Quantum Logical Structures For Identical Particles

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Abstract

In this work we discuss logical structures related to indistinguishable particles. Most of the framework used to develop these structures was presented in [17, 28] and in [20, 14, 15, 16]. We use these structures and constructions to discuss possible ontologies for identical particles. In other words, we use these structures in order to characterize the logical structure of quantum systems for the case of indistinguishable particles, and draw possible philosophical implications. We also review some proposals available in the literature which may be considered within the framework of the quantum logical tradition regarding the problem of indistinguishability. Besides these discussions and constructions, we advance novel technical results, namely, a lattice theoretical structure for identical particles for the finite dimensional case. This kind of approach was not present in the scarcely literature of quantum logic and indistinguishable particles.

Key words: quantum logic-convex sets-indistinguishable particles

1 Introduction

Among the foundational problems of quantum mechanics (QM), the discussion about identical particles (so denominated for example by van Fraassen [61, chaps.11 and 12]) plays a central role. The case of "indistinguishable" particles is treated separately in almost all introductory books on QM [7, 55], and many works on physics [60, 37, 23, 6, 18, 24, 44] and philosophical debate [59, 13, 20, 47, 33] has been dedicated to this problem.

From the interpretational point of view, one the of the most important open tasks is the right characterization of the word 'identical' as used in this context. On the one hand, the axioms of QM, as standardly formulated, are based on classical logic and mathematics: that mathematics that can be build in a fragment of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) [36, 26, 35]. Thus, it involves the 'classical' theory of identity, which says that there are no indiscernible objects: indiscernible, or indistinguishable, entities must be *the very same object*, and that's all. But on the other hand, the theory does not seem to provide the means to distinguish (or label) quanta in most cases. On the contrary, it seems to point in the direction that quanta cannot be discerned at all. Thus, at the interpretational level, a lot of authors inclined themselves for a point of view in which *quanta cannot* be considered individuals [59, 60, 20, 21, 32, 33, 47]. This position is usually called *the received view*. The question either elementary particles may be indiscernible by 'physical properties' only (but retain individuality), or if they can be absolutely indiscernible, being entities that even in God's mind cannot be discerned, either by physical or logical tools, is not a settled topic (see for example [45] and [46] and [20] for a complete discussion). We will return to this problem in this paper and use logical tools to shed light into it.

Many authors claim that the alleged indiscernibility concerns only with 'physical properties', and usually state that even if quanta cannot be discerned by physical means, they still retain their individuality as some kind of "primitive thisness". Thus, as we shall see, this alternative opens the door for some kind of hidden variables, or parameters (in this case, as a form of "hidden identity"), which would exist at least in the logical domain –logic here encompasses mathematics– and which hide themselves when we try to manipulate them experimentally (as happens with "hidden variables" in Bohm's interpretation). No one knows the implications of this kind of assumption, as for instance, if there is some kind of non-go theorem speaking of 'logical' properties, and confusion usually appears in crucial questions, such as testing experimentally if quanta are individuals or not. In order to settle such questions the problem must be properly formulated and some questions clarified. Logic may be helpful for this task, and we will explore this possibility throughout the paper.

If the received view is accepted, there is in fact a foundational problem of logical nature when the interpretation -which presupposes non individuality- is contrasted with the standard axiomatic formulation of the theory, which as mentioned above, presupposes the classical theory of identity. This situation gave rise to different kinds of criticisms, as the discussion about the "surplus structure" in [51, 52], the demand of a more direct formulation (avoiding the symmetrization postulate) as in [33, 47], or the simpler exploitation of the contradiction in order to support the position that quanta are individuals (at least in a weak form), as in [46]. As we shall see in this work, it is possible to put things clearer by stating the problem properly by logical means, in the case, by changing the underlying logic.

Another important foundational problem in QM is the one of compound systems (of which the indistinguishability problem may be considered as a particular case). In particular, the notion of entanglement [9], which was considered by Schrödinger as "**the** characteristic trait of quantum mechanics"[56, 57, 58]. Entanglement has to do with the properties of quantum systems when they interact, when they are gathered together, and studies which concentrate on entanglement of identical particles have been developed relatively recently. Many questions remain open and, as is well known, correlations originated in entanglement are very different than those originated in statistics (exchange correlations) [24, 44, 6, 18]. The singular features which appear when aggregates of systems of identical particles are studied makes the subject to have its own particular problems. This has to be taken into account when structural properties of quanta aggregates are studied.

In this paper, we shall outline a discussion on certain topics involving indiscernible particles within the scope of different mathematical structures erected from motivations taken from quantum theory. One of them has to do with *quantum logics* (QL) and compound quantum systems of identical particles. The standard quantum logical approach to QM [10, 11, 30, 43, 8, 12, 27], uses the lattice of projections of the Hilbert space of the system as the lattice of propositions (see Section 3). This approach has been useful for the study of structural properties of quantum systems by the characterization of their operational lattices, and to clarify differences with other theories (such as classical mechanics). Recently, an alternative proposal has been developed [17, 28, 29] in order to solve some problems which appear in the study of compound quantum systems (see for example [2]). After reviewing the standard formulation of the formalism of indistinguishable particles and posing its problems in Section 2, we will adapt the constructions presented in [17, 28, 29] to the indistinguishable particle case in section 3. This construction is particularly suitable for an extension of the QL approach to the case of indistinguishable particles, which is difficult to accommodate in the traditional approach, and was not explored in the literature (though see [25]). It provides a new formal framework for the study of compound quantum systems in the indistinguishable case and its entanglement properties, as discussed in section 3.2.

The other formal structure studied in this work is the theory of quasi-sets [33], which is based on a non-classical logic, namely, a non-reflexive logic, and has to do with the problem of the identity of indistinguishable particles[1]. This will be done in section 4. In this section we will review the main characteristics of quasi-set theory and discuss its implications. We will also review how it can be used for an alternative formulation of the Fock-space formalism [15, 16] discussing the implication of such a construction for he interpretation of QM. Next, in Section 5, we pose the problem of identical particles in a new form under the light of the logical structures presented in this work.

Finally, we will present our conclusions in Section 6, where we will try to condense some ontological implications of the discussions posed in Sections 3, 4 and 5.

2 The problem of identical particles

We shall sketch here a small introduction to the standard formulation of the problem of identical particles. We will emphasize the usual mathematical trick that is used to achieve indistinguishability, namely, permutational symmetry. The clarification of that trick opens the door –from the foundational point of view– to the idea that a different mathematical formalism would be in order. We will explore the (old) idea that physics may suggest new logical schemas (see for example [49]).

2.1 States And Compound Quantum Systems

In the standard quantum mechanical formalism, for any system S, a Hilbert space $\mathcal{H}$ is assigned and observables are represented by self adjoint compact operators. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on a suitable Hilbert space $\mathcal{H}$, while the set of bounded self adjoint operators is denoted by $\mathcal{A}$. $\mathcal{B}(\mathcal{H})$ is a well known example of a von Neumann algebra [53].

States will be represented (for either single or compound systems) by the set of positive trace class and self adjoint operators of trace 1,

$$\mathcal{C} = \{\rho \in \mathcal{A} \,|\, \mathrm{tr}(\rho) = 1 \text{ and } \rho \geq 0\}, \tag{2.1.1}$$

where the operators ρ are called *density operators*. They represent the more general available states, and for any observable represented by an hermitian operator A, they assign a real number (which is interpreted as the mean value of the observable) according to the rule

$$\mathrm{tr}(\rho A) = \langle A \rangle \tag{2.1.2}$$

[1]We call 'non-reflexive' those logics which deviate from classical logic in what respects the theory of identity, in particular, by questioning the principle of identity in some formulation. In our case, we question the principle in the form $\forall x(x = x)$ (also called the reflexive law of identity) since we assume that there are entities to which the standard notion of identity (described by the theory governing the symbol '=') does not hold. This assumption is based on some of Schrödinger's opinions –see [59] and [20] for a detailed history and context.

If P is a projection operator (i.e., it satisfies $P^2 = P$) intended to represent an elementary test or event [10, 43], then the real number $\mathrm{tr}(\rho P)$ is interpreted as the probability of obtaining the event P given the system in state ρ. This is no other thing than the Born's rule.

There is a special subclass of states, called pure states, which are the density operators satisfying $\rho^2 = \rho$. They have a representation as normalized vectors in $|\psi\rangle \in \mathcal{H}$ in the form

$$\rho_{pure} = |\psi\rangle\langle\psi| \tag{2.1.3}$$

States which are not pure are called *mixed states.* For pure states, we have the superposition principle: any normalized linear combination of states will yield a new state. In formulae, if $|\psi\rangle$ and $|\varphi\rangle$ are normalized vectors representing pure states, and if α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, then

$$|\phi\rangle = \alpha|\psi\rangle + \beta|\varphi\rangle \tag{2.1.4}$$

will also be a state.

For a compound quantum system formed by two subsystems represented by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we assign a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where "$\otimes$" denotes the *tensor product.* Observables are represented by Hermitian operators in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Let $\{|\varphi_i^{(1)}\rangle\}$ and $\{|\varphi_i^{(2)}\rangle\}$ be orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_1$ respectively. The set $\{|\varphi_i^{(1)}\rangle \otimes |\varphi_j^{(2)}\rangle\}$ forms an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then, a pure state of the composite system can be written as $|\psi\rangle = \sum_{i,j} \alpha_{ij} |\varphi_i^{(1)}\rangle \otimes |\varphi_j^{(2)}\rangle$. Given a state ρ of the composite system, partial states ρ_1 and ρ_2 can be defined for the subsystems. The relation between ρ, ρ_1 and ρ_2 is given by:

$$\rho_1 = tr_2(\rho) \qquad \rho_2 = tr_1(\rho) \tag{2.1.5}$$

where tr_i stands for the partial trace over the i degrees of freedom. A density matrix of the composite system ρ is said to be a convex combination of product states, if there exists $\{p_i\}$ and states $\{\rho_1^i\}$ and $\{\rho_2^i\}$ such that

$$\rho = \sum_{i=1}^{N} p_i \rho_1^i \otimes \rho_2^i \tag{2.1.6}$$

If a state of the composite system can be written as a convex combination of product states (or approximated by a sequence of them), then it is said to be *separable.* If not, it is said to be entangled [9].

2.2 Indistinguishable particles

If particles are identical -in the sense of sharing all their intrinsical properties (for example, a collection in which all particles are electrons)-, we must add the condition that all pure states should be symmetrized. This is the content of the symmetrization postulate [20], and this means that all states must have a definite symmetry with respect to the action of the permutation operator. For example, if $|\varphi\rangle \in \mathcal{H}_1$, $|\psi\rangle \in \mathcal{H}_2$ and $|\varphi\rangle \neq |\psi\rangle$, then the corresponding symmetrized states are

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\varphi\rangle \otimes |\phi\rangle \pm |\phi\rangle \otimes |\varphi\rangle) \tag{2.2.1}$$

where the "+" sign stands for bosons and the "−" sign for fermions. If $\{|\varphi_i\rangle\}_{i\in I}$ and $\{|\phi_j\rangle\}_{j\in I}$ are basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then $\{|\varphi_i\rangle\otimes|\phi_j\rangle\}_{<i,j>\in I\times I}$ is a basis of $\mathcal{H}$. Then, a permutation operator

$$\begin{aligned} P_{12} : \mathcal{H} &\longrightarrow \mathcal{H} \\ P_{12}|\varphi_i\rangle\otimes|\phi_j\rangle &\rightarrow |\phi_j\rangle\otimes|\varphi_i\rangle \end{aligned} \tag{2.2.2}$$

can be defined, because it is defined on each element of this basis (and extended linearly in a trivial way). It can be shown that it is independent of the chosen basis. As $P_{12}^2 = \mathbf{1}$, its eigenvalues are 1 and −1 for Bosons and Fermions respectively. Then, this operator selects two special subspaces of $\mathcal{H}$ according to its eigenvalues

$$\mathcal{H}^+ = \{|\psi\rangle \in \mathcal{H} \,|\, P_{12}|\psi\rangle = |\psi\rangle\} \tag{2.2.3a}$$

$$\mathcal{H}^- = \{|\psi\rangle \in \mathcal{H} \,|\, P_{12}|\psi\rangle = -|\psi\rangle\} \tag{2.2.3b}$$

which will represent the possible (pure) states of the system when indistinguishable particles are involved. All physical states of indistinguishable particles must obey these symmetry conditions. This is an empirical statement, and up to now, no other symmetries where found (see [23, 37] for a discussion of this statement). So, the theory opens the door to "para-statistics", but it seems that none of them where found to have correspondents in nature, and we will not treat this case here.

2.3 How do the symmetrization postulate works and its open problems

It is important to make here the crucial observation of how the scheme of the symmetrization postulate works. First, particles are *labeled* by assigning them normalized vectors $|\psi\rangle$ and $|\phi\rangle$ in their corresponding spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If the state of the compound system would be simply $|\psi\rangle\otimes|\phi\rangle$, then, particles could be distinguished by special observables, i.e., the theory would allow for an asymmetry between both systems. But things are not so, and the symmetrization postulate must erase any obserbable characteristic which allows us to identify the particles. Thus, the state must be symmetrized as in 2.2.1. This is how the symmetrization postulate works: by first imposing a label as a mean to *individuate* each particle, and then erasing it. It is not difficult to realize that this trick is unavoidable in the standard formulation of QM, given that its axiomatic is formulated using standard (Zermelo-Frenkel) mathematics, based on classical theory of identity. Thus, symmetrization postulate *hides* particle identities, living the door open to an interpretation based on non-individuals. This is one of the reasons why many authors support the received view.

But the received view has been criticized in many ways. One of them has to do with contradiction between the method of introducing the adequate symmetries and the interpretation itself. On the one hand, Schrödinger and others tell us that we must give up any intent to provide individuality for elementary particles, and this position has a considerable agreement with experience. But it is also true that particles are labeled in the *symmetrization mechanism*, and thus, they seem to posses some form of individuality. How to reconcile these views?

The problem was discussed and a possible solution was proposed in [51, 52]. The authors, characterized the non-symmetrical parts of the Hilbert space, discarded by the symmetrization postulate, as *surplus structure*, i.e., a mathematical structure which plays no role in the final formulation of the theory and its predicted experience. Indeed, this is true. Next, they show arguments in favor of the Fock-space formulation of quantum mechanics. As is well known, it is possible to use the Fock-space formalism as an alternative approach to QM [54]. The criticism raised against this solution, asserts that the Fock-space mechanism also appeals to particle labeling [20], and so, it has a similar problem to that of the usual symmetrization postulate formulation.

From a different point of view, in [45] and [46], it is argued that fermions *are* individuals, at least in a weak form, because antisymmetry grants that they have opposite properties: think in two electrons in the ground state of a Helium atom, they have opposite spin, and thus, there is a property which one has and not the other. In a later work, they "show" that particles are individuals just because stating the axioms of the standard formulation of QM it can be shown that particles can be identified. As we shall discuss in detail in Section 4, this reduces to the fact that QM is axiomatized using ZF mathematics, and we will argue that the conclusion drawn by the authors is premature and the problem and its alternative solutions are not properly posed.

As we shall see later on this work, there is still a possibility for a reconciliation between the received view and a valid reformulation of QM, but one which encompasses a radical change in the logical framework of the theory. This will be done in Sections 4 and 6, after clarifying the problem, i.e., planting it in a clear logical and ontological form.

One question is still at stack. Is the received view really desirable? Or unavoidable? The assumption of the individuality of quanta seem to play no role in any experience or at least, it can be removed and experiments can be successfully explained. On the contrary, if the assumption of individuality is not taken with special care and protected by suitable hypotheses, it may lead to wrong results. Then, in spite of this, why not still postulate a form of "hidden individuality", playing no role but satisfying a particular metaphysical taste? As we shall see, this can be done, and we will discuss the consequences of this assumption when things are properly formulated.

3 A quantum logical formalism for identical particles

In this section, we introduce the lattice of convex subsets formalism presented in [17, 28, 29]. Next, we apply it to the case of identical particles using the preliminaries introduced above. The traditional approach to quantum logic uses the bounded projection operators (or equivalently, closed subspaces) of the Hilbert space as propositions or properties, using the direct sum as the disjunction, the intersection as the conjunction, inclusion as the order relation and orthogonal complement as negation [10, 11, 27]. Contrary to that, we will use a lattice formed by convex subsets of the state space. This lattice is more suitable in order to define maps which relate states of a compound system to states of the subsystems, and allows for the introduction of mixed states (something almost unavoidable when we have for example, bipartite systems of identical particles) [28, 29]. It is important to remark that no similar construction can be made with the usual projection lattice for the identical particles case (as shall be clear from the discussions below).

3.1 The Lattice of Convex Subsets

According to the orthodox quantum logical approach, the propositions of classical mechanics are the subsets of *the set* of states (classical phase space) [11]. In [28, 29] the *convex* subsets of the *convex* set of states are considered. Convexity is an key feature of quantum mechanics (see for example [38], [39] and [40] for an axiomatization based on convex sets). Let us begin by considering the set of all convex subsets of $\mathcal{C}$ (defined in equation (2.1.1))

Definition 3.1. $\mathcal{L}_{\mathcal{C}} := \{C \subseteq \mathcal{C} \mid C \textit{ is a convex subset of } \mathcal{C}\}$

In order to give $\mathcal{L}_{\mathcal{C}}$ a lattice structure, we introduce the following operations:

Definition 3.2. *For all* $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}}$

(∧) $C_1 \wedge C_2 := C_1 \cap C_2$

(∨) $C_1 \vee C_2 := conv(C_1, C_2)$. *It is again a convex set, and it is included in $\mathcal{C}$ (using convexity).*

(¬) $\neg C := C^{\perp} \cap \mathcal{C}$

($\longrightarrow$) $C_1 \longrightarrow C_2 := C_1 \subseteq C_2$

With the operations of definition 3.2, it is apparent that $(\mathcal{L}_{\mathcal{C}}; \longrightarrow)$ is a poset. If we set $\emptyset = \mathbf{0}$ and $\mathcal{C} = \mathbf{1}$, then, $(\mathcal{L}_{\mathcal{C}}; \longrightarrow; \mathbf{0}; \mathbf{1})$ will be a bounded poset. With the operations defined in 3.5, $\mathcal{L}_{\mathcal{C}}$ will be a bounded, atomic and complete lattice.

It is possible to define maps which connect states of the compound system with states of the subsystems as follows [28]. Given $C_1 \subseteq \mathcal{C}_1$ and $C_2 \subseteq \mathcal{C}_2$, define

$$C_1 \otimes C_2 := \{\rho_1 \otimes \rho_2 \,|\, \rho_1 \in C_1, \rho_2 \in C_2\} \tag{3.1.1}$$

Using the above definition, it is possible to define the map

$$\begin{aligned} \Lambda : \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} &\longrightarrow \mathcal{L}_{\mathcal{C}} \\ (C_1, C_2) &\longmapsto conv(C_1 \otimes C_2) \end{aligned} \tag{3.1.2}$$

If we use partial traces:

$$\begin{aligned} \mathrm{tr}_i : \mathcal{C} &\longrightarrow \mathcal{C}_j \\ \rho &\longmapsto \mathrm{tr}_i(\rho) \end{aligned} \tag{3.1.3}$$

we can also construct the maps

$$\begin{aligned} \tau_i : \mathcal{L}_{\mathcal{C}} &\longrightarrow \mathcal{L}_{\mathcal{C}_i} \\ C &\longmapsto \mathrm{tr}_i(C) \end{aligned} \tag{3.1.4}$$

which link the elements of $\mathcal{L}_{\mathcal{C}}$ (compound system) to the elements of $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathcal{C}_2}$ (subsystems). Next, define the product map

$$\begin{aligned} \tau : \mathcal{L}_{\mathcal{C}} &\longrightarrow \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} \\ C &\longrightarrow (\tau_1(C), \tau_2(C)) \end{aligned} \tag{3.1.5}$$

In [28, 29] it is shown that using Λ and τ it is possible to link states of the compound system to the states of its subsystems (at the lattice level). This cannot be done in the standard QL formalism. In the following we will take this feature of $\mathcal{L}_{\mathcal{C}}$ as an advantage to construct a lattice for the case of identical particles, in which the use of mixtures is unavoidable.

3.2 The Identical Particle Lattice $\mathcal{L}_{\mathcal{C}^{\pm}}$ of Convex Subsets (Bipartite Case)

Let us now define a lattice for the identical particles case. We will restrict to the finite dimensional bipartite case for simplicity. We begin by building the lattice of convex subsets for symmetrized Hilbert spaces. Taking into account the principle of indistinguishability, let

Definition 3.3. $\mathcal{C}^{\pm} = \{\rho : \mathcal{H}^{\pm} \longrightarrow \mathcal{H}^{\pm} \mid t_r(\rho) = 1 \ , \rho^{\dagger} = \rho \ and \ \rho \geq 0\}$

and define (in analogy with $\mathcal{L}_{\mathcal{C}}$)

Definition 3.4. $\mathcal{L}_{\mathcal{C}^{\pm}} := \{C \subseteq \mathcal{C}^{\pm} \mid C \ is \ a \ convex \ subset \ of \ \mathcal{C}^{\pm}\}$

$\mathcal{C}^{\pm}$ can be considered as a convex subset of $\mathcal{C}$, namely $\mathcal{C}^{\pm} \in \mathcal{L}_{\mathcal{C}}$. This is because any matrix in $\mathcal{C}^{\pm}$ can be canonically extended to the whole Hilbert space. In order to provide $\mathcal{L}_{\mathcal{C}^{\pm}}$ with a lattice structure similar to that of $\mathcal{L}_{\mathcal{C}}$, we define the following operations:

Definition 3.5. *For any* $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}^{\pm}}$,

$(\wedge^{\pm})$ $C_1 \wedge^{\pm} C_2 := C_1 \cap C_2$

$(\vee^{\pm})$ $C_1 \vee^{\pm} C_2 := conv(C_1, C_2)$

$(\neg^{\pm})$ $\neg^{\pm} C := C^{\perp} \cap \mathcal{C}^{\pm}$

$(\longrightarrow^{\pm})$ $C_1 \longrightarrow^{\pm} C_2 := C_1 \subseteq C_2$

$C^{\perp}$ is the orthogonal complement of C with respect to the scalar product $\langle A, B\rangle = \mathrm{tr}(A.B^{\dagger})$, namely $C^{\perp} = \{\rho \in \mathbb{C}^{N\times N} \mid \mathrm{tr}(\sigma.\rho^{\dagger}) = 0 \ \ \forall \sigma \in C\}$. With these operations, it follows that $(\mathcal{L}_{\mathcal{C}^{\pm}}; \longrightarrow^{\pm})$ is a partially ordered set. And if we take $\emptyset = \mathbf{0}$ and $\mathbf{1} = \mathcal{C}^{\pm}$, then $(\mathcal{L}_{\mathcal{C}^{\pm}}; \longrightarrow^{\pm}; \mathbf{0}; \mathbf{1})$ is a bounded partially ordered set. We also notice that, because $\mathcal{C}^{\pm} \in \mathcal{L}_{\mathcal{C}}$ and since the operations of $\mathcal{C}^{\pm}$ are inherited from $\mathcal{C}$, then $\mathcal{L}_{\mathcal{C}^{\pm}}$ is a sublattice of $\mathcal{L}_{\mathcal{C}}$.
Let $\mathrm{tr}_i(\mathcal{L}_{\mathcal{C}^{\pm}}) := \{\mathrm{tr}_i(C) \mid C \in \mathcal{L}_{\mathcal{C}^{\pm}}\} \subseteq \mathcal{L}_{\mathcal{C}_j}$ $(i \neq j)$. It is possible to define the canonical projections τ_i and τ for the identical particles case as follows

$$\begin{aligned} \tau_i^{\pm} : \mathcal{L}_{\mathcal{C}^{\pm}} &\longrightarrow \mathrm{tr}_j(\mathcal{L}_{\mathcal{C}^{\pm}}) \\ C &\longmapsto \mathrm{tr}_j(C) \end{aligned} \tag{3.2.1}$$

and the product map

$$\begin{aligned} \tau^{\pm} : \mathcal{L}_{\mathcal{C}^{\pm}} &\longrightarrow \mathrm{tr}_2(\mathcal{L}_{\mathcal{C}^{\pm}}) \times \mathrm{tr}_1(\mathcal{L}_{\mathcal{C}^{\pm}}) \\ C &\longmapsto (\tau_1^{\pm}(C), \tau_2^{\pm}(C)) \end{aligned} \tag{3.2.2}$$

In other words, the maps $\tau_i^{\pm} = \tau_i|_{\mathcal{L}_{\mathcal{C}^{\pm}}}$ and $\tau^{\pm} = \tau|_{\mathcal{L}_{\mathcal{C}^{\pm}}}$ are the restrictions of τ_i and τ respectively to $\mathcal{L}_{\mathcal{C}^{\pm}}$. Given that the subsystems are identical, it follows that $\tau_1^{\pm} = \tau_2^{\pm}$. The map $\tau^{\pm}$ is defined in analogy with (3.1.5), and it allows to link states of the compound system to states of its subsystems.
It would be of great interest to find an extension $\Lambda^{\pm}$ of the canonic map Λ. However, such an extension for the identical particle case is not immediate, because the elements of the set $\mathrm{tr}_2(C_1) \otimes \mathrm{tr}_1(C_2)$ will not be symmetrized states in the general case.
The connection between the lattices $\mathcal{L}_{\mathcal{C}_1}$, $\mathcal{L}_{\mathcal{C}_2}$ and $\mathcal{L}_{\mathcal{C}}$ with the lattice of the identical particles $\mathcal{L}_{\mathcal{C}^{\pm}}$ is shown in Figure 3.2.

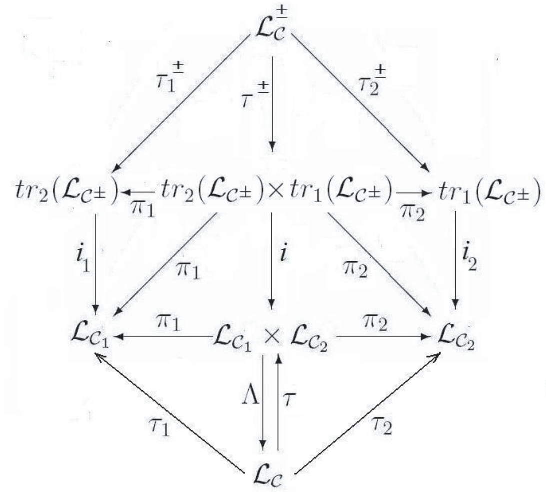

Figure 1: Canonical maps between the lattices $\mathcal{L}_{\mathcal{C}}$, $\mathcal{L}_{\mathcal{C}_1}$, $\mathcal{L}_{\mathcal{C}_2}$ and $\mathcal{L}_{\mathcal{C}^{\pm}}$. The arrows i and π represent inclusions and canonical projections.

4 Quasi-Set Theory

In this section, we basically follow the exposition of [34]; for details, see [21], [20, Chap.7]. Quasi-set theory is a mathematical theory that enables us to deal with collections (quasi-sets) of objects that may be indiscernible without turning to be identical (being the same object). The only way of dealing with objects of this kind in a standard theory such as ZFC is to allow the introduction of some *ad hoc* devices such as the restriction of the lexicon of properties, that is, by taking a language with a finite number of them. This is Quine's famous way of defining identity, namely, by the exhaustion of the chosen (finitely many) predicates [50, Chap.12]. But this just defines indiscernibility relative to the language's predicates, and not identity strictly speaking, for there may exist other predicates not in the language which distinguish among the entities.

In ZFC (the same can be said of most theories with due qualification) given an object a, we can always form the unitary set $A = \{a\}$ and define a unary property (a formula with just one free variable) of a by posing $I_a(x)$ iff $x \in A$. This formula of course distinguishes (or discriminates) a from any *other* object for only a satisfies it. Saying in other words, in the standard mathematics (in the sense mentioned in the Introduction), whenever we have a set with cardinal $\alpha > 1$, its elements *are* distinct. In short, there are no indiscernible objects, except with respect to a few chosen predicates.

Quine says that objects may be (strongly) discriminable when there is a formula with only one free variable that is satisfied by one of the objects but not by the other. Two objects are *moderately discriminable* when there exists a formula with two free variables that is satisfied by the two objects in one order but not in the other order [50, Chap.15]. As he recalls, any two real numbers are strongly discriminable, although they may be 'specified', that is, definable by a formula. But in the interpretation which supports the received view, there is no way to attribute an identity to quanta (let us call this a 'which is which criterion'), something that isolate one of them from the others in such a way that *this* chosen object remains with its identity forever.

Thus, how can we deal with, say, the two electrons of an Helium atom in the fundamental state? We know that they have all the same properties but distinct spins in a given direction, and so they obey an irreflexive and symmetric relation 'to have different spin of' and thus they are moderately discriminable in the sense of Quine. Muller and Saunders think that with this example, they have found objects (the mentioned electrons) that are moderately but not strongly discriminable [41]. Thus apparently we have found a way of speaking about two electrons with different spins and cannot say which is which. But this is a false supposition (see [32]). Within classical logic (and the mentioned authors assume that they are using ZFC), being objects (in some sense of the word) in a finite number, they can always be named, say a and b and the above property 'to belong to $A = \{a\}$', which is true just for a, distinguish them absolutely! In Quine's words, they can be always specified, that is, the language (of ZFC) there is always a formula in one free variable that is uniquely satisfied by a given object [50, p.134]. You may say that this property $I_a(x)$ ('being identical with a') is not a 'legitimate property' of x, but in our opinion this would be a quite arbitrary answer: which would be the legitimate ones? And why such an election (if there is one)?

Quasi-set theory offers a different alternative to treat these questions. Although electrons do present a difference due to Pauli's exclusion principle (for instance, differences in their spins), any permutation of them does not change the relevant probabilities, as is well known; in other words, there is no a which is which criterion. So, it seems that it is better (and apparently most correct) to say that the electrons may be indiscernible but without assuming that this makes them the very same object. A paradigmatic example may be that of the atoms in a Bose-Einstein condensate. Thus, we need to avoid that indiscernibility implies identity (in the philosophical sense of being the same entity). To cope with this idea, we 'separate' the concepts: indiscernibility, or indistinguishability, is a relation that holds for all objects of our domain, but identity is not. Certain objects may be indiscernible without turning to be the same object, as implied by the standard theory of identity, and they may form collections with cardinals greater than one (this is achieved by the postulates of the theory) but in a way that they cannot be identified, named, labeled, counted in the standard way.[2] There is no space here to provide the details of the theory, and we suggest [20] and [21] for detailed references and for the axioms. Anyhow, in the next subsection we outline without the details the main ideas of the theory.

4.1 Basic Ideas Of The Theory $\mathfrak{Q}$

Intuitively speaking, a quasi-set is a collection of objects such that some of them may be indistinguishable without turning to be identical.

As mentioned above, quasi-set theory $\mathfrak{Q}$ has its main motivations in some insights advanced by Schrödinger in that the concept of identity would make no sense when applied to elementary particles [59, pp. 17-18]. Another motivation is (in our opinion) the need, stemming from philosophical worries, of dealing with collections of absolutely indistinguishable items that need not be the same ones. Spatio-temporal differences could be used in this case, you may say, and this serves do distinguish them. Without discussing the role of spatio-temporal properties here (but see [20]), we can argue that these objects are invariant by permutations; in other words, the world does not present differences in substituting them one from the other. Thus, it would be difficult to say that they have some form of identity, for they (in principle) lack any identifying characteristic. Objects of this kind act like those that obey Bose-Einstein statistics, that is, bosons (we should remember that the indiscernibility hypothesis was essential in the derivation of Planck's formula –see [20] once more).

[2] We mean: a series of, say, five objects can be counted by proving that the set having them as elements (and no other element) is equinumerous to a finite ordinal, in the case, the ordinal $5 = \{0, \ldots, 4\}$. We remark that in order to define the bijection, we need to distinguish among the elements being counted –they need to be *individuals*, yet sometimes not specified.

Thus, the first point is to guarantee that identity and indistinguishability (or indiscernibility) will not collapse into one another when the theory is formally developed. We assume that identity (symbolized by '$=$') is not a primitive relation, but we use a weaker primitive concept of indistinguishability (symbolized by '$\equiv$') instead. This is just an equivalence relation and holds among all objects of the considered domain. If the objects of the theory are divided up into groups, namely, the m-objects (standing for 'micro-objects') and M-objects (for 'macro-objects') —these are ur-elements— and quasi-sets of them (an probably having other quasi-sets as elements as well), then identity (having all the properties of standard identity of ZF) can be defined for M-objects and quasi-sets having no m-objects in their transitive closure (this concept is like the standard one). Thus, if we take just the part of the theory obtained by ruling out the m-objects and collections (quasi-sets) having them in their transitive closure, we get a copy of ZFU (ZF with *Urelemente*); if we further eliminate the M-objects, we get a copy of the 'pure' ZF.

Technically, expressions such as $x = y$ are not always well formed, for they are not formulas when either x or y denote m-objects (entities satisfying the unary primitive predicate m). We express that by informally saying that the concept of identity *does not always make sense* for all objects (it should be emphasized that this is just a way of speech). The objects (the m-objects) to which the defined concept of identity does not apply are termed *non-individuals* for historical reasons (see [20]). As a result (from the axioms of the theory), we can form collections of m-objects which have no identity; these collections may have a cardinal (termed its 'quasi-cardinal') but not an associated ordinal. Thus, the concept of ordinal and of cardinal are taken as independent, as in some formulations of ZF proper. So, informally speaking, a quasi-set of m-objects is such that its elements cannot be identified by names, counted, ordered, although there is a sense in saying that these collections have a cardinal (that cannot be defined by means of ordinals, as usual, which presupposes identity).

When $\mathfrak{Q}$ is used in connection with quantum physics, the m-objects are thought of as representing quanta (henceforth, *q-objects*), but they are not necessarily 'particles' in the standard sense—associated with classical physics of even with orthodox quantum mechanics; waves, field excitations (the 'particles' in quantum field theory—QFT), perhaps even strings or whatever entities supposed indiscernible can be taken as possible interpretations of the m-objects. Generally speaking, whatever 'objects' sharing the property of being indistinguishable can also be values of the variables of $\mathfrak{Q}$ (see [19, Chap. 6] for a survey on the various different meanings that the word 'particle' has acquired in connection to quantum physics).

Another important feature of $\mathfrak{Q}$ is that standard mathematics can be developed using its resources, for the theory is conceived in such a way that ZFU (and hence also ZF, perhaps with the axiom of choice, ZFC) is a subtheory of $\mathfrak{Q}$. In other words, the theory is constructed so that it extends standard Zermelo-Fraenkel with *Urelemente* (ZFU); thus standard sets (of ZFU) can be viewed as particular qsets (that is, there are qsets that have all the properties of the sets of ZFU, while the objects in $\mathfrak{Q}$ corresponding to the *Urelemente* of ZFU are termed M-atoms; these satisfy another primitive unary predicate M). The 'sets' in $\mathfrak{Q}$ will be called $\mathfrak{Q}$-sets, or just *sets* for short. An object is a qset when it is neither an m-object nor an M-object and, to make the distinction, the language of $\mathfrak{Q}$ encompasses a third unary predicate Z such that $Z(x)$ says that x is a qset which is also a *set*, and they correspond to those objects erected in the 'classical' part of the theory (without m-objects). It is also possible to show that there is a translation from the language of ZFU into the language of $\mathfrak{Q}$ so that the translations of the postulates of ZFU turn to be theorems of $\mathfrak{Q}$; thus, there is a 'copy' of ZFU in $\mathfrak{Q}$, and we refer to it as the 'classical' part of $\mathfrak{Q}$. In this copy, all the usual mathematical concepts can be stated, as for instance, the concept of ordinal (for sets).

Furthermore, it should be recalled that the postulates for the relation of indiscernibility, when applied to M-atoms or to $\mathfrak{Q}$-sets, collapses into standard identity (of ZFU). The $\mathfrak{Q}$-sets are qsets whose transitive closure (defined as usual) does not contain m-atoms (in other words, they are 'constructed in the classical part of the theory—see Fig. 2).

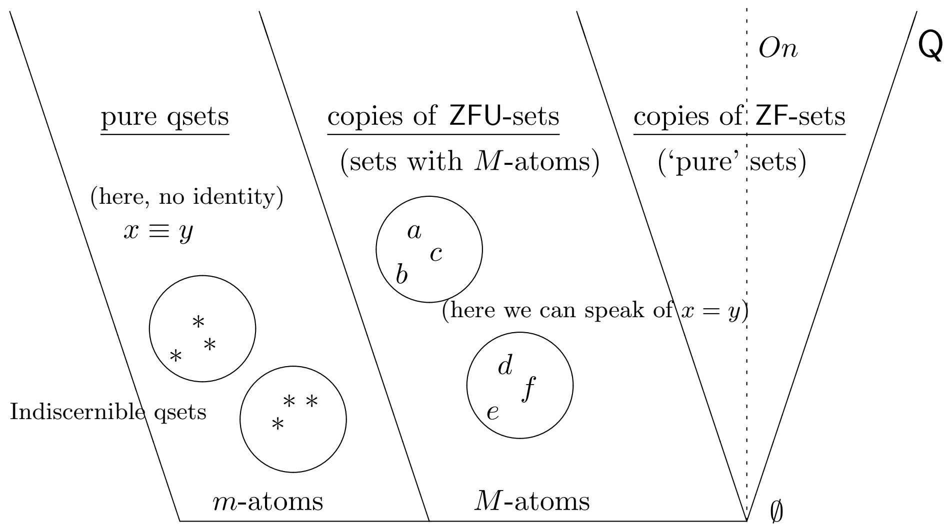

Figure 2: The Quasi-Set Universe Q: *On* is the class of ordinals, defined in the classical part of the theory. See [34]

In order to distinguish between $\mathfrak{Q}$-sets and qsets that may have m-atoms in their transitive closure, we write (in the metalanguage) $\{x : \varphi(x)\}$ for the former and $[x : \varphi(x)]$ for the latter. In $\mathfrak{Q}$, we term 'pure' those qsets that have only m-objects as elements (although these elements may be not always indistinguishable from one another, that is, the theory is consistent with the assumption of the existence of different kinds of m-atoms), and to them it is assumed that the usual notion of identity cannot be applied (that is, let us recall, $x = y$, as well as its negation, $x \neq y$, are not well formed formulas if either x or y stand for m-objects). Notwithstanding, the primitive relation $\equiv$ applies to them, and it has the properties of an equivalence relation.

We have also a defined concept of *extensional identity* and it has the properties of standard identity of ZFU. More precisely, we write $x =_E y$ (read 'x and y are extensionally identical') iff they are both qsets having the same elements (that is, $\forall z(z \in x \leftrightarrow z \in y)$) or they are both M-atoms and belong to the same qsets (that is, $\forall z(x \in z \leftrightarrow y \in z)$). From now on, we shall not bother to always write $=_E$, using simply the symbol "=" for the extensional equality.

Since m-atoms cannot be identified in the formalism, it is not possible in general to attribute an ordinal to qsets of such elements. Thus, for certain qsets, it is not possible to define a notion of cardinal number by means of ordinals. The theory uses a primitive concept of *quasi-cardinal* instead, which intuitively stands for the 'quantity' of objects in a collection.[3] The theory has still an 'axiom of weak extensionality', which states (informally speaking) that those quasi-sets that have the same quantity of elements of the same sort (in the sense that they belong to the

[3]A notion of finite quasi-cardinal can be defined as a derived concept (see [14]).

same equivalence class of indistinguishable objects) are indistinguishable by their own. One of the interesting consequences of this axiom is related to the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta (for a discussion on this point, see [22]). In standard set theories, if $w \in x$, then of course $(x - \{w\}) \cup \{z\} = x$ iff $z = w$. That is, we can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. In $\mathfrak{Q}$ we can prove the following theorem, where $[[z]]$ (and similarly $[[w]]$) stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z (respectively, from w –the reader shouldn't think that this element *is identical to either* z or w:

Theorem 1 (Unobservability of Permutations)**.** *Let x be a finite quasi-set such that x does not contain all indistinguishable from z, where z is an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists $[[w]]$ such that*

$$(x - [[z]]) \cup [[w]] \equiv x$$

Informally speaking, supposing that x has n elements, then if we 'exchange' their elements z by corresponding indistinguishable elements w (set theoretically, this means performing the operation $(x - [[z]]) \cup [[w]]$), then the resulting quasi-set remains *indistinguishable* from the one we started with. In a certain sense, it does not matter whether we are dealing with x or with $(x - [[z]]) \cup w'$. So, within $\mathfrak{Q}$, we can express that 'permutations are not observable', without necessarily introducing symmetry postulates, and in particular to derive 'in a natural way' the quantum statistics (see [20, Chap.7]).

4.2 The $\mathfrak{Q}$-space

As we have seen in Section 2, if particles are indistinguishable they can only access symmetrized states. But particles are labeled in this construction, and this procedure was criticized (Section 2.3). It has been claimed that the Fock-space formalism poses a solution to the questions raised by this criticism [51, 52]. But the Fock-space formalism also makes use of particle labeling uses particle labeling in order to obtain the correct states [20].

How can we avoid this problem of the Fock-space formulation of QM? If we could avoid the individuation of the particles at every step of the construction of a Fock-like formulation of QM, we would give a positive answer to the problem posed in [51, 52] (recalled in the Introduction) which is not affected by the criticisms linked to it. Quasi-set theory can be used for this purpose, and in fact, this construction has been done [15, 16]. There, an alternative proposal is presented which resembles that of the Fock-space formalism but based on $\mathfrak{Q}$. And thus, genuinely avoiding artificial labeling. We give a sketch of the construction here, mainly following [15, 16]. And in the following Section, we will also see that this kind of constructions not only allows us to solve the problem posed above, but they serve also to plant interesting foundational issues.

Let us consider a set $\epsilon = \{\epsilon_i\}_{i \in I}$ (that is, a "set" in $\mathfrak{Q}$), where I is an arbitrary collection of indexes (this makes sense in the 'classical part' of $\mathfrak{Q}$). Suppose that the elements ϵ_i to represent the eigenvalues of a physical observable. Next, quasi-functions f are constructed, such that $f : \epsilon \longrightarrow \mathcal{F}_p$, where $\mathcal{F}_p$ is the quasi-set formed of finite and pure quasi-sets. f is a quasi-set formed by ordered pairs $\langle \epsilon_i; x \rangle$ with $\epsilon_i \in \epsilon$ and $x \in \mathcal{F}_p$.

These quasi-functions are chosen in such a way that whenever $\langle \epsilon_{i_k}; x \rangle$ and $\langle \epsilon_{i_{k'}}; y \rangle$ belong to f and $k \neq k'$, then $x \cap y = \emptyset$. It is further assumed that the sum of the quasi-cardinals of the quasi-sets which appear in the image of each of these quasi-functions is finite. This means that $qc(x) = 0$ for every x in the image of f, except for a finite number of elements of ϵ. These quasi-functions form a quasi-set called $\mathcal{F}$.

A pair $\langle \epsilon_i; x \rangle$ is interpreted as the statement "the energy level ϵ_i has occupation number $qc(x)$". Quasi-functions of this kind are represented by expressions such as $f_{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_m}}$. If the symbol ϵ_{i_k} appears j-times the level ϵ_{i_k} has occupation number j. The levels that do not appear have occupation number zero.

At this point of the construction, the indexes appearing in $f_{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}}$ has no meaning at all. But an order can be defined as follows. Given a quasi-function $f \in \mathcal{F}$, let $\{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_m}\}$ be the quasi-set formed by the elements of ϵ such that $\langle\epsilon_{i_k}, x\rangle \in f$ and $qc(x) \neq 0$ $(k = 1\ldots m)$. This quasi-set is denoted $supp(f)$. Consider now the pair $\langle o, f\rangle$, where o is a bijective quasi-function $o : \{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_m}\} \longrightarrow \{1, 2, \ldots, m\}$. Each one of the quasi-functions o defines an order on $supp(f)$. Let $\mathcal{OF}$ denote the quasi-set formed by all the pairs $\langle o, f\rangle$. $\mathcal{OF}$ is the quasi-set formed by all the quasi-functions of $\mathcal{F}$ with ordered support. Using a similar notation as before (and also repeating indexes according to the occupation number), $f_{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}} \in \mathcal{OF}$ refers to a quasi-function $f \in \mathcal{F}$ and a special ordering of $\{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}\}$. But now, the order of the indexes must not be understood as a labeling of particles, because it can be shown that the permutation of particles does not give place to a new element of $\mathcal{OF}$ [15].

Consider next the collection of quasi-functions C which assign to every $f \in \mathcal{F}$ (or $f \in \mathcal{OF}$) a complex number. A quasi-function $c \in C$ is a collection of ordered pairs $\langle f; \lambda\rangle$, where $f \in \mathcal{F}$ (or $f \in \mathcal{OF}$) and λ a complex number. Let C_0 be the subset of C such that, if $c \in C_0$, then $c(f) = 0$ for almost every $f \in \mathcal{OF}$ (i.e., $c(f) = 0$ for every $f \in \mathcal{OF}$ except for a finite number of quasi-functions). A sum and a product can be defined in C_0 as follows

Definition 4.1. *Given $\alpha, \beta, \gamma \in \mathcal{C}$, and $c, c_1, c_2 \in C_0$, then*

$$(\gamma * c)(f) := \gamma(c(f)) \quad \textit{and} \quad (c_1 + c_2)(f) := c_1(f) + c_2(f)$$

Using the above definitions, $(C_0, +, *)$ is endowed with a complex vector space structure. Given a quasi-function $c \in C_0$ such that $c(f_i) = \lambda_i$ $(i = 1, \ldots, n)$ for some finite set of quasi-functions $\{f_i\}$ belonging to $\mathcal{F}$ or $\mathcal{OF}$ the following association is done

$$c \approx (\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n) \tag{4.2.1}$$

Thus, a quasi-function $c \in C_0$ is interpreted as a linear combination of the quasi-functions f_i (representing a quantum superposition).

Scalar products must be introduced in order to reproduce the quantum mechanical machinery of computation of probabilities. It is possible to define two of them, one for bosons ("$\circ$") and one for fermions ("$\bullet$"). In this way (and using norm completion), two Hilbert spaces $(\mathbb{V}_Q, \circ)$ and $(\mathbb{V}_Q, \bullet)$ are obtained. The scalar product for bosons is defined as follows

Definition 4.2. *Let δ_{ij} be the Kronecker symbol and $f_{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}}$ and $f_{\epsilon_{i'_1}\epsilon_{i'_2}\ldots\epsilon_{i'_m}}$ two basis vectors, then*

$$f_{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}} \circ f_{\epsilon_{i'_1}\epsilon_{i'_2}\ldots\epsilon_{i'_m}} := \delta_{nm} \sum_p \delta_{i_1 pi'_1}\delta_{i_2 pi'_2}\ldots\delta_{i_n pi'_n}$$

The sum is extended over all the permutations of the set $i' = (i'_1, i'_2, \ldots, i'_n)$ and for each permutation p, $pi' = (pi'_1, pi'_2, \ldots, pi'_n)$.

and for fermions

Definition 4.3. *Let δ_{ij} be the Kronecker symbol, $f_{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}}$ and $f_{\epsilon_{i'_1}\epsilon_{i'_2}\ldots\epsilon_{i'_m}}$ two basis vectors, then*

$$f_{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}} \bullet f_{\epsilon_{i'_1}\epsilon_{i'_2}\ldots\epsilon_{i'_m}} := \delta_{nm} \sum_p s^p \delta_{i_1 pi'_1}\delta_{i_2 pi'_2}\ldots\delta_{i_n pi'_n}$$

where: $s^p = +1$ if p is even and $s^p = -1$ if p is odd.

These products can be easily extended to all linear combinations. The second product $\bullet$ is an antisymmetric sum of the indexes which appear in the quasi-functions and the quasi-functions must belong to $\mathcal{OF}$. If the occupation number of a product is greater or equal than two, then, it can be shown that the vector has null norm. Thus reproducing Pauli's exclusion principle for fermions [15].

With these constructions within $\mathfrak{Q}$, the formalism of *QM* can be rewritten giving a positive answer to the problem of giving a formulation of *QM* in which intrinsical indistinguishability is taken into account from the beginning, without artificially introducing artificial labels [16, 16].

5 Stating the problem in an adequate form

Once that the formal setting is determined, we are now ready to come back to the questions posed in Section 2.3 in a more formal way. In this Section we state the problem of identical particles from a new perspective.

5.1 Metaphysical undetermination

As is well known, there are several interpretations of *QM*, the received view being only one among others (perhaps, the most popular). The simple fact that there exists an interpretation such as Bohm's –in which particles possess definite trajectories– represents a problem for someone who wants to extract metaphysical constructions out of physical theories: how to reconcile incompatible but plausible interpretations of a given formalism which reproduces laboratory experiences, such as the von Neumann formulation of *QM*? While in the Bohmian interpretation particles have definite trajectories, there are no trajectories at all in the standard interpretation. Particles are individuals for Bohm and non-individuals for the received view. This is the problem of metaphysical underdetermination, discussed in detail in reference [20]. We will review this problem here.

While Schrödinger used the BE and FD statistics as an argument to support the received view, other authors, interpreted these "strange" statistics as a new form of non-local correlation between particles (considered as individuals). While the symmetrization postulate was used as an argument against particle individuality, Muller and Saunders use particle labels to show that quanta are individuals [45, 46]. There seems to be reasonable arguments for both positions: both interpretations, while incompatible, seem to be valid, in the sense that they do not contradict (up to now) empirical data. The received view has historical problems such as stating clearly what does a non-individual mean. But there are concrete solutions to this problem, see for example [20].

In addition to this *metaphysical* undetermination, in Section 4.2 we showed that an alternative formulation of *QM* may be given, but with a different underlying logic, based on quasi-set theory. This implies that there also exists a kind of *logical* underdetermination, i.e., there is no preferred logic, if the objective is to formulate the theory in an axiomatic way.

Regarding *logical underdetermination* it is important to specify what we mean by the word "logical". As is well known, language has different layers. The axioms of a physical theory, stated in a mathematical form, have an underlying logic, which is usually standard set theory, but it may be formulated in a different frame, such as category theory or even higher order logic. But here we shall speak of ZFC set theory only. Then, the word "logic" means the axioms used in the mathematical formulation of the theory. *ZFC* set theory has a deeper logical level, which are the axioms of first order classical logic.

But physics not only concern mathematical formulation. Interpretation may be regarded as part of the theory or not, but it is for sure that it is unavoidable to have, at least, a minimal interpretational framework in order to connect theory with experience (and eventually, to explain it). The language used for speaking (and thinking) about the concepts related to the word "experience", as well as the theoretical terms representing the entities involved in a given interpretation, are not just mathematical, and it is also not an artificial one based on first order classical logic. But this language has its own "underlying logic", which may be not necessarily a formalized one. What does Scrhödinger meant by a non-individual entity, may be not clear or formal, but it is clear that the logic underlying such an interpretation seems to be not the classical one. Quasi-set theory, would provide a formal framework in order to give us a formal logical stuff for that notion. The logic at the level of the axioms of the theory and the underlying logic of the interpretation, may coincide or not. The second one is the case of the received view, susceptible of all the criticisms mentioned above. *QM* formulated in the $\mathfrak{Q}$-space formalism seems to be in harmony with the underlying logic of the received view.

We are thus faced with incompatible alternatives to take, with different possibilities. How to make a choice? Which attitude is to be taken in the view of this fact?

5.2 Ockham's razor revisited

In order to answer the question posed in the previous Section, let us review a traditional example of quantum theory. Bohm's interpretation presupposes individual trajectories for quantum particles, guided by pilot waves. Quantum statistics would thus be ruled out by hidden variables: trajectories exist, but you will never be able to predict them. The same interpretation comes endowed with a "concealment mechanism", which forbids experimental control of the postulated hidden variables. Thus, these hidden variables, while compatible with predicted experience, play no role in any experiment, i.e., they are completely *dispensable* (for the received view). They play only an explanatory role in Bohms approach: fairies or elves may be responsible for the values that these hidden variables take. *But the impossibility of settling this question experimentally is an a priori requirement of the interpretation itself.*

In view of the discussion of the previous Section, the word "explanatory", should be understood as follows: to make experience compatible with an ontological (or metaphysical) "preference" (or "prejudice"). Any interpretation seems to have theoretical terms which may be suppressed in order to endorse a minimal interpretation. But this is not the point that we want to stress: *dispensability is not our problem.* What we want to remark is that hidden variables in Bohm's theory cannot be measured, nor controlled in any laboratory experience. This is precisely an unavoidable requirement of the Bohmian interpretation, in order to be formally equivalent to standard *QM*. Otherwise, if these variables could be controlled, or some crucial experiment based on them could be designed, standard *QM* would be wrong (and the supposed formal equivalence between Bohms theory an the orthodox formulation of QM would no longer hold). Alike a quantum state or the electron charge, or even a field –which are all theoretical terms–, hidden variables cannot be prepared nor measured, simply because this is what "hidden" is intended to mean.

Which is the limit? There is no limit. Given a metaphysical preference (or election), we can always add as many "hidden" theoretical terms as we want, always taking care that they should not make predictions incompatible with experience (which is regulated by formalism plus a minimal interpretation). But it seems reasonable to assume that science should not be concerned with notions which are —as a matter of principle— impossible to control in any experiment, adding neither new predictions nor postulating states of affairs which are *by definition* impossible to regulate experimentally. It could be the case that a notion used to explain phenomena, such as Boltzmans particles, could not be observed or clearly studied in any experimental set up at a certain stage of a theory, *but this impossibility cannot be part of the definition of this notion.*

As is well known, the existence of Bohm's interpretation and the fact that its hidden variables are non-local, led J. Bell to question himself if there may exist an interpretation based on local hidden variables. These questionings originated the well known story about Bell's inequalities and Bell's theorem: any interpretation based on hidden variables compatible with quantum predictions is attained to non-locality, i.e., *hidden variables must be non-local.* And this was tested experimentally (in favor of QM). Thus, Bell's theorem shows that the acceptance of hidden variables leads also to "hidden non-locality", which of course, cannot be used to send information instantaneously.

One of the conclusions that we extract from the story of hidden variables and Bell's theorem, is as follows. It is always possible to make different interpretations of a given theory, and they may result incompatible. Where does this metaphysical underdetermination comes from? We will not discuss this in detail here, but we will only stress one point. It may be possible that metaphysical underdetermination of physical theories is a general characteristic of language itself, which manifests even at the level of simple examples of logic: think about models built within set theory. There may be several models of the same axiomatic, and nothing determines a preferred one. Another example, is one of the Gödel's results, which asserts that any axiomatic system —with a certain degree of complexity and formulated in a certain way— has true but undecidable propositions: there is always something beyond the scope of the axioms. And if this happens already at the logical level, nothing prevents this to happen in more complex languages, such as the one employed in the formulation and interpretation of a physical theory. The complete "language" of a theory involves a complex mixture of laboratory assertions, theoretical concepts (many of them not necessarily completely or rigourously defined), and if we wish, the formal language of the axiomatic level too.

Thus, it may well be that, as well as it happens with Godel's theorem, there will always exist assertions which cannot be decided using the axioms of the theory plus the available experimental data at a given historical moment, and no one knows how to state the problem in order to design the adequate experiments to decide them. But as it happened with hidden variables and Bell's theorem —by adding the adequate experimental evidence— further non trivial information about hidden variables was extracted by stating the problem in an adequate form. Adequate formulation of the problem may involve introducing new axioms and definitions (as well as theoretical constructions), in order to distinguish between several alternatives. This is one of the great merits of Bell.

A similar program may be delineated for indistinguishable particles. Even if we do not know how to make an experimental test in order to decide if quanta are individuals or not, working on formal structures and by considering ontological specification of the involved entities may serve to design new experiments, which if they don't rule out a given possibility, they may impose restrictions on its validity. As happened with Bohmmian mechanics: hidden variables resulted to be non-local, and as this fact is in a certain sense incompatible with restricted relativity theory, it gives us more elements to make an election (thought we are no obliged to consider it). A similar analysis can be made for the Kochen-Specker theorem.

Even now, we have at hands examples of how this ideas work for identical particles. In order to explain BE and FD statistics, one may assume non-individuality, as usual. But other authors explain this phenomenon by postulating non-independent correlations (which are different for Fermions and Bossons) [62]. In spite of these small steps, we think that an analogue to Kochen-Speker or Bell's theorem for non-individuality is in order. This should be added to the program for the investigation of identical particles in another work.

It is worthy to analyze what happens with individuality in CM. Trajectories can be measured in CM, and thus, one may postulate that any trajectory, it corresponds to one particle or to a body. By means of this association, individuality plays a direct role in observation. Of course, nothing prohibits us to break the link between particles and trajectories and postulate that a strange (unobservable) particle permutation may happen between the trajectories, and then, individuality could be questioned. We could go further and presuppose that classical particles of the same mass and form, charge, etc., are completely indistinguishable, and that they do not have individuality at all, but their trajectories do. In this case, we force the interpretation in order to satisfy a certain metaphysical preference, but none of these extra assumptions can be manipulated experimentally. This example in CM is the reverse of that one in the quantum case: as well as Bohmian Hidden Variables, individuality of quanta is not only dispensable, but it is also impossible to manipulate experimentally.

A possible attitude towards this "metaphysical freedom" could be: use Ockham's razor and discard individuality of quanta. But there exists an alternative and (we think) more fruitful possibility: given extra assumptions (such as the individuality of quanta), we must provide more precise definitions, add extra postulates and to design adequate crucial experiments in order to discriminate and discard between possible metaphysical alternatives.

Before entering into the general conclusions of this work, let us make an interesting remark. Bohm's interpretation postulates trajectories, hidden variables and pilot waves. At the end of the story, when these assumptions are fully analyzed, we find that hidden variables where non-local, and that the pilot waves suffer of similar problems than that of the traditional Scrödingers wave functions. It is as if the "problematic" aspects of the standard interpretation of quantum mechanics reemerge in the Bohmian interpretation in a new fashion, or as if there was a "principle of conservation of problematic aspects". Again, this kind of analysis of the "failure" of the Bohmian's program (i.e., the failure to recover a completely classical picture), does not suffice to discard the whole interpretation. But it sheds light to the consequences of our metaphysical preferences, and by studying these consequences, we gain a lot of information of how things really work. Perhaps the interesting question is not "which is the true ontology or interpretation?"; but questions such as "which are the consequences of each of them?" and "which of them are untenable (according to experience) and which of them are compatible?". This is perhaps the most interesting attitude towards the "metaphysical freedom" originated in metaphysical underdetermination.

6 Final discussion

As we have seen, non-classical logics and algebraic structures may arise also from insights taken from science. Non-reflexive logics, in the sense posed above, constitute a typical example. But, what can we say about ontology? A traditional philosopher may guess that this question is not well posed, for ontology is the study of the basic stuff of the world and it would be indifferent to the theory we are using in such an investigation. We may say that, in this old sense, a philosopher would say that ontology is the study of the basic furniture of the world, and in this sense all we need is to put some light on this world's stuff. But, at least since Quine [63], we become familiarized with the talk of an ontology associated to a theory and in order to speak about ontology, we need to look to our better theories and consider what they say about the world. This *naturalized ontology* is today well accepted by most philosophers of science, and we believe that this way of speech is not contrary with the usual assumptions made by the physicists. Thus, taking into account quantum theory (either relativistic or non-relativistic versions — quantum field theories), we can have some insights for instance about the very nature of the basic constituents of the world, the 'elementary particles'. By 'elementary particles' we mean whatever entity postulated or assumed by the theory in its grounds. We can even name then: electrons, protons, neutrinos, quarks, and so on, and refer to them indiscriminately as *quanta*.

As well as logical constructions may be inspired in physical theories, if we adopt the above point of view about a *naturalized ontology*, we may use these logical constructions to draw conclusions about the possible ontological commitments of physical theories. Let us discuss next the implications of the existence of the formal structures presented in this paper.

6.1 $\mathfrak{Q}$ and non-individuality

Regarding the discussion we have made in this paper, we can take some facts from granted, as for instance:

1. Standard formulations or both non-relativistic and relativistic quantum mechanics use both classical logic and standard mathematics (say, that one built in ZFC). Hence, no theory founded on such a basis can contradict its theorems (that is, classical logic and standard mathematics).

2. Quanta of the same species may be (in certain situations) absolutely indiscernible by all means provided by the theory.

3. There is a sense in saying that, in certain situations, quanta cannot be said to have an identity, a permanent label that distinguishes each one of them from others even of the similar species.

From these three items, we can conclude some basic facts. From (1), standard formalism of QM always distinguishes between two quanta, even if they are of the same species and regarded as indistinguishable. If the distinction cannot be achieved in the physical theory proper (the specific postulates of QM we are using), they can be discerned by the underlying mathematics. In fact, two distinct things can be put within two disjoined open sets, and this distinguishes them absolutely. Thus, even the atoms in a Bose-Einstein condensate, when represented in the mathematical model (QM) constructed this way, can be distinguished. From the logical point of view, we cannot agree *in totum* with the Nobel Prize winner Wolfgang Ketterle, when he says that

> "[i]f we have a gas of ideal gas particles at high temperature, we may imagine those particles to be billiard balls (...). They race around in the container and occasionally collide. This is a classical picture. However, if we use the hypothesis of de Broglie that particles are matter waves, then we have to think of particles as wave packets. The size of a wave packet is approximately given by the de Broglie wavelength λ_{dB}, which is related to the thermal velocity v of the particles as $\lambda_{dB} = h/mv$. Here m is the mass of the particles and h is Plancks constant. Now, as long as the temperature is high, the wavepacket is very small and the concept of indistinguishability is irrelevant, because we can still follow the trajectory of each wavepacket and use classical concepts. However, a real crisis comes when the gas is cooled down: the colder the gas, the lower the velocity, and the longer the de Broglie wavelength. When individual wave packets overlap, then we have an identity crisis, because we can no longer follow trajectories and say which particle is which. At that point, quantum indistinguishability becomes important and we need quantum statistics."[31]

In fact, within standard logic and mathematics, there is no identity crisis! This leads us to (2) and (3), which may be taken as emphasizing the same question, namely, the way to go around standard logic (and mathematics) in order to acknowledge that there is in fact an identity crisis involving quanta in certain situations. As we have said above, there are two options: to confine ourselves to a certain protected region within standard ZFC (say a mathematical structure) and speak of *some* properties and relations only. Thus the structure may be nonrigid and we can define indiscernible objects as those which are lead to one another by one of the non-trivial automorphisms of the structure. This is the classical solution: symmetric and anti-symmetric vectors do the job, and assuming identity as defined *a la* Quine completes the crime [64][4]. But, let us recall, any structure built in ZFC can be extended to a rigid structure, that is, a structure whose only automorphism is the identity function. In this structure, the very nature of our

[4]Quine defines identity by the exhaustion of all predicates of the language, taken always in a finite number. In our opinion, this strategy defines only indiscernibility regarding the chosen predicates.

alleged indiscernible quanta would be revealed, and in the background, we can see them as individuals, as entities having identity.

Thus, we conclude that if we aim at to speak of an ontology of really indiscernible quanta, we need to go out from classical frameworks and adopt an alternative logic, and quasi-set theory is one of the options. And in fact, as we have shown in section 4.2, the construction presented can be used to reformulate QM in a different logical background than that of ZFC (meaning standard mathematics and classical logic). An important consequence of the existence of such a reformulation is that the conclusions about the identity of quanta posed in (1) above are not a characteristic of *any* formulation of QM, but only of those formulations based on ZFC or similar "classical" theories. The construction sketched above show us that the answer to the problem posed in [51, 52] and recalled in the Introduction is in the affirmative, and that our construction is more in accordance with the interpretations of QM which claim that particles are not individuals. The formulation of QM discussed in section 4.2 *supports* this "received view" (see [20]).

It is not our aim in this article to deny hidden variable theories (or haecceities). We only stress the point that there are different interpretations of QM and according to this fact, different mathematical formulations, each of which is more or less compatible with a given interpretation.

6.2 Quantum logic for compound systems of identical particles

The quantum logical approach to physics does not restrict itself to QM. It is a general operational framework which allows us to include a huge family of physical theories. Using the method developed by Piron [43], it is possible to define questions on an arbitrary system, in such a way that these questions form propositions. It is possible to show that these propositions form a lattice. By imposing suitable axioms on this lattice, one may recover QM, CM, or, in principle, any arbitrary theory. This was called the Operational Quantum Logic approach to physics (OQL). This kind of approach, allows us to study the structure of the propositional lattices of any given system, in particular, it allows us to study the structure formed by elementary tests in QM, which as is well known since the work of Birkoff and von Neumann [10], they are isomorphic to the projection lattice $\mathcal{P}(\mathcal{H})$. At the same time, this structural characterization allows for a comparison between theories: although CM and QM are very different theories, the OQL approach allows to compare them in a same formal framework (that is, lattice theoretical) in order to look for analogies and differences. The most striking one is perhaps that the propositional lattice of QM is no-distributive.

The difficulties appear when we realize that the OQL approach has problems when applied to compound systems. It is true that, from a foundational point of view, it gives useful information about the structure of compound systems, but it works in a negative way: this is the content of the results of Aerts and others [1, 2, 4, 5]. The fact that no product of lattices exist, tell us a lot about the structure of compound quantum systems, but this is because of the incapability of the approach to describe them. This incapability comes from the fact that when we have an entangled state of the compound system, the reduced state will not be pure, and thus, there will be not possible to link the state of the compound system to the states of the subsystems at the level of lattices. This happens simply because the (possibly) mixed states of the subsystems cannot be represented as elements of the corresponding lattices [17, 28, 29]. And this is critically true for the case of indistinguishable particles: if we depart from a pure symmetrized state of the bipartite system of fermions, we will always obtain mixed states for the subsystems. This is essentially the reason why the OQL approach presents some disadvantages for the study of entanglement and, even more, for the case of identical particles.

The constructions presented in [17, 28, 29] overcome this difficulty, by incorporating mixed states as atoms of a new lattice. This allows to link states of the compound system to states of the subsystems, and show us the structure of the propositions thus formed. Convex subsets of the convex set of states may be interpreted as probability spaces [29]: our constructions allow us to look at the structure of these probability spaces. This was not possible using the traditional OQL approach.

In this work, we have shown that a quantum logical structure for compound quantum systems of identical particles can be realized, something which was not present in the literature excepting for scarce examples [25, 3]. Our structure captures the maps which link the states of a compound system formed of two identical particles to the states of its subsystems. Thus, providing a formal framework in which we can study how compound quantum systems of identical particles behave. In particular, they allow us to see the divergences with classical structures. We hope that these structures allow us to study the formal structure of compound quantum systems of identical particles in future work.

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