Bell(δ) inequalities derived from separate common causal explanation of almost perfect EPR anticorrelations

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Abstract

It is a well known fact that a *common* common causal explanation of the EPR scenario which consists in providing a local, non-conspiratorial *common* common cause system for a set of EPR correlations is excluded by various Bell inequalities. But what if we replace the assumption of a *common* common cause system by the requirement that each correlation of the set has a local, non-conspiratorial *separate* common cause system? In the paper we show that this move does not yield a solution by providing a general recipe how to derive *any* Bell(δ) inequality—that is an inequality differing from some Bell inequality in a term of order of δ—from the assumption that an appropriate set of *almost* perfect anticorrelations has a separate common causal explanation.

1 Introduction

Consider the Bohm version of the EPR experiment with a pair of spin-$\frac{1}{2}$ particles prepared in the singlet state $|\Psi_s\rangle$. Let a_i denote the event that the measurement apparatus is set to measure the spin in direction $\vec{a}_i$ in the left wing where i is an element of an index set I of spatial directions; and let $p(a_i)$ stand for the probability of a_i. Let b_j and $p(b_j)$ respectively denote the same for direction $\vec{b}_j$ in the right wing where j is again in the index set I. (Note that $i = j$ does not mean that $\vec{a}_i$ and $\vec{b}_j$ are parallel directions.) Furthermore, let $p(A_i)$ stand for the probability that the spin measurement in direction $\vec{a}_i$ in the left wing yields the result "up" and let $p(B_j)$ be defined in a similar way in the right wing for direction $\vec{b}_j$. According to quantum mechanics the quantum probability of getting "up" in both directions $\vec{a}_i$ and $\vec{b}_j$ is

$$Tr\left(W_{|\Psi_s\rangle}\left(P_{A_i}\otimes P_{B_j}\right)\right) \quad = \quad \frac{1}{2}\sin^2\left(\frac{\theta_{a_ib_j}}{2}\right) \tag{1}$$

whereas the quantum probability of getting "up" in direction $\vec{a}_i$ disregarding the outcome in direction $\vec{b}_j$; and the quantum probability of getting "up" in direction $\vec{b}_j$ disregarding the outcome in direction $\vec{a}_i$ respectively are

$$Tr\left(W_{|\Psi_s\rangle}\left(P_{A_i}\otimes I\right)\right) \quad = \quad \frac{1}{2} \tag{2}$$

$$Tr\left(W_{|\Psi_s\rangle}\left(I\otimes P_{B_j}\right)\right) \quad = \quad \frac{1}{2} \tag{3}$$

where Tr is the trace function; $W_{|\Psi_s\rangle}$ is the density operator pertaining to the pure state $|\Psi_s\rangle$; P_{A_i} and P_{B_j} denote projections on the eigensubspaces with eigenvalue $+1$ of the spin operators associated with directions $\vec{a}_i$ and $\vec{b}_j$ respectively; and $\theta_{a_ib_j}$ denotes the angle between directions $\vec{a}_i$ and $\vec{b}_j$.

The standard way to interpret quantum probabilities is to identify them with conditional probabilities as follows:

$$p(A_iB_j|a_ib_j) = Tr\left(W_{|\Psi_s\rangle}\left(P_{A_i}\otimes P_{B_j}\right)\right) \tag{4}$$
$$p(A_i|a_ib_j) = Tr\left(W_{|\Psi_s\rangle}\left(P_{A_i}\otimes I\right)\right) \tag{5}$$
$$p(B_j|a_ib_j) = Tr\left(W_{|\Psi_s\rangle}\left(I\otimes P_{B_j}\right)\right) \tag{6}$$

where the events A_i, B_j, a_i and b_j $(i,j \in I)$ respectively are elements of a classical probability measure space (X,S,p) and the conditional probabilities are defined in the usual way. With this identification quantum mechanics predicts correlation between classical conditional correlations for non-perpendicular directions $\vec{a}_i$ and $\vec{b}_j$:

$$p(A_iB_j|a_ib_j) \neq p(A_i|a_ib_j)p(B_j|a_ib_j) \tag{7}$$

Specially, if the measurement directions $\vec{a}_i$ and $\vec{b}_j$ are parallel then there is a perfect anticorrelation between the outcomes A_i and B_j:

$$p(A_iB_j|a_ib_j) = 0 \tag{8}$$

A further consequence of (5)-(6) is the so-called surface locality that is for any $i,i',j,j' \in I$ the following relations hold

$$p(A_i|a_ib_j) = p(A_i|a_ib_{j'}) \tag{9}$$
$$p(B_j|a_ib_j) = p(B_j|a_{i'}b_j) \tag{10}$$

Now, let (A_i,B_j) $(i,j \in I)$ denote a pair correlating conditionally according to (7) and let Δ^I stand for the set $\{(A_i,B_j)\}_{i,j\in I}$ of correlating pairs pertaining to the index set I. What does a common causal explanation of the correlations in Δ^I consists in? To this question one can have a stronger and a weaker answer. The stronger explanation is called the *common* common causal explanation; the weaker one is called the separate common causal explanation.

1. Common common causal explanation. If we take a common causal explanation to be a *common* common causal explanation then we have to provide a so-called *common* common cause system which satisfies three demands: it screens all correlations off, it is local and no-conspiratorial. Let us see them in turn.

Screening-off. The first characterization of the common cause by the screening-off property (plus some extra requirements) is due to Reichenbach (1956). A lot of work has been done since then especially concerning the generalization of the common cause concept for situations where there are more than one causes present. We call such a system of cooperating common causes a common cause system. To be more specific, a common cause system of a correlation $(A_i,B_j)\in\Delta^I$ is a screener-off that is a partition $\{C_k\}_{k\in K}$ of S such that the following factorization holds for all $k\in K$:

$$p(A_iB_j|a_ib_jC_k) = p(A_i|a_ib_jC_k)p(B_j|a_ib_jC_k) \tag{11}$$

where $|K|$, the cardinality of K is said to be the *size* of the common cause system. A common cause system of size 2 is called a common cause. To find common cause systems for each correlation of Δ^I does not mean to find a *common* common cause system for the whole set. A *common* common cause system is a single screener-off such that it fullfills (11) for *every* pair in Δ^I and in this sense it is a stronger notion than that of separate common cause systems defined below.

Locality. Locality is the probabilistic expression of the direct causal independence of certain events due to their spatiotemporal arrangement. Since events A_i and a_i are located spatiotemporally such that they are spatially separated from events B_j and b_j, the following factorizations are to hold for every $(A_i, B_j) \in \Delta^I$ and $C_k \in S$ $(k \in K)$:

$$p(A_i|a_ib_jC_k) = p(A_i|a_iC_k), \qquad p(B_j|a_ib_jC_k) = p(B_j|b_jC_k) \tag{12}$$

No-conspiracy. Finally, no-consiracy is the expression of the conviction that the choice of the measurement setting is causally not influenced by the common cause system (and *vica versa*) that is for every a_i, b_j, and C_k in S $(i, j \in I,\ k \in K)$ the following independence is to hold:

$$p(a_ib_jC_k) = p(a_ib_j)p(C_k) \tag{13}$$

A common causal explanation taken in this strong sense is unfeasible since (11), (12), and (13) famously result in various Bell inequalities which are violated in the EPR experiment for appropriate measurement settings. Consequently, EPR correlations fall short of local, non-conspiratorial, *common*-common-cause-system-type explanation. One premise has to be given up.

2. Separate common causal explanation. The idea to abandon the first premise is due to Szabó (2000). The core of the idea was to replace the concept of *common* common cause system with that of separate common cause systems and to modify the requirement of locality and non-conspiracy accordingly. A separate common causal explanation of the correlations in Δ^I consists in finding a separate partition $\{C_k^{ij}\}$ of S for *each* correlation (A_i, B_j) in Δ^I such that the partition screens (A_i, B_j) off and every partition is local and non-conspiratorial in the sense that for every $i, j \in I$; $k(ij) \in K(i, j)$ the following prescriptions hold:

$$p(A_iB_j|a_ib_jC_k^{ij}) = p(A_i|a_ib_jC_k^{ij})p(B_j|a_ib_jC_k^{ij}) \tag{14}$$

$$p(A_i|a_ib_jC_k^{ij}) = p(A_i|a_iC_k^{ij}), \qquad p(B_j|a_ib_jC_k^{ij}) = p(B_j|b_jC_k^{ij}) \tag{15}$$

$$p(a_ib_jF) = p(a_ib_j)p(F) \tag{16}$$

In the last equation F is an element of the algebra $S' \subset S$ generated by all separate common cause systems. To motivate why it is important to demand no-conspiracy in this strong sense namely for any element of the generated algebra and not just for the C_k^{ij} elements, let it suffice to refer to Szabó's (2000) paper. Here the author presented a local separate common causal explanation for the EPR correlations that was non-conspiratorial in the sense the every a_i and b_j were independent of every C_k^{ij}—still it was conspiratorial in the sense that a_i and b_j correlated with some *disjunctions* of elements of separate common cause systems such as $C_k^{ij} \cup C_{k'}^{i'j'}$.

It is important to be aware of the consequences of replacing the notion of *common* common cause system with that of separate common cause systems. In the *common* common causal explanation one has only one partition $\{C_k\}_{k \in K}$ of the algebra whereas in the separate common causal explanation one has a set of partitions $\{C_k^{ij}\}$ of S, one for each correlation. The combination of these separate partitions into a finer partition however does not result in a *common* common cause system since the elements of this finer partition does not generally satisfy screening-off (11) and locality (12). Based on this fact an anonymous referee of this paper has formulated the following objection against the cogency of the separate common causal explanation. Since the separate partitions are the "ultimate" partitions in the sense that they can not be combined into *common* common cause system *and* since we regard these partitions as *states* or *properties* of the hidden variable therefore we are forced to say that in the separate common causal explanation the hidden states or properties are *coarse-grained* in the sense that in some run of the experiment a hidden variable can (actually must) be in more than one state or it can (must) instantiate more than one property. But then the separate common cause systems are *less complete* than the (in this respect fine-grained) quantum state we intended to explain by means of them.

Although by the end of the paper it will turn out that there exists no sepatate common causal explanation of the EPR scenario, I am not convinced that the above reasoning can query the soundness of such a project. In my understanding the task of a probabilistic common causal explanation of the EPR scenario consists simply in postulating hidden elements of reality such that *grouping* them in different event classes these classes satisfy screening-off, locality and no-conspriracy. The common common causal explanation is more stringent in the grouping of these singular events: one singular event can only fall into one group. The separate common causal explanation relaxes this strict condition of grouping, tolerating events falling into two different event classes. This tolerance might seem strange at first sight but we have no *a priori* reason to exclude this possibility. If the resulting frequencies of the groupings satisfy screening-off, locality and no-conspriracy then both the *common* and the separate common causal explanations have done its job; at this purely statistical level no more can be expected.

Now, what are the prospects for a local, non-conspiratorial separate common causal explanation of the EPR correlations in Δ^I? Before turning to this question in the next Sections here we briefly sketch the history of the separate common causal explanation of the EPR correlation.

The notion of the common cause, as mentioned above, was first defined by Reichenbach in his *The Direction of Time* (1956). A number of important probabilistic features of the Reichenbachian common cause have been investigated in a series of papers by Hofer-Szabó, Rédei, and Szabó (1999, 2002). Hofer-Szabó and Rédei generalized the notion Reichenbachian common cause to Reichenbachian common cause systems in (2004, 2006). The conceptual difference between *common* common cause and *separate* common cause was first recognized by Belnap and Szabó (1996). Szabó was also the first to apply the concept of separate common cause for the EPR situation in (2000). Here Szabó concluded with the conjecture that EPR can not be given any local, non-conspiratorial, separate-common-cause-model. Grasshoff, Portmann and Wüthrich (2005) have proved Szabó's conjecture by deriving Bell inequalities from Szabó's assumptions. However the derivation was based on perfect correlations. In (2008) Hofer-Szabó has shown that the assumption of perfect correlation reduces the derivation of Grasshoff and al. to a common-screener-off derivation. In the same paper Hofer-Szabó has presented a derivation of Bell inequalities from local, non-conspiratorial separate common causes. Since a common cause is a special common cause system (a common cause system of size 2) the result was not general enough. In (2007) Portmann and Wüthrich have derived the Clauser-Horne inequality from local, non-conspiratorial separate common cause systems. In the present paper we intend to give a general recipe how to derive *any* Bell(δ) inequality—that is a Bell-like inequality differing from some Bell inequality in a δ term—from the assumption that each correlation in a s*pecial subset* of Δ^I has a local, non-conspiratorial separate common cause system.

In Section 2 we prove two Propositions which will play a crucial role in the subsequent construction of the different Bell(δ) inequalities. In Section 3 these Propositions will be applied to the Wigner-Bell and the Clauser-Horne scenario yielding a so-called Wigner–Bell(δ) inequality and a Clauser–Horne(δ) inequality respectively. In the Conclusions we give a general recipe for deriving *any* Bell(δ) inequality and show that these derivations are not as general as they could be since they remain in a 'δ-neighborhood' of some *common* common causal explanation. We conclude the paper with the open question whether one can do it better.

2 Separate common cause systems and Bell(δ) inequalities

Since in the present and the next Section we are to develop a strategy for deriving *any* Bell(δ) inequality close to some Bell inequality, we have to define first what type of Bell inequalities we are concerned with. The general classification of the Bell inequalities is a subtle task since it depends on the various assumptions contained in the premisses of the derivation. For our purpose, however, a rough characterization will suffice. Consider a set Δ^I and suppose that all correlations $(A_i, B_j) \in \Delta^I$ have a local, non-conspiratorial, *common* common cause system in the sense of (11)-(13). Take all the *marginal* probabilities such as $p(A_i|a_ib_j)$ and $p(B_j|a_ib_j)$, and *joint* probabilites such as $p(A_iB_j|a_ib_j)$ pertaining to the correlations $(A_i, B_j) \in \Delta^I$. Now we call "Bell inequality" any constraint among these marginal and joint probabilities which can be derived from (11)-(13).

This characterization of the Bell inequalities is fairly standard. It implies not only that the elements A_i, B_j, a_i, b_j etc. are elements of a classical probability measure space but includes the assumption of locality and no-conspiracy as well. It excludes joint probabilities of more than two events such as $p(A_iB_jD_k|a_ib_jd_k)$ but this is again usual. In what follows we will use the term 'Bell inequality' in the above sense.

In this Section we take some preparatory steps for the general construction of various Bell(δ) inequalities. First we prove two Propositions concerning correlations close to perfect anticorrelation. In Proposition 1 we approximate the marginal probabilities of such correlations by the probabilities of (an appropriate combination of) the elements of the local, non-conspiratorial separate common cause system of the correlation. In Proposition 2 we do the same for the joint probabilities.

Let δ be a non negative real number and let (A_i, B_j) be a correlating pair in Δ^I such that

$$p(A_iB_j|a_ib_j) \leqslant \delta \tag{17}$$

Call such a correlation an *almost perfect anticorrelation* since in the case $\delta = 0$ the correlation satisfying (17) is a perfect anticorrelation. Suppose that (A_i, B_j) has a local, non-conspiratorial separate common causal explanation. For such a correlation the following Proposition will hold.

Proposition 1. Let (X, S, p) be a classical probability measure space and let (A_i, B_j) be a correlating pair in Δ^I satisfying (17), where A_i, B_j, a_i and b_j are elements of S. Suppose furthermore that (A_i, B_j) has a local, non-conspiratorial separate common cause system $\{C_k^{ij}\}$ $(k(ij) \in K(i,j))$ that is a partition of S such that (14)-(16) are satisfied. Then there exist a vector $\varepsilon^{ij} \in \{0,1\}^{|K|}$ ($|K|$ is the cardinality of K) such that defining C^{ij} and $C^{ij\perp}$ as

$$C^{ij} \equiv \bigcup_{k\in K} \varepsilon_k^{ij}\, C_k^{ij}; \qquad C^{ij\perp} \equiv \bigcup_{k\in K} (1-\varepsilon_k^{ij})\, C_k^{ij} \tag{18}$$

the following inequalities hold:

$$|p(C^{ij}) - p(A_i|a_ib_j)| \leqslant 4\delta \tag{19}$$

$$|p(C^{ij\perp}) - p(B_j|a_ib_j)| \leqslant 4\delta \tag{20}$$

We refer to the partition $\{C^{ij}, C^{ij\perp}\}$, or simply to C^{ij} as the *quasi separate common cause* of (A_i, B_j).

Proposition 1 states that if a correlating pair (A_i, B_j) in Δ^I is sufficiently close to perfect anti-correlation *and* it has a local, non-conspiratorial separate common cause system then there exists an appropriate combination of the elements of the common cause system *via* (18), the *quasi* separate common cause $\{C^{ij}, C^{ij\perp}\}$ such that the probability of the conditional probabilities of the outcomes A_i and B_j respectively can be sufficiently approximated by the probability of the partition $\{C^{ij}, C^{ij\perp}\}$. The term "*quasi*" expresses the fact that although C^{ij} and $C^{ij\perp}$ are constructed out of the elements of the local, non-conspiratorial separate common cause system $\{C_k^{ij}\}$ they do not satisfy the screening-off condition (14) (however they satisfy locality (15) and no-conspiracy (16). To see these, just observe that each *quasi* separate common cause is a disjoint sum of elements satisfying (15); and each *quasi* separate common cause is in the algebra S' generated by the separate common cause systems and hence it satisfies (16) by definition).

Proof: For the proof first consider the following

Lemma. Let $\alpha_k, \beta_k \in [0,1]$, $p_k \in (0,1)$ for any $k \in K$ and let $\sum_k p_k = 1$. Let furthermore assume that

$$\begin{aligned} \sum_{k\in K} \alpha_k \beta_k p_k &\leqslant \delta &(21)\\ \sum_{k\in K} (1-\alpha_k)(1-\beta_k) p_k &\leqslant \delta &(22)\end{aligned}$$

Then there exists a vector $\varepsilon \in \{0,1\}^{|K|}$ ($|K|$ is the cardinality of K) such that

$$\left|\sum_{k\in K}(\varepsilon_k - \alpha_k)p_k\right| \leqslant 4\delta \quad \text{and} \quad \left|\sum_{k\in K}(1-\varepsilon_k - \beta_k)p_k\right| \leqslant 4\delta \tag{23}$$

Proof of the Lemma: Let us define the term q_k as follows:

$$q_k \equiv \max\{\alpha_k\beta_k,\, (1-\alpha_k)(1-\beta_k)\} \tag{24}$$

From (21)-(22) it follows that $\sum_k q_k p_k \leqslant 2\delta$ and from (24) it is obvious that

$$\begin{aligned} \alpha_k\beta_k &\leqslant q_k &(25)\\ (1-\alpha_k)(1-\beta_k) &\leqslant q_k &(26)\end{aligned}$$

Solving (25)-(26) for α_k, β_k and using the bound $1-4q_k \leqslant \sqrt{1-4q_k}$ we obtain that for any $k \in K$

$$\left\{\begin{array}{rcl} 0 \leqslant & \alpha_k & \leqslant 2q_k \\ 1-2q_k \leqslant & \beta_k & \leqslant 1 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{rcl} 0 \leqslant & \beta_k & \leqslant 2q_k \\ 1-2q_k \leqslant & \alpha_k & \leqslant 1 \end{array}\right\}$$

which means that there exist an $\varepsilon_k \in \{0,1\}$ for any $k \in K$ such that

$$|\varepsilon_k - \alpha_k| \leqslant 2q_k \quad \text{and} \quad |1-\varepsilon_k-\beta_k| \leqslant 2q_k$$

Multiplying by p_k and summing up for k we get that

$$\sum_{k\in K} |\varepsilon_k - \alpha_k|\, p_k \leqslant \sum_{k\in K} 2q_k p_k \leqslant 4\delta \quad \text{and} \quad \sum_{k\in K} |1-\varepsilon_k-\beta_k|\, p_k \leqslant \sum_{k\in K} 2q_k p_k \leqslant 4\delta \tag{27}$$

Finally, using the fact that

$$\left|\sum_{k\in K}(\varepsilon_k-\alpha_k)p_k\right| \leqslant \sum_{k\in K}|\varepsilon_k - \alpha_k|\, p_k$$

we obtain (23) which was to be proven. Now we turn to the proof of Proposition 1.

Consider the correlating pair $(A_i^\perp, B_j^\perp)$. First note that

$$p(A_i^\perp B_j^\perp | a_i b_j) \leqslant \delta \tag{28}$$

if and only if (17) holds. Moreover, a partition $\{C_k^{ij}\}$ $(k(ij) \in K(i,j))$ is a local, non-conspiratorial separate common cause system for (A_i, B_j) if and only if it is that for $(A_i^\perp, B_j^\perp)$.

Now, introduce the following notation for the conditional probabilities of the separate common cause system:

$$\alpha_k^{ij} = p(A_i|a_i C_k^{ij}) \tag{29}$$
$$\beta_k^{ij} = p(B_j|b_j C_k^{ij}) \tag{30}$$

With this notation using (14)-(16) and the theorem of total probability the conditional probabilities read as follows

$$p(A_i|a_i b_j) = \sum_{k\in K} \alpha_k^{ij} p(C_k^{ij}) \tag{31}$$
$$p(B_j|a_i b_j) = \sum_{k\in K} \beta_k^{ij} p(C_k^{ij}) \tag{32}$$
$$p(A_i B_j|a_i b_j) = \sum_{k\in K} \alpha_k^{ij}\beta_k^{ij} p(C_k^{ij}) \tag{33}$$
$$p(A_i^\perp B_j^\perp|a_i b_j) = \sum_{k\in K} (1-\alpha_k^{ij})(1-\beta_k^{ij}) p(C_k^{ij}) \tag{34}$$

and hence (17) and (28) can be formulated as

$$\sum_{k\in K} \alpha_k^{ij}\beta_k^{ij} p(C_k^{ij}) \leqslant \delta \tag{35}$$
$$\sum_{k\in K} (1-\alpha_k^{ij})(1-\beta_k^{ij}) p(C_k^{ij}) \leqslant \delta \tag{36}$$

Applying our Lemma to (35)-(36) we get that there exists a vector $\varepsilon^{ij} \in \{0,1\}^{|K|}$ for every $i, j \in I$ such that

$$\left| \sum_{k\in K} (\varepsilon_k^{ij} - \alpha_k^{ij}) p(C_k^{ij}) \right| \leqslant 4\delta \quad \text{and} \quad \left| \sum_{k\in K} (1-\varepsilon_k^{ij} - \beta_k^{ij}) p(C_k^{ij}) \right| \leqslant 4\delta \tag{37}$$

Now, defining C^{ij} and $C^{ij\perp}$ as in (18) the inequalities in (37) read as follows

$$|p(C^{ij}) - p(A_i|a_i b_j)| \leqslant 4\delta \quad \text{and} \quad |p(C^{ij\perp}) - p(B_j|a_i b_j)| \leqslant 4\delta \tag{38}$$

which fulfills the proof. □

Now, suppose we have two correlating pairs (A_i, B_j) and $(A_{i'}, B_{j'})$ in Δ^I *both* satisfying (17) and *both* having a local, non-conspiratorial separate common causal explanation. What can be said for the cross correlation, say, $(A_i, B_{j'})$? This question is answered by the following Proposition.

Proposition 2. Let (X, S, p) be a classical probability measure space and let (A_i, B_j) and $(A_{i'}, B_{j'})$ be two correlating pairs in Δ^I satisfying (17). Suppose furthermore that both (A_i, B_j) and $(A_{i'}, B_{j'})$ have a local, non-conspiratorial separate common cause system $\{C_k^{ij}\}$ and $\{C_{k'}^{i'j'}\}$, respectively $(k(ij) \in K(i,j); k'(i'j') \in K'(i',j'))$ that is two partitions of S such that (14)-(16) are satisfied. Then the following inequality holds:

$$|p(C^{ij} C^{i'j'\perp}) - p(A_i B_{j'}|a_i b_{j'})| \leqslant 8\delta \tag{39}$$

where C^{ij} and $C^{i'j'}$ are the *quasi* common causes of (A_i, B_j) and $(A_{i'}, B_{j'})$ respectively.

Proof: For the proof first we show that for a correlating pair (A_i, B_j) satisfying (17) and having a *quasi* separate common cause also stronger inequalities than those in (38) apply; namely

$$p(C^{ij}\,\Delta\, A_i|a_ib_j) \leqslant 4\delta \quad \text{and} \quad p(C^{ij\perp}\,\Delta\, B_j|a_ib_j) \leqslant 4\delta \tag{40}$$

where $C^{ij}\,\Delta\, A_i \equiv (C^{ij}A_i^{\perp}) \cup (C^{ij\perp}A_i)$ is the symmetric difference of C^{ij} and A_i; and $C^{ij\perp}\,\Delta\, B_j$ is defined similarly. To show this just apply inequalities (27) of the Lemma in the proof of Proposition 1 to (35)-(36) to get

$$\sum_{k\in K} |\varepsilon_k^{ij} - \alpha_k^{ij}|\, p(C_k^{ij}) \leqslant 4\delta \quad \text{and} \quad \sum_{k\in K} |1-\varepsilon_k^{ij} - \beta_k^{ij}|\, p(C_k^{ij}) \leqslant 4\delta \tag{41}$$

Summing up separately for the index sets $\{k\,|\,\varepsilon_k^{ij}=1\}$ and $\{k\,|\,\varepsilon_k^{ij}=0\}$ and using no-conspiracy (16) (for $p(C^{ij}) = p(C^{ij}|a_ib_j)$ and $p(C^{ij\perp}) = p(C^{ij\perp}|a_ib_j)$ among others) one readily obtains that

$$\begin{aligned} p(C^{ij}|a_ib_j) - p(C^{ij}A_i|a_ib_j) + p(C^{ij\perp}A_i|a_ib_j) &\leqslant 4\delta & (42)\\ p(C^{ij\perp}|a_ib_j) - p(C^{ij\perp}B_i|a_ib_j) + p(C^{ij}B_i|a_ib_j) &\leqslant 4\delta & (43)\end{aligned}$$

which are just the inequalities (40).

For the correlating pair $(A_{i'}, B_{j'})$ similar inequalities hold. Putting them together and using locality (15) one gets for the pairs (A_i, B_j) and $(A_{i'}, B_{j'})$ the following four inequalities:

$$\begin{aligned} p(C^{ij}\,\Delta\, A_i|a_ib_{j'}) \leqslant 4\delta \quad &\text{and} \quad p(C^{ij\perp}\,\Delta\, B_j|a_ib_{j'}) \leqslant 4\delta & (44)\\ p(C^{i'j'}\,\Delta\, A_{i'}|a_ib_{j'}) \leqslant 4\delta \quad &\text{and} \quad p(C^{i'j'\perp}\,\Delta\, B_{j'}|a_ib_{j'}) \leqslant 4\delta & (45)\end{aligned}$$

But then it is a straightforward consequence of the properties of the symmetric difference that the conditional probability $p(C^{ij}C^{i'j'\perp}\,\Delta\, A_iB_{j'}|a_ib_{j'})$ composed from the *intersections* of the respective terms of the first and the last inequality has to be smaller than 8δ that is

$$p(C^{ij}C^{i'j'\perp}\,\Delta\, A_iB_{j'}|a_ib_{j'}) \quad \leqslant \quad 8\delta \tag{46}$$

from which (39) follows immediately. □

The moral of Proposition 2 is that given two correlating pairs (A_i, B_j) and $(A_{i'}, B_{j'})$ in Δ^I sufficiently close to perfect anticorrelation *and* each having a local, non-conspiratorial separate common cause system the joint probability $p(A_iB_{j'}|a_ib_{j'})$ of the cross correlation $(A_i, B_{j'})$ can suitably be approximated by the probability of the conjunction of the appropriate *quasi* common causes C^{ij} and $C^{i'j'\perp}$ respectively. (For the joint probability $p(A_{i'}B_j|a_{i'}b_j)$ of the other cross correlation $(A_{i'}, B_j)$ the situation is similar.)

Now, let us apply the results of Propositions 1 and 2 generally. The application of both Propositions for a given set of correlations has two conditions: the correlations of the set have to satisfy (17) and they all need to have a local, non-conspiratorial separate common cause system. Denote by Δ_δ^I the subset of Δ^I which contains only correlations satisfying (17). Since *ex hypothesi* the set Δ^I has a separate common causal explanation therefore both Propositions 1 and 2 apply to the subset Δ_δ^I; Proposition 1 holds for any correlation in Δ_δ^I and Proposition 2 holds for any pair of correlations in Δ_δ^I.

To be more specific about the subset Δ_δ^I, consider the correlations (A_i, B_i) of Δ^I that is correlations pertaining to the same index $i \in I$ and *stipulate* that all these correlations are consisted in Δ_δ^I. In other words, let us suppose that the (A_i, B_i) elements of Δ^I are almost perfect correlations. As it will be pointed out in the Conclusions this stipulation turns out to be crucial in the subsequent derivation of the Bell(δ) inequalities. The logical steps of this derivation are the following.

Consider a correlation (A_i, B_j) in Δ^I and the two correlations (A_i, B_i) and (A_j, B_j) in Δ^I_δ pertaining to it. Due to surface locality (9)-(10) the marginal probabitity $p(A_i|a_ib_j)$ pertaining to the correlation (A_i, B_j) is the same as the marginal probabitity $p(A_i|a_ib_i)$ pertaining to the correlation (A_i, B_i); and similarly, the marginal probabitity $p(B_j|a_ib_j)$ pertaining to the correlation (A_i, B_j) is the same as the marginal probabitity $p(B_j|a_jb_j)$ pertaining to the correlation (A_j, B_j). But the correlations (A_i, B_i) and (A_j, B_j) are in Δ^I_δ and thus Proposition 1 applies to them. This means that the marginal probability $p(A_i|a_ib_i)$ can suitably be approximated *via* (19)-(20) by the probability $p(C^{ii})$ of the *quasi* common cause of the correlation (A_i, B_i); and similarly, the marginal probabitity $p(B_j|a_jb_j)$ can be approximated by the probability $p(C^{jj\perp})$ of the *quasi* common cause of (A_j, B_j). *A fortiori* the marginal probabitities $p(A_i|a_ib_j)$ and $p(B_j|a_ib_j)$ pertaining to the correlation (A_i, B_j) can suitably be approximated by the probabilities $p(C^{ii})$ and $p(C^{jj\perp})$.

In the same way, due to surface locality and Proposition 2 any joint probability $p(A_iB_j|a_ib_j)$ can suitably be approximated by the probability $p(C^{ii}C^{jj\perp})$ of the conjunction of the appropriate *quasi* common causes *via* (39). However, these marginal and joint probabilities $p(A_i|a_ib_j)$, $p(B_j|a_ib_j)$ and $p(A_iB_j|a_ib_j)$ are the ones that turn up in the Bell inequalities characterized at the beginning of this Section. Approximating these terms by the *quasi* common causes and their conjunctions we obtain various Bell(δ) inequalities that is Bell-like inequalities differing from the appropriate Bell inequalities in a term of magnitude δ where the exact size of this term is the function of the approximation. In the next Section we derive two special Bell(δ) inequalities depending on the choice of the set Δ^I.

3 A Wigner–Bell(δ) and a Clauser–Horne(δ) inequality

Wigner–Bell(δ) inequality. Let Δ^{WB} be the set $\{(A_i, B_j)\}_{i,j\in\{1,2,3\}}$ and let Δ^{WB}_δ be the subset $\{(A_i, B_i)\}_{i\in\{1,2,3\}}$ of Δ^{WB} and suppose that any correlation in Δ^{WB}_δ has a separate common cause system. Due to Proposition 1 each correlation (A_i, B_i) has a *quasi* separate common cause C^{ii} as well.

Now, consider the terms $C^{11}C^{22\perp}$, $C^{11}C^{33\perp}$ and $C^{33}C^{22\perp}$. Elementary calculation shows that for any such events the following inequality holds

$$p(C^{11}C^{22\perp}) \quad \leqslant \quad p(C^{11}C^{33\perp}) + p(C^{33}C^{22\perp}) \tag{47}$$

Approximating each term of (47) by the appropriate conditional probability *via* (39) we arrive at the following Bell-type inequality

$$p(A_1B_2|a_1b_2) - 24\delta \quad \leqslant \quad p(A_1B_3|a_1b_3) + p(A_3B_2|a_3b_2) \tag{48}$$

Let us call (48) a *Wigner–Bell(δ) inequality* since it is a 'perturbation' of the original Bell–Wigner inequality

$$p(A_1B_2|a_1b_2) \quad \leqslant \quad p(A_1B_3|a_1b_3) + p(A_3B_2|a_3b_2) \tag{49}$$

Clauser–Horne(δ) inequality. Let Δ^{CH} be the set $\{(A_i, B_j)\}_{i,j\in\{1,2,3,4\}}$ and let Δ^{CH}_δ be the subset $\{(A_i, B_i)\}_{i\in\{1,2,3,4\}}$ of Δ^{CH} and suppose again that any correlation in Δ^{CH}_δ has a separate common cause system C^{ii}_k and hence due to Proposition 1 a *quasi* separate common cause C^{ii}.

Now, consider the four events C^{11}, C^{22}, $C^{33\perp}$ and $C^{44\perp}$. For these events the following simple probabilistic constraint applies:

$$p(C^{11}C^{33\perp}) + p(C^{11}C^{44\perp}) + p(C^{22}C^{44\perp}) - p(C^{22}C^{33\perp}) - p(C^{11}) - p(C^{44\perp}) \leqslant 0 \tag{50}$$

Due to Proposition 1 the terms $p(C^{11}C^{33\perp})$, $p(C^{11}C^{44\perp})$, $p(C^{22}C^{44\perp})$ and $p(C^{22}C^{33\perp})$ in (50) respectively can be replaced by the the terms $p(A_1B_3|a_1b_3)$, $p(A_1B_4|a_1b_4)$, $p(A_2B_4|a_2b_4)$ and $p(A_2B_3|a_2b_3)$

respectively causing an error of magnitude not greater than $4 \times 8\delta$. Due to Proposition 2 the terms $p(C^{11})$ and $p(C^{44\perp})$ in (50) respectively can be replaced by the the terms $p(A_1|a_1b_1)$ and $p(B_4|a_4b_4)$ causing an error of magnitude not greater than $2 \times 4\delta$. Finally, using surface locality (9)-(10) and putting all these together we get

$$p(A_1B_3|a_1b_3) + p(A_1B_4|a_1b_4) + p(A_2B_4|a_2b_4) - p(A_2B_3|a_2b_3) - p(A_1|a_1b_3) - p(B_4|a_2b_4) - 40\delta \leqslant 0 \quad (51)$$

to which we refer as a *Clauser–Horne(δ) inequality* since (51) is the modification of the Clauser–Horne inequality

$$p(A_1B_3|a_1b_3) + p(A_1B_4|a_1b_4) + p(A_2B_4|a_2b_4) - p(A_2B_3|a_2b_3) - p(A_1|a_1b_3) - p(B_4|a_2b_4) \leqslant 0 \quad (52)$$

To show that both the Wigner–Bell(δ) inequality (48) and the Clauser–Horne(δ) inequality (51) can be violated we now construct the appropriate Δ^{WB} and Δ^{CH} sets. For this let us come back for a moment to the meaning of inequality (17). First set $\delta = 0$. In this case the correlation (A_i, B_i) is a perfect anticorrelation. According to quantum mechanics, perfect anticorrelation obtains when the measuring directions in the two wings of the EPR-Bohm scenario are set parallelly. Now, if the measurement directions deviate from parallel setting the resulting correlation (A_i, B_i) will also deviate (continuously) from perfect anticorrelation. To hold the deviation of the correlation under the threshold δ, for the angle $\theta_{a_ib_i}$ between the measuring directions $\vec{a}_i$ and $\vec{b}_i$ the following is to hold:

$$|\theta_{a_ib_i}| \quad \leqslant \quad 2\arcsin\sqrt{2\delta} \quad (53)$$

Thus, in this reading inequality (17) expresses the *deviation of the measurement directions* in the two wings of the measurement *from parallel setting*.

However, we can read (17) in a more experimental way as well. Suppose we set the measuring directions parallelly but *due to experimental imperfections* we are able to test perfect anticorrelations in this parallel setting only with δ precision. In this case the δ expresses directly the limit of our measuring capacity.

We can construct the Δ^{WB} and Δ^{CH} sets according to both readings of (17). In the first reading of δ we choose three pairs of measurement directions in the Wigner–Bell case such that all the angles $\theta_{a_ib_i}$ satisfy (53) *and* we exclude measurement imperfections. With this choice of measurement directions any correlation (A_i, B_i) $(i = 1, 2, 3)$ will be an almost perfect anticorrelation and hence the three pairs of measurement directions will define a Δ^{WB} set. According to the second reading of δ we choose three pairs of *parallel* measurement directions *and* we admit measurement imperfections of magnitude δ. Again any three pairs of parallel measurement directions plus a δ tolerance in measurement imperfection will define a Δ^{WB} set. For the Clauser–Horne case the construction is similar except we have to choose four measurement directions instead of three. Let us continue the construction according to this second reading of (17).

To give a Δ^{WB} and a Δ^{CH} set which *violates* the appropriate Bell(δ) inequality we simply have to fix the angle $\theta_{a_ib_j}$ between the measurement directions pertaining to *different* indices in an appropriate way. For the Wigner–Bell case a measurement setting which violates (48) can be given by the following measurement angles:

$$\theta_{a_1b_2} = \frac{3\pi}{4}, \quad \theta_{a_1b_3} = \frac{\pi}{4}, \quad \theta_{a_3b_2} = \frac{\pi}{2} \quad (54)$$

Since the original Wigner–Bell inequality (49) is maximally ($\sqrt{2} < 1!$) violated for the setting

$$\theta_{a_1b_2} = \frac{3\pi}{4}, \quad \theta_{a_1b_3} = \frac{\pi}{4}, \quad \theta_{a_3b_2} = \frac{\pi}{2} \quad \text{and} \quad \theta_{a_ib_i} = 0 \quad (55)$$

the new Wigner–Bell(δ) inequality (48) will also be violated for the measurement setting (54) as long as

$$\delta \quad < \quad \frac{\sqrt{2}-1}{24} \tag{56}$$

Hence, assuming that δ is smaller than approximately $1.73 \cdot 10^{-2}$ the Wigner–Bell(δ) inequality (48) will be violated for the setting (54) which in turn excludes the existence of a local, non-conspiratorial separate common causal explanation of the correlations of Δ_δ^{WB}.

For the violation of the Clauser–Horne inequality let the setting be:

$$\theta_{a_1b_3} = \theta_{a_2b_4} = \frac{3\pi}{4}, \quad \theta_{a_1b_4} = \frac{5\pi}{4}, \quad \theta_{a_2b_3} = \frac{\pi}{4} \tag{57}$$

Again, since for the measurement setting (58) the original Clauser–Horne inequality (52) is maximally ($\frac{\sqrt{2}-1}{2} < 0!$) violated (for the upper bound discussed here) for the setting

$$\theta_{a_1b_3} = \theta_{a_2b_4} = \frac{3\pi}{4}, \quad \theta_{a_1b_4} = \frac{5\pi}{4}, \quad \theta_{a_2b_3} = \frac{\pi}{4} \quad \text{and} \quad \theta_{a_ib_i} = 0 \tag{58}$$

the new Clauser–Horne(δ) inequality (51) will also be violated for the setting (57) as long as

$$\delta \quad < \quad \frac{\sqrt{2}-1}{80} \tag{59}$$

Again, if δ is smaller than approximately $5.18 \cdot 10^{-3}$ the violation of the Clauser–Horne(δ) inequality (51) excludes the existence of a local, non-conspiratorial separate common causal explanation of the correlations in Δ_δ^{CH}.

4 Conclusions

In the last Section we have derived two special Bell(δ) inequalities from the assumption that an appropriate set of almost perfect anticorrelations has a local, non-conspiratorial separate common causal explanation. Here we give a general recipe for deriving *any* Bell(δ) inequality composed of marginal probabilities $p(A_i|a_ib_j)$, $p(B_j|a_ib_j)$ and joint probabilities $p(A_iB_j|a_ib_j)$ from the same assumptions.

(i) Consider a Bell inequality resulting from the local, non-consipratorial *common* common causal explanation of a set of correlations. Form a Δ^I set such that Δ^I contains every correlation (A_i, B_j) pertaining to the events A_i or B_j which appear in either a marginal or a joint probability in the Bell inequality *plus (if not already contained in Δ^I) one almost perfect anticorrelation (A_i, B_i) for every $i \in I$. In other words, extend Δ^I by a Δ_δ^I subset. Suppose that any correlation in Δ_δ^I has a local, non-conspiratorial separate common cause system in the sense (14)-(16).*

(ii) Approximate the marginal probalitity $p(A_i|a_ib_j)$ and $p(B_j|a_ib_j)$ of the correlating events in Δ^I by the probability $p(C^{ii})$ and $p(C^{jj\perp})$ of the *quasi* common causes of correlations in Δ_δ^I according to Proposition 1; and approximate the joint probabilities $p(A_iB_j|a_ib_j)$ by the probability $p(C^{ii}C^{jj\perp})$ of the conjunction the *quasi* common causes according to Proposition 2.

(iii) This results in a Bell(δ) inequality differing from the original Bell inequality in a term of order of δ where the exact magnitude of this term is the function of the approximation. Choose the setting which violates the Bell inequality maximally. If the δ term is smaller than the violation of the original Bell inequality than the new Bell(δ) inequality will also be violated excluding a common causal explanation of the correlations pertaining to the setting.

The crucial point of the above recipe is the italicized part of point (i). This requires that the correlations for which we are looking for local, non-conspiratorial separate common cause sytems form a Δ^I_δ set. These correlation *do not even need to turn up in the Bell(δ) inequalities* which we intend to derive. Actually, this requirement is the essence of the whole construction since all the *quasi* common causes needed for the derivation of the appropriate Bell(δ) inequality are constructed out of the separate common cause systems pertainig to some correlation in Δ^I_δ. Both the derivation of the weak Clauser–Horne inequality in (Portmann, Wüthrich, 2007) and the derivation of the Wigner–Bell inequality in (Hofer-Szabó, 2008) have been based on this strategy.

However, the correlations in Δ^I_δ are almost perfect anticorrelations. This fact places the above derivation in the 'δ-neighborhood' of a local, non-conspiratorial *common* common causal derivation of some Bell inequality. To see this, set δ equal to 0 and consider a set Δ^I of correlations such that the correlations pertaining to the same index are in $\Delta^I_{\delta=0}$. Now, suppose that for a given $\Delta^I_{\delta=0}$ set each correlation has a local, non-conspiratorial separate common cause sytem C^{ii}_k. In this case Proposition 1 states that for any $i \in I$

$$p(C^{ii}) = p(A_i|a_ib_i) \quad \text{and} \quad p(C^{ii\perp}) = p(B_i|a_ib_i) \tag{60}$$

which means that any correlation in $\Delta^I_{\delta=0}$ has a local, non-conspiratorial separate common *cause* as well satisfying (14)-(16) (not just a *quasi* common cause!). Now, consider the partition $\{C_l\}$ $(l \in L)$ generated by the conjunction of all the different common causes C^{ii} and their complements $C^{ii\perp}$ for all $i \in I$. We claim that this partition $\{C_l\}$ forms a *deterministic*, local, non-conspiratorial *common* common cause sytem for *all* correlations in Δ^I. As far as screening-off (11) and locality (12) are concerned just observe that for any correlation $(A_i, B_j) \in \Delta^I$ and for any element C_l of the partition the following equations hold:

$$\begin{aligned} \varepsilon_{il}\varepsilon_{jl} = p(A_iB_j|a_ib_jC_l) \quad &= \quad p(A_i|a_ib_jC_l)p(B_j|a_ib_jC_l) = \varepsilon_{il}\varepsilon_{jl} && (61)\\ \varepsilon_{il} = p(A_i|a_ib_jC_l) = p(A_i|a_iC_l) = \varepsilon_{il}, \qquad & \varepsilon_{jl} = p(B_j|a_ib_jC_l) = p(B_j|b_jC_l) = \varepsilon_{jl} && (62) \end{aligned}$$

where ε_{il} (and similarly ε_{jl}) is defined as follows:

$$\varepsilon_{il} \quad = \quad \begin{cases} 1 & \text{if } C_l \in C^{ii} \\ 0 & \text{otherwise} \end{cases} \tag{63}$$

To see that no-conspiracy (13) also holds recall that any C_l is contained in the algebra $S' \subset S$ generated by all the elements of the different separate common cause systems and hence it fulfills (13) by definition. (For a detailed proof see (Hofer-Szabó, 2008).)

Now, since all correlations in Δ^I have a local, non-conspiratorial *common* common cause system and the separate common cause systems are just disjunctions of elements of $\{C_l\}$ therefore any derivation of some Bell inequality from local, non-conspiratorial separate common cause systems is a derivation from a local, non-conspiratorial *common* common cause system as well.

Now, in the $\delta \neq 0$ case the partition $\{C_l\}$ $(l \in L)$ constructed from the *quasi* common causes C^{ii} and their complements $C^{ii\perp}$ is *not* a local, non-conspiratorial *common* common cause sytem. Generally, it neither satisfies screening-off (11) nor locality (12); no-conspiracy (13) is stipulated again by definition. Still, we are not sure that the derived Bell(δ) inequalities are not some *indirect* consequence of the fact that we are in the 'δ-neighborhood' of an underlying *common* common cause sytem. To settle the problem we should be able to derive the Bell(δ) inequalities from local, non-conspiratorial separate common cause systems of correlations which are *not* almost perfect anticorrelations. Again, without the implementation of a Δ^I_δ set in Δ^I we could not even set up the derivation. So we conclude our paper with an

Open question: Does there exist a derivation of a Bell or Bell-like inequality from the assumption that each correlation in a set Δ^I has a local, non-conspiratorial separate common cause system *without assuming* that Δ^I has a Δ^I_δ subset?

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