg E(\tau') \in \Delta^{0,\kappa}_{w*x}$ then $\neg E(\tau) \in \Delta^{0,\kappa}_{w*x}$; similarly if $\neg E(\tau) \in \Delta^{0,\kappa}_{w*x}$ then $\neg E(\tau') \in \Delta^{0,\kappa}_{w*x}$; so $E(\tau) \in \Gamma^1_{w*x}$ iff $E(\tau') \in \Gamma^1_{w*x}$.

Suppose that $\neg\Box\psi_x \in \Gamma_w$. We'll define $\Gamma^{0,y}_{w*x}$ and $\Delta^{0,y}_{w*x}$ for $y < \kappa$; let $(\Gamma^{0,0}_{w*x}, \Delta^{0,0}_{w*x}) = (\Gamma^0_{w*x}, \Delta^0_{w*x})$. Suppose that $(\Gamma^{0,y}_{w*x}, \Delta^{0,y}_{w*x})$ has been defined and is consistent. If $(\Gamma^{0,y}_{w*x} \cup \{E(\tau_y)\}, \Delta^{0,y}_{w*x} \cup \{E(\tau_y)\})$ is consistent, let it be $(\Gamma^{1,y+1}_{w*x}, \Delta^{1,y+1}_{w*x})$; otherwise let $\Gamma^{1,y+1}_{w*x} = \Gamma^{1,y}_{w*x}$, $\Delta^{1,y+1}_{w*x} = \Delta^{1,y}_{w*x}$. Where y is a limit $\leqslant \kappa$, let $\Gamma^{0,\kappa}_{w*x} = \bigcup_{z<y} \Gamma^{0,z}_{w*x}$, $\Delta^{0,y}_{w*x} = \bigcup_{z<y} \Delta^{0,z}_{w*x}$. Let $\Gamma^1_{w*x} = \Gamma^{0,\kappa}_{w*x}$. So $(\Gamma^1_{w*x}, \Delta^{0,\kappa}_{w*x})$ is consistent. Furthermore if $E(\tau_y) \notin \Gamma^1_{w*x}$, then: $\Gamma^1_{w*x}, \Delta^{0,\kappa}_{w*x} \vdash^w \neg E(\tau_y)$. Let:

$$\Delta^1_{w*x} = \Delta^{0,\kappa}_{w*x} \cup \{\neg E(\tau_y) : E(\tau_y) \notin \Gamma^1_{w*x}\}.$$

Then $(\Gamma^1_{w \bullet x}, \Delta^1_{w \bullet x})$ is consistent. Suppose that $(\tau \approx \tau') \in \Gamma_w$, $E(\tau') \in \Gamma^1_{w \bullet x}$ and $E(\tau) \notin \Gamma^1_{w \bullet x}$. Where $\tau' = \tau_y$: $\Gamma^{1,y}_{w \bullet x}, \Delta^{1,y}_{w \bullet x} \vdash^w \neg E(\tau)$. So for some $\sigma_0, \ldots, \sigma_{n-1}$ with $E(\sigma_0), \ldots, E(\sigma_{n-1}) \in \Gamma^{1,y}_{w \bullet x} - \Gamma^0_{w \bullet x}$, we have:

$$\Box^{-1}\Gamma_w \cup \{\neg\psi_x, E(\sigma_0), \ldots, E(\sigma_{n-1})\}, \Box^{-1}\Delta_w \cup \{\neg\psi_x, E(\sigma_0), \ldots, E(\sigma_{n-1})\} \vdash^w \neg E(\tau).$$

Therefore:

$$\Box^{-1}\Gamma_w, \Box^{-1}\Delta_w \vdash^w (\neg\psi_x \,\&\, E(\sigma_0) \,\&\, \ldots \,\&\, E(\sigma_{n-1})) \supset \neg E(\tau);$$

thus:

$$\Gamma_w, \Delta_w \vdash^w \Box((\neg\psi_x \,\&\, E(\sigma_0) \,\&\, \ldots \,\&\, E(\sigma_{n-1})) \supset \neg E(\tau)).$$

By (WC) we have:

$$\Gamma_w, \Delta_w \vdash^w \Box((\neg\psi_x \,\&\, E(\sigma_0) \,\&\, \ldots \,\&\, E(\sigma_{n-1})) \supset \neg E(\tau'));$$

therefore:

$$(\neg\psi_x \,\&\, E(\sigma_0) \,\&\, \ldots \,\&\, E(\sigma_{n-1})) \supset \neg E(\tau') \in \Delta^0_{w \bullet x}.$$

Thus $\Gamma^0_{w \bullet x}, \Delta^0_{w \bullet x} \vdash^w \neg E(\tau')$. This violates the consistency of $(\Gamma^1_{w \bullet x}, \Delta^1_{w \bullet x})$. Thus for $(\tau \approx \tau') \in \Gamma_w$: if $E(\tau') \in \Gamma^1_{w \bullet x}$ then $E(\tau) \in \Gamma^1_{w \bullet x}$; similarly if $E(\tau) \in \Gamma^1_{w \bullet x}$ then $E(\tau') \in \Gamma^1_{w \bullet x}$.

We then expand $(\Gamma^1_{w \bullet x}, \Delta^1_{w \bullet x})$ to $(\Gamma_{w \bullet x}, \Delta_{w \bullet x})$ so that the latter is consistent, $\neg$-complete and $\exists$-complete; this runs as usual.

CASE 2. '$\approx_s$' $\in$ lem$_y$. Here the previous elaborate construction of $(\Gamma^1_{w \bullet x}, \Delta^1_{w \bullet x})$ is unnecessary; similarly if 'T' $\in$ lex$_y$. Since

$$\{(\tau_0 \approx_s \tau_1)\}, \{(\tau_0 \approx_s \tau_1)\} \vdash \Box(E_w(\tau_0) \supset E_s(\tau_1)),$$

the fact that D is $\Box$-normal will make D proper without extra effort.

The rest of the construction is as in the proof of Theorem *p*. 2.; details are left to the reader.

Where $y = 0, \ldots,$ the difficulty faced before with defining $\underline{\mathbf{K}}^{sp}_x(L_y)$ to be sound and complete for $\mathbf{K}^{sp}_x$ is compounded when we consider $\mathbf{K}^a_x$. First of all, we seem to have to define $\underline{\mathbf{K}}^a_x(L_y)$ so that the diagram constructed in the proof of the model-existence theorem would be settled. The most likely axioms to introduce with this in mind are:

$$\text{(actualism*)} \quad \{\ \}, \{\neg E(\tau)\} \vdash^{w} \underline{\Box}\neg E(\tau);$$

$$\text{(actualism*}_s\text{)} \quad \{\neg E_s(\tau)\}, \{\neg E_s(\tau)\} \vdash \underline{\Box}\neg E_s(\tau).$$

However when D is consistent and $\underline{\Box}$-normal and '$\approx$' $\in$ lex$_y$, (actualism*) will not insure that D is settled, since $\underline{\Box}$-normality doesn't require that $\underline{\Box}^{-1}\Delta_w \subseteq \Delta_{w*x}$ for $w*x \in W$. Fortunately, where '$\approx_s$' $\in$ lex$_y$, (actualism*$_s$) will insure that D is settled. However even in this case, a proof of Lemma a. 1 faces an obstacle which appears formidable. The natural variation of Axiom (14_s), namely:

$$\{(\tau_0 \not\approx_s \tau_1)\}, \{(\tau_0 \not\approx_s \tau_1)\} \vdash^{w} \underline{\Box}(\tau_0 \not\approx_s \tau_1)$$

does not play the role that (14_s) played in the previous proof of Lemma a. 1; once again, $\underline{\Box}$-normality will not insure that $(\tau_0 \not\approx_s \tau_1) \in \Delta_{w*x}$ given that $\underline{\Box}(\tau_0 \not\approx_s \tau_1) \in \Delta_w$. So the problem with defining $\underline{\mathbf{K}}^a_x(L_{0,s})$ or $\underline{\mathbf{K}}^a_x(L_{0,\mathbf{u},s})$ seems to boil down to finding an appropriate version of Axiom (14_s).

Of course if y is 0, T or 0, T, $\mathbf{u}$, these difficulties vanish, as they did in our discussion of $\mathbf{K}^{sp}_x$. We replace (actualism) and (14) by:

$$\{\neg TE(\tau)\}, \{\neg TE(\tau)\} \vdash \underline{\Box}\neg TE(\tau);$$

$$\{(\tau_0 \not\approx \tau_1)\}, \{(\tau_0 \not\approx \tau_1)\} \vdash \underline{\Box}\neg T(\tau_0 \approx \tau_1);$$

Furthermore (S'$\Box$'I) and (W'$\Box$'I) are replaced by (S'$\underline{\Box}$'I*) and (W'$\underline{\Box}$'I*). A soundness and completeness proof for the resulting system is left to the reader.

7. ON $\mathbf{L}^p_x$, $\mathbf{L}^{sp}_x$ AND $\mathbf{L}^a_x$ WHEN $\mathbf{L}$ IS STRONGER THAN $\mathbf{K}$

We'll consider the following axiom schemata:

(sr) $\quad \{\Box\phi\}, \{\Box\phi\} \vdash \phi;$

(wr) $\quad \{\ \}, \{\Box\phi) \vdash^{w} \phi;$

(sr*) $\quad \{\underline{\Box}\phi\}, \{\underline{\Box}\phi\} \vdash \phi;$

(st) $\quad \{\Box\phi\}, \{\Box\phi\} \vdash \Box\Box\phi$:

(*wt*) $\{\ \}, \{\Box\phi\} \vdash^w \Box\Box\phi$;

(*st**) $\{\underline{\Box}\phi\}, \{\underline{\Box}\phi\} \vdash \underline{\Box}\,\underline{\Box}\phi$;

(*ss*) $\{\phi\}, \{\phi\} \vdash \Box\Diamond\phi$;

(*ws*) $\{\ \}, \{\phi\} \vdash^w \Box\Diamond\phi$;

(*ss**) $\{\phi\}, \{\phi\} \vdash \underline{\Box}\,\underline{\Diamond}\phi$;

(*ws**) $\{\ \}, \{\phi\} \vdash^w \underline{\Box}\,\underline{\Diamond}\phi$;

(*se*) $\{\Diamond\phi\}, \{\Diamond\phi\} \vdash \Box\Diamond\phi$;

(*we*) $\{\ \}, \{\Diamond\phi\} \vdash \Box\Diamond\phi$;

(*se**) $\{\underline{\Diamond}\phi\}, \{\underline{\Diamond}\phi\} \vdash \underline{\Box}\,\underline{\Diamond}\phi$.

Where '*x*' is replaced as usual, '*z*' is replaced by '*p*' or '*sp*', and $y = 1, \ldots, [= 0, \ldots]$ for $L = L_y$, we form the sequence-calculus on the left by adding all instances, for $\phi \in fml(L)$, of the axiom-schemes indicated on the right to $\underline{\mathbf{K}}^z_x(L)$;

$\mathbf{T}^z_x(L)$: (*sr*) and (*wr*)[(*sr**)];

$\underline{\mathbf{K}}\,\underline{\mathbf{4}}^z_x(L)$: (*st*) and (*wt*)[(*st**)];

$\underline{\mathbf{S}}\,\underline{\mathbf{4}}^z_x(L)$: (*sr*), (*wr*), (*st*) and (*wt*)[(*sr**) and (*st**)];

$\underline{\mathbf{B}}^{-z}_x(L)$: (*ss*) and (*ws*)[(*ss**) and (*ws**)];

$\underline{\mathbf{B}}^z_x(L)$: (*sr*), (*wr*), (*ss*) and (*ws*)[(*sr**), (*ss**) and (*ws**)];

$\underline{\mathbf{S}}\,\underline{\mathbf{5}}^z_x(L)$: (*st*), (*wt*), (*se*), (*we*)[(*st**), (*ss**) and (*se**)].

THEOREM 3. If **L** is either **T**, **K4** or **S4**, then $\underline{\mathbf{L}}^z_x$ is sound and complete relative to $\mathbf{L}^z_x$; similarly if **L** is either $\mathbf{B}^-$, **B** or **S5** and $y = 1, \ldots$. The soundness of $\underline{\mathbf{L}}^z_x$ relative to $\mathbf{L}^z_x$ is easily proved by the usual induction on the length of derivations.

The following pecularities deserve notice; for appropriate choice of ϕ:

($\{\ \}, \{\underline{\Box}\phi\}, \phi$) is not weakly $\mathbf{T}^z_x$ (or $\mathbf{S4}^z_x$ or $\mathbf{B}^z_x$ or $\mathbf{S5}^z_x$)-valid;

($\{\ \}, \{\underline{\Box}\phi\}, \underline{\Box}\,\underline{\Box}\phi$) is not weakly $\mathbf{K4}^z_x$ (or $\mathbf{S4}^z_x$ or $\mathbf{S5}^z_x$)-valid;

($\{\ \}, \{\underline{\Diamond}\phi\}, \underline{\Box}\,\underline{\Diamond}\phi$) is not weakly $\mathbf{S5}^z_x$-valid.

These facts might make the weak $\mathbf{B}_x^{-z}$ (and $\mathbf{B}_x^z$ and $\mathbf{S5}_x^z$)-validity of all instances of (*ws**) surprising.

As usual, to prove completeness, it suffices to show that for any $\Gamma \subseteq \Delta \subseteq fml(L)$:

> if (Γ, Δ) is $\underline{\mathbf{L}}_x^z$-consistent then there is an $\mathbf{L}_x$-model $\mathfrak{M} = (\mathfrak{A}, \langle\ \rangle)$ and an $\mathfrak{A}$-assignment α so that $\mathfrak{M} \models_z \Gamma[\alpha]$ and $\mathfrak{M} \models_z^w \Delta[\alpha]$.

Since the argument will be the same when 'z' is replaced by 'sp', we'll consider the case in which 'z' is replaced by 'p'.

First we consider the case in which **L** is either **T** or **K4** or **S4**. The construction in §5 yields a $\mathbf{K}_x$-model $\mathfrak{M} = (\mathfrak{A}, \langle\ \rangle)$ and an $\mathfrak{A}$-assignment α so that $\mathfrak{M} \models_p \Gamma[\alpha]$ and $\mathfrak{M} \models_p^w \Delta[\alpha]$. Let $(W, R) = \text{frame}(\mathfrak{A})$. If **L** is **T**, form $\mathfrak{M}' = (\mathfrak{A}', \langle\ \rangle)$ by replacing R by the reflexive closure of R; using (*sr*) and (*wr*)[(*sr**)] we can show that for all $\phi \in fml(L)$ and $w \in W$:

(*) $(\mathfrak{A}, w) \models_p \phi[\alpha]$ iff $(\mathfrak{A}', w) \models_p \phi[\alpha]$; $(\mathfrak{A}, w)\ {}_p\!\dashv \phi[\alpha]$ iff $(\mathfrak{A}', w)\ {}_p\!\dashv \phi[\alpha]$;

thus $\mathfrak{M}'$ and α are as desired. If **L** is **K4**, form $\mathfrak{M}'$ and $\mathfrak{A}'$ by replacing R with the transitive closure of R; by (*st*) and (*wt*)[(*st**)], (*) holds; so $\mathfrak{M}'$ and α are as desired. If **L** is **S4**, form $\mathfrak{M}'$ and $\mathfrak{A}'$ by replacing R with the reflexive transitive closure of R; as above (*) holds; so $\mathfrak{M}'$ and α are as desired.

Where **L** is either $\mathbf{B}^-$, **B** or **S5** and $y = 1, \ldots$, construct the $\mathbf{K}_x$-model $\mathfrak{M}$ and α as above. If **L** is $\mathbf{B}^-$, form $\mathfrak{M}'$ and $\mathfrak{A}'$ by replacing R with its symmetric closure; if **L** is **B**, replace R by R' = the reflexive symmetric closure of R. Again, we can prove (*); so $\mathfrak{M}'$ and α are as desired. A familiar proof shows that (*sr*) and (*ss*) are theorems of $\underline{\mathbf{S5}}_x^p(L)$ and that (*wr*) and (*ws*) are weak theorems of $\underline{\mathbf{S5}}_x^p(L)$. Thus if **L** is **S5**, construct $\mathfrak{M}'$ and $\mathfrak{A}'$ as in the case for **B**; again (*) holds; form $\mathfrak{M}'' = (\mathfrak{A}'', \langle\ \rangle)$ by replacing R' with its transitive closure. Then for all $\phi \in fml(L)$ and $w \in W$:

$$(\mathfrak{A}', w) \models_p \phi[\alpha] \text{ iff } (\mathfrak{A}'', w) \models_p \phi[\alpha];$$

$$(\mathfrak{A}', w)\ {}_p\!\dashv \phi[\alpha] \text{ iff } (\mathfrak{A}'', w)\ {}_p\!\dashv \phi[\alpha].$$

Since $\mathfrak{M}''$ is an $\mathbf{S5}_x$-model, $\mathfrak{M}''$ and α are as desired.

Where **L** is either $\mathbf{B}^-$, **B** or **S5** and $y = 0, \ldots$, it's easy to see that $\underline{\mathbf{L}}^z_x$ is sound relative to $\mathbf{L}^z_x$; but when we consider the completeness of $\underline{\mathbf{L}}^z_x$, the strategy used above will not work. Suppose **L** is $\mathbf{B}^-$ and $\mathfrak{A}'$ is constructed by replacing R with its symmetric closure; since we may have $(\mathfrak{A}, w^*x) \models_p \underline{\Box}\phi[\alpha]$ and $(\mathfrak{A}, w) \not\models_p \phi[\alpha]$, (*) does not follow. The lesson here is that we can't simply take the $\mathfrak{M} = (\mathfrak{A}, \langle\ \rangle)$ produced by the construction from §5. Conjecture: where $y = 0, \ldots$, and **L** is either $\mathbf{B}^-$, **B** or **S5**, $\underline{\mathbf{L}}^z_x$ is complete relative to $\mathbf{L}^z_x$.

We'll now consider actualistic logics. If $y = 1, \ldots [= 0, \ldots]$ form $\underline{\mathbf{T}}^a_x(L)$ from $\underline{\mathbf{K}}^a_x(L)$ by adding as axioms all instances of (*sr*) and (*wr*)[(*sr**)]. Then $\underline{\mathbf{T}}^a_x$ is sound and complete relative to $\mathbf{T}^a_x$. The proof of this is a straightforward use of the approach just considered to $\underline{\mathbf{T}}^a_x$ and $\underline{\mathbf{T}}^{sp}_x$.

It's important to realize that the formalization of the actualistic logics based on the other logic-defining classes discussed above is not as simple as one might have expected. Indeed the following should come as a disturbing surprise.

OBSERVATION:

(1) (*st*) and (*st**) are not $\mathbf{K4}^a_x$ (or $\mathbf{S4}^a_x$ or $\mathbf{S5}^a_x$)-valid; (*wt*) is not weakly $\mathbf{K4}^a_x$ (or $\mathbf{S4}^a_x$ or $\mathbf{S5}^a_x$)-valid;

(2) (*ss*) and (*ss**) are not $\mathbf{B}^{-a}_x$ (or $\mathbf{B}^a_x$ or $\mathbf{S5}^a_x$)-valid; (*ws*) and (*ws**) are not weakly $\mathbf{B}^{-a}_x$ (or $\mathbf{B}^a_x$ or $\mathbf{S5}^a_x$)-valid;

(3) (*se*) and (*se**) are not $\mathbf{S5}^a_x$-valid; (*we*) is not weakly $\mathbf{S5}^a_x$-valid.

For example, let $W = \{0, 1, 2\}$, $R = W^2$, $U = \bar{U}(0) = \{a, b\}$, $\bar{U}(w_1) = \{a\}$, $\bar{U}(w_2) = \{b\}$, $\mathcal{N}(\mathbf{a}) = a$, $\mathcal{N}(\mathbf{b}) = b$, and $\mathfrak{A} = (W, R, U, \bar{U}, \mathcal{N})$; $\mathfrak{M} = (\mathfrak{A}, 0)$ is a $\mathbf{K4}_a$-model (in fact an $\mathbf{S4}_a$ and an $\mathbf{S5}_a$-model) and $\mathfrak{M} \models \Box(E(\mathbf{a}) \vee E(\mathbf{b}))$; but $\mathfrak{M}^{\langle 1,2\rangle} \not\models (E(\mathbf{a}) \vee E(\mathbf{b}))$; so $\mathfrak{M}^1 \not\models \Box(E(\mathbf{a}) \vee E(\mathbf{b}))$; so $\mathfrak{M} \not\models \Box\Box(E(\mathbf{a}) \vee E(\mathbf{b}))$; thus $\mathfrak{M}$ invalidates (*st*). Another example: let $W = \{0, 1\}$, $R = W^2$, $U = \bar{U}(0) = \{a\}$, $\bar{U}(1) = \{\ \}$, $\mathcal{N}(\mathbf{a}) = a$ and $\mathfrak{A} = (W, R, U, \bar{U}, \mathcal{N})$; then $\mathfrak{M} = (\mathfrak{A}, 0)$ is a $\mathbf{B}^-_a$-model (in fact a $\mathbf{B}_a$ and an $\mathbf{S5}_a$-model) and $\mathfrak{M} \models E(\mathbf{a})$; but $\mathfrak{M}^1 \not\models \Diamond E(\mathbf{a})$; so $\mathfrak{M} \not\models \Box\Diamond E(\mathbf{a})$; so $\mathfrak{M}$ invalidates (*ss*). Similar counterexamples may be constructed to prove the remaining claims.

It appears to be impossible to construct sequent calculi, along the lines pursued in §5, §6 and above for $\mathbf{K4}_x^a$, $\mathbf{S4}_x^a$, $\mathbf{B}_x^{-a}$, $\mathbf{B}_x^a$ and $\mathbf{S5}_x^a$. There is one exception: where $\mathbf{L}$ is $\mathbf{B}^-$ or $\mathbf{B}$ and we have $y = 1$, T or $y = 2$; this rather peculiar case will be discussed in §8.

Fortunately, we can use the translations t from §2 and our formalization of the possibilistic logics to replace a direct formalization of these actualistic logics. Clearly:

> (Γ, Δ, ϕ) is $\mathbf{L}_x^a$-valid [weakly L_x^a-valid] iff $(t''\Gamma, t''\Delta, t(\phi))$ is $\mathbf{L}_x^p$-valid [weakly $\mathbf{L}_x^p$-valid].

Using the results of §5 we have:

> (Γ, Δ, ϕ) is $\mathbf{L}_x^a$-valid iff $(t''\Gamma, t''\Delta \cup \Delta', t(\phi))$ is $\mathbf{L}_{x*}^p$-valid;
>
> (Γ, Δ, ϕ) is weakly $\mathbf{L}_x^a$-valid iff $(t''\Gamma, t''\Delta \cup \Delta', t(\phi))$ is weakly $\mathbf{L}_{x*}^p$-valid;

where 'x' is replaced by '*a*', '*at*', '*a* & *nn*' or '*at* & *nn*' and 'x*' is replaced by '*ea*', '*eat*', *ea* & *nn*' or '*eat* & *nn*' respectively. This shows the instrumental value for an individual-actualist of "speaking like", or pretending to be, an individual-possibilist; a "detour" through possibilistic logics makes it easier to derive conclusions from premises which are all understood actualistically. Non-actual possible objects are analogous to what Hilbert called "ideal elements" in proofs of finitary conclusions from finitary premises. (Of course this analogy is not perfect; Hilbertian ideal elements were not objects, but rather quantifiers with infinite ranges; but the Hilbertian "detour through the infinite" is analogous to the detour through possibilistic logics mentioned above.)

8. FORMALIZING $\mathbf{B}_x^{-a}$ AND $\mathbf{B}_x^a$

In this section we'll construct sequent calculi $\underline{\mathbf{B}}_x^{-a}(L)$ and $\underline{\mathbf{B}}_x^a(L)$, where L is $L_{1,T}$ or L_2. By the closing remarks in §7, we don't really need such a formalization. But the fact that there are such sound and complete calculi for $\text{lex}_{1,T}$ and lex_2, and that it's very hard (or perhaps impossible) to find them for other lexica, is surprising and intriguing. Furthermore the problem of proving the completeness of these sequent calculi should be sufficiently challenging to make "mere technical interest" interesting. The reader who is not mathematically intrepid will lose little by skipping this section. Hereafter L is $L_{1,T}$ or L_2.

Given ϕ, $\psi \in fml(L)$, let:

$$\vec{\tau} = (\tau_0, \ldots, \tau_{l-1}),\ \vec{\sigma} = (\sigma_0, \ldots, \sigma_{m-1})$$

be sequences of terms of L with $l < \omega$, $0 < m < \omega$; let:

$$\vec{\nu} = (\nu_0, \ldots, \nu_{p-1}),\ \vec{\mu} = (\mu_0, \ldots, \mu_{q-1}),$$
$$\vec{\varrho} = (\varrho_0, \ldots, \varrho_{r-1})$$

be sequences of variables; let none of these five sequences overlap. Where order doesn't matter, we'll regard these sequences as sets of terms. Suppose that:

$$\text{Param}(\phi) \subseteq \vec{\tau} \cup \vec{\sigma} \cup \vec{\nu} \cup \vec{\mu}$$

$$\text{Param}(\psi) \subseteq \vec{\tau} \cup \vec{\sigma} \cup \vec{\varrho} \cup \vec{\mu}$$

Where $\mathscr{F} = \{\eta_0, \ldots, \eta_{s-1}\} \subseteq \vec{\nu} \cup \vec{\mu} \cup \vec{\varrho}$, let:

$\phi_{\mathscr{F}}$ = the result of replacing each free occurrence in ϕ of each variable in $(\vec{\nu} \cup \vec{\mu}) - \mathscr{F}$ by σ_0;

this notation hereby surplanting that used in §2;

$$\psi^{\mathscr{F}} = (\exists \eta'_0) \ldots (\eta'_{t-1})\psi,$$

where $\{\eta'_0, \ldots, \eta'_{t-1}\} = (\vec{\mu} \cup \vec{\varrho}) - \mathscr{F}$;

$$\theta(\mathscr{F}) = (\exists \eta_0) \ldots (\exists \eta_{s-1})(\phi_{\mathscr{F}} \ \& \ \Diamond(\psi^{\mathscr{F}})).$$

Let '$(\exists \vec{\nu} \cup \vec{\mu} \cup \vec{\varrho})$' abbreviate:

$$(\exists \nu_0) \ldots (\exists \nu_{p-1})(\exists \mu_0) \ldots (\exists \mu_{q-1})(\exists \varrho_0) \ldots (\exists \varrho_{r-1}).$$

Finally, let $E(\vec{\tau})$ and $E_s(\vec{\tau})$ abbreviate respectively;

$$\wedge\{E(\tau_i) : i < l\},\ \wedge\{TE(\tau_i)\ i\ < l\}.$$

Consider the following axioms:

(Strong Actualistic Symmetry):

$$\{\Diamond(\exists \vec{\nu} \cup \vec{\mu} \cup \vec{\varrho})(\Box\phi \ \& \ \psi \ \& \ E(\vec{\tau}))\},$$

$$\{\Diamond(\exists \nu \cup \vec{\mu} \cup \vec{\varrho})(\Box\phi \ \& \ \psi \ \& \ E(\vec{\tau})), \neg E(\sigma_i) : i < m\} \vdash^{w}$$
$$\vee\{\theta(\mathscr{F}) : \mathscr{F} \subseteq \vec{\nu} \cup \vec{\mu} \cup \vec{\varrho}\};$$

' (Weak Actualistic Symmetry):

$$\{\ \}, \{\Diamond(\exists \vec{v} \cup \vec{\mu} \cup \vec{\varrho})(\Box\phi \ \&\ \psi \ \&\ E_s(\vec{\tau})), \neg E(\sigma_i) : i < m\} \vdash^w \vee \{\theta(\mathscr{F}) : \mathscr{F}\ \vec{v} \cup \vec{\mu} \cup \vec{\varrho}\}.$$

In this section, '*x*' may be replaced by '*a*', '*a* & *nn*', '*at*' or '*at* & *nn*'. Form $\underline{\mathbf{B}}_x^{-a}(L)[\underline{\mathbf{B}}_x^a(L)]$ by adding all cases of (WAS) to $\underline{\mathbf{K}}_x^a(L)[\underline{\mathbf{T}}_x^a(L)]$. By straightforward manipulation of 'T', we then have that all instances of (SAS) are theorems of $\underline{\mathbf{B}}_x^{-a}(L)[\underline{\mathbf{B}}_x^{-a}(L)]$. (Note: we could instead have added all cases of (SAS) as our axioms, and then have had all cases of (WAS) as weak theorems.)

THEOREM 4. $\underline{\mathbf{B}}_x^{-a}[\underline{\mathbf{B}}_x^a]$ is sound and complete relative to $\mathbf{B}_x^{-a}[\mathbf{B}_x^a]$. For soundness, the following will suffice.

LEMMA 1. (WAS) *is weakly* $\mathbf{B}_x^{-a}$*-valid* [*and therefore weakly* $\mathbf{B}_x^a$*-valid*]. Suppose $\mathfrak{M} = (\mathfrak{A}, w)$ is a $\mathbf{B}_x^{-a}$-model for L with frame(W, R), and α is an $\mathfrak{M}$-assignment, $\mathfrak{M} \models^w \neg E(\sigma_i)[\alpha]$ for all $i < m$, and:

$$\mathfrak{M} \models^w (\exists \vec{v} \cup \vec{\mu} \cup \vec{\varrho})(\Box\phi \ \&\ \psi \ \&\ E_s(\vec{\tau}))[\alpha]$$

So for some $u \in W$ with wRu:

$$\mathfrak{M}^u \models^w (\exists \vec{v} \cup \vec{\mu} \cup \vec{\varrho})(\Box\phi \ \&\ \psi \ \&\ E_s(\vec{\tau}))[\alpha^u].$$

Fixed $\vec{a} = (a_0, \ldots, a_{p-1}) \in \bar{U}(u)^p$ and $\vec{b} = (b_0, \ldots, b_{q-1}) \in \bar{U}(u)^q$, $\vec{c} = (c_0, \ldots, c_{r-1}) \in \bar{U}(u)^r$ and $\beta = (\alpha^u)^{\vec{v}, \vec{\mu}, \vec{\varrho}}_{\vec{a}, \vec{b}, \vec{c}}$ so that:

$$\mathfrak{M}^u \models^w \Box\phi \ \&\ \psi \ \&\ E_s(\vec{\tau}))[\beta].$$

Since Frame$(\mathfrak{A})$ is symmetric, uRw. So $\mathfrak{M}^{\langle u,w\rangle} \models^w \phi[\beta^w]$; furthermore $\mathfrak{M}^{\langle u,w\rangle} = (\mathfrak{A}^u, w)$. Let:

$$\mathscr{F} = \{\eta \in \vec{v} \cup \vec{\mu} \cup \vec{\varrho} : \beta^w(\eta)\downarrow\}.$$

Since $\mathfrak{M} \models^w \neg E(\sigma_0)[\alpha]$, $\mathfrak{M}^{\langle u,w\rangle} \models^w \neg E(\sigma_0)[\beta^w]$; so $(\mathfrak{A}^u, w) \models^w \phi_{\mathscr{F}}[\beta^w]$. Where $\mathscr{F} = \{\eta_0, \ldots, \eta_{s-1}\}$, let $\alpha^{\eta_0, \ldots, \eta_{q-1}}_{\beta(\eta), \ldots, \beta(\eta_{s-1})} = \gamma$.

CLAIM. For $\tau \in \vec{\tau} \cup \vec{\sigma} \cup \mathscr{F}$: den$(\mathfrak{A}^u, \beta^w, \tau) \simeq$ den$(\mathfrak{A}, \gamma, \tau)$. For $i < l$, den$(\mathfrak{A}^u, \beta, \tau_i)\downarrow$; since $\tau_i \notin \vec{v} \cup \vec{\mu} \cup \vec{\varrho}$, den$(\mathfrak{A}^u, \beta, \tau_i) \in \bar{U}(u) \cap \bar{U}(w)$; so:

$$\text{den}(\mathfrak{A}^u, \beta^w, \tau_i) = \text{den}(\mathfrak{A}^u, \beta, \tau_i) = \text{den}(\mathfrak{A}, \gamma, \tau_i).$$

For $i < n$, den($\mathfrak{A}$, γ, σ_i)$\uparrow$ and den($\mathfrak{A}^u$, β^w, σ_i)$\uparrow$. For $i < s$, $\beta^w(\eta_i) = \beta(\eta_i) = \gamma(\eta_i)$; so the claim holds. Therefore $\mathfrak{M} \vDash^w \phi_{\mathscr{F}}[\gamma]$.
Since $\mathfrak{M}^u \vDash^w \psi[\beta]$:

$$\mathfrak{M}^u \vDash \psi^{\mathscr{F}}[(\alpha^u)^{\eta_0,\ldots,\eta_{s-1}}_{\beta(\eta_0),\ldots,\beta(\eta_{s-1})}];$$

furthermore:

$$(\alpha^u)^{\eta_0,\ldots,\eta_{s-1}}_{\beta(\eta_0),\ldots,\beta(\eta_{s-1})} = \gamma^u.$$

So $\mathfrak{M} \vDash^w \Diamond(\psi^{\mathscr{F}})[\gamma]$; so $\mathfrak{M} \vDash^w \theta(\mathscr{F})[\alpha]$; so $\mathfrak{M} \vDash^w \vee \{\theta(\mathscr{F}) : \mathscr{F} \subseteq \vec{v} \cup \vec{\mu} \cup \vec{\varrho}\}[\alpha]$. Perhaps the reader will find (*WAS*) more intelligible if it's thought of as a distant descendant of $\{\ \}, \{\Diamond\Box\phi\} \vdash^w \phi$, a contraposed version of (*ws*). QED

Theorem 4 will follow as usual from the following:

THEOREM 5. For $\Gamma \subseteq \Delta \subseteq fml(L)$: if (Γ, Δ) is $\underline{\mathbf{B}}_x^{-a}[\underline{\mathbf{B}}_x^a]$-consistent then there is a $\mathbf{B}_x^-[\mathbf{B}_x]$ model $\mathfrak{M}$ and an $\mathfrak{M}$-assignment α so that $\mathfrak{M} \vDash \Gamma[\alpha]$ and $\mathfrak{M} \vDash^w \Delta[\alpha]$.

LEMMA 2. *Let* $\Gamma \subseteq \Delta \subseteq fml(L)$. *If* (Γ, Δ) is $\underline{\mathbf{B}}_x^{-a}$-consistent and τ is a term not occurring as a parameter in any member of Δ then $(\Gamma, \Delta \cup \{\neg E(\tau)\})$ is $\underline{\mathbf{B}}_x^{-a}$-consistent.

Proof. Let Δ^* be:

$$\{[\wedge\{\neg E(\sigma_i) : i < m\} \;\&\; \Diamond(\exists\vec{v} \cup \vec{\mu} \cup \vec{\varrho})(\Box\phi \;\&\; \psi \;\&\; E_s(\vec{\tau}))] \supset_w (\vee\{\theta(\mathscr{F}) : \mathscr{F} \subseteq \vec{v} \cup \vec{\mu} \cup \vec{\varrho}\}) : \phi, \psi, \vec{\sigma}, \vec{\tau}, \vec{v}, \vec{\mu}, \vec{\varrho} \text{ as above}\}.$$

Then for any formula θ:

$$(\Gamma, \Delta, \theta) \in Th\ \underline{\mathbf{B}}_x^{-a}(L) \text{ iff } (\Gamma, \Delta \cup \Delta^*, \theta) \in Th\ \underline{\mathbf{K}}_x^a(L),$$

by an easy induction on the length of derivations. Let τ be as stated in the lemma. Since (Γ, Δ) is $\underline{\mathbf{B}}_x^{-a}$-consistent, $(\Gamma, \Delta \cup \Delta^*)$ is $\underline{\mathbf{K}}_x^a$-consistent; so by Theorem 2a there is a $\mathbf{K}_x$-model $\mathfrak{M}$ and an $\mathfrak{M}$-assignment α so that $\mathfrak{M} \vDash \Gamma[\alpha]$, $\mathfrak{M} \vDash^w \Delta \cup \Delta^*[\alpha]$. Without loss of generality, $\mathfrak{M} \nvDash E(\tau)[\alpha]$. So $(\Gamma, \Delta \cup \Delta^* \cup \{\neg E(\tau)\})$ is $\underline{\mathbf{K}}_x^a$-consistent; so $(\Gamma, \Delta \cup \{\neg E(\tau)\})$ is $\underline{\mathbf{B}}_x^{-a}$-consistent. QED

Hereafter, "consistent" shall mean "$\underline{\mathbf{B}}_x^{-a}$-consistent." Where W is a tree and D is a diagram on W for L, let D be an **0**-diagram iff for all

$w \in W$, $\neg E(\mathbf{0}) \in \Delta_w$. The previous lemma will make our life easier by permitting us to introduce a new constant $\mathbf{0}$ not occurring in a given (Γ, Δ) and to then construct a $\mathbf{0}$-diagram. Let D be consistent iff for each $w \in W$, $(D_0(w), D_1(w))$ is consistent. We'll replace the notions of $\Box$-normality and $\Diamond$-completeness as follows. Let:

$\Lambda_w = \{\phi : \phi \in fml(L^u)$ for some $u \in W$, and for every $\tau \in \text{Param}(\phi) - \{\mathbf{0}\}$, $E(\tau) \in D_0(w)\}$;

$D_i(w)^* = D_i(w) \cap \Lambda_w$ for $i < 2$;

D is $\Box$-normal* iff for all $w^*x \in W$ and all $i < 2$: $\Box^{-1} D_i(w^*) \subseteq D_i(w^*x)$.

D is $\Diamond$-complete* iff for every $w \in W$ and $i < 2$ if $\neg\Box\phi \in D_i(w)^*$ then for some $w^*z \in W$, $\phi \in D_i(w^*z)$.

Suppose that for all $w \in W$, $D_1(w) \subseteq fml(L')$ where $L' = L_y(\mathbf{Pred}, \mathscr{C}')$; then:

D is $\neg$-complete for $\mathscr{C}'$ iff for every $w \in W$, $(D_0(w), D_1(w))$ is $\neg$-complete for L';

D is $\exists$-complete for $\mathscr{C}'$ iff for every $w \in W$, $(D_0(w), D_1(w))$ is $\exists$-complete for L';

D is complete* for $\mathscr{C}'$ iff D is $\neg$-complete and $\exists$-complete for $\mathscr{C}'$ and $\Diamond$-complete*.

Let D be symmetric iff for every $w \in W$ with $|w| \geqslant 1$ and all $i < 2$: $\Box^{-1}(D_i(w)^*) \subseteq D_i(w^-)$.

The following lemma will permit us to regard $\mathbf{0}$ as the only term whose undefinedness need be considered in the $\mathbf{0}$-diagram.

LEMMA 3. *Where ϕ' results from replacing some parameter-occurrences of σ in ϕ by* $\mathbf{0}$:

$\{\phi\}, \{\phi, \neg E(\sigma), \neg E(\mathbf{0})\} \vdash \phi'$;

$\{\ \}, \{\phi, \neg E(\sigma), \neg E(\mathbf{0})\} \vdash^w \phi'$;

$\{\phi'\}, \{\phi', \neg E(\sigma), \neg E(\mathbf{0})\} \vdash \phi$;

$\{\ \}, \{\phi', \neg E(\sigma), \neg E(\mathbf{0})\} \vdash^w \phi$;

Proof. By induction on the construction of ϕ. QED

Suppose that D is a consistent, $\neg$-complete, $\Box$-normal* 0-diagram. The previous lemma allows the fact that $\Box\phi \in D_i(w^-)$ to influence $D_i(w)$; for if ϕ' is formed from ϕ by replacing each parameter-occurrence of each τ such that $\neg E(\tau) \in D_1(w^-)$ by $\mathbf{0}$ then $\Box\phi' \in D_i(w^-)^*$, and so $\phi' \in D_i(w)$, even though we might have $\phi \notin D_i(w)$. Furthermore, D is proper, by the argument used in §7 for the case in which '$\approx_s$' or 'T' $\in$ lex$_y$; but notice that we can't conclude that D is settled using only $\Box$-normality*.

When W is a tree, let W' be a subtree of W with root w_0 iff $w_0 \in W' \subseteq W$, for all $w \in W'$ $w_0 \subseteq w$, and for any $w, w' \in W'$: if $w \subseteq u \subseteq w'$ then $u \in W'$; let D be good iff for every term τ, $\{w \in W: E(\tau) \in D_0(w)\}$ is a subtree of W.

Suppose that D is a good $\Box$-normal*, consistent, complete* for $\mathscr{C}$, symmetric 0-diagram for W. Let $U_0 = \mathbf{Var} \cup \mathscr{C}'$ and let:

$$U_1 = \{\tau \in U_0: \text{for some } w \in W,\ E(\tau) \in D_0(w)\};$$

$$\tau \sim \tau' \text{ iff for some } w \in W,\ (\tau \approx \tau') \in D_0(w).$$

Clearly dom$(\sim) = U_1$. Furthermore, since D is complete* for $\mathscr{C}'$, $\sim$ is transitive by Lemma *a*. 2. from §6 applied to $\underline{\mathbf{B}}_x^{-a}$. So $\sim$ is an equivalence relation. Let:

$$R = \{(w^-, w), (w, w^-): w \in W\};$$

$$U = U_1/\sim\ = \{[\tau]: \tau \in U_1\}, \text{ where } [\tau] \text{ is the } \sim\text{-class of } \tau;$$

$$\bar{U}(w) = \{[\tau]: E(\tau) \in D_0(w)\};$$

$$\mathscr{E}(\mathbf{P})(w, [\tau_0], \ldots, [\tau_{n-1}]) \simeq \begin{cases} 1 & \text{if } \mathbf{P}(\tau_0, \ldots, \tau_{n-1}) \in D_0(w); \\ 0 & \text{if } \neg\mathbf{P}(\tau_0, \ldots, \tau_{n-1}) \in D_0(w); \end{cases}$$

$$\mathscr{N}(\mathbf{c}) \simeq [\mathbf{c}] \text{ for } \mathbf{c} \in \mathscr{C}';$$

$$\alpha(v) \simeq [v] \text{ for } v \in \mathbf{Var};$$

$$\mathfrak{A} = (W, R, U, \bar{U}, \mathscr{E}, \mathscr{N}).$$

Firstly, we must show that if $\tau \sim \tau'$ then for every $w \in W$, $(\tau \not\approx \tau') \notin D_0(w)$. Since D is good and the intersection of subtrees of W is itself a subtree of W, $\{w: E(\tau), E(\tau') \in D_0(w)\}$ is a subtree of W; let w_0 be its root. It suffices to show:

$$\text{if } (\tau \approx \tau') \in D_0(w_0) \text{ then for all } w \in W', (\tau \approx \tau') \in D_1(w);$$

$$\text{if } (\tau \not\approx \tau') \in D_0(w_0) \text{ then for all } w \in W', (\tau \not\approx \tau') \in D_1(w);$$

This follows by induction on $|w| - |w_0|$, using $\Box$-normality*, Axiom (14), and this theorem: $\{(\tau \approx \tau')\}, \{(\tau \approx \tau')\} \vdash^w \Box(\tau \approx \tau')$. Secondly, using the notation from §5, Lemma *p.* 4., we need to show that for all $w \in W$ with $|w| \geqslant 1$:

$$\text{if } \Box\psi' \in D_0(w) \text{ then } (\mathfrak{A}^{\langle w, w^- \rangle}, w^-) \models \psi[\alpha'^{\langle w, w^- \rangle}];$$

$$\text{if } \Box\psi' \in D_1(w) \text{ then } (\mathfrak{A}^{\langle w, w^- \rangle}, w^-) \models^w \psi[\alpha'^{\langle w, w^- \rangle}].$$

Form $\overline{\psi}$ from ψ by replacing every parameter τ in ψ other than $v_0, \ldots, v_{n-1}$ such that $\neg E(\tau) \in D_1(w)$ by $\mathbf{0}$; form $\overline{\psi'}$ from ψ' by replacing every parameter τ in ψ' such that $\neg E(\tau) \in D_1(w)$ by $\mathbf{0}$. For $i < n$, let:

$$\tau_i' = \begin{cases} \tau_i & \text{if } E(\tau_i) \in D_0(w); \\ \mathbf{0} & \text{if} \neg E(\tau_i) \in D_1(w). \end{cases}$$

Thus $\overline{\psi'}$ is $\overline{\psi}(v_0, \ldots, v_{n-1}/\tau_0', \ldots, \tau_{n-1}')$. Notice that for $i < 2$:

$$\Box\psi' \in D_i(w) \text{ iff } \Box\overline{\psi'} \in D_i(w)^*,$$

by Lemma 3. Let $\bar{\alpha}$ be $\alpha^{v_0, \ldots, v_{n-1}}_{[\tau_0'], \ldots, [\tau_{n-1}']}$.

Then:

$$(\mathfrak{A}^{w^-}, w^-) \models \overline{\psi}[\bar{\alpha}^{w^-}] \text{ iff } (\mathfrak{A}^{(w, w^-)}, w^-) \models \overline{\psi}[\bar{\alpha}^{w, w^-)}] \text{ iff } (\mathfrak{A}^{(w, w^-)}, w^-) \models \psi[\alpha'^{(w, w^-)}];$$

$$(\mathfrak{A}^{w^-}, w^-) \models^w \overline{\psi}[\bar{\alpha}^{w^-}] \text{ iff } (\mathfrak{A}^{(w, w^-)}, w^-) \models^w \overline{\psi}[\bar{\alpha}^{w, w^-)}] \text{ iff } (\mathfrak{A}^{(w, w^-)}, w^-) \models^w \psi[\alpha'^{(w, w^-)}].$$

Of $\Box\psi' \in D_0(w)$, since D is symmetric, $\overline{\psi'} \in D_0(w^-)$; by induction hypothesis $(\mathfrak{A}^{w^-}, w^-) \models \overline{\psi}[\bar{\alpha}^{w^-}]$; so by the preceding $(\mathfrak{A}^{\langle w, w^- \rangle}, w^-) \models \psi[\alpha'^{\langle w, w^- \rangle}]$, as required. The second conditional follows similarly.

LEMMA 4. *If* (Γ, Δ) *is consistent then there is a good,* $\Box$*-normal*, complete* symmetric consistent 0-diagram D on a tree W so that* $\Gamma \subseteq D_0(\langle\ \rangle)$, $\Delta \subseteq D_1(\langle\ \rangle)$. Proof of this lemma shall require some work.

We'll construct D as the limit of the sequence $\langle D^n \rangle_{n<\omega}$ of good $\Box$-normal* complete* consistent diagrams on W^n for $\bar{L}^n$. Let $\kappa = \text{card}(fml(L))$. For each $n < \omega$ fix a name-array $\mathscr{C}^n$ on $(\kappa \cdot (n + 1))^{<\omega}$ so that $\mathbf{0} \in \mathscr{C}^0(\langle\ \rangle) - \mathbf{C}$ and for $-1 \leqslant n < \omega$:

$$\bar{\mathscr{C}}^n \subseteq \mathscr{C}^{n+1}(\langle\ \rangle);\ \text{card}(\mathscr{C}^{n+1}(\langle\ \rangle) - \bar{\mathscr{C}}^n) = \kappa;$$

where $\bar{\mathscr{C}}^{-1} = \mathbf{C}$.

Recall that $\bar{\mathscr{C}}^n = \cup\{\mathscr{C}^n(w) : w \in (\kappa \cdot (n + 1))^{<\omega}\}$. Let:

$$\bar{L}^{n+1} = L(\text{Pred}, \bar{\mathscr{C}}^n);$$

so $\bar{L}^0 = L$.

We'll have $W^n \subseteq (\kappa \cdot (n + 1))^{<\omega}$ and have D^n be $\mathscr{C}^n \mid W^n$-strict. The transition from D^n to D^{n+1} will occur in two big stages: first we work "towards the root $\langle\ \rangle$", expanding D^n to a diagram D^{n+} on W^n; then we work "away from the root $\langle\ \rangle$", expanding D^{n+} and W^n to D^{n+1} and $W^{n+1} \subseteq (\kappa \cdot (n + 2))^{<\omega}$.

We'll begin with a detailed overview of the construction of D^{n+} and then carry out that construction. For $w \in W_n$, let:

$$\mathscr{C}^{n+}(w) = \cup\{\mathscr{C}^n(u) : w \subseteq u \in W^n\}.$$

$$L^{w,n+} = L(\mathbf{Pred}, \mathscr{C}^{n+}(w)).$$

We'll construct D^{n+} with $D^{n+}(w) \subseteq fml(L^{w,n+})$ for all $w \in W_n$ and meeting these three conditions:

(a) for $w \in W^n$ and $\mathbf{c} \in \mathscr{C}^{n+}(w)$: either $E(\mathbf{c}) \in D_0^{n+}(w)$ or $\neg E(\mathbf{c}) \in D_1^{n+}(w)$;

(b) D^{n+} is good.

Notice that (a) alone will prevent D^{n+} from being $\mathscr{C}^n$-strict. The third condition will require some definitions. Let a term τ be troublesome for w, z by n iff $E(\tau) \in D_0^{n+}(w{*}z)$ and $\neg E(\tau) \in D_1^{n+}(w)$. For $\phi, \psi \in fml(L^{w,n+})$, let:

$\phi_{w,z}$ = the result of replacing each parameter in ϕ that is troublesome for w, z by n with $\mathbf{0}$;

$\psi^{w,n}$ = the result of existentially quantifying-out of ψ exactly those parameters troublesome for w, z by n.

(c) for $i < 2$ and $w^*z \in W^n$:
if $\Box\phi \in D_i^{n+}(w^*z)^*$ then $\phi_{w,z} \in D_i^{n+}(w)$;
if $\psi \in D_i^{n+}(w^*z)$ then $\Diamond(\psi^{w,z}) \in D_i^{n+}(w)$.

By imposing (c) on D^{n+} we'll make D symmetric.

D^{n+} shall itself be the limit of a sequence of diagrams $\langle D^{n,k}\rangle_{k<\omega}$ with $D^{n,0} = D^n$. For $w \in W^n$ let:

$$\mathscr{C}^{n,0}(w) = \mathscr{C}^n(w);$$

$$\mathscr{C}^{n,k+1}(w) = \mathscr{C}^{n,k}(w) \cup \bigcup \{\mathscr{C}^{n,k}(w^*z) : w^*z \in W^n\};$$

$$L^{w,n,k} = L(\mathbf{Pred}, \mathscr{C}^{n,k}(w)).$$

For $w \in W^n$, we'll have $D_1^{n,k}(w) \subseteq fml(L^{w,n,k})$, and the following:

(a′) for $\mathbf{c} \in \mathscr{C}^{n,k}(w)$, either $E(\mathbf{c}) \in D_0^{n,k}$ or $\neg E(\mathbf{c}) \in D_1^{n,k}(w)$;

(b′) for $w^*z \in W^n$: if $\neg E(\tau) \in D_1^{n,k}(w^*z)$ then $\neg E(\tau) \in D_1^{n,k+1}(w)$;

(c′) for $w^*z \in W^n$:
if $\Box\phi \in D_i^{n,k}(w^*z)^*$ then $\phi_{w,z} \in D_i^{n,k+1}(w)$;
if $\psi \in D_i^{n,k}(w^*z)$ then $\Diamond(\psi^{w,z}) \in D_i^{n,k+1}(w)$.

Then (a′) and (c′) respectively will insure (a) and (c), and (b′) will help insure (b).

For each $z \leqslant \kappa$ we'll construct a diagram $D^{n,k,z}$; we'll take $D^{n,k}$ to be the $\underline{\mathbf{B}}_x^{-a}$-closure of $D_i^{n,k,\kappa}$, that is, $D^{n,k}$ = the minimal D' so that for all $w \in W_n$ and $i < 2$, $D_i^{n,k,\kappa}(w) \subseteq D_i'(w)$ and for all $\phi \in fml(L^{w,n,k})$:

$$\text{if } D_0'(w), D_1'(w) \vdash \phi \text{ then } \phi \in D_0'(w);$$

$$\text{if } D_0'(w), D_1'(w) \vdash^w \phi \text{ then } \phi \in D_1'(w).$$

For each $z \leqslant \kappa$, $w \in W_n$ and $i < 2$, we'll have:

$$\text{if } z' < z \text{ then } D_i^{n,k,z'}(w) \subseteq D_i^{n,k,z}(w);$$

$$\text{if } z \text{ is a limit then } D_i^{n,k,z}(w) = \cup\{D_i^{n,k,z'}(w) : z' < z\};$$

if $z' \leqslant z$ and $\mathbf{c} \in \mathscr{C}^{n,k}(w^*z')$ then either $E(\mathbf{c}) \in D_0^{n,k,z}(w)$ or $\neg E(\mathbf{c}) \in D_1^{n,k,z}(w)$.

We'll take $D^{n,k,0} = D^{n,k}$.

Finally, each $D^{n,k,z}$ shall be the limit of the sequence $\langle D^{n,k,z,t} \rangle_{t<\kappa}$, where $D^{n,k,0,0} = D^{n,k,0}$, $D^{n,k,z+1,0} = D^{n,k,z,\kappa}$, and where t is a limit:

$$D_i^{n,k,z,t}(w) = \cup\{D_i^{n,k,z,t'}(w) : t' < t\}.$$

Let:

$$\mathscr{C}^{n,k,z}(w) = \cup\{\mathscr{C}^{n,k}(w^*z') : z' < z\};$$

$$L^{w,n,k,z} = L(\mathbf{Pred}, \mathscr{C}^{n,k,z}(w)).$$

Fix $\langle \mathbf{c}_t^{w,n,k,z} \rangle_{t<\kappa}$ to be a κ-ordering without repetition of $\mathscr{C}^{n,k}(w^*z) - \mathscr{C}^{n,k}(w)$. Let:

$$\mathscr{C}^{n,k,z,t}(w) = \{\mathbf{c}_{t'}^{w,n,k,z} : t' < t\};$$

$$L^{w,n,k,z,t} = L(\mathbf{Pred}, \mathscr{C}^{n,k,z,t}(w) \cup \mathscr{C}^{n,k}(w)).$$

Letting $D^{n,k,z,t'} = D^{\cdots t'}$ for $t' \leqslant t$, we'll have:

$$D_1^{\cdots t}(w) \subseteq fml(L^{w,n,k,z,t}) \cup fml(L^{w,n,k,z});$$

$$\text{if } t' < t \text{ then } D_i^{\cdots t'}(w) \subseteq D_i^{\cdots t}(w);$$

$$\text{if } t \text{ is a limit then } D_i^{\cdots t}(w) = \cup\{D_i^{\cdots t'}(w) : t' < t\}.$$

(i) $(D_0^{\cdots t}(w), D_1^{\cdots t}(w))$ is consistent.

Letting $\mathbf{c}_{t'} = \mathbf{c}_{t'}^{w,n,k,z}$ for $t' < t$ we'll also insure the following:

(ii) either $E(\mathbf{c}_{t'}) \in D_0^{\cdots t}(w)$ or $\neg E(\mathbf{c}_{t'}) \in D_1^{\cdots t}(w)$;

(iii) if $\neg E(\mathbf{c}_{t'}) \in D_1^{n,k}(w^*z)$ then $\neg E(\mathbf{c}_{t'}) \in D_1^{\cdots t}(w)$.

Furthermore for all $\phi, \psi \in fml(L^{w,n,k,z,t})$:

(iv) if $\Box\phi \in D_i^{n,k}(w^*z)^*$ then $\phi_{w,z} \in D_i^{\cdots t}(w)$;

(v) if $\psi \in D_i^{n,k}(w^*z)$ then $\Diamond(\psi^{w,z}) \in D_i^{\cdots t}(w)$.

Note: where (ii) is satisfied, $\phi_{w,z}$ and $\psi^{w,z}$ may be computed from $D^{\cdots t}$.

For $\mathbf{c} \in \mathscr{C}^{n+1} - \bar{\mathscr{C}}^n$, let $u(\mathbf{c})$ be the minimal w so that $\mathbf{c} \in \mathscr{C}^{n+1}(w)$. For $\mathbf{c} \in \cup\{\bar{\mathscr{C}}^n : n < \omega\}$ with $w \subsetneq u(\mathbf{c})$, let $ind(w, \mathbf{c}) = (n_0, k_0, z, t)$, where:

$$n_0 = \text{the least } n \text{ so that } \mathbf{c} \in \bar{\mathscr{C}}^n;$$

$$k_0 = |u(\mathbf{c})| - |w| - 1;$$

so k_0 is the least k with $\mathbf{c} \in \mathscr{C}^{n_0,k+1}(w)$;

z is the unique ordinal so that $\mathbf{c} \in \mathscr{C}^{n_0,k_0}(w^*z)$; $\mathbf{c}$ is $\mathbf{c}_t^{w,n_0,k_0,z}$

Notice that the possible values of $ind(w, \mathbf{c})$ are well-ordered under the lexicographic ordering. Let $\mathbf{c}$ be troublesome for w, z by n, k, t iff $\neg E(\mathbf{c}) \in D_1^{\cdots t}(w)$, for some $z' E(\mathbf{c}) \in D_0^{n,k}(w^*z')$, and $ind(\mathbf{c}) < (n, k, z, t)$. For $\mathbf{c} = \mathbf{c}_t^{w,n,k,z}$, we'll have:

$\mathbf{c}$ is troublesome for w, z by n iff $\mathbf{c}$ is troublesome for w, z by n, k, $t + 1$.

Furthermore if $\mathbf{c}$ as above is troublesome for w, z by n, k, $t + 1$ then we'll have associated $\mathbf{c}$ with (A_0, A_1, B_0, B_1), to be called the negative witness for $(w, \mathbf{c})$, where $A_i, B_i \subseteq fml(L^{w,n,k,z,t})$ for $i < 2$, $A_0 \subseteq A_1$, $B_0 \subseteq B_1$, and meeting the following conditions for any constant $\mathbf{c}'$ not occurring in $D_1^{\cdots t}(w)$:

(vi) for $\phi \in A_i$, $\Box\phi(v/\mathbf{c}) \in D_i^{n,k}(w^*z)^*$;

(vii) for $\psi \in B_i$, $\psi(v/\mathbf{c}) \in D_i^{n,k}(w^*z)$;

(viii) $(D_0^{\cdots t}(w) \cup \{E(\mathbf{c}')\} \cup A_{0,wz}(v/\mathbf{c}') \cup \Diamond B_0^{w,z}(v/\mathbf{c}'),$ $D_1^{\cdots t}(w) \cup \{E(\mathbf{c}')\} \cup A_{1,w,z}(v/\mathbf{c}') \cup \Diamond B_1^{w,z}(v/\mathbf{c}'))$ is inconsistent;

where

$$A_{i,w,z}(v/\mathbf{c}') = \{\phi_{w,z}(v/\mathbf{c}') : \phi \in A_i\},$$

$$\Diamond B_i^{w,z}(v/\mathbf{c}') = \{\Diamond(\psi^{w,z}(v/\mathbf{c}') : \psi \in B_i\},$$

and where v is a variable for which these substitutions make sense, and for which each free occurrence of v in ϕ or ψ remains free in $\psi_{w,z}$ and $\phi^{w,z}$. Note: if $(A_0, \ldots, B_1)$ is as above, $\mathbf{d}$ occurs in it (i.e. in some member of A_1 or B_1) and $w \subsetneq u(\mathbf{d})$ then $ind(\mathbf{d}) < ind(\mathbf{c})$ under the lexicographic ordering of indices. The point of (vi)–(viii) will become visible later. Suffice now to say that these conditions will require that we meet (ii) by putting $E(\mathbf{c})$ into $D_0^{\cdots t+1}(w)$, unless doing so will destroy consistency when combined with meeting conditions (iv) and (v).

We'll now carry out the construction. By Theorem 3a, using Δ^* from Lemma 2, there is a $W^0 \subseteq \kappa^{<\omega}$ and a $\mathscr{C}^0$-strict $\Box$-normal consistent, complete for $\mathscr{C}^0$ diagram D^0 on W^0 with $\Gamma \subseteq D_0^0(\langle\ \rangle)$ and $\Delta \cup \{\neg E(\mathbf{0})\} \subseteq D_1^0(\langle\ \rangle)$; $\Box$-normality makes D^0 a settled $\mathbf{0}$-diagram. Thus no member of **Var** $\cup$ **C** will ever be troublesome for any w, z; furthermore D^0 is good. Let $D^{0,0,0,0} = D^{0,0} = D^0$; so (ii), (iii), (vi), (vii) and (viii) are trivially satisfied by $D^{0,0,0,0}$.

We'll show that conditions (iv) and (v) are satisfied. For $w{*}0 \in W^0$, $\Box\phi \in fml(L^{w,0,0,0,0}) \cap D_i^{0,0}(w{*}0)^*$, no parameter of ϕ is troublesome for w, 0 by 0, 0, 0, since D^0 is settled; so $\phi_{w,0}$ is ϕ. Let $\vec{\tau}$ be a sequence of all parameters in ϕ other than $\mathbf{0}$; so $E_s(\vec{\tau}) \in D_0^{0,0}(w{*}0)$. Since $D^{0,0}$ is consistent, complete and $\Box$-normal, $\Diamond(\Box\phi \ \&\ E_s(\vec{\tau})) \in D_i^{0,0}(w)$. Using (SAS) or (WAS), as i is 0 or 1, and regarding σ_0 is $\mathbf{0}$, ψ as '$\neg\bot$' and $\vec{v} \cup \vec{\mu} \cup \vec{\varrho} = \{\ \}$, we have $\theta(\{\ \}) = \phi \in D_i^{0,0}(w)$; so condition (iv) is satisfied. For $\psi \in fml(L^{w,0,0,0,0}) \cap D_i^{0,0}(w{*}0)$, $\psi^{w,0}$ is ψ as above; so as above $\Diamond(\psi^{w,0}) \in D_i^{0,0}(w)$; so (v) is satisfied.

Now suppose we have constructed $D^{n,k,z,t}$, as required by our preview of the construction, with $t < \kappa$. Let $D^{\cdots t}$ be $D^{n,k,z,t}$ and $L^{w\cdots t}$ be $L^{w,n,k,z,t}$. For $w \in W^n$ and $i < 2$ we must construct $D_i^{\cdots t+1}(w)$. Let $\mathbf{c}_t$ be $\mathbf{c}_t^{w,n,k,z}$; by construction $\mathbf{c}_t$ doesn't occur in $D_i^{\cdots t}(w)$. Let:

$$A_i' = \{\phi \in fml(L^{w\cdots t}) : \Box\phi(v/\mathbf{c}_t) \in D_i^{n,k}(w{*}z)^*\};$$

$$B_i' = \{\psi \in fml(L^{w\cdots t}) : \psi(v/\mathbf{c}_t) \in D_i^{n,k}(w{*}z)\},$$

where v is a variable for which these substitutions make sense. We'll consider two cases.

CASE 1. $\neg E(\mathbf{c}_t) \in D_1^{n,k}(w{*}z)$. Let:

$$D_0^{\cdots t+1}(w) = D_0^{\cdots t}(w) \cup \{\Diamond(\psi^{w,z}(v/\mathbf{c}_t)) : \psi \in B_0'\};$$

$$D_1^{\cdots t+1}(w) = D_1^{\cdots t}(w) \cup \{\neg E(\mathbf{c}_t)\} \cup \{\Diamond(\psi^{w,z}(v/\mathbf{c}_t)) : \psi \in B_1'\}.$$

Conditions (ii), (iii) and (v) are satisfied. Since $\mathbf{c}_t$ is not troublesome for w, z at n, k, $t+1$, conditions (vi), (vii) and (viii) carry over from $D^{\cdots t}$. If $\mathbf{c}_t$ occurs in $\Box\phi(v/\mathbf{c}_t) \in D_i^{n,k}(w{*}z)^*$, since $\mathbf{c}_t$ isn't $\mathbf{0}$, $E(\mathbf{c}_t) \in D_0^{n,k}(w{*}z)$, contrary to assumption; so $\mathbf{c}_t$ can't occur in such $\Box\phi(v/\mathbf{c}_t)$, i.e. v isn't free in ϕ. So suppose that $\Box\phi \in D_i^{n,k}(w{*}z)^* \cap fml(L^{w\cdots t+1})$; then $\phi \in D_i^{n,k}(w{*}z)^* \cap fml(L^{w\cdots t})$; using (iv) and our induction hypothesis, $\phi_{w,z} \in D_i^{\cdots t}(w)$. So $D^{\cdots t+1}$ satisfies (iv). We must check (i).

By Lemma 2, $(D_0^{\cdots t}(w), D_0^{\cdots t}(w) \cup \{\neg E(\mathbf{c})\})$ is consistent. If $\psi \in B_i$, by Lemma 3, $\psi(v/\mathbf{0}) \in D_i^{n,k}(w^*z)$. But $\psi(v/\mathbf{0}) \in \mathit{fml}(L^{w \cdots t})$; by induction hypothesis, $\Diamond(\psi(v/\mathbf{0})^{w,z} \in D_i^{\cdots t}(w)$. Clearly $\Diamond(\psi(v/\mathbf{0})^{w,z})$ is $(\Diamond(\psi^{w,z}))(v/\mathbf{0})$ and $\Diamond(\psi^{w,z}(v/\mathbf{c}_t))$ is $(\Diamond(\psi^{w,z}))(v/\mathbf{c}_t)$; so by Lemma 3, the consistency of $(D_0^{\cdots t}(w), D_1^{\cdots t}(w) \cup \{\neg E(\mathbf{c}_t)\})$ yields the consistency of $(D_0^{\cdots t+1}(w), D_1^{\cdots t+1}(w))$, as desired.

CASE 2. $E(\mathbf{c}_t) \in D_0^{n,k}(w^*z)$. For $i < 2$, let:

$$\bar{D}_i(w) = D_i^{\cdots t}(w) \cup \{\phi_{w,z}(v/\mathbf{c}_t) : \phi \in A_i'\} \cup \{\Diamond(\psi^{w,z}(v/\mathbf{c}_t)) : \psi \in B_i'\}.$$

If $(\bar{D}_0(w) \cup \{E(\mathbf{c}_t)\}, \bar{D}_1(w) \cup \{E(\mathbf{c}_t)\})$ is consistent let that be $(D_0^{\cdots t+1}(w), D_1^{\cdots t+1}(w))$. It's easy to see that conditions (i)–(v) are satisfied; since $\mathbf{c}$ is not troublesome for w, z at n, z, $t + 1$, so are (vi)–(viii).

Suppose now that the previous pair is inconsistent. Then $\mathbf{c}$ will have to be troublesome for w, z by n, z, $t + 1$. Let:

$$D_0^{\cdots t+1}(w) = D_0^{\cdots t}(w) \cup \{\phi_{w,z}(v/\mathbf{0}) : \phi \in A_0'\} \cup \{\Diamond(\exists v)(\psi^{w,z}) : \psi \in B_0'\};$$

$$D_1^{\cdots t+1}(w) = D_1^{\cdots t}(w) \cup \{\neg E(\mathbf{c}_t)\} \cup \{\phi_{w,z}(v/\mathbf{0}) : \phi \in A_1'\} \cup \{\Diamond(\exists v)(\psi^{w,z}) : \psi \in B_1'\},$$

where v is as before. By syntactic compactness, fix finite sets $A_i \subseteq A_i'$, $B_i \subseteq B_i'$ for $i < 2$ so that $A_0 \subseteq A_1$, $B_0 \subseteq B_1$ and the following is inconsistent:

$$(D^{\cdots t}(w) \cup \{\phi_{w,z}(v/\mathbf{c}_t) : \phi \in A_0\} \cup \{\Diamond(\psi^{w,z}(v/\mathbf{c}_t) : \psi \in B_0\} \cup \{E(\mathbf{c}_t)\},$$

$$D_1^{\cdots t}(w) \cup \{\phi_{w,z}(v/\mathbf{c}_t) : \phi \in A_1\} \cup \{\Diamond(\psi^{w,z}(v/\mathbf{c}_t) : \psi \in B_1) \cup \{E(\mathbf{c}_t)\}).$$

Since $\mathbf{c}_t$ doesn't occur in $D_1^{\cdots t}(w)$, (A_0, A_1, B_0, B_1) satisfy (vi)–(viii); so let it be the negative witness for $(w, \mathbf{c}_t)$.

Suppose that $\Box\phi' \in \mathit{fml}(L^{w \cdots t+1}) \cap D_i^{n,k}(w^*z)^*$; then ϕ' is $\phi(v/\mathbf{c}_t)$, relabelling bound variables if necessary, for $\phi \in A_i$; also $\phi'_{w,z}$ is $\phi_{w,z}(v/\mathbf{0})$. Since $\phi'_{w,z} \in D_i^{n \cdots t+1}(w)$, (iv) is satisfied. A similar argument shows that (v) is satisfied. To show that (i) is satisfied, it will suffice to show:

(ix) if $\phi \in A_0$ and $\psi \in B_0$ then: $D_0^{\cdots t}(w), D_1^{\cdots t}(w) \vdash \phi_{w,z}(v/\mathbf{0})$ & $\Diamond(\exists v)(\psi^{w,z})$;

(x) If $\phi \in A_1$ and $\psi \in B_1$ then: $D_0^{\cdots t}(w), D_1^{\cdots t}(w) \vdash^{w} \phi_{w,z}(v/\mathbf{0})$ & $\Diamond(\exists v)(\psi^{w,z})$.

The point here is that given (ix) and (x), the consistency of $D^{\cdots t}$ makes $(D_0^{\cdots t+1}(w), D_1^{\cdots t+1}(w) - \{\neg E(\mathbf{c}_t)\})$ consistent, using the fact that A_0', A_1', B_0' and B_1' are closed under '&'; since $\mathbf{c}_t$ doesn't occur in the above pair, by Lemma 2, adding $\neg E(\mathbf{c}_t)$ on the right doesn't destroy consistency.

We'll now prove (x); here we use negative witnesses. Fix $\phi \in A_1$, B_1. For $j < \omega$ and $i < 2$ let:

$\mathscr{R}_0 =$ $\{\mathbf{c} : \mathbf{c}$ occurs in $A_1 \cup B_1$ and $\mathbf{c}$ is troublesome for w, z by n, k, $t\}$;

$\mathscr{S}_0^i =$ $\{\phi'(v/\mathbf{c}_t) : \phi' \in A_i\}$;

$\mathscr{T}_0^i =$ $\{\psi'(v/\mathbf{c}_t) : \psi' \in B_i\}$;

$\mathscr{S}_{j+1}^i =$ $\{\phi'(v/\mathbf{c}) : \mathbf{c} \in \mathscr{R}_j$, some $(\bar{A}_0, \bar{A}_1, \bar{B}_0, \bar{B}_1)$ is the negative witness for $(\mathbf{c}, w)$ and $\phi' \in \bar{A}_i\}$;

$\mathscr{T}_{j+1}^i =$ $\{\psi'(v/\mathbf{c}) : \mathbf{c} \in \mathscr{R}_j$, some $(\bar{A}_0, \bar{A}_1, \bar{B}_0, \bar{B}_1)$ is the negative witness for $(\mathbf{c}, w)$ and $\psi' \in \bar{B}_i\}$;

$\mathscr{R}_{j+1} =$ $\{\mathbf{c} : \mathbf{c}$ occurs in $\mathscr{S}_{j+1}^1 \cup \mathscr{T}_{j+1}^1$ and $\mathbf{c}$ is troublesome for w, z by n, k, $t\}$;

$\mathscr{R} =$ $\cup\{\mathscr{R}_j : j < \omega\}$;

$\mathscr{S}^i =$ $\cup\{\mathscr{S}_j^i : j < \omega\}$;

$\mathscr{T}^i =$ $\cup\{\mathscr{T}_j^i : j < \omega\}$.

Clearly $\mathscr{S}_j^0 \subseteq \mathscr{S}_j^1$, $\mathscr{T}_1^0 \subseteq \mathscr{T}_j^1$. $\mathscr{R}_0$ is finite. So by induction all $\mathscr{R}_j$, $\mathscr{S}_j^i$ and $\mathscr{T}_j^i$ are finite. By the remark following the statement of condition (viii):

$$\max\{\text{ind}(w, \mathbf{c}) : \mathbf{c} \in \mathscr{R}_{j+1}\} < \max\{\text{ind}(w, \mathbf{c}) : \mathbf{c} \in \mathscr{R}_j\},$$

interpreted in the lexicographic ordering of indices. So for sufficiently large j, $\mathscr{R}_j$, $\mathscr{S}_j^i$ and $\mathscr{T}_j^i$ are empty; so $\mathscr{R}$, $\mathscr{S}^i$ and $\mathscr{T}^i$ are finite.

In what follows, we seem forced to use the supposition that 'T' $\in$ lex_y^m, in order to get into a position from which we can use (WAS). Let:

$$\hat{\phi} = \wedge\{T\phi' : \phi' \in \mathscr{S}^0\} \;\&\; \wedge\mathscr{S}^1;$$

$$\hat{\psi} = \wedge\{T\psi' : \psi \in \mathscr{T}^0\} \;\&\; \wedge\mathscr{T}^1.$$

By easy manipulations of '$\Box$', '&' and 'T':

$$\Box(\hat{\phi} \;\&\; \phi(v/\mathbf{c}_t)) \in D_1^{n,k}(w^*z)^*;$$

$$\hat{\psi} \;\&\; \psi(v/\mathbf{c}_t) \in D_1^{n,k}(w^*z).$$

Fix $\vec{\mathbf{d}} = (\mathbf{d}_0, \ldots, \mathbf{d}_{p-1})$, $\vec{\mathbf{e}} = (\mathbf{e}_0, \ldots, \mathbf{e}_{q-1})$, $\vec{\mathbf{f}} = (\mathbf{f}_0, \ldots, \mathbf{f}_{r-1})$ to be sequences of constants so that:

$$\vec{\mathbf{d}} \cup \vec{\mathbf{e}} \cup \vec{\mathbf{f}} = \mathscr{R};$$

$$\vec{\mathbf{d}} \cup \vec{\mathbf{e}} = \mathscr{R} \cap \text{Param}(\hat{\phi} \;\&\; \phi);$$

$$\vec{\mathbf{e}} \cup \vec{\mathbf{f}} = \mathscr{R} \cap \text{Param}(\hat{\psi} \;\&\; \psi).$$

Clearly any parameter of $\hat{\phi}$, ϕ, $\hat{\psi}$ or ψ that's troublesome for w, z by n belongs to $\mathscr{R}$. Fix $\vec{\tau} = (\tau_0, \ldots, \tau_{l-1})$ and $\vec{\sigma} = (\sigma_0, \ldots, \sigma_{m-1})$ be sequences so that

$$\vec{\tau} \cup \vec{\sigma} = ((\text{Param}(\hat{\phi} \;\&\; \phi) \cup \text{Param}(\hat{\psi} \;\&\; \psi)) - \mathscr{R}) \cup \{\mathbf{0}\};$$

$$E(\tau_j) \in D_0^{n,k}(w^*z) \text{ for all } j < l;$$

$$\neg E(\sigma_j) \in D_1^{n,k}(w^*z) \text{ for all } j < m;$$

where σ_0 is '$\mathbf{0}$' and all τ_j and σ_j are distinct from v. Fix sequences of variables $\vec{v}$, $\vec{\mu}$ and $\vec{\varrho}$ all mutually disjoint, disjoint from $\{v\}$, $\vec{\tau}$ and $\vec{\sigma}$, and so that for appropriate choices of $\check{\phi}$, $\bar{\phi}$, $\check{\psi}$ and $\bar{\psi}$:

$$\hat{\phi} = \check{\phi}(\vec{v}/\vec{\mathbf{d}}, \vec{\mu}/\vec{\mathbf{e}});\; \phi = \bar{\phi}(\vec{v}/\vec{\mathbf{d}}, \vec{\mu}/\vec{\mathbf{e}});$$

$$\hat{\psi} = \check{\psi}(\vec{v}/\vec{\mathbf{e}}, \vec{\mu}/\vec{\mathbf{f}});\; \psi = \bar{\psi}(\vec{v}/\vec{\mathbf{e}}, \vec{\mu}/\vec{\mathbf{f}}).$$

Let:

$$\psi^* = (\exists v)(\Box(\check{\phi} \;\&\; \bar{\phi}) \;\&\; (\check{\psi} \;\&\; \bar{\psi}) \;\&\; E_s(\vec{\tau}))(\vec{v}/\vec{\mathbf{d}}, \vec{\mu}/\vec{\mathbf{e}}, \vec{\varrho}/\vec{\mathbf{f}});$$

Since $\psi^* \in D_1^{n,k}(w^*z)$, $\Diamond(\psi^{*w,z}) \in D_1^{n,k}(w)$ by condition (v) and the induction hypothesis; this formula is:

$$\Diamond(\exists\vec{v} \cup \vec{\mu} \cup \vec{\varrho})(\exists v)(\Box(\phi \mathbin{\&} \bar{\phi}) \mathbin{\&} (\psi \mathbin{\&} \bar{\psi}) \mathbin{\&} E_s(\vec{\tau})).$$

By (*WAS*):

$$\vee\{\theta(\mathscr{F}) : \mathscr{F} \subseteq \{v\} \cup \vec{v} \cup \vec{\mu} \cup \vec{\varrho}\} \in D_0^{n,k}(w).$$

Keep in mind that where $\mathscr{F} = \{\eta_0, \ldots, \eta_{s-1}\}$, $\theta(\mathscr{F})$ is

$$(\exists\eta_0) \ldots (\exists\eta_{s-1})(\check{\phi} \mathbin{\&} \bar{\phi})_{\mathscr{F}} \mathbin{\&} \Diamond((\check{\psi} \mathbin{\&} \bar{\psi})^{\mathscr{F}})).$$

CLAIM. If $\mathscr{F}$ is non-empty then $(D_0^{n,k}(w), D_1^{n,k}(w) \cup \{\theta(\mathscr{F})\})$ is inconsistent. Suppose $\mathscr{F}$ is non-empty and $(D_0^{n,k}(w), D_1^{n,k}(w) \cup \{\theta(\mathscr{F})\})$ is consistent. Let $\mathbf{b} = (\mathbf{b}_0, \ldots, \mathbf{b}_{s-1})$ be a sequence of distinct constants not appearing in $D_1^{\cdot\cdot\prime}(w)$, $\check{\phi}$, $\bar{\phi}$, $\check{\psi}$ or $\bar{\psi}$. Then:

$$(*) \qquad (D_0^{\cdot\cdot\prime}(w) \cup \{E(\vec{\mathbf{b}})\}, D_1^{\cdot\cdot\prime}(w) \cup \{E(\vec{\mathbf{b}})\} \cup \{\check{\phi} \mathbin{\&} \bar{\phi})_{\mathscr{F}}(\vec{\eta}/\vec{\mathbf{b}}) \mathbin{\&} \Diamond((\check{\psi} \mathbin{\&} \bar{\psi})^{\mathscr{F}}(\vec{v}/\vec{\mathbf{b}}))\})$$

is also consistent. Where $\mathscr{F}' =$

$$\{\mathbf{d}_j : v_j \in \mathscr{F}\} \cup \{\mathbf{e}_j : \mu_j \in \mathscr{F}\} \cup \{\mathbf{f}_j : \varrho_j \in \mathscr{F}\},$$

let:

$$\bar{\mathscr{F}} = \begin{cases} \mathscr{F}' \cup \{v\} & \text{if } v \in \mathscr{F}; \\ \mathscr{F}' & \text{otherwise.} \end{cases}$$

Let $\mathbf{c} \in \bar{\mathscr{F}}$ be of minimal index, i.e. for all $\mathbf{c}' \in \bar{\mathscr{F}}$, $ind(\mathbf{c}') \leqslant ind(\mathbf{c})$. Let $(\bar{A}_0, \bar{A}_1, \bar{B}_0, \bar{B}_1)$ be the negative witness for $\mathbf{c}$. Then for any $\mathbf{c}' \in \mathscr{R}$ occurring in $\bar{A}_1 \cup \bar{B}_1$, $ind(\mathbf{c}') < ind(\mathbf{c})$; so $\mathbf{c}' \notin \bar{\mathscr{F}}$; so the variable corresponding to $\mathbf{c}'$ doesn't belong to $\mathscr{F}$. Let:

$$\bar{A}_2 = T\bar{A}_0 \cup \bar{A}_1;\ \bar{B}_2 = T\bar{B}_0 \cup B_1.$$

Let ξ be: v_j if $\mathbf{c}$ is $\mathbf{d}_j$, μ_j if $\mathbf{c}$ is $\mathbf{e}_j$, and ϱ_j if $\mathbf{c}$ is $\mathbf{f}_j$. Then:

$\wedge \bar{A}_2(\xi/\mathbf{c})$ is a conjunct of $\hat{\phi}$;

$\wedge \bar{B}_2(\xi/\mathbf{c})$ is a conjunct of $\hat{\psi}$.

Suppose $\wedge \bar{A}_2(\xi/\mathbf{c})$ is $\delta(\vec{v}/\vec{\mathbf{d}}, \vec{\mu}/\vec{\mathbf{e}})$ and $\wedge \bar{B}_2(\xi/\mathbf{c})$ is $\lambda(\vec{\mu}/\vec{\mathbf{e}}, \vec{\varrho}/\vec{\mathbf{f}})$; then δ is a conjunct of $\check{\phi}$ and λ is a conjunct of $\check{\psi}$. Let:

δ' = the result of replacing in δ each free occurrence of each v_i and μ_i other than ξ by $\mathbf{0}$;

then δ' is a conjunct of $(\check{\phi}\ \&\ \bar{\phi})_{\mathscr{F}}$. Let **b** be $\mathbf{b}_i$ where ξ is η_i for $i < s$. Then $\delta'(\xi/\mathbf{b})$ is a conjunct of $(\check{\phi}\ \&\ \bar{\phi})_{\mathscr{F}}(\vec{v}/\vec{\mathbf{b}})$. Let:

> λ' = the result of existentially quantifying-out of λ all free occurrences of each μ_i and ϱ_i other than ξ.

Then by easy manipulation of '$\Diamond$' and '&':

$$\{\ \}, \{\Diamond((\check{\phi}\ \&\ \bar{\phi})^{\mathscr{F}}(\xi/\mathbf{b})\} \vdash^{w} \Diamond\lambda'(\xi/\mathbf{b}).$$

So by (*):

$$(D_0^{\cdots t}(w) \cup \{E(\mathbf{b})\}, D_1^{\cdots t}(w) \cup \{E(\mathbf{b}), \delta'(\xi/\mathbf{b}), \Diamond\lambda'(\xi/\mathbf{b})\})$$

is consistent. But:

$$\wedge T\bar{A}_{0,w,z}(\xi/\mathbf{b})\ \&\ \wedge \bar{A}_{1,w,z}(\xi/\mathbf{b}) \text{ is } \delta'(\xi/\mathbf{b});$$

$$\{\ \}, \{\Diamond\lambda'(\xi/\mathbf{b})\} \vdash^{w} \wedge T\bar{B}_0^{w,z}(\xi/\mathbf{b})\ \&\ \wedge \bar{B}_1^{w,z}(\xi/\mathbf{b}).$$

So:

$$(D_0^{\cdots t}(w) \cup \{E(\mathbf{b})\} \cup \bar{A}_{0,w,z}(\xi/\mathbf{b}) \cup \bar{B}_0^{w,z}(\xi/\mathbf{b}),$$

$$D_1^{\cdots t}(w) \cup \{E(\mathbf{b})\} \cup \bar{A}_{1,w,z}(\xi/\mathbf{b}) \cup \bar{B}_1^{w,z}(\xi/\mathbf{b}))$$

is consistent. This contradicts (viii) and the choice of $(\bar{A}_0, \bar{A}_1, \bar{B}_0, \bar{B}_1)$, and establishes our claim.

That claim now yields: $D_0^{n,k}(w), D_1^{n,k}(w) \vdash \theta(\{\ \})$. Clearly $(\check{\phi}\ \&\ \bar{\phi})_{\{\ \}}$ is $\check{\phi}_{\{\ \}}\ \&\ \bar{\phi}_{\{\ \}}$, and $\bar{\phi}_{\{\ \}}$ is $\phi_{w,z}(v/\mathbf{0})$. Furthermore $\Diamond((\check{\psi}\ \&\ \bar{\psi})^{\{\ \}}$ is $\Diamond(\exists\vec{\mu} \cup \vec{\varrho})(\exists v)(\check{\psi}\ \&\ \bar{\psi})$, and:

$$\{\ \}, \{\Diamond((\check{\psi}\ \&\ \bar{\psi})^{\{\ \}}) \vdash^{w} \Diamond(\exists v)(\exists\vec{\mu} \cup \vec{\varrho})\bar{\psi},$$

and $(\exists\vec{\mu} \cup \vec{\varrho})\bar{\psi}$ is $\psi^{w,z}$. Thus:

$$D_0^{\cdots t}(w), D_1^{\cdots t}(w) \vdash^{w} \phi_{w,z}(v/\mathbf{0})\ \&\ \Diamond(\exists v)(\psi^{w,z}),$$

as claimed in (x). By straightforward manipulation of 'T', we obtain (ix) from (x). Thus (i) has been established, and $D^{\cdots t+1}$ is as required.

The rest of the construction of $D^{n,k,z+1}$, and then $D^{n,k+1}$, and finally D^{n+}, is as described in the preview. Condition (iii) on $D^{n,t,z,t}$ makes sure that D^{n+} satisfied condition (b).

We'll now construct W^{n+1} and D^{n+1} on W^{n+1}, so as to be consistent, $\Box$-normal*, good and complete* for $\mathscr{C}^{n+1}$.

Form $D_0^{n+1}(\langle\ \rangle) \subseteq D_1^{n+1}(\langle\ \rangle) \subseteq fml(L^{\langle\ \rangle,n+1,0})$ so that $D_i^{n+}(\langle\ \rangle) \subseteq D_i^{n+1}(\langle\ \rangle)$ for $i < 2$ and $(D_0^{n+1}(\langle\ \rangle), D_1^{n+1}(\langle\ \rangle))$ is consistent and $\neg$-complete and $\exists$-complete for $L^{\langle\ \rangle,n+1,0}$. This is by the usual Henkin construction. Let $W_0^{n+1} = \{\langle\ \rangle\}$.

Now suppose that $W_m^{n+1} = W^{n+1} \cap \{w \in (\kappa \cdot (n+2))^{<\omega}: |w| \leqslant m\}$ has been defined, and that for all $w \in W_m^{n+1}$, $D_i^{n+1}(w)$ for $i < 2$ have been defined. Fix $w \in W_m^{n+1}$ with $|w| = m$.

Suppose that $w \in W^n$ and $w{*}z \in W^n$. For $i < 2$, let:

$$D_i^{(n+1)-}(w{*}z) = D_i^{n+}(w{*}z) \cup \Box^{-1}(D_i^{n+1}(w)^*).$$

CLAIM. $(D_0^{(n+1)-}(w{*}z), D_1^{(n+1)-}(w{*}z))$ is consistent.

Suppose not. By syntactic compactness there are finite sets $E_i \subseteq D_i^{n+}(w{*}z)$ and $F_i \subseteq \Box^{-1}(D_i^{n+1}(w)^*)$ so that $E_0 \subseteq E_1$, $F_0 \subseteq F_1$, and $(E_0 \cup F_0, E_1 \cup F_1)$ is inconsistent. Let:

$$\psi = \wedge TE_0 \,\&\, \wedge E_1;\ \phi = \wedge TF_0 \,\&\, \wedge F_1.$$

Then $(\{\ \}, \{\phi, \psi\})$ is inconsistent. Clearly $\Diamond(\psi^{w,z}) \in D_i^{n+1}(w)$ and $\Box\phi \in D_1^{n+1}(w)^*$, by straightforward manipulations of '$\Box$', 'T' and '&'; also:

$$\{\ \}, \{\Box\phi, \Diamond(\psi^{w,z})\} \vdash^w \Diamond(\phi \,\&\, \psi^{w,z}).$$

But $(\{\ \}, \{\phi, \psi^{w,z}\})$ is also inconsistent, since it's exactly the terms troublesome for w, z by n that are quantified-out of ψ in forming $\psi^{w,z}$, and none of these occur in ϕ. Thus $(\{\ \}, \{\Box\phi, \Diamond(\psi^{w,z})\})$ is inconsistent, contrary to the consistency of $(D_0^{n+1}(w), D_1^{n+1}(w))$.

Let:

$$\mathscr{G}_{w,z}^n = \{\mathbf{c} : \mathbf{c} \in \mathscr{C}^{n+}(w) - \mathscr{C}^n(w),\ u(c) \text{ is incompatible with } w{*}z \text{ and } \neg E(\mathbf{c}) \in D_1^{n+}(w)\};$$

$$\mathscr{H}_w^{n+1} = \{\mathbf{c} : \mathbf{c} \in \mathscr{C}^{n+1}(w) - \mathscr{C}^{n+}(w) \text{ and } \neg E(\mathbf{c}) \in D_1^{n+1}(w)\}.$$

No member of $\mathscr{G}_{w,z}^n \cup \mathscr{H}_w^{n+1}$ occurs in $D_1^{(n+1)-}(w{*}z)$. So by Lemma 2:

$$(D_0^{(n+1)-}(w{*}z), D_1^{(n+1)-}(w{*}z) \cup \{\neg E(\mathbf{c}): \mathbf{c} \in \mathscr{G}_{w,z}^n \cup \mathscr{H}_w^{n+1}\})$$

is consistent. Let $(D_0^{n+1}(w{*}z), D_1^{n+1}(w{*}z))$ be the result of expanding the above pair to be consistent, $\neg$-complete and $\exists$-complete for $L^{w*z,n+1,0}$.

If for some ϕ we have $\neg\Box\phi \in D_1^{n+1}(w)^* - D_1^n(w)^*$, then we must put new successors of w into W_{m+1}^{n+1}. Where $\langle \phi_z \rangle_{z<\kappa}$ is a listing of such ϕ, let:

$$D_1^{(n+1)-}(w^*(\kappa \cdot (n+1) + z)) = \Box^{-1}(D_1^{n+1}(w)^*) \cup \{\neg\phi_z\};$$

$$D_0^{(n+1)-}(w^*(\kappa \cdot (n+1) + z)) = \begin{cases} \Box^{-1}(D_0^{n+1}(w)^*) \cup \{\neg\phi_z\} \\ \text{if } \neg\Box\phi_z \in D_0^{n+1}(w)^*; \\ \Box^{-1}(D_0^{n+1}(w)^*) \quad \text{otherwise.} \end{cases}$$

As usual:

$$(D_0^{(n+1)-}(w^*(\kappa \cdot (n+1) + z)),$$
$$D_1^{(n+1)-}(w^*(\kappa \cdot (n+1) + z)) \cup \{\neg E(\mathbf{c}) : \mathbf{c} \in \mathscr{H}_w^{n+1}\})$$

is consistent. We expand this to:

$$(D_0^{n+1}(w^*(\kappa \cdot (n+1) + z)), D_1^{n+1}(w^*(\kappa \cdot (n+1) + z))$$

as usual; obviously $w^*(\kappa \cdot (n+1) + z)$ is put into W_{m+1}^{n+1}.

If $w \notin W^n$ and for some ϕ, $\neg\Box\phi \in D_1^{n+1}(w)^*$ then for all $z < \kappa$ and $n' \leqslant n + 1$ we put $w^*(\kappa \cdot n' + z)$ into W_{m+1}^{n+1} and construct $(D_0^{n+1}(w^*z), D_1^{n+1}(w^*z))$; the construction is like that just described for $w^*(\kappa \cdot (n+1) + z)$.

Letting $W^{n+1} = \cup\{W_m^{n+1}: m < \omega\}$, we have D^{n+1} on W^{n+1} so as to be complete*, $\Box$-normal* and consistent. Our use of $\mathscr{G}_{w,z}^n$ and $\mathscr{H}_w^{n+1}$ insures that D^{n+1} is good.

Where $W = \cup\{W^n: n < \omega\}$, let $D_i(w) = \cup\{D_i^n(w): n < \omega\}$ for $w \in W$ and $i < 2$. It's easy to see that D is as desired. Theorem 5 follows for $\underline{\mathbf{B}}_x^{-a}$ from Lemma 4. To prove Theorem 5 for $\underline{\mathbf{B}}_x^a$, we replace $\underline{\mathbf{B}}_x^{-a}$ by $\underline{\mathbf{B}}_x^a$ throughout the previous discussion, and reflexivize the final model.

Can we define a sound and complete formalization of B_x^{-a} or $\mathbf{B}_x^a$ when L is $L_{1,s}$ or $L_{1,\blacksquare,s}$? Since the sole use of 'T' in (WAS) was in the construction of $E_s(\vec{t})$, and this could be done with '$\approx_s$' alone, it might seem as if $\underline{\mathbf{B}}_x^{-a}(L_{1,s})$ and $\underline{\mathbf{B}}_x^a(L_{1,s})$ would be such formalizations. However I can't see how to prove this. Nor do I have a plausible candidate for a $\mathbf{B}_x^{-a}$-valid or weakly $\mathbf{B}_x^{-a}$-valid sequent in $L_{1,s}$ that is not a theorem or, respectively, a weak theorem of $\underline{\mathbf{B}}_x^{-a}(L_{1,s})$.

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