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Stewart Shapiro's *Philosophy of Mathematics*[*]

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This book presents a realistic structuralist account of the subject-matter of contemporary mathematics, one "much like traditional Platonism." (p. 4)[1] It also has things to say about the history of mathematics[2] and constructivism, and it proposes extending structuralism beyond the ontology of mathematics. I'll concentrate on Chapters 3 and 4, which presents the book's most central concepts and theses.

Two slogans define structuralism: contemporary mathematics studies structures; mathematical objects are places in those structures. Shapiro's version of structuralism posits abstract objects of three sorts. A *system* is "a collection of objects with certain relations" (p. 73) between these objects. "An extended family is a system of people with blood and marital relationships." A baseball defense, e.g., the Yankee's defense in the first game of the 1999 World Series, is a also a system, "a collection of people with on-field spatial and 'defensive-role' relations" (pp. 73-74). "A *structure* is the abstract form of a system" (p. 74); it consists of "a collection of *places* [sometimes called *positions* or *offices*] and a finite[3] collection of functions and relations on these places" (p. 93).

Shapiro introduces the relation of being the-abstract-form-of and its converse, exemplification, by examples. The Baseball Defense (hereafter *BD*)

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[*] Shapiro, Stewart, *Philosophy of Mathematics: Structure and Ontology* (New York : Oxford University Press, 1997).

[1] It isn't Platonism because, on Shapiro's reading of the distinction between arithmetic and logistic (e.g. in *Gorgias*), Plato maintained "that one can state the *essence* of each number without referring to the other numbers." (p. 72) But structuralism asserts: "there is no more to the individual numbers 'in themselves' than the relations they bear to each other." (p. 73)

[2] Shapiro maintains that mathematics evolved towards being about structures; e.g., geometry wasn't about structures until late in the 19th century. Presumably the arithmetic of the natural numbers was about a structure much earlier than that.

[3] The finiteness constraint here seems gratuitous, and didn't occur in the definition on p. 73.

is a structure, and the Pitcher, the Catcher, etc.[4] are places in it. The Yankee's defense in the first game of the 1999 World Series exemplifies *BD*, with e.g. the Pitcher and the Catcher occupied by Hernandez and Pesada respectively. Any omega sequence with its successor function exemplifies the Natural Number structure, with the former's third element (starting the Naturals with 0) occupying the place commonly called "the natural number 2".

When we consider a structure with an eye towards its exemplifications, we take "the places-are-offices perspective" (p. 82) towards that structure. But we can also treat "places as objects in their own right, at least grammatically" (p. 83). In describing the structure of the federal government of the United States,[5] we might say that the Vice President is President of the Senate, and in so doing we don't refer to whoever is currently Vice President, but rather to the office itself, a place in the Federal-Government structure. This is to take "the places-are-objects perspective". And herein lies one distinctive element of Shapiro's view: he takes this way of speaking "literally, at face value." "Places in structures are bona fide objects." "Bona fide singular terms, like 'vice president,' 'shortstop,' and '2' denote bona fide objects" (p. 83). In particular, mathematical objects are places in structures.

Each place "lives in" a unique home-structure in which it is a place. (This doctrine implies that each constitutive relation of a structure is also tied to a unique home-structure, since all of its relata are so tied.) Relational statements, including identity statements, concerning places in the same structure and invoking the constitutive relations of that structure have truth-values. What about a relational statement such that one of its singular terms designates a place in a given structure and the other doesn't? I take it that Shapiro thinks that such a statement lacks a truth-value. But we can "stipulate" a truth-value (subject to certain constraints) for such statement.[6]

This "ante rem" view of structures[7] contrasts with the "in re" view, which doesn't take the places-are-objects perspective at face value, maintaining that we should take discourse within this perspective as merely a way of expressing "generalizations in the places-are-offices perspective" (p. 85). This natu-

4 When I use an English noun-phrase to refer to a position in a structure, I'll capitalize; Shapiro frequently uses italics.

5 I take it that the American "system" of government is a structure, exemplified by various administrations.

6 About questions regarding the truth of such statements, he says "a philosophy of mathematics should show why these questions need no answers, ...there is no answer to be discovered." "Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not, and neither is identity between numbers and the positions of other structures" (p. 79). This requires that places be peculiar abstract objects, not of a sort with which straightforward Platonists like Frege would be comfortable.

7 Branded by Dummett as "mystical structuralism" in M. Dummett (1991).

rally leads to what Charles Parsons called "eliminative structuralism", since if structures are exemplified only by systems that are not structures, we might go on to say "Talk of structures generally[8] is convenient shorthand for talk about systems" (p. 85).

Many mathematical theories are about infinite structures, each of which require infinitely many objects to occupy places in any of its exemplifications. To avoid vacuity for such a theory, there had better be a system[9] exemplifying the structure that the theory is about. Unlike in re structuralism, ante rem structuralism doesn't demand that a structure be exemplified "in the non-structural realm." Every structure is itself a system, and at least every mathematical structure exemplifies itself.[10] This neatly answers the worry "Are there enough objects?", perhaps too neatly. For example, the Natural-Numbers structure by itself insures that there are at least countably many objects.

What is distinctive about mathematical structures? The objects in any exemplification of *BD* must be people,[11] who are playing ball, the pitcher pitching (or ready to pitch when her team takes the field) to the catcher, etc. "Imagine a system [call it bd_1] that consists of nine people placed in the configuration of a baseball defense but hundreds of miles apart... This system does not exemplify the structure of baseball defense" (p. 98). Similarly, "[i]magine a system that consists of a ballpark with nine piles of rocks, or nine infants, placed [call it bd_2] where the fielders usually stand.... The system of rock piles...can perhaps be said to model or simulate the baseball defense structure, but [it does] not exemplify it" (p. 99-100). In contrast, an exemplification of the Natural-Numbers structure need only be isomorphic to the Natural Numbers under successor. Shapiro recognizes two aspects to this contrast: formality and freestandingness.

Somehow *BD* carries with it "the requirement that the officeholders be people prepared to play..."; bd_2 fails this requirement. This constraint on exemplification of *BD* "is not described [sic., expressible] solely in terms of relations among the offices and their occupants." (p. 99) Because exemplification of *BD* is constrained by such a condition, *BD* is not what Shapiro calls "freestanding". "In contrast, mathematical structures are *freestanding*. Every office is characterized completely in terms of how its occupant relates to the occupants of the other offices of the structure, and any object can occupy any of its places." (p. 100)

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8 Meaning, I suppose, "always", rather than indicating an allowance for exceptions.

9 For the eliminativist, a system "in a non-structural realm" (p. 89), since the defining idea of the eliminative camp is that structures are not serious posits.

10 The book says "So, in a sense, each structure exemplifies itself" (p. 89). The force of the hedge "in a sense" is unexplained.

11 Well, at least agents capable of doing the actions "basic" to baseball: throwing and catching a baseball, running bases, swinging a bat, etc.

The system bd_1 doesn't exemplify *BD* because of "an implicit requirement that the player at first base be within a certain distance of first base, the pitcher, and so forth ..." (p. 98), a requirement somehow built into the constitutive relations of *BD*. Since being approximately 90 feet apart, etc., are not formal relations,, *BD* is non-formal. But mathematical structures are formal.

Shapiro defines being a formal relation as follows: "it can be completely defined in a higher-order language, using only terminology that denotes Tarski-logical notions[12] and the other objects and relations of the system, with the other objects and relations completely defined at the same time. All relations in a mathematical structure are formal in this sense." (p. 99) I couldn't make satisfying sense this. What is this "system"? Is formality an absolute property of relations (as the surrounding material suggests), or is it relative to systems? And if the latter, to which systems? Or is there only one relevant system? and if so what is it? Perhaps that structure itself? What resources are allowed for this definition in a higher-order language? Without answers to these questions, Shapiro's explanation of formality remains unilluminating.

Shapiro sets out "to articulate a relation among systems that amounts to 'have the same structure'" (p. 90).[13] He rejects isomorphism: "Isomorphism is too fine-grained for present purposes. Intuitively one would like to say that the natural numbers with addition and multiplication exemplify the same structure as the natural numbers with addition, multiplication and less-than" (p. 91). I suspect that many readers, like myself, will not have this "intuition". Instead of isomorphism, Shapiro offers Resnik's relation of structure-equivalence as a necessary and sufficient condition for exemplification of the same structure. Unfortunately, the book omitted a crucial clause in the definition of structure-equivalence![14] Shapiro's use of 'the' in phrases like 'the abstract form of a system' seem to indicate that he subscribes to the following abstraction principle:

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12 Tarski's "... idea is that a notion is logical if its extension is unchanged under every permutation of the domain" (p. 99).

13 Notice the definite article.

14 Under the definition given in the book, any systems with the same number of constituent objects are structure-equivalent. Here is the intended definition. For systems *s* and *r*, "[d]efine *s* to be a full subsystem of *r* if they have the same objects...and if every relation [constitutive] of *r* can be defined in terms of the relations [constitutive] of *s*" and (here's the missing clause) every constitutive relation of *s* is a constitutive relation of *r*. Then let systems *s* and *s′* be structure-equivalent iff "there is a system *r* such that *s* and *s′* are each isomorphic to a full subsystem of *r*" (p. 91). Shapiro does not tell us what logical resources are allowed for his "can be defined in terms of"; without specifying these, we don't have a well-defined relation. Fortunately, we have enough to prove that structure-equivalence is transitive, a worthwhile exercise.

(Abstr$_1$) for any systems s_0 and s_1, there is a unique structure exemplified by both s_0 and s_1 iff s_0 and s_1 are structure-equivalent.

This leads to serious problems which I won't pursue here. Shapiro has informed me that he didn't intend to take on this commitment. Without this principle, it's unclear what work he wants structure-equivalence to do. If we can make clear sense of formality and freestandingness, perhaps this is the principle Shapiro would favor:

(Abstr$_2$) for any systems s_0 and s_1, there is a unique formal freestanding structure exemplified by both s_0 and s_1 iff s_0 and s_1 are structure-equivalent.

Certain mathematical theories (elementary number theory, real analysis, ZF set-theory) are usually taken to be about specific structures.[15] Why believe that a given theory or (using Shapiro's word) formula has a particular structure to be about? One way to persuade oneself of the existence of a structure is to get to know a system of concrete or quasi-concrete objects[16] that exemplifies it, and then "abstract". But such abstraction won't secure all the structures we want. Perhaps the best we can do is to assume axioms that generalize from the posits justified by abstraction.

Shapiro's main axiom, the "central (albeit vague) principle of structuralism" (p. 105), is Coherence: "if Φ is a coherent formula in a second order language, then there is a structure that satisfies Φ". He says that the concept of coherence is "a primitive, intuitive notion, not reducible to something formal, so I do not venture a rigorous definition" (p. 135). What are we given by way of non-rigorous definition, or at least explication? How can we understand this use of 'coherent' so as to make informative inferences of the form "This theory is coherent; therefore it describes a structure"? Is coherence a quasi-observational property of formulas and theories, one that someone who has never heard or thought of structures could come to recognize? Could one recognize it independently of one's inclination to think that a given formulas or theories describes a structures?

Clearly Shapiro relies on our "everyday" understanding of 'coherent'. Is this notion robust enough to make Coherence a usable axiom? We're told that satisfiability (presumably the set-theoretic relation holding between a

15 Other mathematical theories, e.g. group theory, are "algebraic" in that they are not about a single structure, but rather a class of appropriately similar structures. (See p. 73, note 2; for a model-theoretic version of this distinction, see p. 50.)

16 Shapiro explains quasi-concreteness by quoting Parsons, who introduced the notion [C. Parsons (1990)]; they are abstract objects that "are directly 'represented' or 'instantiated' in the concrete. Examples might be geometric figures (as traditionally conceived), symbols whose tokens are physical utterances or inscriptions, and perhaps sets or sequences of concrete objects" (p. 101).

formula and models in which it is satisfied) "models" coherence; but this didn't help me. We're also told that satisfiability is to coherence as recursiveness (and Turing Computability, etc.) are to effective computability (p. 135). My understanding of 'coherent' is will-o'-the-wisp in comparison to my understanding of 'effectively computable'. Before Church's thesis, mathematicians' understanding of 'effectively computable' was robust enough for it to be uncontroversial that the Ackerman function was effectively computable [S. C. Kleene (1952), pp. 272-73], and that computations relying on probabilistic events or (if there ever is an "open future") on undetermined future events are not effective. The development of various rigorous explications of effective computability, followed by proofs that they were all equivalent, provided a sort of evidence that the original concept was robust. Can we reasonably expect to achieve a similar explication of coherence? I would not hold my breath.

As a realist about abstract objects, Shapiro needs to answer or dissolve the structuralist's versions of Benacerraf's Question (how can we know about, or even refer to, places in a structure?) and the Caesar Question (how are places in a given structure related to other things, e.g.—as a bow to Frege—Julius Caesar?).

Shapiro approaches the former question by considering an account of abstraction[17] due to Robert Kraut (see [R. Kraut 1980]). Kraut proposed that we can talk of more by saying less, using what Shapiro calls "the sublanguage procedure." (p. 129) This is a linguistic version of the mathematical construction of factoring by an equivalence relation: given such a relation, one limits one's vocabulary to expressions that do not distinguish between equivalent objects. For example, an economist might talk about income in a fragment of English

> ... that does not have the resources to distinguish between people with the same income. Anything she says about a person P applies equally well to anyone else Q who has the same income as P.... Someone who interprets the economist's language might...conclude that for those stuck with the impoverished resources [namely those of this fragment], P = Q. That is, from the standpoint of the economist's scheme, people with the same income are identified and treated as a single object.... Nothing is lost by interpreting her language as being about income levels and not people (assuming sharp boundaries between levels, of course). A singular term like "the Jones woman", denotes an income level. (p. 121)

Shapiro likes this idea, but finds Kraut's description of it "far-fetched". He complains that Kraut requires that the parties to this discourse forget or suspend the background language "to speak one of these impoverished

17 Thus a mechanism of abstraction differs from the familiar Aristotelian mechanism, viz. putting on blinders to all but one of the quality-particulars (these days called tropes) in a given object, to gain access to the quality-universal instanced by the remaining quality-particular.

languages": no one does either thing (p. 122). Surely no one forgets one's "background language"; but why be so sure that no one ever suspends it? Shapiro seems to have been bothered, quite rightly, by Kraut's description of this suspension as leaving the speaker "stuck with the impoverished resources." After all, is our economist, while he's at work, "stuck with" the impoverished fragment of English that he deploys in this case? Shapiro writes: "Of course in the full background language, English, the two people [viz. people of the same income-level] can be distinguished in lots of ways.... By hypothesis, however, these resources are not available to our economist (while at work)." (p. 121) Perhaps Kraut should have been more circumspect: these resources remain available, but merely unused; what matters is what the economist actually says, not what she could say.

Shapiro seems to think that Kraut has described the mechanism by which we achieve referential and cognitive access to places in structures. He discusses two examples: income-levels and cardinal numbers. His overarching purpose is to show that Benacerraf's Question is pseudo: it doesn't arise as long as we take the places-are-offices perspective. The Krautian description of how we adopt the places-are-objects perspective is supposed to show this. But however we read pp. 122-23, Shapiro doesn't come close to making this idea clear, let alone persuasive. This entire discussion seems to me to be a failure.

Shapiro accepts a sort of ontological relativity for places, relativity to the impoverished sublanguage that gives one referential access to places-as-objects:

> "Mathematical object" is to be understood as relative *to a theory*, or, loosely, to a background framework. Natural numbers are objects *of arithmetic*, but "natural numbers" may not designate [sic., apply to] objects in another theory or framework. In particular, natural numbers may not be objects in the original background language from which we began. They may be offices. (p. 126)

In maintaining this relativity of objecthood, and with it of identity, to theory or framework (or to conceptual scheme or context[18]), Shapiro takes himself to be following Putnam, who rejects "the idea of a single universe of discourse, fixed once and for all" (p. 128).[19] This relativity lies behind Shapiro's response to questions like 'Is Julius Caesar identical to the number 2?', 'Is the natural number 2 a member of 4?', 'Is the Shortstop a better hitter than the Catcher?' These questions are illegitimate, they "need no answers" (p. 79). Presumably this is because the corresponding propositions, that Julius Caesar = 2, that $2 \in 4$, and that the Shortstop is a better hitter than the Catcher, lack truth-value. And why is that? For the first two, Shapiro says

18 "My proposal is that there is a determinate statement of identity, one with a truth-value to be discovered, only if the context is held fixed..." (p. 126).

19 Shapiro quotes Putnam's claim that it would be a mistake to "single out one use of the existential quantifier...as the only metaphysically serious one" (p. 128).

"Roughly speaking, mathematical objects are tied to the structures that constitute them."[20] (p. 80) Identity for places in a structure is structure-bound. So not only the above propositions, but also the proposition that The natural number 2 = the rational number 2 lacks truth-value.[21]

The above involves rough speaking only because "mathematicians sometimes find it convenient, and even compelling, to identify the positions of different structures." (p. 81) For example, mathematicians identify the natural number 2 with the integer 2 with the rational number 2 with the real number 2. Furthermore, such stipulations of identity change (or replace) the discourse (or the context, or the language, or something) into (or by) one in which such identities have truth-values.[22]

How should we understand such identification? Identification is stipulation of identities. Shapiro only discusses identifications of places with places. Can we identify the natural number 2 with Julius Caesar? Why should stipulating that natural number is identical to a Roman general[23] be less legitimate than stipulating that it is identical to {{},{{}}}? Setting this question aside, how does Shapiro understand stipulation? In places it seems that the stipulator had the astonishing power to make true what gets stipulated and its logical consequences.

Finally, a complaint about the writing in this book. Shapiro's prose is full of hedges that raise unaddressed questions. Those who don't understand the reason for these hedges (I usually didn't) will find this frustrating. The most frustrating hedge of all comes when Shapiro contrasts *ante rem* structuralism with straightforwardly eliminative structuralism and Hellman's modal eliminativism: "there is very little to choose among the options. In a sense they all say the same thing, using different primitives." (p. 97) So Shapiro is really an ecumenical structuralist—well, except for those unspecified little choices.

But are these differences that little? Consider an analogy with physicalism. Many who take interest in the philosophy of mind these days, probably most, are physicalists. The philosophical action on the mind-body problem is with understanding what version of physicalism to accept. The philosophical action among structuralists would, I think, be about what sort of structuralism to accept. Perhaps disputants on this matter are speaking past one another; but that would needs to be shown by arguments of a sort that this book doesn't give.

Thanks to Jeffrey Roland, Zoltan Szabo and Jessica Wilson for very helpful discussion and comments.

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20 I take it that the structure that constitutes a place is that place's "home structure".

21 'The natural number' and 'the rational number' indicate the home structure.

22 On this matter, Shapiro follows Parsons in C. Parsons (1990).

23 Shapiro refers to Julius Caesar as "a monarch" (p. 80); at best this is misleading.

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