DEDUCTIVE PLURALISM [1]

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ABSTRACT

This paper proposes an approach to the philosophy of mathematics, deductive pluralism, that is designed to satisfy the criteria of inclusiveness of and consistency with mathematical practice. Deductive pluralism views mathematical statements as assertions that a result follows from logical and mathematical foundations and that there are a variety of incompatible foundations such as standard foundations, constructive foundations, or univalent foundations. The advantages of this philosophy include the elimination of ontological problems, epistemological clarity, and objectivity. Possible objections and relations with some other philosophies of mathematics are also considered.

1 INTRODUCTION

This paper proposes an approach to the philosophy of mathematics, deductive pluralism, that is designed to be inclusive of existing mathematics and consistent with mathematical practice. Here mathematical practice refers to mathematical statements, such as definitions, examples, and theorems. We will also show that deductive pluralism is consistent with many of the attitudes expressed by mathematicians towards the questions of the absolute or relative nature of concepts such as consistency, existence, or truth in mathematics – see section 1.1 for a discussion of terminology and concepts as used in this paper. Without inclusiveness a decision would need to be made about what to exclude, creating a partial philosophy of mathematics, and without any generally acceptable criteria for what is to be excluded. Without consistency with mathematical practice a philosophy of mathematics would be incompatible with mathematics, an unacceptable position for a purported philosophy of mathematics. The argument of this paper is that there are varieties of mathematics that have incompatible mathematical or logical foundations, sometimes implicit, and thus to satisfy the inclusiveness criterion a pluralist approach is required. By inclusiveness none of the varieties can be considered as true in an absolute sense (otherwise the others would be rejected) and so within a variety the statements need to be viewed as implications, requiring a deductivist approach. As we will see in the discussion of the attitudes of mathematicians and the reports by philosophers about these attitudes, modern mathematics has moved towards attitudes consistent with deductivism and pluralism. Thus a modern philosophy of mathematics should reflect these changes.

Several varieties of mathematics will be discussed in the next section, including: "standard mathematics" which has as foundations the intended interpretation of Zermelo-Fraenkel set theory with the axiom of choice (ZFC) and with

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First Order Predicate Calculus (FOPC) as the logic; constructive mathematics; univalent foundations; and inconsistent mathematics. The mathematical or logical foundations of a variety, sometimes called a framework for the variety, have been systematized to varying extents: some have been axiomatized for a century, but others are works in progress and with different approaches within a variety.

In deductive pluralism mathematical assertions state that a conclusion follows from assumptions, ultimately from the logical and mathematical foundations after a long development of definitions and intermediate results. Thus deductive pluralism is a form of deductivism but may differ from other forms by allowing both the logical and the mathematical foundations to vary, by not requiring the foundations to be purely formal uninterpreted axioms, or by allowing foundations other than set theory.

Section 1.1 below discusses some terminology and concepts used in this paper, and then section 2 discusses some varieties of mathematics that are distinguished by incompatible mathematical or logical foundations, highlighting these inconsistencies. The attitudes of leading mathematicians developing or using a variety are cited to show substantial compatibility with deductive pluralism. Section 3 considers deductive pluralism as a philosophy of mathematics by discussing its ontology and epistemology as well as its consistency with mathematical practice and attitudes. Section 4 considers some possible objections to deductive pluralism as a variety of deductivism. Section 5 then considers deductive pluralism as it relates to some other philosophies of mathematics. Since logical assumptions are part of a foundation for a variety there is an appendix on relevant logical concepts which may be referred to as needed. There is also an appendix giving some examples of the historical development of mathematics towards an axiomatic (thus deductive) viewpoint.

1.1 Terminology and Concepts as Used in This Paper

This section will discuss some terminology and concepts, with illustrations from standard mathematics. As used in this paper a *variety* of mathematics is not merely a theory: within a variety there may be many mathematical theories but these theories have the same foundation and thus these theories are not classified as varieties. Within a variety different theories are applicable to and illuminate other theories. For example in standard mathematics number theory is consistent with and uses results from analysis (analytic number theory) and algebra (algebraic number theory). A criterion for consideration as a variety is that the mathematics appears in professional publications such as journals or books. Since contemporary mathematics subsumes historical mathematics, clarifying and generalizing its implicit assumptions and foundations, this criterion embraces mathematics as it has been done throughout history.

In this paper a *fully formal proof* is one that can be checked step by step, in particular by a computerized proof checker. Such a proof is objective in that mathematicians favoring any variety of mathematics would agree that a fully formalized proof within another variety does establish that the conclusion follows from the logical and mathematical assumptions within that variety. We will consider a *rigorous proof* as one that can be fully formalized in a relatively straightforward manner, such as by filling in details. This concept of rigor is necessarily imprecise since it will vary between mathematicians, between areas of mathematics, and in different historical periods.

A useful distinction applicable to a variety of mathematics is between *syntax* and *semantics*: that is, between the axiomatic, uninterpreted formalism (the syntax) and the interpretations of the formalism (the semantics). The formalism is used in fully formal or rigorous theorems and proofs, providing some advantages including: explicit assumptions (axioms); clarification of relationships between systems of axioms; and applicability of results to all interpretations. The semantics will be in a system that is assumed to be better understood, more basic, or have other advantages over the formal system. An uninterpreted formal system usually needs a semantics in order to provide intuition, examples, or a basis for deciding such questions as existence, validity or satisfaction. In order to do this an interpretation requires a satisfaction predicate. An interpretation of a formal system is called a *model* for the system if the axioms of the system are satisfied. By Gödel's completeness theorem a first order system has a model if and only if it is consistent. Since some assumptions are necessary – nothing comes from nothing, *ex nihilo nihil fit* – to avoid infinite regress the search for semantics or interpretations must stop somewhere with a satisfaction predicate that is assumed to be consistent. This distinction is best developed in standard mathematics in which model theory studies formal uninterpreted axioms and their interpretations in set theory. Thus the foundation of standard mathematics must include both the formal axioms of ZFC and a set theory, such as the intended interpretation. Since ZFC and other first order theories containing Dedekind-Peano arithmetic cannot prove their own consistency by Gödel's incompleteness theorem, consistency of both formal ZFC and its intended interpretation are usually implicitly assumed. The assumption of consistency then allows new axiomatically defined structures to be shown to be consistent relative to that theory. For example the Dedekind-Peano axioms for the natural numbers were proven by Dedekind to be consistent and unique up to isomorphism within ZFC and its intended interpretation. (Sometimes a concept is unique up to a unique isomorphism, as in the universal diagram definitions within category theory.)

The next concept is that of *truth*. Since mathematics deals with abstracta, attributing truth to existential assertions can be problematical, so some relevant meanings of truth will be considered here. Standard mathematics has a concept of truth within model theory: a sentence in an axiomatic system is true if it is true in all models, and truth in a model is defined in terms of the interpretation of the axiomatic system within set theory. For example Gödel's sentence is true in the intended interpretation of Dedekind-Peano arithmetic but not true in all interpretations. When mathematicians state that a sentence is true they may be using (possibly implicitly) one of several concepts: the mathematical (model theoretic) concept, so that in a first order theory true is equivalent to provable; true in the intended interpretation of ZFC but not necessarily provable (as with Gödel's sentence); or some concept which is independent of models or proofs and thus will be referred to as an absolute concept of truth rather than the mathematical concept of truth which is relative to model-theoretic interpretations. Since the ideas of truth may vary when discussing mathematical concepts, it may be necessary to clarify which concept is meant.

Many mathematical statements assert the existence of a mathematical object, e.g., the empty set exists. The object may be asserted to exist relative to some explicit or implicit assumptions, e.g., given ZFC then there is an empty set, or absolutely. In this paper the use of the term "object" will reflect common mathematical usages and will not imply either absolute or relative existence.

The final concept is the distinction between *relative* or *absolute consistency*. The assertion that system A is consistent relative to system B means that if B is consistent then A is consistent (i.e., in the context of foundational logical and mathematical assumptions the consistency of B implies the consistency of A). We will say that a system is *absolutely consistent* if it is consistent as such, independent of the consistency of other systems. In the case of starting points for deductions where only one system is under consideration, such as the foundations for a variety of mathematics, the distinction is somewhat different. In these cases an absolute view would be, e.g., that ZFC is absolutely consistent while a contrasting position might be that it is reasonable to assume the consistency of ZFC.

The concepts of truth, consistency and existence are often closely related. A mathematical statement to which truth may be assigned often asserts the existence of a concept or the consistency of a theory. Also, existence is sometimes defined in terms of consistency: as we will see in section 3.1 Hilbert wrote that a mathematical concept exists if it is consistent.

2 VARIETIES OF MATHEMATICS

Most of mathematics as practiced, both pure and applied, is standard mathematics, which constitutes the great majority of what is taught in educational institutions, appears in publications, and is used in applications. Since standard mathematics is so dominate and extensive most other varieties of mathematics, including those discussed below, are careful to include many of the same or similar theories and theorems as standard mathematics.

2.1 Nonstandard Analysis

Nonstandard analysis is an extension of standard mathematics that provides for infinitesimals and was developed by Abraham Robinson to put them on a rigorous foundation. The logic is the same as in standard mathematics, and there are many approaches to developing the infinitesimals. Nonstandard analysis is a conservative extension of standard mathematics in that any proposition stated in the language of standard mathematics that can be proven using nonstandard analysis can also be proven using standard mathematics. An example of this is the nonstandard proof by Bernstein and Robinson [1966] that every polynomially compact operator has a non-trivial invariant subspace, which appeared back to back with a standard proof. In his article Bernstein wrote that "[t]he proof is within the framework of Nonstandard Analysis" [Bernstein and Robinson, 1966, p 421], which illustrates that when a variety of mathematics other than standard mathematics is used the foundations are made explicit, especially if the work is in a journal containing standard mathematics in which standard foundations would otherwise be implicitly assumed.

2.2 Tarski-Grothendieck Set Theory

Tarski-Grothendieck set theory (TG or ZFCU) is a nonconservative extension of ZFC using FOPC. A motivation is to provide a basis for category theory and in particular for Grothendieck's work in algebraic geometry. Many categories of interest, such as the category of all topological spaces, are proper classes. To allow for these TG set theory adds an axiom U to ZFC, giving ZFCU, stating that every set is an element of a Grothendieck universe, where a Grothendieck universe is a set defined so that it is closed under the usual set operations such as the power set. A Grothendieck universe is equivalent to an inaccessible cardinal, where an inaccessible cardinal is one that cannot be reached from below by the usual set operations. Since a Grothendieck universe acts as an internal model for ZFC the consistency of TG implies the consistency of ZFC and so by Gödel's second incompleteness theorem (which implies that ZFC cannot prove its own consistency) TG must be a nonconservative extension of ZFC. Thus a Grothendieck universe is an object that exists in ZFCU but not in ZFC.

In spite of the conceptual clarity provided by Grothendieck universes (and the prestige of Grothendieck) there is a reluctance to go beyond ZFC even within algebraic geometry. The Stacks Project [2014], an open source collaborative ongoing textbook on algebraic stacks and the required algebraic geometry, explicitly avoids the use of universes. This is an example of the reluctance of mathematicians to add axioms to ZFC, which is supported by the fact that extensions of ZFC generally increase the possibility of an inconsistency and is contrary to the admonition of Ockham's razor that entities should not be multiplied beyond necessity.

2.3 Constructive Mathematics

Constructive mathematics is an example of a variety of mathematics in which the mathematical assertions and logic have both rules and interpretations different from standard mathematics. The basic idea is that the existence of a mathematical object can only be asserted if there is a method of constructing the object. This requires that intuitionistic logic be used in which the Law of the Excluded Middle (LEM) fails: if P is an assertion then $P \vee \neg P$ can be asserted only when there is a constructive method of asserting P or a constructive method of asserting $\neg P$, which is not always possible. Similarly an assertion that P implies Q is interpreted as stating that there is a constructive way of transforming the construction for P into a construction for Q. The main version of constructivism was developed from the work of Bishop [1967], in which standard mathematics is a proper extension of constructive mathematics. Thus all theorems of constructive mathematics are also theorems of standard mathematics, but not conversely. An example of an object familiar to most mathematicians that exists in standard mathematics but not in constructive mathematics is the Dirichlet (or comb) function, which is defined on the unit interval so that it is 1 on the rational numbers and 0 on the irrational numbers in the interval. It cannot be defined constructively [Bridges and Palmgren, 2013], but in standard mathematics it is an important example of a function that is Lebesgue integrable but not Riemann integrable.

2.4 Univalent Foundations

The univalent foundations program, currently under active development, is an example of a variety of mathematics not based on set theory. It has as its basis an extension of the predicative, intuitionistic Martin-Löf type theory with additional axioms such as univalence. Just as standard set theory assumes the existence of the empty set and has axioms that assert the existence of new sets given existing sets (e.g., unions), univalent foundations assumes the needed types, such as the natural number type. The logic is intuitionistic and in this approach there are several primitive concepts including type, identity of types, function types, and ordered pairs. The motivating interpretation is homotopy theory in which types are considered as spaces and with constructions as homotopy invariants. The univalence axiom implies that isomorphic structures can be identified. Identifying structures up to isomorphism is common in standard mathematics, e.g., the von Neumann, Zermelo, and other interpretations of the natural numbers are isomorphic in standard set theory and thus can be considered identical as a type. However in standard mathematics isomorphic objects are not necessarily identified. For example the singleton sets $\{0\}$ and $\{1\}$ are isomorphic as sets (and by a unique isomorphism) but if they are identified then by extensionality the elements would be the same and so as a consequence $0 = 1$. Thus univalent foundations are incompatible with standard set theory. Univalent foundations does, however, define a class of types that behave in a similar manner to classical sets in many applications. Unlike other versions of constructivism the univalence approach does not deny the Law of Excluded Middle in principle, but uses variations on it as needed in theorems. Additional assumptions and particular care in the presentations of the theory are required due to the predicative nature of the type theory, as when presenting impredicative concepts such as the power set or the least upper bound. Another interesting feature is the use of the Coq proof assistant, which implements the logic. With regard to interpretations and consistency, the authors of the univalent foundations book [The Univalent Foundations Program Authors, 2013, p. 11] wrote:

> As with any foundational system, consistency is a relative question: consistent with respect to what? The short answer is that all of the

> constructions and axioms considered in this book have a model in the category of Kan complexes, due to Voevodsky ... . Thus, they are known to be consistent relative to ZFC (with as many inaccessible cardinals as we need nested univalent universes).

This quotation illustrates the common view, which also holds in deductive pluralism, that statements about consistency are relative rather than absolute.

Since univalent foundations uses category theory, among other theories, as a basis for interpretation and consistency, it is appropriate to now consider it as a possible foundation.

2.5 Category Theory

There have been proposals that some variety of category theory (CT) be a foundation for mathematics as an alternative to set theory. This approach is similar to univalent foundations in that the primary objective is usually a different foundation rather than a substantially different mathematics. It is also similar in that categorical foundations use topoi, which are a generalization of sets and whose logic is, in general, intuitionistic logic. Linnebo and Pettigrew [2011] survey some possibilities for using category theory as a foundation with some criteria, e.g., requiring independence from set theory and requiring some existential assertions (as ZFC asserts the existence of the empty set). Some theories are rejected: Synthetic Differential Geometry (SDG) as too narrow and the Category of Categories As Foundations (CCAF) as not independent of set theory. They then consider the Elementary Theory of the Category of Sets (ETCS) as a case study. ETCS is significantly different from set theory. In it everything is defined in terms of (category theoretic) arrows, including membership, which presents problems for set membership, e.g., an element cannot be a member of more than one set, extensionality does not hold for sets, and there are multiple (isomorphic) empty sets. In addition, although ETCS may be logically independent of set theory, it requires prior set theory for interpretations, for examples, and thus for comprehension.

2.6 Inconsistent Mathematics

Inconsistent mathematics is mathematics in which some contradictions are allowed [Mortensen, 1995]. If a contradiction implies all statements then the system is trivial, thus the logic used cannot be standard logic. The most common alternative is some kind of relevant logic. Most of the work in this area has been in the logical foundations and their immediate consequences, although suggestions have been made for other possible applications including inconsistent databases, inconsistent pictures (such as those by Escher), earlier mathematics (such as infinitesimals), alternative accounts of the differentiability of delta functions, or solutions of inconsistent sets of equations. Inconsistent set theory is one of the most widely studied topics within inconsistent mathematics. The objective is often to have a set theory based on two assumptions: unrestricted comprehension (for any predicate P, $\exists z \forall x(x \in z \leftrightarrow P(x))$) and extensionality ($y = z \leftrightarrow \forall x(x \in y \leftrightarrow x \in z)$). As is well known the former leads to Russell's paradox by setting $P(x) = (x \notin x)$, and so to avoid triviality, in which all predicates hold, a non-explosive logic must be used.

In this section we have briefly examined several varieties of mathematics. The list is not meant to be exhaustive: some varieties not discussed are various versions of finitism. However the above varieties should be enough for the following discussion. If a philosophy of mathematics is to be inclusive of mathematical practice then it must accommodate these varieties, which have different logical assumptions (e.g., FOPC, intuitionistic), different set theoretic foundations (e.g., ZFC, ZFCU) or foundations not using set theory (e.g., univalent foundations, category theory), and even different approaches towards consistency (e.g., inconsistent mathematics). As a consequence objects, such as the Dirichlet comb function, may exist in one variety of mathematics but not in another variety. The discussions of the above varieties show that no single logical or mathematical foundation is feasible and have also given illustrations of the attitudes of mathematicians concerned with foundations that are compatible with deductive pluralism.

3 DEDUCTIVE PLURALISM AS A PHILOSOPHY OF MATHEMATICS

As shown in the previous sections there are varieties of mathematics with incompatible logical or mathematical foundations. Deductive pluralism proposes that the simplest way to view mathematics with respect to the requirements of inclusiveness of and consistency with mathematical practice and attitudes is to allow for a plurality of varieties and with a form of deductivism within each variety. The pluralistic component of deductive pluralism automatically satisfies the criterion of inclusiveness. Within the context of a variety the definitions, theorems, proofs, and examples (which in this paper are referred to as mathematical practice) hold whether the foundations are considered as true in some absolute sense or as useful assumptions. In practice little or no reference is made to standard previous results, much less to the foundational assumptions, such as ZFC. However, when an alternative foundation is used then a reference is made, as in the example of Bernstein's article discussed in section 2.1. Thus the deductive component of deductive pluralism satisfies the criterion of compatibility with mathematical practice. This section will concentrate on showing that deductive pluralism is consistent with the attitudes of mathematicians towards their work and with applications. Not all mathematicians will have the same attitude and there is no survey of attitudes, so what we need to show is that a substantial proportion, possibly a majority, of their attitudes are consistent with deductive pluralism. But before doing this we will discuss ontological and epistemological considerations which are relevant to any philosophy of mathematics.

3.1 Ontology and Epistemology

One of the advantages of any version of deductivism is the elimination of ontological problems since no variety is considered as true in some absolute sense and the basic statements are assertions that the assumptions (ultimately the foundations) imply the conclusions. Thus there are no problematic questions about the existence of abstract objects. For example the assumptions of standard set theory immediately imply the existence, within that variety, of the empty set. This is similar to Carnap's view that the "reality" of abstract entities can only be considered within a linguistic framework. Mathematicians working within standard mathematics will implicitly assume standard set theory and thus will use the empty set and set theoretic constructions without mentioning the foundational assumptions.

Any attempt to go beyond deductivism requires confronting the problematic question of the existence of abstract objects. There are many views, such as that of Balaguer [1998, p. 22] who considered the question as essentially meaningless:

> Now I am going to motivate the metaphysical conclusion by arguing that the sentence – there exist abstract objects; that is there are objects that exist outside of space-time (or more precisely, that do not exist in space-time) – does not have any truth condition... .

One of the clearest approaches to abstract objects within mathematics is that of Hilbert who equated existence of such objects with consistency in his 1900 address introducing the Hilbert Problems when he stated:

> If contradictory attributes be assigned to a concept, I say that *mathematically the concept does not exist.* ... But if it can be proved that the attributes assigned to the concept can never lead to a contradiction by the application of a finite number of logical inferences, I say that the mathematical existence of the concept ... is thereby proved. [Hilbert, 1902, pp. 9–10]

From Gödel's results we know that most mathematical systems of interest cannot prove their own consistency thus this condition must generally be replaced by relative consistency. In addition, Hilbert's condition is explicitly violated in the case of inconsistent mathematics considered above in section 2.6. In order to include inconsistent mathematics the condition of consistency might be replaced by non-triviality.

In deductive pluralism mathematical statements take the form of assertions that the assumptions, ultimately the foundations, imply the conclusions. With this approach the assertions (i.e., implications) are also objectively true in that mathematicians favoring different varieties of mathematics can agree that given the assumptions and a correct deduction from these then the conclusion follows. Thus the question of epistemology for deductive pluralism centers on the reliability of these assertions. The assertions are usually supported by rigorous, but not fully formal, proofs. There can be considerable disagreement on when a published proof has sufficient detail, but, as discussed in section

1.1 above, a common idea is that it should be possible to expand such a published proof to obtain a fully formal proof, e.g., one which can be checked by a computer proof verification program. The proof verification system Mizar (www.mizar.org) uses Tarski-Grothendieck set theory as its basis and the results are in the *Journal of Formalized Mathematics*. As an example Gödel's completeness theorem has been verified using Mizar. The univalent foundations program uses Coq (coq.inria.fr) in a much more extensive way, using proof assistants "not only in the formalization of known proofs, but in the discovery of new ones. Indeed, many of the proofs described in this book were actually first done in a fully formalized form in a proof assistant..."[The Univalent Foundations Program Authors, 2013, p. 8]. According to Mackenzie [2001, p. 323] mechanization of proofs in the mathematical literature has supported the belief that these rigorous, semi-formal proofs are reliable:

> Research for this book has been unable to find a case in which the application of mechanized proof threw doubt upon an established mathematical theorem, and only one case in which it showed the need significantly to modify an accepted rigorous-argument proof. This is testimony to the robustness of "social processes" within mathematics.

Nothing is perfect and there are errors in published proofs which may lie undetected for many years, especially in those which are seldom examined. However mechanical checking, as with Coq or Mizar, substantially reduces the chance for error and provides a robust check on mathematics.

The questions of mathematical ontology and epistemology are related to how mathematics is viewed: is it discovery or creation. Deductive pluralism provides a clear perspective on this question. A mathematician works within the context of a variety of mathematics with foundational mathematical and logical assumptions, definitions, and previous results. Within this context necessary consequences are discovered. Sometimes a mathematician generalizes and abstracts out features of existing examples to create a new definition, such as the development of the abstract group concept in the nineteenth century. Or a mathematician may extend an existing variety to accommodate mathematical requirements, such as the extension of ZFC to ZFCU by Grothendieck, or develop a new variety such as constructive mathematics. These activities can be viewed as the creation of new theories or varieties of mathematics. Thus mathematics involves both discovery of new mathematical results (from existing mathematics) and creation of new concepts (by generalization, unification, and abstraction).

3.2 Consistency with Attitudes of Mathematicians

This section will consider the consistency of deductive pluralism with the attitudes of mathematicians towards foundations – do mathematicians regard some variety or its foundations as true in some absolute sense? If this were so, then there would be a conflict between deductive pluralism and the attitudes of mathematicians. Almost all the work in mathematics, past or present, is within standard mathematics and for those within this tradition there is no need to consider or mention the foundational assumptions – FOPC and standard set theory. If a mathematician uses another foundation then that is usually mentioned, as was illustrated in the above section 2.1 on nonstandard analysis. Also, the attitudes of contemporary mathematicians towards foundations tend to be consistent with deductive pluralism in that when foundations are considered they are not viewed as true or false in some absolute sense. Some examples will be given from leading mathematicians when they consider foundational questions. The univalent foundations group wrote that "we therefore believe that univalent foundations will eventually become a viable alternative to set theory as the 'implicit foundation' for the unformalized mathematics done by most mathematicians" [The Univalent Foundations Program Authors, 2013, p. 1], thus demonstrating both a pluralistic and deductive attitude. Mumford [2000, p. 208] has suggested that statistical random variables should be a primitive concept with stochastic set theory as a foundation for mathematics. In order to do this he made explicit some assumptions about standard mathematics when he wrote: "This calls for the most difficult part of this proposed reformulation of the foundations: we need to decide how to define stochastic set theory. Clearly we must drop either the axiom of choice or the power set axiom." If they can be dropped, then they cannot be regarded as true in some absolute sense.

Philosophers have commented on the attitudes of mathematicians towards foundations. Maddy [1989, p. 1223–4] generalized about the attitude of mathematicians when she wrote that "[w]hat you hear from the mathematician intent on avoiding philosophy often sounds more like this: 'All I'm doing is showing that this follows from that. Truth has nothing to do with it. Mathematics is just a study of what follows from what.'" Of course, from the point of view of deductive pluralism the characterization of mathematics as studying "what follows from what" is not an avoidance of philosophy but an assertion of philosophy, i.e, some form of deductivism. In a similar vein Clarke-Doane [2013, p. 470] wrote that "[m]athematicians are overwhelmingly concerned with questions of logic — questions of what follows from what" and Hellman and Bell [2006, p. 65] express a compatible view that "[t]o be sure, classical practice itself does not imply endorsement of Platonism, as many mainstream mathematicians, if pressed, fall back on some kind of formalism or fictionalism." These views are also supported by Hersh [1997, p. 39] who wrote: "Writers agree: The working mathematician is a Platonist on weekdays and a formalist on Sundays." This can be interpreted as stating that when doing mathematics (on weekdays) within the context and implicit assumptions of a variety a mathematician can assert existence, e.g., of the empty set, but when reflecting on mathematics or considering foundational questions (on Sundays) a more deductivist view is adopted.

The above examples of specific statements by mathematicians when considering foundational questions show that there is support for deductivism and pluralism. Also, if the above statements by philosophers and others discussing the views of mathematicians are correct, then attitudes consistent with deductivism are widespread. For some arguments supporting a form of absolute truth or consistency, see section 4.1 below.

3.3 Consistency with Applied Mathematics

We will now consider the consistency of deductive pluralism with applications of mathematics. Deductivism views mathematical statements as asserting that certain conclusions follow from the assumptions within a variety. There is something of an analogy in applications which use models of natural systems and derives conclusions from these models using mathematical theory. In more detail, a natural system, physical or social, is modelled by selecting some components that are relevant to the scientist. This model is often designed with regard to the available mathematical techniques and a correspondence is set up between mathematical elements and natural elements. Mathematical deduction then produces consequences that map back to the natural system, thus giving supporting or disconfirming evidence for the model when compared to data.

Usually the mathematical theory used is part of standard mathematics since it was axiomatized to be consistent with existing mathematical practices including applications. However the use of other varieties is possible, e.g., there has been some interest in using nonstandard analysis in applications such as by Albeverio et. al. [1985]. Sometimes within science the term "model" is explicitly used: e.g., the "standard model" in particle physics, the "Hodgkin-Huxley model" in biology, the "General Circulation Model" in climatology, and the "Gibbs model" in thermodynamics. The models are not viewed as true in some absolute sense, but as approximations; e.g., when a better model is found it replaces the previous model as when General Relativity replaced Newtonian gravitational theory. The consistency of deductivism with applied mathematics was supported by Resnik who wrote "it [deductivism] appears to account nicely for the applicability of mathematics, both potential and actual; for when one finds a physical structure satisfying the axioms of a mathematical theory, the application of that theory is immediate" [Resnik, 1980, p. 118].

One factor that allows immediate application of a theory is the fact that sometimes the mathematical theory and its applications are developed together by the same person or as part of a long tradition. Some examples in physics of interaction between mathematical theory and physical theory are the Newtonian gravitational model which was developed by Newton along with the calculus; Einstein's General Relativity of the early twentieth century which relied on Riemann's theory of differential manifolds from the mid-nineteenth century, but which also spurred research on semi-Riemannian manifolds; the interaction between the development of quantum mechanics and operator theory; and string theory which has had major interactions with new mathematics such as Calabi-Yau manifolds and mirror symmetry. As an example of the conjoined development of models and theory in biology and statistics, Ronald Fisher has been called a founder of modern statistics and the greatest biologist since Darwin by Dawkins [2011]: "Not only was he the most original and constructive of the architects of the neo-Darwinian synthesis, Fisher also was the father of modern statistics and experimental design." These examples of the joint development of mathematical theory and natural system models will be referred to below in section 4.2 when objections to deductivism based on applications are considered.

This section has shown that deductive pluralism is consistent with mathematical practice, applications and attitudes about mathematics. Mathematicians work within a variety of mathematics and thus their assertions, either formal or informal, implicitly assume the foundations of that variety. But when considering the foundations, especially in recent times, mathematicians do not view the foundations as true in some absolute sense. In applications a variety of mathematics is applied to a model of a natural system to deduce consequences and compare with data. Some criticisms of deductivism related to applications are discussed below in section 4.2.

4 POSSIBLE CRITICISMS

This section will consider some possible objections to deductive pluralism. Since deductive pluralism can be viewed as an extension of previous versions of deductivism (if-thenism) some objections to earlier versions of deductivism will be discussed as they may apply to the philosophy presented here.

4.1 Objections Based on Absolute Views

Some objections are based on the view that some foundation is true or false in an absolute sense rather than merely in the sense within mathematical model theory, and mathematics more broadly, in which a sentence is true if and only if it is true in all models. An example is Platonism, a strong version of which considers the entities and concepts as eternal, acausal, objectively true, and mind independent. There are also weaker versions of objections based on absolute truth or consistency. Resnik [1997, p. 142] wrote that "[deductivism] is an unsatisfactory doctrine. Mathematicians want to know that their systems have models; and they want to know this absolutely, and not just relative to a metaphysical theory." Wants cannot always be satisfied: "if wishes were horses, beggars would ride." However, contrary to Resnik's assertion, we have seen that mathematicians who consider foundational questions accept relative consistency, e.g., as quoted above in section 2.4: "[a]s with any foundational system, consistency is a relative question" [The Univalent Foundations Program Authors, 2013, p. 11]. More generally this view contradicts the previously discussed assertions by Maddy, Clarke-Doane, and individual mathematicians that mathematicians are concerned with "what follow from what." From a more technical point of view the absolute existence of a model would conflict with Gödel's result that having a model implies consistency, and (first order) systems of the power needed cannot prove their own consistency. Thus such views require some form of Platonism in which consistency is assumed absolutely rather than relatively or implicitly. This contradicts our requirement of inclusiveness since adherents of different varieties of mathematics want contradictory things: users of TG set theory want it to be consistent, while strict constructivists may not believe that ZFC, much less TG, is consistent. Even Resnik in the pages preceding this assertion in a discussion of mathematical practice wrote that "[t]he real issue concerns what is true if [the axioms] are true, and in the course of proving theorems one provides conclusive evidence for such conditional truths" [Resnik, 1997, p. 140].

As another example of an objection relying on an absolute concept of truth Hellman [1989, p. 26] wrote that a "decisive objection" to if-thenism is to suppose that an arithmetic sentence is implied by some assumptions but that the antecedent is false, e.g., that there is no natural number sequence. Then using FOPC, in which a false sentence implies all statements, the assumptions would imply all sentences. There is an implicit assumption that the assertion that there is a natural number sequence can be classified as true or false in some absolute sense. How can this be done? A natural number sequence is an abstract object, so we return to the vexed question of conditions for the existence of abstracta. For example, using Hilbert's criterion for non-existence, which is that the concept leads to a contradiction, the only way that it can be determined that there is no natural number sequence is to find a contradiction in the Dedekind-Peano axioms, which is possible but seems very unlikely. Mathematicians do sometimes look for such contradictions. For example in 2013 a well-known mathematician, Edward Nelson, posted a claim that he had found a contradiction within the Dedekind-Peano axioms, but an error in his reasoning was soon found and the claim was withdrawn. This example illustrates the fact that although consistency is generally (implicitly) assumed mathematicians sometimes look for contradictions within the standard foundations, and the failures of these explicit efforts give additional support to the assumption that the standard foundations are consistent. If such a contradiction were found then a likely result would be a modified set of axioms that avoids the contradiction and preserves (almost all) mathematics as occurred with the discovery of Russell's paradox.

Some philosophers argue against deductivism on the basis of absolute views about sets. For example in discussing the continuum hypothesis (CH), which states that any infinite subset of the reals must have the same cardinality as (be equinumerous with) either the reals or the natural numbers, Maddy [1989, p. 1124] wrote that "if we move to the idea of second order consequence, the Continuum Hypothesis becomes a real question in its own right, in the sense that it either follows or doesn't follow from second order ZF. But CH is just the sort of question If-thenism hopes to count as meaningless." A problem with this objection is that for second order ZF (which assumes proper classes) to determine CH requires an absolute concept of sets. Jané [2005, p. 797] wrote that "claiming that canonical second-order consequence is determinate requires taking a strong realist view of set theory." Such a strong realist view assumes existential conditions on abstracta that are hard to justify and that are unnecessary from the point of view of deductive pluralism. In practice there is little or no use of CH outside of logic. If it were needed then deductive pluralism could view ZFC+CH as a reasonable foundation for mathematics.

Also, if-thenism (or deductivism, or deductive pluralism) would not view CH as meaningless but as indeterminate using standard axiom systems.

Some mathematicians do have attitudes that assert the absolute existence of abstract objects, especially in set theory. An example is a possible extension of ZFC by the Axiom of Constructibility, which asserts that the universe of sets (V) is identical to all constructible sets (L), i.e., $V = L$. This axiom resolves some major questions in set theory, in particular the continuum hypothesis: ZFC+V=L implies CH. However ZFC+V=L is inconsistent with many of the large cardinal axioms (although it is consistent with Grothendieck Universes). Thus Hauser and Wooden [2014, p. 13] wrote: "In fact the assertion V = L itself is almost certainly false because among other things it rules out the existence of measurable cardinals." More generally, Hamkins [2014, p. 25] wrote that this is a common view: "Set theorists often argue against the axiom of constructibility V = L on the basis that it is restrictive." But he also wrote that this view is based on an absolute set concept. Such absolute attitudes are inconsistent with deductive pluralism since they would rule out those with other views, for example those who would accept ZFC+V=L.

The belief in the absolute existence of some mathematical object contradicts deductive pluralism since such a belief would require that contradictory assumptions be rejected, thus violating pluralism. Such a belief may provide motivation for research, but does not affect mathematical statements since these statements assert that an implication holds: an assumption implies a conclusion. If the mathematical argument is valid, then the implication holds whether or not the assumption is viewed as an absolute truth. For example, in the case of extending ZFC with the large cardinal axiom of measurable cardinals the rigorous proof that the existence of a measurable cardinal implies that $V \neq L$ holds whether or not one believes in the absolute truth of the existence of measurable cardinals.

4.2 Objections Based on Applications

Other objections view applications as determining the validity of foundations: the existence of applications of mathematics is sometimes used not only to justify mathematics but to allow attribution of absolute truth or falsity to mathematical statements. This view would contradict pluralism since varieties, or theories within varieties, not supported by applications would be viewed as false. As an example Resnik [1997, p. 99] wrote: "On my account, *ultimately* our evidence for mathematics and mathematical objects is their usefulness in science and practical life." Similarly Azzouni [1994, p. 84] wrote: "In particular, the truth or falsity of a particular branch of mathematics or logic turns rather directly on whether it is applied to the empirical sciences." First let us consider what portion of mathematics is relevant to applications to the empirical sciences. Physics is the area of science most often discussed in the philosophy of mathematics, but the mathematical physicist Roger Penrose [2005, p. 18] wrote that "[it] is certainly the case that the vast preponderance of the activities of pure mathematicians today has no obvious connection with physics." Thus if

Penrose is even approximately correct any philosophy of mathematics that requires applicability will be unable to satisfy the condition that a philosophy of mathematics be inclusive. Another problem is that this view has mathematical objects flickering in and out of existence. As an example of this applicability criterion for mathematical existence Riemann's differentiable manifolds, developed in the nineteenth century, flickered into existence in the twentieth century with Einstein's General Relativity, and entire branches of mathematics may flicker out of existence if theories such as loop quantum gravity or the speculations by Einstein and Feynman that space and time are discrete result in superior discrete models replacing continuous models in physics. Few people would reject a field of mathematics merely on ephemeral considerations of applicability.

Other objections also centered on applications criticize deductivism. For example Maddy [1989, p. 1124–1125] wrote that:

> [b]ut for all this, the argument that seems to have clinched the case against If-thenism for Russell and Putnam is a version of Frege's problem, a problem about applications. Reformulated for the If-thenist, it becomes: how can the fact that one mathematical sentence follows from another be correctly used to derive true physical conclusions from true physical premises?[2]

Consider a natural model, such as Newtonian gravitation. It is not a physical "truth": it is a model of physical reality, which is now an approximation to an improved model, General Relativity. Objections to deductivism that rely on applicability to natural systems seem to often assume, sometimes implicitly, that physical theories are absolutely true rather than approximate models: models of reality should not be conflated with reality. It also should be noted that mathematical deductions sometimes give results applicable to natural system models because they are designed to do so since, as the previous section on consistency with applied mathematics illustrated, in many cases the mathematical theory and applications to natural systems are developed together by an individual or by a research community.

In this subsection we have seen that objections based on applications do not hold. Some objections are based on the mistaken belief that models of natural systems are true in some absolute sense; other objections are based on an extreme view of mathematics as necessarily playing a subordinate role to ephemeral models of natural systems.

[2]It is not clear that Russell abandoned his original view. In the preface to the second edition of *Principles of Mathematics* Russell [1937, p. v] wrote: "The fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never since seen any reason to modify." This logicism is Russell's version of if-thenism: "PURE Mathematics is the class of all propositions of the form 'p implies q';" [Russell, 1937, p. 3]. What Russell did criticize in the second edition is strict formalism in which the symbols are uninterpreted. However deductive pluralism (and possibly if-thenism) does not require uninterpreted symbols.

4.3 Objections Based on Mathematical Practice and Attitudes

Objection to deductivism are sometimes based on mathematical practice. Maddy [1989, p. 1124] wrote that "we need to ask what mathematicians were doing before arithmetic was axiomatized. Was it not mathematics?" It *was* mathematics, which has been expanded, rationalized, and given additional interpretations throughout history. These changes have incorporated previous mathematics. For example the study of natural numbers assumes they are infinite (or potentially infinite) and is abstracted from experience with finite collections of discrete persistent objects. The Dedekind-Peano axiomatization of the natural numbers in the 1880s incorporated this experience and since then the elementary number theoretic results are consequences of these axioms. This is an example of the axiomatization of mathematics which has occurred over many decades and has made implicit assumptions explicit. Deductivism might be viewed as an incorporation of this development into philosophy: just as the properties of the natural numbers follow from the Dedekind-Peano axioms, so do the properties of a variety of mathematics follow from the foundational mathematical and logical axioms.

Resnik [1980, pp. 133–136] wrote that "deductivism is a powerful and appealing philosophy of mathematics", but he expressed concerns about "loose ends" related to mathematical practice. The first concern was that the deductivist "would need to explain why realism is acceptable in nuclear physics but not in mathematics." Some concepts of realism will be discussed later, but the basic answer to this objection is that physics develops models of space-time objects and processes while mathematics does not, although it may be applied to such models as previously discussed. This objection also suggests the error discussed above in section 4.2 in which models of reality are conflated with reality. Another of Resnik's concerns was that "deductivism may be unable to present a satisfactory epistemology for deductive reasoning itself." As has been noted, different varieties of mathematics have different views about the rules for deductive reasoning (e.g., the acceptance of LEM), so in deductive pluralism the logic is part of the foundational assumptions. Resnik also wrote that according to the deductivist the "sincere affirmations of the mathematician that a certain mathematical structure exists and that certain statements are true are elliptical" and that the mathematician denies that they are elliptical. However such statements are made in a context of implicit assumptions, such as standard set theory, definitions, results, and methods. In the given context the statements are true in that they follow from the implicit assumptions. In addition, Resnik's claim about the attitude of mathematicians is inconsistent with the statements cited in section 3.2 by mathematicians and by philosophers that mathematicians are concerned with "what follows from what."

A final objection along related lines is that deductivism is incomplete. Hellman [1989, p. 9] wrote that "a straightforward formalist or deductive approach is ruled out by the Gödel incompleteness theorems: no consistent formal system can generate all sentences standardly interpreted as truths 'about the intended type of structures(s).'" This objection has several problems: it primarily applies to deductivism when the foundations are fixed unlike in deductive pluralism where the foundations vary; the incompleteness theorems apply to most philosophies of mathematics and deductive pluralism's pluralistic component allows it to handle incompleteness as well as other philosophies; and a problematic absolute concept of truth seems to be used since what is considered as true will vary, e.g., in set theory is CH true? does a Grothendieck Universe exist? – questions which most mathematicians do not even consider since they do not impinge on their work and where there is no common view.

5 RELATED PHILOSOPHIES

This section considers the relationship between deductive pluralism and some other philosophies of mathematics. One problem of discussing these is that there are often multiple versions of each philosophy. Thus only some features of other philosophies most relevant to deductive pluralism are considered.

5.1 Fictionalism

Fictionalism is a variety of nominalism since it asserts the non-existence of abstracta. Balaguer [2013] wrote that the basic tenets of fictionalism are that (1) mathematical theorems and theories assert the existence of abstracta, (2) abstracta do not exist, (3) and thus mathematical theorems and theories are false. Deductive pluralism denies this syllogism since (1) is not accepted: mathematical theorems and theories are about "what implies what." As has been shown in section 3.2 this is consistent with the attitudes of mathematicians and philosophers (e.g., Mumford, Clarke-Doane, Maddy, and univalent foundations) and with the fact that mathematicians leave as implicit the foundations, especially when they use standard mathematics, but make them explicit when using an alternative variety (e.g., in Bernstein and Robinson's paper quoted in section 2.1). Balaguer also discussed another fictionalist slogan that asserts mathematical statements are "true in the story of mathematics." This use of the word "story" asserts an analogy to fiction, and adds unnecessary baggage to nominalism. Literary fictions deal with events in imaginary space-times, e.g., Sherlock Holmes in London, which is not the case for mathematical objects such as numbers. As Burgess [2004, p. 35] wrote in his conclusion to a discussion of fictionalism: "I think that in view of this radical difference between mathematics and novels, fables, or other literary genres, the slogan 'mathematics is a fiction' not very appropriate, and the comparison of mathematics to fiction not very apt." In any case, the slogan "true in the story of mathematics" can be given an interpretation consistent with deductive pluralism. To do this we consider a "story" to be a variety of mathematics and the assertion that "a statement is true in a story of mathematics" becomes "a statement is implied within a variety of mathematics."

5.2 Realism

Some philosophies of mathematics have a realistic view of mathematical concepts or entities. Platonism is a strong realism since the entities and concepts are viewed as eternal, acausal, objectively true, and mind independent. Such views usually contradict deductive pluralism since they reject incompatible varieties. However there are many versions of realism, including the one given by Putnam [1975, pp. 69–70] who wrote:

> I am indebted to Michael Dummett for the following very simple and elegant formulation of realism: A realist (with respect to a given theory or discourse) holds that (1) the sentences of that theory or discourse are true or false; and (2) that what makes them true or false is something *external* – that is to say, it is not (in general) our sense data, actual or potential, or the structure of our minds, or our language, etc.

In deductive pluralism the fully formalized statements are implications that are true or false, possibly automatically verified. Also, these statements depend only on the logical and mathematical syntax. Thus the statements of deductive pluralism may satisfy Putnam's the criteria for realism, depending on the interpretation of the second condition.

5.3 Other Forms of Pluralism

Various forms of pluralism have been advocated. Rudolf Carnap in *The Logical Syntax of Language* [Carnap, 1937, p. xv] wrote:

> Let any postulates and any rules of inference be chosen arbitrarily; then this choice, whatever it may be, will determine what meaning is to be assigned to the fundamental logical symbols. By this method, also, the conflict between the divergent points of view on the problem of the foundations of mathematics disappears ... . The standpoint which we have suggested – we will call it the *Principle of Tolerance* ... [thus] before us lies the boundless ocean of unlimited possibilities.

Koellner [2009, p. 98] considered Carnap's position as too radical and that "[t]he trouble with Carnap's entire approach (as I see it) is that the question of pluralism has been detached from actual developments in mathematics." Koellner then went on to consider pluralism with respect to additional axioms for ZFC with the general view that the choices are not arbitrary and that there is a question of truth other than model-theoretic truth. (His paper used the last lyrical phrase of the quotation from Carnap as an epigraph and coda.) Since both postulates and rules of inference are included in Carnap's position it can be viewed as a generalization of deductive pluralism. However since deductive pluralism is based on actual mathematical practice, it avoids Koellner's criticism.

Another form of pluralism was advocated by Pedeferri and Friend [2011]. Their proposal was a form of methodological pluralism, allowing "deviant" proofs "where mathematicians use steps which deviate from the rigorous set of rules methodologies and axioms agreed to in advance." Rigorous proofs were not required to be fully formal: there can be missing steps that in principle can be filled by relatively routine work in to produce a formal proof, which is consistent with the usage of this paper. They claimed that there are many deviant proofs and gave as the central case study the classification of finite simple groups. The basis for the claim that a portion of the classification was deviant was an interview with Serre [Raussen and Skau, 2004] in which, according to Pedeferri and Friend, Serre found that deviant methods were used to overcome an impasse. This does not correctly represent the issue, which was the classification of "quasi-thin" groups and which at one point relied on an unpublished manuscript. Those who considered that the classification was complete at that time viewed the quasi-thin case as having been satisfactorily dealt with by the manuscript. Serre considered it as a substantial gap. The question was not one of "deviant" methodology: all the classification was carried out with standard mathematics and methods. The question was whether the manuscript was sufficient. As it turned out Serre was correct and the quasi-thin case was completed at about the time of the Serre interview. Methodological pluralism was considered as part of a larger program of pluralism in Friend [2013]. In this work Friend advocated pluralism with respect to mathematics, including inconsistent mathematics. She did not consider foundations containing both mathematical and logical components. Instead she suggested the use of some paraconsistent logic when the varieties of mathematics are compared. No specific version of the many types of paraconsistent logic was advocated, and no example of its use was given. There is also the problem that any overarching logic used to compare and contrast the varieties of mathematics must include intuitionistic logic (as in constructive mathematics) or predicative mathematics (as in the univalent foundations approach) as well as other possible logics. When the mathematical and logical foundations are considered together, as in deductive pluralism, the attempt to use an overarching logic is unnecessary.

There are also advocates for pluralism of two varieties of mathematics or for pluralistic extensions of an existing variety. Davies [2005] discussed standard (called "classical" in the paper) and constructive mathematics, with an emphasis on the justification of constructive mathematics. The paper viewed each of these two varieties as valid within its own context. He wrote [Davies, 2005, p. 272] that "[o]ne should simply accept each mathematical theory on its merits, and judge it according to the non-triviality and interest of the results proved within it." This is pluralism with respect to two varieties and the phrase "proved within it" contains a suggestion of deductivism. Thus deductive pluralism is compatible with this view, extending it to general varieties of mathematics and grounding them in an explicitly deductivist format. An example of pluralism within a particular area is the approach to set theory developed by Hamkins [2014], which he calls the set-theoretic multiverse, in which there are many distinct concepts of set, each instantiated in a corresponding set-theoretic universe.

This section has considered some related work in the philosophy of mathematics and has shown that some approaches are consistent with pluralism or deductivism. Thus deductive pluralism as advocated in this paper provides a systematic approach that encompasses much of this other work.

6 CONCLUSION

This paper shows that deductive pluralism is inclusive of and consistent with mathematical practice and attitudes. It is inclusive of mathematical practice since it allows various logical and mathematical foundations, and is flexible enough to allow for future developments. Its consistency with mathematical practice and attitudes is shown in several ways: by the statements of mathematicians who base their work on something other than standard mathematics who explicitly state their foundations (such as nonstandard analysis); by the expressed view of mathematicians who consider altering the standard foundations (such as Mumford and those working in Univalent Foundations); and by the statements of philosophers of mathematics who report that mathematicians are concerned with "what follows from what."

Deductive pluralism also has significant philosophical advantages. Mathematical statements take the form of deductions, ultimately from the foundations. As a consequence the ontological problem of the existence of abstract objects is eliminated and the problem of epistemology is reduced to the validity of proofs. Also, given the validity of a proof, possibly verified by a proof assistant, then the statement is objectively true in that mathematicians supporting any variety of mathematics would agree that within another variety the conclusion follows from the assumptions.

7 APPENDIX: LOGIC

This appendix will present in more detail some logical assumptions that differ between the varieties of mathematics and will discuss some logical results used in the discussion of these varieties. There is a distinction between syntax (primarily form) and semantics (related to meaning or truth). Thus when a statement is considered as true, it is implicitly meant as true in some interpretation. As an introduction to interpretations of formal systems some examples of interpretations of logics in terms of sets will also be given.

7.1 Classical Sentential Logic

Most of mathematics uses classical sentential logic and its extension to First Order Predicate Calculus (FOPC). Propositions are combined using conjunction $\wedge$, disjunction $\vee$, negation $\neg$, and other connectives into new propositions. If a formula has a free variable, e.g., $P(x)$, the universal quantifier $\forall$ or existential quantifier $\exists$ can be used to bind the free variables, e.g., $\forall x P(x)$, producing a sentence, which by definition has no free variables. The main deductive rule is *modus ponens*: if P holds and if $P \to Q$ holds then Q holds. In classical logic implication is defined as "material implication": $P \to Q$ is equivalent to (or defined as) $\neg P$ holds or Q holds, i.e., $\neg P \vee Q$. In this logic a false sentence implies every sentence, since if P is false, $\neg P$ is true, $\neg P \vee Q$ holds, and so $P \to Q$ ("explosion" is when a false statement implies every statement). Non-classical logics often retain *modus ponens* but do not use material implication. A second element of classical sentential logic that varies is the Law of Excluded Middle (LEM): for any sentence P either P holds or $\neg P$ holds and so $P \vee \neg P$ always holds.

An interpretation of sentential logic can be given in which a sentence corresponds to a set in the Boolean algebra of all subsets of a fixed set U (the universe). In this interpretation $\vee$ corresponds to set union $\cup$, $\wedge$ corresponds to set intersection $\cap$, and negation $\neg$ corresponds to set complement. When discussing interpretations the same letter will used for a sentence and its interpretation to simplify notation if there is no danger of confusion.

7.2 Intuitionistic Logic

Intuitionistic logic is used in several varieties of mathematics, including constructive mathematics. This logic rejects LEM and consequently rejects the general form of proof by contradiction $\neg\neg P \to P$. However some particular proofs by contradiction still go through since by a theorem of Brouwer $\neg\neg\neg P \to \neg P$ holds in intuitionistic logic.

An interpretation of intuitionistic logic can be given in which a sentence corresponds to an open set in a fixed topological space U where $\vee$ and $\wedge$ are as in the Boolean set interpretation of classical sentential logic (since the union and intersection of two open sets are both open), but negation corresponds to the interior of the set complement $\text{int}(A^c)$ (since the complement of an open set is not generally open) and instead of material implication, where $A \to B$ is defined as $\neg A \vee B$, the intuitionistic interpretation takes the interior: $A \to B$ corresponds to $\text{int}(A^c \cup B)$. Since *false* corresponds to the empty set and *true* corresponds to its complement, U, LEM corresponds to $A \cup \text{int}(A^c) = U$, which need not hold for all A. Thus LEM fails as desired in this interpretation of intuitionistic logic.

7.3 Paraconsistent Logic

A paraconsistent logic is one that does not allow the derivation of all sentences in the case that some sentence and its negative have both been derived. In classical logic if both P and $\neg P$ are asserted, then any sentence Q can be asserted – from a contradiction everything follows –*ex contradictione quodlibet* (ECQ). Thus a paraconsistent logic must change classical logic to prevent this explosion and thus triviality (in which all statements can be derived). Various proposals have been made for paraconsistent logic; one of the most common is *relevant logic* in which the conclusion of a deduction must be relevant to the assumption. A way of doing this is to require both A and B to have a common term as a precondition for the assertion of $A \to B$. In ECQ the conclusion need not be relevant to the assumption, so relevant logic blocks explosion.

An interpretation of paraconsistent logic is closed set logic, a dual to the interpretation of intuitionistic logic. In this approach a sentence corresponds to a closed set in a fixed topological space. As with the interpretation of intuitionistic logic, $\vee$ corresponds to union and $\wedge$ corresponds to intersection. The interesting case is again negation. Since in general the complement of a closed set is not closed, negation corresponds to the closure of the complement $\overline{A^c}$. In parallel with the intuitionistic case $A \wedge \neg A$ corresponds to $A \cap \overline{A^c}$, which need not be empty (i.e., *false*).

7.4 Model Theory

A few results are used from FOPC (in which there is only one type of variable), model theory, and Gödel's theorems.

Let L_0 be a logic, in this case FOPC. A first order language L is an extension of L_0 obtained by adding relation, function, and constant symbols. (These can all be considered relation symbols, e.g., a constant symbol is a 0-ary relation symbol.) One of these relation symbols will be the binary equivalence relation of equality, if it is not considered to be part of the logic. A first order L-theory T is L together with a collection of sentences, which can be viewed as axioms, in the language L. (Sometimes the term "theory" is used for both the axioms and all sentences that can be deduced from them.) If S is a collection of sentences and a sentence ϕ can be deduced from S by a finite number of applications of the rules of deduction (such as *modus ponens*) then ϕ is a syntactic (or deductive) consequence of S, which is written symbolically as $S \vdash \phi$. A collection S of sentences is inconsistent if there is some sentence ϕ such that both ϕ and $\neg\phi$ can be deduced, i.e., $S \vdash \phi$ and $S \vdash \neg\phi$.

Standard model theory uses sets, often not in the context of a specific set theory. In this approach an interpretation of L is an L-structure: a set (or domain) over which the variables range together with assignments sending constant, relation and function symbols to constants, functions, and relations on the domain. Thus we have four elements: a logic, a language, a theory (all three formal and generally uninterpreted), and an interpretation of the language. The L-structure interpreting T is assumed to have a consistent way of determining if a relation is satisfied. The logic, language, and theory are together referred to as a (first order) deductive system. An L-structure M is said to be a model of an L-theory T, or M satisfies T, if all the sentences of T interpreted in M are satisfied in M. Symbolically this is written $M \models T$, read as M models T. A sentence ϕ in the language L is defined to be true or semantically valid (or model-theoretically valid) if it is satisfied in all interpretations, i.e., $M \models \phi$ for all interpretations M. Thus "true" in model theory (and more generally in mathematics) means true in all models. The models symbol is also used in the slightly different form $S \models \phi$ where S is a collection of sentences in L, ϕ is a sentence in L, and $S \models \phi$ means that every model of S is also a model of ϕ. When $S \models \phi$ holds we say that ϕ is a semantic consequence of S. Thus there are two versions of consequence: syntactic consequence $S \vdash \phi$ and semantic consequence $S \models \phi$.

The following results from logic and model theory are used:

- Gödel's completeness theorem for first order systems implies that the two notions of consequence agree: $S \models \phi$ if and only if $S \vdash \phi$.

- Gödel's completeness theorem and the Gödel-Mal'cev theorem imply that a first order theory is consistent if and only if it has a model. Thus an interpretation should not be referred to as a model unless consistency is proven (or assumed).

- Gödel's first incompleteness theorem and its extensions imply that in any consistent formal system containing arithmetic there are statements in the language of the system such that neither the statement nor its negative can be proven in that system.

- Gödel's second incompleteness theorem implies that any consistent first order system containing arithmetic cannot prove its own consistency. Thus most results are about relative consistency rather than consistency. Note that if a system is inconsistent then in FOPC any statement can be proven, including the statement that the system is consistent.

- The compactness theorem implies that if every finite subset of a first order system with countably many variables has a model, then the system as a whole has a model.

- The Löwenheim-Skolem theorem implies that a first order system has a model with a countably infinite domain if and only if it has a model with an uncountably infinite domain.

As an example of these concepts we will consider the first order Dedekind-Peano axiomatization of the natural numbers (with intended interpretation $\mathbb{N} = \{0, 1, ...\}$). The formal language L_N of the natural numbers is $(S, 0, =)$ where S is a function symbol (interpreted as successor), 0 is a constant symbol, and $=$ is the equivalence relation of equality. The theory PN of the natural numbers adds to the language L_N the Dedekind-Peano axioms:

i. $\forall x \neg(S(x) = 0)$
ii. $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$
iii. $(\phi(0) \text{ and } \forall x(\phi(x) \rightarrow \phi(S(x)))) \rightarrow \forall x \phi(x)$

Axiom (iii) is the axiom schema of induction where, for simplicity, ϕ is assumed to be any unary predicate formula. (In general n-ary predicate formulas are used.) The arithmetic operations can be defined using these three axioms to give the full set of axioms for the formal first order theory of Dedekind-Peano arithmetic, PA.

The formal theory PA has the intended interpretation $(\mathbb{N}, S, 0, =)$ (where for simplicity the relations in this interpretation are again given the same names as the formal relation symbols). By the Löwenheim-Skolem theorem if there is a countable model for a first order theory, then there are models of all infinite cardinalities. This is an example of the inability of first order theories to distinguish orders of infinity. By the second incompleteness theorem if PA is consistent it cannot prove its own consistency, and thus by the completeness theorem the intended interpretation $(\mathbb{N}, S, 0, =)$ cannot be proven to be a model of PA (without additional assumptions).

Assume that PA is consistent and so has a model M. Then a nonstandard model of PA can be constructed from it by adding a new natural number constant symbol c to L_N giving L'_N with symbols $(S, 0, =, c)$. (The constant c can be interpreted as an infinite number.) The theory T'_N is defined to have the same sentences as PA with the addition of the countable set of sentences $\neg(c = 0), \neg(c = S(0)), \neg(c = S(S(0))), \ldots$. Let F be a finite subtheory of T'_N. Then F has a model with c interpreted as a suitable element of the domain of M not corresponding to any element of F. So by the compactness theorem for first order logic there is a model for the infinite theory T'_N, and thus for PA. This model is a nonstandard model that is not isomorphic to M.

Since proofs in standard mathematics apply FOPC to the axioms of ZFC, a (fully formalized) proof holds in all interpretations. This can cause some seeming contradictions. For example the Löwenheim-Skolem theorem implies that a first order system such as ZFC has a model (i.e., is consistent) with a countably infinite domain if and only if it has a model with an uncountably infinite domain. So, assuming consistency, the real numbers can be defined and proven to be uncountable in any interpretation. This appears to be a contradiction to the Löwenheim-Skolem theorem, but it is resolved by recalling that a set is countable if and only if there is a one-to-one function from the natural numbers onto the set. Thus from the (internal) perspective of an interpretation there may not exist enough such one-to-one functions so that a set is uncountable, while from the (external) perspective of another interpretation such a one-to-one function exists. Thus every interpretation "thinks" that it is the intended interpretation. From a deductive perspective this does not matter since a deduction from the axioms of ZFC applies to all interpretations.

7.5 Second Order Logic

Some considerations concerning second order logic are needed in this paper. In second order logic there are two types of variables, first order variables ranging over the elements of the domain and second order variables ranging over sets of elements. The second order variables are sometimes considered as properties, but we will take an extensional approach in which a set corresponds to all elements having that property. The standard (or canonical) interpretation of second order logic is to use "all" subsets of a domain, although there is a problem in deciding what "all" means. The model-theoretic results listed above do not generally hold for second order logic: second order logic is not complete, since $S \models \phi$ may hold but not $S \vdash \phi$; the compactness theorem does not hold; and the Löwenheim-Skolem theorem does not hold.

Quine famously referred to second order logic as "set theory in sheep's clothing" [Quine, 1970, p. 66], and Shapiro wrote that "second-order logic, as understood through standard semantics, is intimately bound up with set theory", [Shapiro, 2012, p. 305]. Considering the problems of second order logic such as incompleteness, its close relation to set theory, its use of sets in its model-theoretic semantics, its relative lack of development compared with FOPC, and no clear mathematical advantages, mathematicians have generally stuck with the traditional approach of standard set theory with FOPC rather than use second order logic.

8 Appendix: Historical Examples

Mathematics has been practiced for thousands of years. Over this period mathematicians have abstracted, generalized, reinterpreted and axiomatized past work. This section gives two examples.

One of the oldest practices is natural number arithmetic. The use of the natural numbers grew over many centuries in many cultures, initially used for counting and then in some cultures for arithmetic. Often counting is done algorithmically, without any assumptions about the nature of the numbers. For example natural numbers may be learned as one-to-one correspondences with number names (or fingers!). This one-to-one approach is now the basis for equinumerosity in standard set theory. Definitions of the natural numbers have been given since early times. For example, Euclid [1908], Book VII, definition 1 states that "a unit is that by virtue of which each of the things that exist is called one" and definition 2 states that "[a] number is a multitude composed of units." The definition of unit is unclear or circular, and multitude is not defined. Of course, not all concepts can be defined if infinite regress is to be avoided. Euclid also uses implicit assumptions, and there have been various proposals on how to fill in the gaps. When it comes to proof Euclid interprets numbers as geometrical line segments. For example, proposition 1, in which a condition is given for two numbers to be prime to one another, begins "[f]or, the less of two unequal numbers AB, CD . . .", where these are line segments. Thus Euclid is an early example of the use of definitions, interpretations (as line segments), and implicit assumptions. Newton [1769, p. 2] defined numbers, including rationals and irrationals, by abstracting from ratios: "By number we understand not so much a multitude of unities, as the abstracted ratio of any quantity, to another quantity of the same kind, which we take for unity." By the end of the nineteenth century the widely used properties of the natural numbers were axiomatized by the Dedekind-Peano axioms, and by their extension to Peano Arithmetic, PA. The applicability of the natural numbers is thus to be expected since PA is based on the natural practice of cultures with discrete, stable, numerous (but finite) objects. The finiteness property is a notable difference between many applied uses of numbers and the axioms of PA which might lead to inconsistency: the inductive axiom produces an infinity, potential or actual, of natural numbers. As noted in the above discussion of standard mathematics, some mathematicians have believed that PA is inconsistent due to the inductive axiom.

As another example of the growth of mathematical concepts consider the group concept. As discussed in Kleiner [1986] the concept developed from a variety of sources: in the eighteenth century Euler studied modular arithmetic and Lagrange studied permutations of solutions to algebraic equations; in the nineteenth century Jordan defined isomorphisms of permutation groups and Cayley extended the study of groups beyond permutations to other examples, such as matrices. Although Cayley was ahead of his time in abstracting the concepts to sets of symbols, group elements were usually considered as transformations until the twentieth century. The first study of groups without assuming them to be finite, without making any assumptions as to the nature of their elements, and formulated as an independent branch of mathematics may have been the book "Abstract Group Theory" by O. Shmidt in 1916. Thus analogous to the axiomatization of the natural numbers the axiomatization of group theory occurred as the result of a long period of development.

In these and other examples history shows that basic mathematical concepts can arise over a long period of gradual development, abstraction, generalization, and eventual axiomatization. These concepts are not arbitrarily selected variations on existing concepts, and in many cases the development is intertwined with applications so that the rigorous definition is naturally applicable.

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