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Logic, Geometry And Probability Theory

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Abstract:
We discuss the relationship between logic, geometry and probability theory under the light of a novel approach to quantum probabilities which generalizes the method developed by R. T. Cox to the quantum logical approach to physical theories.

Keywords:
Quantum Logic; lattice Theory; Geometry of Quantum Mechanics; Quantum Probability

1. INTRODUCTION

The formalism of quantum mechanics (QM) achieved its rigorous formulation after a series of papers by von Neumann, Jordan, Hilbert and Nordheim [1]. Its final form was accomplished in the monumental work of von Neumann [32]. But an interpretation of quantum mechanics is still lacking, despite the efforts of many researchers during the years.

In the axiomatic approach of von Neumann projection operators play a key role. The spectral decomposition theorem [10, 33] allows to associate a projection valued measure to any quantum observable represented by a self adjoint operator [10, 32]. It turns out that the set of projection operators can be endowed with a lattice structure; more specifically, they form an orthomodular lattice [25].

The subsequent developments turn the attention of von Neumann to the theory of *rings of operators*, better known as *von Neumann algebras* [10]. It was an attempt of generalizing certain algebraic properties of Jordan algebras [1]. But it turned out that the theory of von Neumann algebras was strongly related to lattice theory: in a series of papers, Murray and von Neumann provided a classification of factors[1] using orthomodular lattices [17–20]. Time showed that all kinds of factors would find physical applications, as is the case of type II factors in statistical mechanics or type III factors in the rigorous axiomatic approach to Quantum Field Theory (QFT) [10, 11]. It is important to remark that essentially all the information needed to develop a physical theory out of these algebras is contained in the logico-algebraic structure of their lattices of projection operators [11].

On the other hand, lattice theory is deeply connected to geometry: projective geometry can be described in terms of lattices and related also to vector spaces [22]. As an example, any vector space has associated a projective geometry and a lattice of subspaces. In particular, projection operators of the Hilbert spaces used in QM form a lattice and their sets of pure states form projective geometries. But von Neumann was

[1] *von Neumann algebras whose center is formed by the multiples of the identity operator*

not only interested in Hilbertian projection lattices; as the investigation continued, he turned his attention to more general geometries, namely, continuous geometries [3, 35]. That is, the geometries associated to the type II_1 factors found in the classification theory of Murray-von Neumann. As an example of the exotic characteristics of the more general factors, type II_1 algebras are non-atomic and the type III contain no non-trivial finite projections. In this way, it could be said that the generalization of algebras studied by von Neumann points in the direction of a rather radical generalization of geometry. Using the words of von Neumann:

> "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert-space (as far as quantum-mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of "conserving the validity of all formal rules". This is very clear, if you consider the axiomatic-geometric definition of Hilbert-space, where one simply takes Weyl's axioms for a unitary-Euclidean-space, drops the condition on the existence of a finite linear basis, and replaces it by a minimum of topological assumptions (completeness + separability). Thus Hilbert-space is the straightforward generalization of Euclidean space, if one considers the vectors as the essential notions.
>
> Now we begin to believe, that it is not the vectors which matter but the lattice of all linear (closed) subspaces. [...]
>
> But if we wish to generalize the lattice of all linear closed subspaces from a Euclidean space to in infinitely many dimensions, then one does not obtain Hilbert space, but that configuration, which Murray and I called "case II_1". (The lattice of all linear closed subspaces of Hilbert-space is our "case I_∞".) And this is chiefly due to the presence of the rule
>
> $$a \leq c \longrightarrow a \vee (b \wedge c) = (a \vee b) \wedge c \text{ [modularity!]}$$
>
> This "formal rule" would be lost, by passing to Hilbert space!"[7]

In this way, we see how deeply connected is the quantum logical approach to physics developed by von Neumann to the development of geometry. But what is the meaning of *Logic* and *Geometry* in this context? In this short article, we will explore a possible answer to this question and relate it to a *generalized probability theory* [11].

Later on, the quantum logical approach of Birkhoff and von Neumann was developed further by other researchers, giving rise to monumental foundational works (as examples see [4, 12, 23, 24, 26, 27]). The problem of compound quantum systems in the QL approach was studied first in [29–31]. For complete expositions of the QL approach see for example [6, 25, 28]. It is very important to remark that C. Piron showed that any propositional system can be coordinatized in a generalized Hilbert space [24]. A later result by Solèr asserts that, under reasonable conditions, it can only be a Hilbert space over the fields of the real numbers, complex numbers or quaternions [34]. In this way, an operational propositional system with suitably chosen axioms can be represented in a generalized Hilbert space.

Probability measures can be defined in general von Neumann algebras [4, 11]. A generalized non-kolmogorovian probability calculus can be developed including Kolmogorovian probabilities as a particular case (i.e., when the algebra is commutative) [11]. Thus, the approach developed by von Neumann and others leads to an interesting connection between logic, geometry and probability theory.

In Section 2 we discuss a problem posed by von Neumann regarding the foundations of logic and probability theory. We largely quote von Neumann because, on the one hand, we think that these unpublished works are not too well known. On the other hand, the content of the quotation is useful for our purposes: our aim is to outline an interpretation of it. In Section 3 we review and discuss the main features of quantum probabilities (and compare them to the classical ones). In Section 3.3 we discuss a novel approach to generalized probability theory [21], and then, in Section 4 we discuss its consequences for the problem posed by von Neumann providing the general features of an interpretation.

An important final remark is in order before starting. While it is certainly true that there exist many different interpretations of the quantum formalism, and that there exists the problem of metaphysical underdetermination (a point of view that we endorse), in this work we will restrict to what was called the “standard” (or “Copenhaguen”) interpretation. Of course, the so called “standard” interpretation, is not a consistent interpretation at all: it is a collection of ideas regarding the quantum formalism and a discussion between its founding fathers. While there are many points which can be identified clearly (like the absence of determinism, and the quantum jumps), it is not in general a consistent set of ideas. The content of the present work goes in the direction of developing a perspective which has the standard interpretation as a starting point, and tries to provide more consistency to it, and we will not discuss other interpretations. By no means is an aim of this work to present a definitive point of view on the problem of the interpretation of the quantum formalism in detriment of other interpretations.

2. A PROBLEM POSED BY VON NEUMANN

As is well known, any boolean algebra can be represented in a set theoretical framework (as subsets of a given set). With regard to this relationship, von Neumann asserted that

> “And one also has the parallelism that logics corresponds to set theory and probability theory corresponds to measure theory and that a given system of logics, so given a system of sets, if all is right, you can introduce measures, you can introduce probability and you can always do it in very many different ways.”(unpublished work reproduced in [9], pp. 244).

In this way, the connection between Logic, Set Theory, and Probability Theory is clear (see also Sections 3 and 4 of this work). What does this means? The definition of Cantor of a set reads

> “A set is a gathering together into a whole of definite, distinct objects of our perception [Anschauung] or of our thought —which are called elements of the set.” [5]

A set is a collection of *objects*, and the internal logic governing them is *classical logic*. And of course, this also applies to *things in space*, because the last is just a particular case of a set theoretical approach to collections of objects: in the ultimate level the classical organization of experience in an Euclidean space-time —as well as in the curved background of General Relativity— is an expression of classical logic.

But the things chance radically in the quantum formalism, as von Neumann pointed out

> “In the quantum mechanical machinery the situation is quite different. Namely instead of the sets use the linear sub-sets of a suitable space, say of a Hilbert space. The set theoretical situation of logics is replaced by the machinery of projective geometry, which is in itself quite simple.
>
> However, all quantum mechanical probabilities are defined by inner products of vectors. Essentially if a state of a system is given by one vector, the transition probability in another state is the inner product of the two which is the square of the cosine of the angle between them. In other words, probability corresponds precisely to introducing the angles geometrically. Furthermore, there is only one way to introduce it. The more so because in the quantum mechanical machinery the negation of a statement, so the negation of a statement which

> is represented by a linear set of vectors, corresponds to the orthogonal complement of this linear space." (unpublished work reproduced in [9], pp. 244).

von Neumann continues

> "And therefore, as soon as you have introduced into the projective geometry the ordinary machinery of logics, you must have introduced the concept of orthogonality. This actually is rigorously true and any axiomatic elaboration of the subject bears it out. So in order to have logics you need in this set a projective geometry with a concept of orthogonality in it.
>
> In order to have probability all you need is a concept of all angles, I mean angles other than 90. Now it is perfectly quite true that in geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, leave all the angles intact, in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logics are fixed under some isomorphic transformation, all of probability theory is already fixed." (unpublished work reproduced in [9], pp. 244).

Now we ask: what is the meaning of the connection between Geometry and Logic in the above quotations? It is clear that in confronting with the empirical propositions of QM we are facing essentially a *Geometry*, which is at the same time a *Logic*. But this Geometry is not the geometry of classical space-time. Quite on the contrary, *is the geometrical form in which quantum events are organized*. And of course, *this geometrical form has an internal logical structuration, which is the quantum logic*.
It is important to remark that this logic does not necessarily denies the classical logic that we use when we think. The word *logic* above refers to the *organization of experience (phenomena)*. But what is the connection of all this with probability theory? von Neumann suggested a clue as follows

> "This means, however, that one has a formal mechanism, in which logics and probability theory arise simultaneously and are derived simultaneously." (unpublished work reproduced in [9], pp. 245).

In the rest of this work we will discuss the implications of a novel derivation of QL using the algebraic properties of the propositional lattice of QM [21].

3. GENERALIZED PROBABILITY THEORY

3.1 Kolmogorov

In this Section we introduce classical probability theory using the axioms of Kolmogorov [13]. Given an outcome set Ω, consider a σ-algebra Σ of subsets of Ω. Then, a probability measure will be given by a function μ such that

$$\mu : \Sigma \to [0,1] \tag{1a}$$

which satisfies

$$\mu(\emptyset) = 0 \tag{1b}$$

$$\mu(A^c) = 1 - \mu(A), \tag{1c}$$

where $(\ldots)^c$ means set-theoretical-complement and for any pairwise disjoint denumerable family $\{A_i\}_{i\in I}$

$$\mu(\bigcup_{i\in I} A_i) = \sum_i \mu(A_i) \tag{1d}$$

The triad (Ω, Σ, μ) is called a *probability space* (to which we refer as a *Kolmogorovian probability*). It is possible to show that if (Ω, Σ, μ) is a Kolmogorovian probability space, all usual properties of classical probability can be derived. Of particular importance for this work is the *inclusion-exclusion principle*

$$\mu(A\cup B) = \mu(A) + \mu(B) - \mu(A\cap B) \tag{2}$$

which can be derived from 1. The logical version of (2) reads

$$\mu(A\vee B) = \mu(A) + \mu(B) - \mu(A\wedge B) \tag{3}$$

due to the direct correspondence between the connectives of classical logic ("$\vee$" and "$\wedge$") and set theoretical union and intersection. This is a clear expression of what is said in our first quotation to von Neumann in Section 2.

3.2 Quantum probabilities

Let $\mathscr{P}(\mathscr{H})$ be the orthomodular lattice of projection operators in a separable Hilbert space. In order to define quantum probabilities, the following axioms on a function s must be *postulated* [10]

$$s : \mathscr{P}(\mathscr{H}) \to [0;1] \tag{4a}$$

such that:

$$s(\mathbf{0}) = 0 \ (\mathbf{0} \text{ is the null subspace}). \tag{4b}$$

$$s(P^{\perp}) = 1 - s(P), \tag{4c}$$

and, for a denumerable and pairwise orthogonal family of projections P_j

$$s(\sum_j P_j) = \sum_j s(P_j). \tag{4d}$$

How do we know that these axioms capture all the desired features of quantum probabilities? Gleason's theorem [37] gives us the answer: if $dim(\mathscr{H}) \geq 3$, for any measure s satisfying (4) there exists a positive Hermitian trace class operator (of trace one) ρ_s, such that

$$s(P) := \mathrm{tr}(\rho_s P) \tag{5}$$

And also the converse is true; using Eqn. (5), any positive trace class Hermitian operator of trace one defines a measure satisfying (4).

A generalized probability calculus can be extended to all orthomodular lattices [11] and even to σ-orthocomplemented orthomodular posets [4] in an analogous way (as in Eqns. (4)). Classical probabilities are a particular case when the algebra is commutative [4, 11]. In this way, observables can be defined in theories more general than Hilbertian QM.

One of the main differences between the axioms 1 and 4 is that the σ-algebra in (1) is boolean, while $\mathscr{P}(\mathscr{H})$ is not. In this sense, the measures defined by Eqns. 4 are called *non-kolmogorovian (or non-boolean) probability measures*.
One of the expressions of the fact that quantum and classical probabilities are different, is that Eq. (2) is no longer valid in QM. Indeed, in QM it may happen that

$$s(A)+s(B) \leq s(A \vee B) \tag{6}$$

von Neumann considered that Eq. (2) was crucial for the interpretation of $\mu(A)$ and $\mu(B)$ as relative frequencies [35] in a frequentistic interpretation. But as explained in [8, 35] one of the main dissatisfactions of von Neumann was that Eq. (2) was not generally valid in the quantum case, making the frequentistic interpretation untenable. This was one of the reasons that led him to search for generalizations of the algebra of projections in Hilbert space, and type II_1 factors were good candidates for this objective [8].

3.3 A new approach to quantum probabilities

In a recent work [21] it was shown that the approach to probability theory of R. T. Cox [14, 15] can be applied to lattices more general than Boolean. And in particular, that quantum probabilities and the generalized probability theory can be obtained by applying a variant of this method. In the rest of this work we explore possible interpretations of this fact under the light of the problem posed by von Neumann (Section 2 of this work)[2]. We don't have place here to introduce all the details (for which we refer to [21]) and just limit ourselves to describe the general method:

- Our starting point is an orthomodular lattice $\mathscr{L}$.
- Next, we assume that $\mathscr{L}$ represents the propositional structure of a given system.
- It is reasonable to assume[3] that there is a definite state of affairs determined by the preparation of the system. This preparation could be natural or artificial, this is not relevant. But the system has its own definite history as a constitutive feature of its own actuality.
- Define a function $s : \mathscr{L} \longrightarrow \mathbb{R}$ such that $s(a) \geq 0$ $\forall a \in \mathscr{L}$ and it is order preserving ($a \leq b \longrightarrow s(a) \leq s(b)$). This function is intended to represent the degree of likelihood about what would happen in the different (contextual) future situations. But it is important to remark that this measure is a manifestation of a structured actual state of affairs: its origin is ontological and there are no hidden variables.

It can be shown that under the above rather general assumptions, a probability theory can be developed [21] following a variant of R. T. Cox approach [14, 15]. In other words, it is possible to show that:

$$s(\bigvee\{a_i\}_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} s(a_i) \tag{7a}$$

[2] *See also [16] for a very interesting but different perspective of the R. T. Cox approach and quantum probabilities.*
[3] *As a precondition of physical science, something which is not necessarily true in any field of experience. A similar remark holds for the existence of —at least— statistical regularities: if such regularities are not present, mathematical description of phenomena is untenable. We are not asserting that any phenomena could be subsumed into this condition, but that it is a precondition of mathematical physics.*

$$s(\neg a) = 1 - s(a) \tag{7b}$$

$$s(\mathbf{0}) = 0 \tag{7c}$$

(where the $\{a_i\}$ in Eqn. (7a) form a denumerable and orthogonal family). Let us see an example of how the Cox's machinery works. If $a, b \in \mathscr{L}$ and $a \perp b$, we have that $a \wedge b = \mathbf{0}$. Next, it is reasonable to assume that $s(a \vee b)$ can only be a function of $s(a)$ and $s(b)$. In this way, $s(a \vee b) = f(s(a), s(b))$, with f an unknown function to determine. Due to associativity of "$\vee$", $s((a \vee b) \vee c) = s(a \vee (b \vee c))$ for any $a, b, c \in \mathscr{L}$. If a, b and c are orthogonal, we will have $s((a \vee b) \vee c) = f(f(s(a), s(b)), s(c))$ and $s(a \vee (b \vee c)) = f(s(a), f(s(b), s(c)))$. But then $f(f(s(a), s(b)), s(c)) = f(s(a), f(s(b), s(c)))$. Or put in a more simple form, we are looking for a function f such that

$$f(f(x,y),z) = f(x, f(y,z)) \tag{8}$$

But Eqn. (8) is a functional equation [36] whose solution —*up to rescaling*[4]— is $f(x,y) = x + y$. In this way we arrive at $s(a \vee b) = s(a) + s(b)$. Eqns. (7) follow in a similar way [21].
As explained above, it is possible to show that the probability theory defined by Eqns. 7 is non classical in the general case. If $\mathscr{L}$ is not Boolean, it may happen that $s((a \wedge \neg b) \vee (a \wedge b)) = s(a \wedge \neg b) + s(a \wedge b) \leq s(a)$, but any Kolmogorovian probability satisfies $s(a) = s(a \wedge b) + s(a \wedge \neg b)$ [21].

4. PROBABILITY THEORY AS ARISING AS THE LOGICAL AND GEOMETRICAL STRUCTURATION OF PHENOMENA

Maybe it is not just a coincidence that the discussion posed by von Neumann on Logic, Probability and Geometry appeared in the axiomatization of quantum theory. QM seems to pose a problem in the interpretation of space-time, as is expressed, for example, in the impossibility of defining trajectories for the particles. In this way, a new kind of structure of experience underlies the quantum mechanical description.

Space-time —as considered by modern physics— is not a naturally given structure: it was a great achieving to develop geometry up to the point to which it is possible to give the mathematical description of reality provided for example, by classical mechanics (CM) or general relativity (GR). The continuous description of experience provided by Euclidean geometry, has as a precondition to have definite logical objects: mathematical objects such as numbers, geometrical figures, and all of this related to things of our experience. Our experience in not a complete chaos and can be structured in such a way. But we must never forget that the fact that we can organize our experience in a space-time description is just an assumption whose consistency is to be tested empirically. General relativity, shows us that one can use a more elegant and more powerfully predictive description of experience than the one provided by the flat space-time of the Euclidean geometry of classical physics. But the limits and success of these descriptions are not granted in advance: they must be confronted with their capability of defining a consistent experience.

But we are committed to the space time description in the following sense: we need definite things and objective things to happen in order to even *speak* about an experiment. An example of this is a pointer of an instrument yielding a value in a given outcome set (which could be, for example, the set of real numbers, but could also be more general, like the set formed by $\{+, -\}$). The fact that an outcome set

[4] *For a discussion about the rescaling we refer to [15]*

always forms a set and the events will be represented by its subsets (forming a σ-algebra), ties us to a very specific kind of logic (classical) and a very specific form of spaciality (example, Euclidean geometry, or a curved space-time). This is the real content of the observations of N. Bohr: the very possibility of exerting experiments ties us to classical logic and a set theoretical organization of experience. Space-time description is just a particular case of this more general regulative logical machinery.

But there is absolutely nothing granting us that this Boolean description (thought necessary to exert experiments) will exhaust the scenario in which phenomena appear. And this lies at the heart of the existence of complementary (and incompatible) contexts in quantum mechanics: in order to determine the state of the system, a quantum tomography must be exerted, and thus, we are obliged to study the system in different incompatible contexts. While the structuration of experience in CM can be reduced to a boolean algebra (and thus, to a set theoretical description provided by an outcome set), in QM this is no longer possible. In classical mechanics, the description of an object can be equated with its space-time representation: form the point of view of classical mechanics, the main goal is to describe continuous motion of material bodies inside space. That is why motion (and change) can be described as the solutions of deterministic differential equations. And this feature is much more general than the usual description of a particle moving through space under the action of forces. Any quantity of interest taking continuous values, if it is classical, will have associated a time derivative, and thus the description reduces to the motion of a system in a phase space obeying deterministic differential equations. Quite contrarily, quantum mechanics is characterized by *jumps*, by *discontinuous* and *unpredictable* behavior. That is why the organization of experience in QM comes endowed with a probabilistic description: it is impossible to predict the future events with complete certainty, and thus, the actual state of affairs *is just a probability distribution*.

In this way, QM fails to give a spatio-temporal description of phenomena. In other words, *QM shows us that the spatio-temporal description is just a part (or perspective) of the whole scenario of the organization of phenomena*; one of the most important consequences of quantum mechanics is that space-time can no longer considered as an exhaustive scenario in which physical events take place. Quite on the contrary, experience can be structured as a logic, and at the same time as a specific kind of geometry, as von Neumann explained. Different models of event structures represent different organizations of phenomena.

The results of [21] show that once the logic-algebraic properties of the structured experience are determined, to great extent, the whole probability theory is determined. In this way, we have a concrete step in the solution of the problem posed by von Neumann. Experience is not complete chaos, but on the contrary, it can be structured. This organization of phenomena may have a definite *logical form* (as is the case in the CM or the QM descriptions), and this *form* is expressed as a *geometry*.

But in QM this geometry must not be confused with the geometrical background of space-time (Euclidean space or the curved background of general relativity); space-time description is just an aspect or perspective of a more general state of affairs. In the general case, physical events can be organized as lattices[5] much more general than the boolean case and a similar assertion holds.

Thus, once that the structure of experience is determined as a Logic-Geometry, a probability calculus follows. If the description is boolean, then probabilities will be Kolmogorovian, and deterministic equations of motion can be used (in principle) to govern the laws of motion. But if the logic is (ontologically speaking) non-boolean, then the description will fail to be deterministic because an ignorance interpretation of probabilities will be untenable. Thus, deterministic equations of motion (as the Schrödinger

[5] *Or even more generally, σ-orthocomplemented orthomodular posets [4], which are not lattices in the general case.*

equation) must be complemented with "jumps" (as the quantum jumps) and the concomitant *processes* that they trigger.

Now, a crucial question is in order: which kind of objects fit with this notion of structured experience? In other words, which kind of objects are compatible with the organization of events provided by QM (or more general non-boolean lattices)? Our answer is that in the QM description, objects appear as a partial aspect of a particular description [6]. As an example, think about a room full of objects. Each object (the door, the walls, the chairs, a source, a photon counter, etc) has a definite position and is situated in a definite relationship with respect to the others. But which is the nature of the room itself? The room itself, as it presents to us, comes into being as an *organized structure* of objects: *everything is correlated in some way*. To presuppose that the description of the room can be reduced to the relative positions of objects in Euclidean space (or in a more general spatio-temporal setting), is a metaphysical assumption which is not necessarily valid for the description of all phenomena[7]. A quantum system is just as real as the *place* in which objects are situated and structured, but it is not an object: *it is the organization of phenomena itself*. There is an actual state of affairs in the room, which has its own history (state), and the set-theoretical-spatio-temporal description as a collection of objects in space is just an *aspect* of it. And this structure is logical form expressed as a particular geometry. A quantum setup in a laboratory cannot be reduced to the classical description: this is at the heart of the complementarity principle. In this way the QM description manifests itself as the study of probability distributions, which are of course, objective and (at least in principle), experimentally controllable.

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