What is classical mereology?

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Abstract

Classical mereology is a formal theory of the part-whole relation, essentially involving a notion of mereological fusion, or sum. There are various different definitions of fusion in the literature, and various axiomatizations for classical mereology. Though the equivalence of the definitions of fusion is provable from axiom sets, the definitions are not logically equivalent, and, hence, are not interchangeable when laying down the axioms. We examine the relations between the main definitions of fusion and correct some technical errors in prominent discussions of the axiomatization of mereology. We show the equivalence of four different ways to axiomatize classical mereology, using three different notions of fusion. We also clarify the connection between classical mereology and complete Boolean algebra by giving two "neutral" axiom sets which can be supplemented by one or the other of two simple axioms to yield the full theories; one of these uses a notion of "strong complement" that helps explicate the connections between the theories.

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The idea of a *mereological fusion* or *mereological sum* has become a commonplace in philosophical literature. Those who use the notion casually may do so without giving an exact definition. Some very rough explanation like "the fusion of some things is what you get when you put them together" is enough for some purposes. Something more substantial and precise is often wanted, however, for there are appeals to such principles as

> If every proper part of x is part of y, and every proper part of y is part of x, then $x = y$

which are supposed to follow from some axioms in which the notion of fusion plays a central role.

When definitions of fusion are given, they are not always the same. In fact, there are many slightly different definitions, of which two are quite common in the literature. These two definitions are often run together, but they are logically distinct. It is true that once we have "the correct" axioms in place, or any equivalent set of axioms, then the two definitions can be shown to be equivalent (i.e., their equivalence logically follows from those axioms). But when actually giving an axiomatization intended to yield "the correct" theory, which we will call *classical mereology*, the difference matters. There is fairly universal agreement on what the *theorems* of classical mereology ought to be—not on whether they are *true*, but on what they are. (Roughly, they are the same as the theorems derived from the axioms for complete Boolean algebra, except without a zero element.) The difference between the definitions of fusion makes for a difference in how one can get those theorems. It turns out that Peter Simons' system **SC** in *Parts* does not suffice to get the desired theorems. Casati and Varzi's definition of system **GEM** in *Parts and Places* suffers from an unintended ambiguity; on one disambiguation, we do get the desired theorems, on the other, we do not. These mistakes, first addressed in the literature by Carsten Potow in [10], are fairly easily fixed, however, once they are noticed: we will see that, as in [10], one way is to replace a weak "supplementation" axiom with a stronger one; we also show that another way is to replace the "weaker" definition of fusion with the other one.[1]

We will also consider an alternative axiom set that does not directly use either common definition of fusion; rather it splits a fusion existence axiom into two parts and uses the notion of *minimal upper bound* in place of fusion, gaining, perhaps, in intuitive appeal what it loses in brevity. Using a related axiom set, we will give a very clear picture of the close connection

[1] I wish to express my gratitude to Pontow for very useful comments on an earlier draft of this paper.

between mereologies and complete Boolean algebras. The connection was known to Tarski (see [15] and [16]) and has been given a recent treatment in [11]. The treatment given here differs from others in that it crucially uses the concept of a "strong complement" in the axiomatizations, which sheds an alternative light on the roles of the "supplementation" axioms of mereology and the complement and "distribution" axioms of Boolean algebra. Along the way, we will correct an axiomatization found in the work of Fred Landman and in the work of Manfred Krifka that uses the notion of minimal upper bound.

We presuppose no substantial knowledge of mereology or Boolean algebra, and the technical arguments are intended to be accessible to non-specialists interested in a fairly self-contained, careful treatment. The paper is almost purely technical in nature; we do not address the question of whether classical mereology is a plausible theory.

Part One: Definitions of fusion

We begin with an explication of the devices needed for a formal language in which classical mereology might be expressed. Suppose we have a first-order (or higher) language[2] that includes a special 2-place predicate $\leq$, meant to represent "is part of" or "is a part of".[3] Thus $\forall x(\mathsf{Cat}(x) \rightarrow \exists y(\mathsf{Tail}(y) \wedge y \leq x))$ says that every cat has a tail as part. For any terms *s*

[2]Strictly speaking, when we get to axioms and theorems, we will be interested in not a *single* axiomatic theory of mereology, but rather any system that results from introducing a new relation symbol $\leq$ into a system by (augmenting its language and) adding certain axioms and axiom-schemes. For our model examples, below, we assume we are working in *pure unrestricted* mereology: *pure*, meaning $\leq$ is the only non-logical expression in the language; *unrestricted* meaning that the quantifiers of the mereology axioms are unrestricted. For most of our purposes, we may assume unrestrictedness (the uniform imposition of explicit restriction being a routine matter) and what other expressions there are in the language will not matter. For informal examples, we will often assume our language contains predicates like 'is a cat' and 'is a dog'. The availability of set-theory or higher-order logical devices in the language will be addressed below.

[3]Sharvy suggests in [13] (cf. [12]) that "is part of" and "is a part of" have rather different meanings, but classical mereology treats a single relation.

and t, pick a variable v not free in s or in t and stipulate:[4]

$s \circ t$ abbreviates $\exists v(v \leq s \wedge v \leq t)$

$s \wr t$ abbreviates $\neg s \circ t$

$s \ll t$ abbreviates $s \leq t \wedge \neg s = t$

'$101 \circ 102$' can be paraphrased as 'Rooms 101 and 102 have a common part' or 'Rooms 101 and 102 overlap.' '$101 \wr 102$' says that they do not overlap, or are disjoint, and '$102 \ll 101$' says that Room 102 is a proper part of room 101; it is a part, but is not the whole.[5] One could, instead, take $\circ$ or $\wr$ or $\ll$ as primitive, and define $\leq$ and the others in terms of the primitive, but this substantially affects the axiomatization, as we will see later in this paper. It seems most natural to take $\leq$ as primitive.

Schematic fusion-definitions

We now look at the two common definitions of fusion. According to the first, roughly put, a fusion of the F's is a thing x such that for every thing y, y overlaps x iff y overlaps one of the F's. We will first look at a way of formalizing this that uses an open sentence Fx in place of the notion of "the F's."

We will use the expression $\phi(\mathsf{y})$ to stand for any wff (well-formed formula) whose free variables may or may not include y, and so on for any variable. For any variable x, any wff $\phi(x)$, and any term t distinct from the variable x, find a variable y that does not occur free in $\phi(x)$ or in t, and stipulate that

SCHEMATIC TYPE-1 FUSION

$$Fu_1(t, [x \mid \phi(x)]) \text{ abbreviates } \forall y(y \circ t \leftrightarrow \exists x(\phi(x) \wedge y \circ x))$$

(read "t is a fusion of the first type, of the condition $\phi(x)$" or, perhaps, "t fuses the ϕ's"). For example,

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[4]We will use lower-case italic letters (s, t, x, etc.) as meta-language variables meant to stand for terms and variables of the object language; the object language will be in sans-serif font ($\mathsf{x} \leq \mathsf{y}$, etc.). We will be a little loose with use/mention.

[5]One might complain about the fact that in formal mereology, everything is treated as part of itself. The usual reply is that this is a mere formal convenience, eliminable in principle.

$$Fu_1(\mathsf{a}, [\mathsf{x} \mid \exists \mathsf{z}(\mathsf{Cat}(\mathsf{z}) \wedge \mathsf{Loves}(\mathsf{x}, \mathsf{z}))])$$

abbreviates

$$\forall \mathsf{y}(\mathsf{y} \circ \mathsf{a} \leftrightarrow \exists \mathsf{x}(\exists \mathsf{z}(\mathsf{Cat}(\mathsf{z}) \wedge \mathsf{Loves}(\mathsf{x}, \mathsf{z})) \wedge \mathsf{y} \circ \mathsf{x}))$$

and says that a fuses (in the first sense) the things that love a cat.

Note that we do not, with our notation, take for granted that there is at most one fusion of cat-lovers. Roughly put, "fusing" is a relation between a thing (the fusion) and a condition, or between a thing (the fusion) and some things (that get "fused"). But since we here assume only a first-order language, no such "relation" can be explicitly mentioned; its logical type would be beyond the type associated with first-order relation symbols. Note, for example, that though we know how to say a fuses the cats, it is not immediately evident how we might say that a fuses some cats: we want something like

$$\exists \psi\, (\exists \mathsf{x}\, \psi(\mathsf{x}) \wedge \forall \mathsf{x}(\psi(\mathsf{x}) \rightarrow \mathsf{Cat}(\mathsf{x})) \wedge \\ \forall \mathsf{y}(\mathsf{y} \circ \mathsf{a} \leftrightarrow \exists \mathsf{x}(\psi(\mathsf{x}) \wedge \mathsf{y} \circ \mathsf{x})))$$

but, of course, this is nonsense, unless 'ψ' here is being used as a second-order or plural variable; we will consider this possibility in more detail momentarily.

Further, the expression "*the* fusion of cat-lovers" has to be justified by showing that our axioms entail that if some things are fused by z and also by w, then $z = w$. Yet, since we are using schemes in a standard first-order setting, we have another kind of uniqueness for free. If every ϕ is a ψ, and vice-versa, then anything that fuses the ϕ's fuses the ψ's:

$$\forall \mathsf{x}(\phi(\mathsf{x}) \leftrightarrow \psi(\mathsf{x})) \rightarrow \\ \forall \mathsf{z}\, (Fu_1(\mathsf{z}, [\mathsf{x} \mid \phi(\mathsf{x})]) \leftrightarrow Fu_1(\mathsf{z}, [\mathsf{x} \mid \psi(\mathsf{x})]))$$

For the second notion of fusion: for any $\phi(x)$, t, x, as above (in the following we will often suppress qualifications like these), find y as above and stipulate

TYPE-2 FUSION $Fu_2(t, [x \mid \phi(x)])$ abbreviates
$$\forall x(\phi(x) \rightarrow x \leq t) \wedge \forall y(y \leq t \rightarrow \exists x(\phi(x) \wedge y \circ x))$$

("t is a fusion of the second type, of $\phi(x)$").

Roughly the second notion of fusion is the one used by Alfred Tarski in

[16] and David Lewis in [7]. The former notion is used by Simons [14], (see his SD9 on p. 37) and Casati and Varzi [2], p. 46. Casati and Varzi seem to assume that the difference does not matter in their reference to Tarski's system on p. 47.

Schematic vs. non-schematic

We had to say "roughly" in connection with Tarski and Lewis because their definitions are non-schematic. It is possible, and sometimes desirable[6], to use sets, second-order quantification, plural quantification, or some other auxilliary device in place of the schematic $[\mathsf{x} \mid \phi(\mathsf{x})]$ that we used, to give definitions of fusion to similar effect. E.g., if we were helping ourselves to set theory, then we would define Type-2 fusion like this:

SET-THEORETIC TYPE-2 FUSION $Fu_2(t, s)$ abbreviates

$$\forall x(x \in s \rightarrow x \leq t) \wedge \forall y(y \leq t \rightarrow \exists x(x \in s \wedge y \circ x))$$

(Tarski gives an obviously equivalent definition of what is called 'sum', in the translation, in [16].) Going the plural route, Lewis would replace '$x \in s$' with 'x is one of Xs'; one could also aim to get the intended effect using monadic second-order variables. In the case of sets, it is common and natural to take the quantifiers in the mereology axioms (formulated in a language that contains both $\leq$ and $\in$) to be restricted to a set (and thus to give a single axiom of fusion-existence instead of an axiom scheme). To see the expressive power of the use of auxilliaries, note that it is easy to say that a is a set-theoretic type-1 fusion of a set of cats:

$$\exists \psi\, (\mathsf{Set}(\psi) \wedge \exists \mathsf{x}\, \mathsf{x} \in \psi \wedge \forall \mathsf{x}(\mathsf{x} \in \psi \rightarrow \mathsf{Cat}(\mathsf{x})) \wedge \\ \forall \mathsf{y}(\mathsf{y} \circ \mathsf{a} \leftrightarrow \exists \mathsf{x}(\mathsf{x} \in \psi \wedge \mathsf{y} \circ \mathsf{x})))$$

with 'ψ' just another first-order variable.

Using auxilliaries, we get an "explicit" definition of the fusion relationship, as in something of the form "for all x and y, x fuses y just in case..." or of the form "for any x and any Ys, x fuses Ys just in case..." In the case

[6] And sometimes not desirable. E.g., the nominalist might wish to to avoid commitment to sets in defining fusions; also, one may wish to consider what happens when unrestricted fusion axiom-schemes are added to something else, like an already given first-order theory, e.g., a modal formal language, or set theory. Cf. Uzquiano's discussion of the difficulties of combining set theory and mereology, in [17].

of set theory, the fusion relationship acquires the logical type of a standard relation between objects: fuser and fused are both objects (things in the range of the first-order quantifiers). With plural logic, the logical type is of a relation between an object and some objects. As we noted, with a schematic definition of fusion, no such relation is even hinted at (except perhaps in our abbreviatory conventions), and no "explicit" definition is possible: there is nothing to put in the blank in "for all x and all __, x fuses __ just in case…"[7]

Thus we have a second kind of ambiguity in the notion of mereological fusion, among the purely schematic and alternative non-schematic versions. Fortunately, most of the issues we discuss arise in parallel for all of these alternatives, so, for our purposes, it usually does not matter which is chosen. Informally, we will ignore the differences among the schematic, set-theoretic, and plural versions, when the differences do not matter. Formally, we will finesse the issue by adopting the notation

$$Fu_2(t, \phi_{\mathsf{x}})$$

in place of the schematic $Fu_2(t, [\mathsf{x} \mid \phi(\mathsf{x})])$ or the set theoretic $Fu_2(t, \phi)$ (where ϕ is taken as a first-order variable whose range includes sets). Officially, ϕ_x is an abbreviation to be unabbreviated differently according to whether one wants to proceed schematically or by sets, or by plural variables, etc. Similarly for Fu_1. For example, $Fu_1(t, \phi_{\mathsf{x}})$ is *always* partially unabbreviated as

$$\forall y(y \circ t \leftrightarrow \exists \mathsf{x}(\phi_{\mathsf{x}} \wedge y \circ \mathsf{x})),$$

but the occurrence of 'ϕ_{x}' in this will be (partially) unabbreviated as '$\phi(\mathsf{x})$' on a schematic treatment, and (completely) unabbreviated as '$\mathsf{x} \in \phi$' on a set-theoretic treatment (with ϕ a first-order variable), and as 'x is one of the ϕs' on a plural variable treatment (with ϕs a plural variable), and so forth.

[7] It is worth noting that even if we use auxilliaries to define fusion, schemes will still be invoked when the auxilliary theory is axiomatized (as in the Separation scheme of set theory, or the Comprehension schemes of plural and second-order logic) and the resulting notions of fusion will thus logically link back to these schemes. Basically, utilitzing set theory, our schematic '$[\mathsf{x} \mid \phi(\mathsf{x})]$' will be linked to '$\{\mathsf{x} : \phi(\mathsf{x})\}$'; utilizing plural quantification, with 'Xs' a plural variable, it gets linked to 'Xs such that x is one of them if and only if $\phi(\mathsf{x})$'.

Minimal Upper Bounds

Now, it is easy enough to say 'z is a fusion of all lovers of cats', but if we are required to spell out (in English) the defined notion in terms of the part-whole relation, we are left with quite a mouthful; without a lot of training, it is far from easy to understand just what is being said.

There is a perhaps more intuitive notion that, in conjunction with the right axioms, is basically equivalent: the notion of a *minimal upper bound*. It is rather intuitive that if z is the fusion of all cats, then, whatever else it is, it has every cat as a part. That is to say, it is an "upper bound" on the cats:

$$\forall \mathsf{x}(\mathsf{Cat}(\mathsf{x}) \rightarrow \mathsf{x} \leq \mathsf{z})$$

But it is not just any upper bound. According to classical mereology, there is some object which is the fusion of all objects, call it *the universe*, and of which everything is a part. Thus, every cat is part of the universe, so the universe is an upper bound on the cats. But the fusion of cats should be something smaller than the universe; no dogs should be part of it, for example. What's special about z, the fusion of the cats, is that it is a *minimal* upper bound (mub), a part of any upper bound on the cats:

$$\forall \mathsf{w}((\mathsf{Cat}(\mathsf{x}) \rightarrow \mathsf{x} \leq \mathsf{w}) \rightarrow \mathsf{z} \leq \mathsf{w})$$

For a compact notation for mubs, stipulate

Min Upper Bound $Mub(t, \phi_x)$ abbreviates

$$\forall x(\phi_x \rightarrow x \leq t) \wedge \forall w(\forall x(\phi_x \rightarrow x \leq w) \rightarrow t \leq w)$$

We use the term *minimal* instead of *least* so as not to build uniqueness into our very definition. The axiom of Anti-symmetry (see below) is enough, however, to guarantee that any mub of ϕ_x is identical with every mub of ϕ_x, so with Anti-symmetry in place, *minimal* amounts to (uniquely) *least*. (The terms *supremum* and *join* are sometimes used for formally the same notion.) We will eventually see that in classical mereology, for any ϕ_x, *if, and only if,* there is an x with ϕ_x, there is exactly one type-1 fusion of ϕ_x, exactly one type-2 fusion, and exactly one minimal upper bound, and they are all the same thing. Hence, once the right axioms are in place, one could use the notion of least upper bound (supremum, join) in place of fusion.[8]

[8] Acknowledgment is due to Tony Martin for directing my attention to the notion of least upper bound in connection with the notion of fusion; see footnote 16.

The use of the notion of least upper bound in place of type-1 or type-2 fusion can be found in the formal linguistic literature in connection with the semantics of mass nouns and plurals, e.g., in Krifka [3], Landman [4] and [5], and Link [8]. Landman and Krifka intend to capture classical mereology with their axiomatizations, but they do not quite succeed, as we note below when we show how to use mubs in place of fusions to axiomatize classical mereology. Richard Sharvy uses the notion of least upper bound as his central fusion-like concept, but favors a notion of *quasi-mereology*, which is weaker than classical mereology; see p. 234 of [13] and footnote 8 of [12].

Part Two: Axiomatizations short of classical mereology

Adopting nomenclature from Casati and Varzi[9], let us have the system **M** (Ground Mereology) be the set of axioms

Reflexivity $\forall x\, x \leq x$
Anti-symmetry $\forall x \forall y((x \leq y \wedge y \leq x) \rightarrow x = y)$
Transitivity $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$

These say $\leq$ is (in mathematician's parlance) a partial ordering.

We will show shortly that the two notions of fusion are not equivalent, in the sense that we cannot derive, using first-order logic alone

$$\forall z(Fu_1(z, \phi_x) \leftrightarrow Fu_2(z, \phi_x))$$

without further assumptions. In the presence of Transitivity, however, the right-to-left direction can be derived. The second type of fusion thus may be said to be the *stronger* notion of fusion.

Fusion existence axioms

Consider now the system **GM1** that results from adding to **M** instances of a scheme (or, if one is using auxiliaries, a single axiom) asserting the existence of type-1 fusions. For any wff ϕ_x, if the variable z is not free in ϕ_x then

9 [2] and [18]

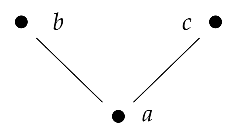
Fusion1E $\exists \mathsf{x}\, \phi_{\mathsf{x}} \rightarrow \exists \mathsf{z}\, Fu_1(\mathsf{z}, \phi_{\mathsf{x}})$

is an axiom of **GM1**. (If desired, take universal closures instead of allowing free variables in the axioms.)

Similarly, **GM2** is the system that results from adding an existence axiom or axiom scheme for type-2 fusions to **M**:

Fusion2E $\exists \mathsf{x}\, \phi_{\mathsf{x}} \rightarrow \exists \mathsf{z}\, Fu_2(\mathsf{z}, \phi_{\mathsf{x}})$

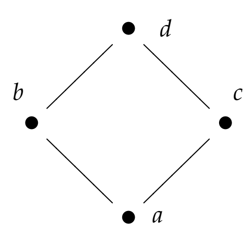
GM1 is a very weak system, in the sense that it imposes very little structure on the part-whole relation; much less than in classical mereology. For example, the following is a model of **GM1**:



M. 1

In this and our other diagrams, the lines connecting dots are to be thought of as holdings of a relation R from the lower dot to the higher dot, and the interpretation of the $\leq$ symbol in the model is to be the reflexive and transitive closure of R.

In M. 1, everything overlaps everything, so for any things, any thing is a type-1 fusion of those things. This example also allows us to see the logical independence of the two notions of fusion, for it is not a model of **GM2**. There is no type-2 fusion of $\{b, c\}$[10] in this model, for there is nothing of which both b and c are parts. (The fact that each of a, b, and c is a type-1 fusion of $\{b, c\}$ underscores the relative weakness of the notion of type-1 fusion.) But **GM2** is also quite weak, for the following is a model of **GM2**:



M. 2

[10] or, if one is being schematic, of the condition $\mathsf{x} = \mathsf{b} \vee \mathsf{x} = \mathsf{c}$ with respect to x when b is the value of the term b and c is the value of the term c. We will suppress such subtleties below.

Again, everything overlaps everything, and everything is part of d, so, for any things, d is a type-2 fusion of them.

Weak Supplementation

To get to full classical mereology, one course is to add a "supplementation" axiom that forbids such a situation as that in M. 1, and M. 2, where a is a proper part of b, and yet b has no other proper parts. The most obvious one says that a thing with a proper part x has some other, supplementing, part that is disjoint from x:

WeakSup $\forall x \forall y (x \ll y \rightarrow \exists z (z \leq y \wedge x \wr z))$

It is called "weak" because there is an alternative that is found in the literature that is "stronger"; we will return to it later.

Let **MM** be the system that results from adding WeakSup to **M**, and let **WGM1** be **GM1** plus WeakSup.[11] Anticipating, we will call the system that results from adding WeakSup to **GM2**, **CLM** (Classical Mereology).[12]

Neither M. 1 nor M. 2 are models of **MM**, since, in both of them, $a \ll b$, but there is no part of b that a does not overlap. It is worth briefly noting that **MM** by itself is quite weak, since the following is a model of **MM**:

M. 3

In M. 3, nothing is a proper part of anything else, so WeakSup is trivially satisfied. Neither fusion axiom is satisfied, however, since there is no fusion for $\{a, b\}$.

We now show that **WGM1**, the result of adding fusion-1 existence and WeakSup to the partial ordering axioms, yields a surprisingly weak system, and does not in fact yield classical mereology. This fact has been addressed in print by Carsten Pontow, in [10]. Pontow's discussion considers

[11] We call it '**WGM1**' and do not use Casati and Varzi's term '**GEM**', since, given the situation, that term is not well defined by their introduction of it on p. 46 of [2].

[12] We call it '**CLM**' instead of '**CM**' to avoid collision with Casati and Varzi's use of '**CM**' for what they call 'Closure Mereology'.

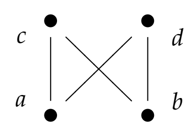
only the type-1 definition of fusion, however, and so he (reasonably, given only the one definition) concludes that "only the Strong Supplementation Principle is sufficient to fit the theories with a strong kind of extensionality," and suggests Strong Supplementation in place of WeakSup as the way to mend the errors in the literature. We explore some different paths here, on which WeakSup remains central. We tend to "place the blame" more on the weakness of the type-1 definition of fusion than on WeakSup. Perhaps it is worth noting that some, including Peter Simons, find WeakSup much more plausible as a *basic* truth about the part-whole relation than StrongSup; see, e.g., p. 116 of [14]. Consider first the following two propositions, desired as theorems of classical mereology,

Product $\forall x \forall y (x \circ y \rightarrow \exists z \forall w (w \leq z \leftrightarrow (w \leq x \wedge w \leq y)))$

and

BLUB $\forall x \forall y \exists z \forall w (z \leq w \leftrightarrow (x \leq w \wedge y \leq w))$

There is a model of **WGM1**, and hence **GM1**, in which both Product and BLUB (binary least upper bound) fail:

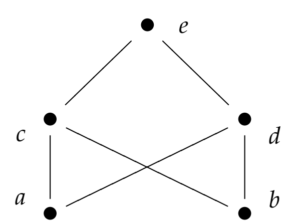


M. 4

Here, c and d each overlap everything, but they have no product, since, the things that are parts of both of them are $\{a, b\}$, and yet every thing of which both a and b are parts has a part that fails to be a part of both c and d. To see that Fusion1E is true, we need to show that for each non-empty subset of the domain (that is definable with a ϕ_x; and since the domain is finite, all of its subsets are definable) has at least one type-1 fusion. Observe that for each singleton, its member is a fusion of it (in fact, one can see that this is true in all models, since $\forall z(z, Fu_1(z, [x \mid x = z]))$ is true by first-order logic alone). For any other non-empty set of things in the domain of the model, if it includes either c or d, then everything in the domain overlaps a member of the set, and so c fuses the set (and so does d). The only set left over is $\{a, b\}$, but everything overlaps either a or b, so, again c and d each fuse this set. One can confirm WeakSup by noting that the only things that are proper parts of anything are a and b, and everything, that either is a proper part of, has both as parts; since they do not overlap, WeakSup is satisfied.

Simons claims that his system **SC**, which is equivalent with **WGM1**, yields both Product and BLUB, and, indeed, a host of other things that do seem to be theorems of CLM.[13] Casati and Varzi claim that **GM1** is enough to get Product.[14] This is the source of their claim that **WGM1** is equivalent with the system **SGM1** that results from adding the axiom StrongSup (discussed below) to **GM1**. These three claims are incorrect. In the next part, we will show that **SGM1** is indeed equivalent with **CLM** (= **GM2** + WeakSup). Thus, classical mereology is indeed obtained by adding, to the partial ordering axioms, a fusion-existence axiom (scheme) and a supplementation principle: if we use type-1 fusion, we need StrongSup, but if we use type-2 fusion, we need only WeakSup.

One might wonder whether Product is a theorem of **GM2** (partial ordering plus fusion-2 existence); we have not shown otherwise, since M. 4 is not a model of **GM2**, since $\{c, d\}$ has no type-2 fusion, and Product holds in M. 2. But it can be "extended" to M. 5, a model of **GM2** in which Product fails (for $\{c, d\}$):



M. 5

Part Three: Classical mereology

Now, M.5 is not a model of **CLM**, since, in it, c is a proper part of e and yet both of them overlap everything, so WeakSup fails. If we add a proper part to e that does not overlap c, so as to try to satisfy WeakSup while leaving the failure of Product in place, we will find that we need more fusions, involving the new thing and the old things, which WeakSup will then constrain; a lot of structure is imposed. Product cannot be made to

[13] See pp. 37–40 of [14].

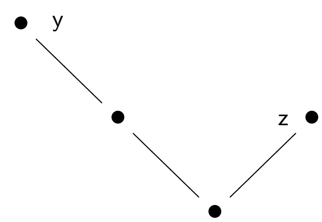
[14] [2] p. 46. Cf. [18] section 4.2.

fail. We will show that in **CLM**, Fusion2E and WeakSup work together to yield the uniqueness of type-2 fusions. This is a powerful theorem, and a linchpin for other strong theorems; Product, for example, is a fairly easy consequence. This uniqueness theorem is a corollary of our main theorem, which says that type-2 fusions are minimal upper bounds.

Before proceeding to the main theorem, it helps to note first a minor lemma that is often wanted in reasoning about fusions:

Overlap Lemma $\forall z \forall y(\exists x(x \leq y \wedge x \circ z) \rightarrow y \circ z)$

This lemma is easy to prove using Transitivity; the following diagram gives one a feel for it:



Now suppose that we have $\exists x\ \phi_x$. By Fusion2E, we will have a fusion z of the ϕ's. We can prove that this object is a minimal upper bound: it is part of anything that all the ϕ's are part of.

Formally, given **CLM**, for any variable x and any ϕ_x, and any variables y and z that do not occur free in ϕ_x, we can derive[15] (the universal closure of)

Fu2MUB $\forall z(Fu_2(z, \phi_x) \rightarrow Mub(z, \phi_x))$

Sketch of derivation: Pick an arbitrary z with $Fu_2(z, \phi_x)$. After unabbreviation, it should be clear that the main task is to show that, given an arbitrary upper bound y, i.e., a y with $\forall x(\phi_x \rightarrow x \leq y)$, we have $z \leq y$.

Use Fusion2E to obtain v with $Fu_2(v, [x \mid x = y \vee x = z])$ (or $Fu_2(v, \{y, z\})$ etc.). Get that $z \leq v$. If $v = y$ we will have the desired formula $z \leq y$. So suppose for *reductio* that $y \neq v$. Since $y \leq v$, $y \ll v$; apply WeakSup to get s

[15] Strictly: if we are using the schematic Fusion2E axiom, then we can derive this, within any standard deduction system that includes first-order logic. If we are using a non-schematic formulation, then we must take advantage of certain basic assumptions about the replacements of the schemes (the sets or pluralities or what have you) e.g., that there is a set $\{y, z\}$. Similar remarks go for all of our derivations below.

with $\mathsf{s} \le \mathsf{v}$ and $\mathsf{y} \wr \mathsf{s}$. Since $\mathsf{s} \le \mathsf{v}$, by the definition of type-2 fusion we can get that $\mathsf{s} \circ \mathsf{y} \vee \mathsf{s} \circ \mathsf{z}$; the former disjunct is ruled out, so $\mathsf{s} \circ \mathsf{z}$. Get w with $\mathsf{w} \le \mathsf{s}$ and $\mathsf{w} \le \mathsf{z}$.

Since $\mathsf{w} \le \mathsf{z}$, unabbreviating and applying the fusion clause on z in our assumptions, get $\exists x(\phi_x \wedge \mathsf{w} \circ x)$, and instantiate to a so we have $\phi_{\mathsf{a}} \wedge \mathsf{w} \circ \mathsf{a}$. Since ϕ_{a}, given our assumption that y is an upper bound on the ϕs, $\mathsf{a} \le \mathsf{y}$. But $\mathsf{w} \circ \mathsf{a}$; from these last two we can get that $\mathsf{w} \circ \mathsf{y}$, applying the Overlap Lemma. But $\mathsf{w} \le \mathsf{s}$; this leads to a contradiction by another application of the lemma, since we had $\mathsf{y} \wr \mathsf{s}$ above. So, by *reductio*, $\mathsf{y} = \mathsf{v}$; thus $\mathsf{z} \le \mathsf{y}$; universally generalize and we are done. ■

In **CLM**, type-2 fusions are minimal upper bounds. It is basically built into the definition of type-2 fusion that a type-2 fusion is *an* upper bound; it is not trivial that they are minimal, and that is what WeakSup is for.[16]

Since we have Anti-symmetry, there is at most one minimal upper bound for any ϕ_x. So, as a corollary to Fu2MUB, we get a crucial theorem-scheme:

Fu2Uniqueness $\forall z \forall y((Fu_2(z, \phi_x) \wedge Fu_2(y, \phi_x)) \rightarrow z = y)$

This justifies our speaking of *the* mereological fusion of ϕ_x. Now, in **CLM** one can derive that if $\exists \mathsf{x} \exists \mathsf{y}\ \mathsf{x} \neq \mathsf{y}$ then there is no mub for ϕ_x unless $\exists x\ \phi_x$ (see the discussion below of the connection to Boolean algebras). Hence we can derive a slightly qualified equivalence between type-2 fusion and mubs: for any ϕ_x,

$$(\exists \mathsf{x} \exists \mathsf{y}\ \mathsf{x} \neq \mathsf{y} \vee \exists x\ \phi_x) \rightarrow \forall z(Fu_2(z, \phi_x) \leftrightarrow Mub(z, \phi_x))$$

Using mubs instead of fusions

The theory of **CLM** could have been axiomatized using the notion of minimal upper bound instead of fusion, given a couple of minor adjustments. First, notice that we can re-locate the "difference" between the definitions of minimal upper bound and of type-2 fusion, extracting a somewhat intuitive axiom to the effect that if something is part of a minimal upper bound on the ϕ's, then it overlaps some ϕ.

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[16] Thanks to Tony Martin for helpfully suggesting that Fu2MUB be brought to the center of the discussion of **CLM**. This suggestion re-oriented and simplified an earlier presentation of mine of the route to Fu2Uniqueness in **CLM**.

Filtration $\forall y \forall z((y \leq z \wedge Mub(z, \phi_x)) \rightarrow \exists x(\phi_x \wedge y \circ x))$

Now suppose that we modify **CLM** by taking Filtration as an axiom, and replacing Fusion2E with

MubE $\exists x\, \phi_x \rightarrow \exists z\, Mub(z, \phi_x)$.

A little unabbreviation shows that Fusion2E is then easily derivable.

In fact, it is easy to see that we can then drop Reflexivity and Anti-symmetry, as derivable. For the former, given y, get a z with $Mub(z, [x \mid x = y])$ (or $Mub(z, \{y\})$); $y \leq z$, so if $y \neq z$, $y \ll z$; apply WeakSup and then Filtration to get a contradiction. The latter can be had by a fairly simple *reductio* and applications of WeakSup, Reflexivity and Transitivity. These last two arguments are also possible in **CLM**, of course, but they are perhaps a little easier to understand when Fusion2E is split into Filtration and MubE.

The set-theoretic version of the modified axiom set (partial ordering, or just Transitivity, plus WeakSup, Filtration, and MubE) is close to Landman's definition of a *part-of structure*.[17] Almost the same definition is used by Krifka for his notion of *lattice sort*.[18] These definitions are clearly intended to generate classical mereology, since the authors (incorrectly) claim that the defined structures are, in general, complete Boolean algebras with the zero element removed.

Basically, a part-of structure is defined as a structure that satisfies the partial-ordering axioms, set-theoretic MubE (the existence of a mub (*join*) for each non-empty set), WeakSup, and something called *Distributivity* which governs mubs of two-element sets (binary joins). Where $x + y$ denotes the z with $Mub(z, \{x, y\})$, the axiom is

$$x \leq y + z \rightarrow (x \leq y \vee x \leq z \vee \exists y' \leq y\, \exists z' \leq z(x = y' + z')).$$

This does not by itself yield Filtration (though it does if the domain is finite) and we do not get classical mereology.

To see this, let A be any infinite set and let B be

$$\{S | S \subseteq B \text{ and } S \text{ is non-empty and finite }\} \cup \{A\}.$$

[17]See p. 315 of [4] (cf. the beginning of Lecture Four of [5] and the use of Landman's notion by Link in Chapter 8 of [9]).

[18]In [3].

Then $\langle B, \subseteq \rangle$ is a part-of structure, but not a model of classical mereology.[19]

If *Distributivity* is expanded to an infinitary analog, then it is equivalent (in the presence of the other axioms) with the set-theoretic version of Filtration. Letting $\bigvee \phi$ denote the z such that $Mub(z, \phi)$, the axiom would be:

$$z \leq \bigvee \phi \rightarrow \exists \psi (\forall y (\psi_y \rightarrow \exists x (\phi_x \wedge y \leq x)) \wedge z = \bigvee \psi).$$

This proposition is not really statable with schemes (because of the $\exists \psi$) though one can use Filtration to show that

$$[y \mid \exists x (\phi_x \wedge x \circ z \wedge Mub(y, [w \mid w \leq x \wedge w \leq z]))]$$

"witnesses" the requirement.[20] For simplicity and neutrality (with respect to the scheme-versus-set issue) Filtration seems to be the superior axiom.

Strong Supplementation

We have seen that we get the desired uniqueness of fusions with the type-2 notion in **CLM**, using WeakSup. Consider now a stronger supplementation proposition:

StrongSup $\forall z \forall y (\forall x (x \leq y \rightarrow x \circ z) \rightarrow y \leq z)$

We can show that StrongSup is derivable in **CLM**. We could show that **CLM** yields Product, and take advantage of Simons' derivation of StrongSup from Product and WeakSup[21], but we can avoid a direct appeal to WeakSup and proceed by taking advantage of Fu2Uniqueness, and the following easy lemma, to the effect that, put set-theoretically, x fuses $\{x\}$;

Lemma 2 $\forall x \, Fu_2(x, \{x\})$

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[19] The latter can be verified by noting that Filtration will fail. One can express "x is a singleton in B" with $\forall y (y \leq x \rightarrow y = x)$; now if x is a singleton, consider the lub of all singletons not identical with x. It would have to be A; but $x \leq A$ and yet x does not overlap any of these things.

[20] The easiest proof of this makes use of StrongSup, to be introduced immediately.

[21] See pp. 30–31 of [14].

Sketch of derivation of StrongSup in **CLM**: take arbitrary a and b and suppose that $\forall \mathsf{x}(\mathsf{x} \leq \mathsf{a} \rightarrow \mathsf{x} \circ \mathsf{b})$; we want $\mathsf{a} \leq \mathsf{b}$. By Fusion2E, $\exists \mathsf{z}\ Fu_2(\mathsf{z}, \{\mathsf{a}, \mathsf{b}\})$; call it z. We will then show that $Fu_2(\mathsf{z}, \{\mathsf{b}\})$; once we have that, given Lemma 2 and Fu2Uniqueness, we then get that $\mathsf{z} = \mathsf{b}$, and since $\mathsf{a} \leq \mathsf{z}$ (easily), $\mathsf{a} \leq \mathsf{b}$ and we are done. To show what we need, we need only show that $\mathsf{b} \leq \mathsf{z}$ and $\forall \mathsf{w}(\mathsf{w} \leq \mathsf{z} \rightarrow \mathsf{w} \circ \mathsf{b})$; these are both fairly easily obtained.

Thus we have derived StrongSup from Lemma 2, the partial ordering axioms (**M**), and Fu2E and Fu2Uniqueness. The payoff is that, since WeakSup easily follows from StrongSup with an appeal to Anti-symmetry, we have done half the work needed to show that we may axiomatize **CLM** with Tarski's surprisingly small system from [16] (or a schematic or other variant), that results basically from taking Fu2Uniqueness as an axiom and along with Transitivity and Fusion2E. Surprisingly, one can derive Reflexivity and Anti-symmetry from these. A sketch of these derivations, the other half of the needed work, is here in a footnote.[22]

Extending WGM1 to classical mereology

Returning to **WGM1** (or Simons' **SC**, one of the disambiguations of Casati and Varzi's **GEM**), we now show that it can be extended to yield classical mereology by using StrongSup instead of WeakSup. We do this by show-

[22]The derivation of reflexivity is somewhat long:

Lemma 1: $\forall \mathsf{x}\ \mathsf{x} \circ \mathsf{x}$

Proof: Let a be the fusion of $\{\mathsf{x}\}$ (i.e. $Fu_2(\mathsf{a}, [y : y = \mathsf{x}])$). Then $\mathsf{x} \leq \mathsf{a}$ and $\forall \mathsf{y}(\mathsf{y} \leq \mathsf{a} \rightarrow \exists \mathsf{z}(\mathsf{z} = \mathsf{x} \wedge \mathsf{z} \circ \mathsf{y}))$. Apply the universal to x.

Lemma 2: $\forall \mathsf{x}\ Fu_2(\mathsf{x}, [y : y \leq \mathsf{x}])$

Proof: Obviously, every part of x is part of x. If $\mathsf{y} \leq \mathsf{x}$, then $\mathsf{y} \circ \mathsf{y}$, so the second condition is met.

Theorem: $\forall \mathsf{x}\ Fu_2(\mathsf{x}, [y : y = \mathsf{x}])$

Proof: Take a with $Fu_2(\mathsf{a}, [y : y = \mathsf{x}])$. Now we show that $Fu_2(\mathsf{a}, [y : y \leq \mathsf{x}])$. (i) if $\mathsf{y} \leq \mathsf{x}$ then $\mathsf{y} \leq \mathsf{a}$ (since $\mathsf{x} \leq \mathsf{a}$). And if $\mathsf{y} \leq \mathsf{a}$, then, by def., $\mathsf{y} \circ \mathsf{x}$. So get a z with $\mathsf{z} \leq \mathsf{x}$ and $\mathsf{z} \leq \mathsf{y}$. Since $\mathsf{z} \circ \mathsf{z}$, get a w with $\mathsf{w} \leq \mathsf{z}$. By transitivity, $\mathsf{w} \leq \mathsf{y}$, so $\mathsf{y} \circ \mathsf{z}$. So (ii) $\forall \mathsf{y}(\mathsf{y} \leq \mathsf{a} \rightarrow \exists \mathsf{z}(\mathsf{z} \leq \mathsf{x} \wedge \mathsf{y} \circ \mathsf{z}))$.

So we have $Fu_2(\mathsf{a}, [y : y \leq \mathsf{x}])$; by Lemma 2, $Fu_2(\mathsf{x}, [y : y \leq \mathsf{x}])$, so by uniqueness of fusions $\mathsf{a} = \mathsf{x}$. (Strictly, we must note also that by Lemma 1, there is some y such that $\mathsf{y} \leq \mathsf{x}$, in order to apply the uniqueness axiom.)

Anti-symmetry is then fairly straightforward using Lemma 2, since $\mathsf{a} \leq \mathsf{b}$ and $\mathsf{b} \leq \mathsf{a}$ together imply $\forall \mathsf{z}(\mathsf{z} \leq \mathsf{a} \leftrightarrow \mathsf{z} \leq \mathsf{b})$.

ing that **GM1** and StrongSup together yield

$$\forall z(Fu_1(z, \phi_x) \rightarrow Fu_2(z, \phi_x))$$

Sketch of derivation: Suppose we have $\forall y(y \circ z \leftrightarrow \exists x(\phi_x \wedge y \circ x))$; we want $\forall x(\phi_x \rightarrow x \leq z) \wedge \forall y(y \leq z \rightarrow \exists x(\phi_x \wedge y \circ x))$. The right conjunct is almost immediate with an appeal to Reflexivity. For the left conjunct, suppose its negation for *reductio*, and derive that there is x with $\phi_x \wedge \neg x \leq z$; call it x. Applying the contrapositive form of StrongSup, get that $\exists w(w \leq x \wedge w \wr z)$; call it w. Since $w \leq x$, we have $\phi_x \wedge w \circ x$, so by our original supposition, $w \circ z$; this contradicts $w \wr z$, so we are done.

Since StrongSup yields WeakSup (if we also have Anti-symmetry), classical mereology can be axiomatized with **GM1** (= partial ordering plus Fusion1E) plus StrongSup. It is not clear whether Reflexivity and Anti-symmetry can be dropped, however, for we appealed to each in the relevant derivations above.

Thus we have seen that classical mereology can be obtained by the partial ordering axioms together with a fusion-existence axiom and a supplementation principle: if we use Fusion1E, we need StrongSup; if we use Fusion2E, we need only WeakSup. Alternatively, we may instead use partial ordering together with WeakSup, Filtration and the existence of minimal upper bounds. Further, in classical mereology, the two notions of fusion and the notion of minimal upper bound all basically coincide; for any non-empty ϕ_x, there is a single thing that is the unique type-1 fusion, type-2 fusion, and least upper bound for ϕ_x.

This completes the main track of this part of the paper; we close this part with a couple of side tracks.

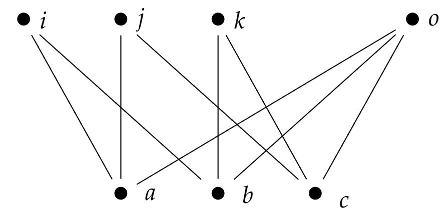
No obvious Tarski-style system for type-1 fusions

The question naturally arises whether one can axiomatize **CLM** in something like the manner of Tarski's compact axiomatization, but with type-1 fusions instead. Suppose we try the most obvious thing: Transitivity plus a universal closure for every instance of

Fusion1UE $\exists x\, \phi_x \rightarrow \exists! z\, Fu_1(z, \phi_x)$

(with z not free in ϕ_x). This won't work, because there is a model of these axioms in which we have one element that is not part of itself.

Suppose then that we add Reflexivity. Anti-symmetry and WeakSup may then be derived. Still, we get unwanted models. For example, consider



M. 6

One can confirm that M. 6 is a model of the resulting system, as follows: let D be the domain of M. 6, and for each non-empty set $S \subseteq D$ write $O(S)$ for the set $\{y \in D : \text{there is some } x \in S \text{ with } x \circ y\}$. Then we have

$$O(\{o\}) = D$$

$$O(\{a\}) = \{a, i, j, o\} \qquad O(\{i\}) = D \setminus \{c\}$$
$$O(\{b\}) = \{b, i, k, o\} \qquad O(\{j\}) = D \setminus \{b\}$$
$$O(\{c\}) = \{c, j, k, o\} \qquad O(\{k\}) = D \setminus \{a\}$$

Let F be the set containing these seven sets. For any singletons $S, T \subseteq D$, if $O(S) = O(T)$, $S = T$, hence if any element x of D fuses some non-empty subset of D, it is the only such element. Thus we have satisfied the uniqueness part of Fusion1U; so, if we can show that for each non-empty $S \subseteq D$, there is something that fuses S, then we are done. We can show this by noting first that each element fuses its singleton, and second that $O(S \cup T) = O(S) \cup O(T)$, while the set F is closed under union. So M. 6 is a model of the system that strives for Tarskian brevity with type-1 fusion. It is easy to check that M. 6 is not a model of **CLM**. One might regard this as a mark in favor of the type-2 notion.

Alternate primitives

Since the notions we have formalized with $\leq$, $\circ$, $\wr$, and the fusion notations are all inter-related, it is possible to take any of them as primitive and define the others with respect to it. In a sense, the choice is a mere matter of convenience; but the details of axiomatization of a theory equivalent with **CLM** are rather different.

An illustrative example is the Calculus of Individuals of Leonard and Goodman [6]. Here, $\wr$ is taken as primitive, and $x \leq y$ is defined as

$\forall z(z \wr y \rightarrow z \wr x)$, and $x \circ y$ is defined as we did above. The notion of fusion that they give, which we notate with Fu_{lg} is as follows:

$Fu_{lg}(t, \phi_x)$ abbrev. $\forall y(y \wr t \leftrightarrow \forall x(\phi_x \rightarrow y \wr x))$

(They use classes for the ϕ_x, rather than schemes.)

Since some quantification is built into the definition of $\leq$, we get Transitivity and Reflexivity as a matter of mere first-order logic. If we add Leonard and Goodman's second axiom (Anti-symmetry) and third axiom

$\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} \circ \mathsf{y} \leftrightarrow \neg(\mathsf{x} \wr \mathsf{y}))$

(which was true by definition in our systems) we get significant further theorems, including WeakSup, the equivalence of $Fu_{lg}(t, \phi_x)$ with each of $Fu_1(t, \phi_x)$ and $Fu_2(t, \phi_x)$, uniqueness for LG-Fusions, i.e.,

$\forall y \forall z(Fu_{lg}(y, \phi_x) \wedge Fu_{lg}(z, \phi_x) \rightarrow y = z)$,

and, hence, Fu2Uniqueness (and the similar result for type-1 fusions). When we add their first axiom, a fusion-existence axiom

$\exists \mathsf{x}\, \phi_\mathsf{x} \rightarrow \exists \mathsf{z}\, Fu_{lg}(\mathsf{z}, \phi_\mathsf{x})$

we get that $Fu_{lg}(t, \phi_x)$ is equivalent with $Mub(t, \phi_x)$ and, hence, we get all the theorems of **CLM**.

Part Four: Strong complements and Boolean algebra

We now show the close connection between classical mereology and the notion of a complete Boolean algebra. The basic result, which seems to go back to [15], is roughly this: every complete Boolean algebra is a classical mereology, except for the presence of a single extra element called 0, an element that is a part of everything; and every classical mereology is a complete Boolean algebra, except for the presence of the 0 element. (In classical mereology, there is no 0, unless there is only one thing; one way to see this is that every object would then be a fusion of $\{0\}$; another is that WeakSup fails, since 0 would be a proper part of anything else, but overlaps everything.)

A qualification on the claim of near-equivalence is in order, regarding the way that the "completeness" of complete Boolean algebra is conceived. The standard conception of completeness is that *every subset* of the domain of the algebra (the domain itself being conceived as a set) has a minimal upper bound. This requirement is intended to be strictly stronger than the analogous requirement imposed by the *schematic* version of **CLM**; with the axiom-scheme we require, in effect, minimal upper bounds only for those subsets of the domain that are *definable* in the language whose formulas are in the "substitution range" for the scheme. Thus there are set-theoretic models of pure first-order schematic **CLM** whose structures are not standardly complete Boolean algebras, even after a 0 is added. (See [11] for a careful discussion of this fact.) Here we have a mismatch between two ways of getting at "every subset of the domain." Schemes get at them only indirectly, as correlates to formulas, and hence only countably many are addressed; the standard notion of complete Boolean algebra gets at them directly, from within the set theory. However, as our discussion will make clear, if the mechanisms for generality are matched, the basic equivalence-except-for-0 result holds perfectly.

First, we will construct a "neutral" axiom set that effectively contains the common core of mereology and complete Boolean algebras. If one adds to the neutral axioms the axiom that if there is more than one thing, then there is not a 0 element, the result is **CLM**; if one adds the axiom that there is a 0 element, the result is (schematic or, with sets, standard) complete Boolean algebra. We will also eventually relate **CLM** to (not necessarily complete) Boolean algebra.

Next, we will find an alternative neutral axiom set, which will make central use of a new notion: that of the *strong complement* of an object: basically, the strong complement of x is something y such that (1) y is disjoint from x; (2) everything disjoint from x is part of y; and (3) everything disjoint from y is part of x. Recall that **CLM** can be axiomatized with the combination of Reflexivity, Anti-symmetry, Transitivity, MubE, WeakSup, and Filtration, and that in fact, Reflexivity and Anti-symmetry can be derived from the other four. It turns out that if we bring Anti-symmetry back in as an axiom, then we can basically capture the combined effect of WeakSup and Filtration with a single axiom about strong complements. The axiom says that *almost* everything has a strong complement: the only exception is the fusion of all things.

While the classic [15] and the recent [11] also address the near-equivalence, our use of the notion of strong complement is, as far as the author knows, unique. The comparison with (not necessarily complete) Boolean algebra and complete Boolean algebra is facilitated by using this notion, and we are led to non-standard axiomatizations of both of those theories, as well as a non-standard axiomatization of **CLM**.

The neutral axiom set

Our first task is to display a neutral axiom set that is obviously very close to **CLM**, and which can be slightly supplemented to yield either **CLM** or complete Boolean algebra.

For neutrality, we must be careful about our defined symbols. We will use the defined notion of *Mub* exactly as we did above. Since we do not want to rule a 0 in or out, we will focus on a predicate 0 (rather than a name) defined as follows:

$$\forall x\ (0(x) \leftrightarrow Mub(x, [y \mid y \neq y]))$$

(or use the empty set or the like if one is using auxiliaries). It is a consequence of the definition alone that

$$\forall x \forall y(0(y) \rightarrow y \leq x)$$

The definition of proper part ($\ll$) remains the same. We use revised notions of overlap and disjointness as follows:

$$s \bullet t \textit{ abbrev. } \exists x(\neg 0(x) \wedge x \leq s \wedge x \leq t)$$

and

$$s \wr t \textit{ abbrev. } \neg s \bullet t$$

(In a Boolean algebra, there is a zero element, so if we use the old notion of overlap, everything overlaps everything, and nothing is disjoint from anything.)

The neutral axiom set **N** is:

ZeroU $\forall x \forall y(0(x) \wedge 0(y) \rightarrow x = y)$
Transitivity $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$
WeakSup^N $\forall x \forall y((\neg 0(x) \wedge x \ll y) \rightarrow \exists z(\neg 0(z) \wedge z \leq y \wedge x \wr z))$
Filtration^N $\forall y \forall z((\neg 0(y) \wedge y \leq z \wedge Mub(z, \phi_x)) \rightarrow \exists x(\phi_x \wedge y \bullet x))$
MubE $\exists x\, \phi_x \rightarrow \exists z\, Mub(z, \phi_x)$

To get **CBA** (complete Boolean algebra), it suffices to add

Zero $\exists x\, 0(x)$

to **N**.

To get **CLM**, add

NoZero $\exists x \exists y(x \neq y) \to \forall x\, \neg 0(x)$

to **N**.

As in **CLM**, we may derive Reflexivity and Anti-symmetry in **N**. The derivations are as before, with the necessary adjustments to take account of the possibility that a thing be 0. (Alternatively, if we took Anti-symmetry as an axiom, then we would not need ZeroU.)

It is easy to confirm that the addition of NoZero to **N** is equivalent with **CLM**. If we assume $\exists x \forall y(x = y)$, then this is straightforward (note that Zero is then derivable); otherwise, derive NoZero in **CLM** and then derive (in each system) that the new definitions of overlap and disjointness are equivalent with the old ones, and then similarly for the axioms. To show the connection with Boolean algebra will require more work, which we postpone for the moment.

N2

First, we introduce and show the utility of the notion of a strong complement. Consider the following variant of **N** that uses strong complements (with the neutral definitions of overlap and disjointness): First define $1(x)$ as $\forall y\, y \leq x$. **N2** is then Anti-symmetry, Transitivity, MubE, and

Strong Complement $\forall x((\neg 1(x) \to \exists z(z \wr x \wedge \forall y((y \wr x \to y \leq z) \wedge (y \wr z \to y \leq x))))$

N from N2

We may derive **N** from **N2**. It is easy to see that there is a unique z with $1(z)$ (viz. the z with $Mub(z, [w|w = w])$), so we may use '1' as if it were a name. Use the expression 'a^*' for "the strong complement of a," as asserted to exist by Strong Complement, when it is clear that a is not 1.[23] (Deduce that strong complements are unique with anti-symmetry; similarly, get ZeroU.)

Reflexivity: Given x with $\mathsf{x} \neq 1$ (otherwise $\mathsf{x} \leq \mathsf{x}$), get strong complement x^* with $\mathsf{x}^* \downharpoonleft \mathsf{x}$. Since $\forall \mathsf{y}$, if $\mathsf{y} \downharpoonleft \mathsf{x}^*$ then $\mathsf{y} \leq \mathsf{x}$, and $\mathsf{x} \downharpoonleft \mathsf{x}^*$, $\mathsf{x} \leq \mathsf{x}$.

Using the definition of $\wr$ as before, note that $0(\mathsf{x})$ implies $\mathsf{x} \downharpoonleft \mathsf{y}$ and $\neg\, \mathsf{x} \wr \mathsf{y}$. Using reflexivity, note that if $\mathsf{x} \leq \mathsf{y}$ and $\neg\, 0(\mathsf{x})$, then $\mathsf{x} \bullet \mathsf{y}$. Next deduce these *Easy Lemmas:*

$0(\mathsf{a}^*) \leftrightarrow \mathsf{a} = 1;\ \mathsf{a}^* = 1 \leftrightarrow 0(\mathsf{a});$
$\mathsf{a} \leq \mathsf{b} \rightarrow \mathsf{b}^* \leq \mathsf{a}^*;$
$\mathsf{a}^{**} \leq \mathsf{a}; \mathsf{a} \leq \mathsf{a}^{**};\ \mathsf{a} = \mathsf{a}^{**};$
$\mathsf{b}^* \leq \mathsf{a}^* \rightarrow \mathsf{a} \leq \mathsf{b};\ \mathsf{a} \leq \mathsf{b}^* \leftrightarrow \mathsf{b} \leq \mathsf{a}^*;\ \mathsf{a}^* \leq \mathsf{b} \leftrightarrow \mathsf{b}^* \leq \mathsf{a}.$

Though *WeakSup*^{N} may be given a more direct derivation, some of the naturalness of **N2** is revealed by first deriving *StrongSup*^{N} and inferring *WeakSup*^{N} by applying anti-symmetry.

StrongSup^N $\quad \forall \mathsf{x} \forall \mathsf{y}(\forall \mathsf{b}(\mathsf{b} \neq 0 \wedge \mathsf{b} \leq \mathsf{x} \rightarrow \mathsf{b} \bullet \mathsf{y}) \rightarrow \mathsf{x} \leq \mathsf{y})$

StrongSup^N follows easily from

Strong Overlap Lemma: $\forall \mathsf{x} \forall \mathsf{y}(\forall \mathsf{b}(\mathsf{b} \bullet \mathsf{x} \rightarrow \mathsf{b} \bullet \mathsf{y}) \rightarrow \mathsf{x} \leq \mathsf{y})$

To show this lemma, assume $\forall \mathsf{b}(\mathsf{b} \bullet \mathsf{x} \rightarrow \mathsf{b} \bullet \mathsf{y})$. We may suppose that $\mathsf{y} \neq 1$, and thus (by an Easy Lemma) $\neg\, 0(\mathsf{y}^*)$. Now, if $\mathsf{x} = 1$, then: $\forall \mathsf{z}\ (\neg\, 0(\mathsf{z}) \rightarrow \mathsf{z} \bullet \mathsf{x})$, hence $\mathsf{y}^* \bullet \mathsf{x}$, so $\mathsf{y}^* \bullet \mathsf{y}$; contradiction. Conclude $\mathsf{x} \neq 1$. By contraposition, $\forall \mathsf{b}(\mathsf{b} \downharpoonleft \mathsf{y} \rightarrow \mathsf{b} \downharpoonleft \mathsf{x})$. But $\mathsf{y}^* \downharpoonleft \mathsf{y}$, so $\mathsf{y}^* \downharpoonleft \mathsf{x}$, hence $\mathsf{y}^* \leq \mathsf{x}^*$. So, using an Easy Lemma, $\mathsf{x} \leq \mathsf{y}$.

Filtration^N*:* Given y with $\neg\, 0(\mathsf{y})$ and $\mathsf{y} \leq \mathsf{z}$ and $Mub(\mathsf{z}, \phi_x)$. (Note that then $\neg\, 0(\mathsf{z})$ and that $\exists \mathsf{x}(\phi_{\mathsf{x}} \wedge \neg\, 0(\mathsf{x}))$. Also, we may suppose that for no x of ϕ_x do we have $\mathsf{x} = 1$, since, if so, we are done.) Suppose for *reductio* that for each x of ϕ_x with $\neg\, 0(\mathsf{x})$, $\neg\, \mathsf{x} \bullet \mathsf{y}$. Then, for each such x, $\mathsf{y} \downharpoonleft \mathsf{x}$, so $\mathsf{y} \leq \mathsf{x}^*$. Thus for each such x, $\mathsf{x} \leq \mathsf{y}^*$; (and $0(\mathsf{b})$ implies $\mathsf{b} \leq \mathsf{y}^*$) so each x of ϕ_x is $\leq \mathsf{y}^*$. Hence $\mathsf{z} \leq \mathsf{y}^*$. Hence $\mathsf{y} \leq \mathsf{y}^*$, so $\mathsf{y} \bullet \mathsf{y}^*$; contradiction.

Thus **N** can be derived from **N2**.

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[23]Officially, uses of '1', 'a^*', and the like are to regarded as definite descriptions, which are abbreviatory devices to be handled in the manner of Russell's theory of descriptions.

N2 from N

We may go the other way.

First derive the Strong Overlap Lemma (SOL) (stated above): Assume $\forall b(b \bullet x \rightarrow b \bullet y)$. Now if $0(x)$ then $x \leq y$, so we may assume not. Now, use MubE to get a with $Mub(a, [z|z \leq x \wedge z \leq y]$. Get that $\neg 0(a)$ (starting from $x \bullet x$). Get that $a \leq x$ and $a \leq y$ by design of a. Now suppose for *reductio* that $a \ll x$ (so that when done we will conclude $a = x$ and hence $x \leq y$, as desired). Use WeakSup^N to get w with $\neg 0(w)$ and $w \leq x$ and $w \upharpoonleft a$. Since $w \bullet x$, $w \bullet y$; get u with $\neg 0(u)$ and $u \leq w$ and $u \leq y$. $u \leq x$, as well, so $u \leq a$ by design; but then we have that $a \bullet w$, a contradiction.

Next, infer StrongSup^N from SOL, and then, to derive Strong Complement: Given x with $x \neq 1$. We must show that x "has a complement" as the axiom describes. If $0(x)$, it is easy to see that 1 is the desired complement. Otherwise: we know that $x \ll 1$, so, by WeakSup^N, $\exists y\, y \upharpoonleft x$ with $\neg 0(y)$. So get z with $Mub(z, [w|w \upharpoonleft x])$. z is the desired complement. If $z \bullet x$, then get some y ($\neg 0(y)$) with $y \leq x$ and $y \leq z$; by Filtration^N, $y \bullet w$ for some w with $w \upharpoonleft x$; but since ($\neg 0(w)$ and) $w \bullet y$ and $y \leq x$, we then have $w \bullet x$; contradiction. Thus $z \upharpoonleft x$ as desired. Next, given arbitrary y with $y \upharpoonleft x$, $y \leq z$ by design. Next, given arbitrary y with $y \upharpoonleft z$, if $0(y)$ then $y \leq x$; else: for any w with $\neg 0(w)$ and $w \leq y$, $\neg w \leq z$, so, by design, $\neg w \upharpoonleft x$. Hence $w \bullet x$. Now apply StrongSup^N and conclude $y \leq x$.

NoZero is an easy theorem of **CLM**. Thus we may conclude that if we add NoZero to **N2**, the result is equivalent with **CLM**. Thus we have an alternate axiomatization of **CLM**, in which Strong Complement, (with the aid of Anti-symmetry) basically has the effect of the combination of WeakSup and Filtration. (We can replace '$\upharpoonleft$' with '$\wr$' in this axiom set, as in the "Fifth way" in the summary below.)

Boolean Algebra

Traditionally, there are two different ways to give axioms for (not necessarily complete) Boolean algebra. (See, for example, the first two sections of the first chapter of [1].)

The "algebraic" way, suggestive of the connection with the Boolean connectives of propositional logic, involves taking as primitive the constants 0 and 1, a binary operation called "join" (symbolized $x + y$ or $x \vee y$), a binary operation "meet" (symbolized $x \cdot y$ or $x \wedge y$), and a singulary operation of "complementation" (symbolized x^* or $\neg x$). The axioms then constrain the behavior of the operations on arbitrary items in the domain (and on 0 and 1). $x \leq y$ is then defined as $x + y = y$ or as $x \cdot y = x$.

The "relational" way is to take $\leq$ as primitive, and define 0 and 1 and all of the algebraic operations in terms of it; e.g., $x + y$ is defined as the least upper bound or supremum of x and y, while $x \cdot y$ is defined as their infimum. Such definitions have to be justified by the axioms stated in terms of $\leq$. In the case of a not-necessarily-complete Boolean algebra, we only care about least upper bound and greatest lower bound on *pairs* of elements. These notions are straightforwardly definable in terms of first-order logic and $\leq$, so no schemes, set theory, or additional primitives are required.

To give axioms for *complete* Boolean algebra, one needs to add an axiom or scheme that uses a generalized (that is, stated with schemes or auxilliary entities) notion of supremum (minimal upper bound), whether one takes as primitive the relation $\leq$ or the algebraic operations and 0 and 1. In the complete case, it is more natural to take $\leq$ as primitive, and, of course, this facilitates comparison with mereology.

A standard axiom set for complete Boolean algebra, **sCBA**, breaks into three groups. The first group says that $\leq$ is a partial ordering. The second group consists of a single axiom (scheme) saying that any ϕ_x has a supremum (mub):

Supremum $\exists \mathsf{z}\, Mub(\mathsf{z}, \phi_x)$

(Recall that standardly, the notion of complete Boolean algebra is defined within set theory, so the ϕ_x's would be set-variables.) Supremum is obviously equivalent to the conjunction of MubE and Zero. With anti-symmetry in place, we get that there is exactly one supremum for any ϕ_x. This justifies our introducing defined complex terms

$Sup(\phi_x)$

(the *supremum* of ϕ_x) for arbitrary ϕ_x, to be treated as the definite description "the minimal upper bound for ϕ_x". Next, we may get that for any ϕ_x, there is a greatest lower bound for ϕ_x. Let

$Inf(\phi_x)$

(the *infimum* of ϕ_x), be treated as $Sup([y \mid \forall x(\phi_x \rightarrow y \leq x)])$.

By definition, $\forall y(\forall x(\phi_x \rightarrow y \leq x) \rightarrow y \leq Inf(\phi_x))$. Also, we have that $\forall z(\forall y(\forall x(\phi_x \rightarrow y \leq x) \rightarrow y \leq z) \rightarrow Inf(\phi_x) \leq z)$. Now every x such that ϕ_x is like z in the antecedent of the main conditional of this last formula; hence we have $\forall x(\phi_x \rightarrow Inf(\phi_x) \leq x)$. Thus $Inf(\phi_x)$ is indeed a greatest lower bound on ϕ_x.

The third group of axioms is stated in terms of 1 and 0 and the algebraic operations of meet and join, all defined in terms of $\leq$. 0 is as above; 1 is $Sup([\mathsf{x} \mid \mathsf{x} = \mathsf{x}])$.

$(s + t)$ abbrev. $Sup([x \mid x = s \vee x = t])$
$(s \cdot t)$ abbrev. $Inf([x \mid x = s \vee x = t])$

The axioms are:

Complement $\forall \mathsf{x} \exists \mathsf{y}(\mathsf{x} + \mathsf{y} = 1 \wedge \mathsf{x} \cdot \mathsf{y} = 0)$

Distributivity $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}(\mathsf{x} + (\mathsf{y} \cdot \mathsf{z}) = (\mathsf{x} + \mathsf{y}) \cdot (\mathsf{x} + \mathsf{z}))$

One can derive that the object asserted to exist by Complement is unique, using some of the important little theorems[24] $\mathsf{x} \cdot \mathsf{y} = \mathsf{x} \leftrightarrow \mathsf{x} \leq \mathsf{y} \leftrightarrow \mathsf{x} + \mathsf{y} = \mathsf{y}$, $(\mathsf{x} \cdot \mathsf{y}) + \mathsf{x} = \mathsf{x}$, $\mathsf{x} \cdot (\mathsf{y} + \mathsf{x}) = \mathsf{x}$, $\mathsf{x} = \mathsf{x} + 0$, $\mathsf{x} = \mathsf{x} \cdot 1$, $\mathsf{x} \cdot \mathsf{y} = \mathsf{y} \cdot \mathsf{x}$, $\mathsf{x} + \mathsf{y} = \mathsf{y} + \mathsf{x}$, $\mathsf{x} + (\mathsf{y} + \mathsf{z}) = (\mathsf{x} + \mathsf{y}) + \mathsf{z}$, and $\mathsf{x} \cdot (\mathsf{y} \cdot \mathsf{z}) = (\mathsf{x} \cdot \mathsf{y}) \cdot \mathsf{z}$. If a and a' are complements of x, i.e., $\mathsf{x} + \mathsf{a} = 1$, $\mathsf{x} \cdot \mathsf{a} = 0$, $\mathsf{x} + \mathsf{a}' = 1$, and $\mathsf{x} \cdot \mathsf{a}' = 0$, then

$$\mathsf{a} \cdot \mathsf{a}' = (\mathsf{a} + (\mathsf{x} \cdot \mathsf{a}')) \cdot \mathsf{a}' = ((\mathsf{a} + \mathsf{x}) \cdot (\mathsf{a} + \mathsf{a}')) \cdot \mathsf{a}' = (\mathsf{a} + \mathsf{a}') \cdot \mathsf{a}' = \mathsf{a}'$$

So $\mathsf{a}' \leq \mathsf{a}$; similarly, get $\mathsf{a} \leq \mathsf{a}'$ and apply Anti-symmetry. One can derive the dual distribution principle, namely

$$\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}(\mathsf{x} \cdot (\mathsf{y} + \mathsf{z}) = (\mathsf{x} \cdot \mathsf{y}) + (\mathsf{x} \cdot \mathsf{z}))$$

without appealing to Complement, by making use of some of the little theorems; begin by using Distribution to get that $(\mathsf{x} \cdot \mathsf{y}) + (\mathsf{x} \cdot \mathsf{z}) = ((\mathsf{x} \cdot \mathsf{y}) + \mathsf{x}) \cdot ((\mathsf{x} \cdot \mathsf{y}) + \mathsf{z})$.

The notion of a (not necessarily complete) Boolean algebra is much weaker than the notion of a complete Boolean algebra. A standard axiom set, **sBA**,

[24] Their use for this purpose is standard, as in [1].

for the general notion of Boolean algebra, also has axioms falling into three groups. The first group is again the partial ordering axioms. The second forms a greatly weakened version of the Supremum axiom: it is only required that each *pair* of things have both a supremum and an infimum (cf. Product and BLUB above). The third group is Distribution (the definitions of join and meet are the same) and an existentially quantified version of Complement above, that says that there exist objects (to be called '0' and '1') such that the Complement axiom above is satisfied. (One can then prove that these objects are unique, that $\forall x\ (0 \leq x \wedge x \leq 1)$, and so on.)

If we do not require completeness (of at least the schematic sort), the close correlation with classical mereology would fail. Every Boolean algebra with finitely many objects is complete, but the two notions come apart if there are infinitely many objects.[25]

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[25]Consider **PSCLM**, "pure" schematic **CLM**, in which $\leq$ is the only non-logical relation. There are non-complete Boolean algebras that cannot be converted into models of **PSCLM** by simply deleting 0 and restricting $\leq$ accordingly; for example, there are Boolean algebras in which the set of atoms has no least upper bound. (An atom of a Boolean algebra is an element such that only 0 and it are $\leq$ it; an atom of a mereology is an element such that only it is part of it.) Since "atom" is easily given a definition in a first-order language with $\leq$, the set of atoms is definable in the relevant sense, and so it is an easy theorem of **PSCLM** that if there is an atom, then there is a least upper bound on the atoms.

That there are such Boolean algebras can be seen by the following argument. First, given that $\langle B, \leq_B \rangle$ is a Boolean algebra, and $A \subseteq B$: if A is closed under (binary) meet and (binary) join and complementation, and $1 \in A$ (where 1 is the "top" element of $\langle B, \leq_B \rangle$), $\langle A, \leq_A (= \leq_B \restriction A) \rangle$ is a Boolean algebra (where $\leq_B \restriction A$ is $\{\langle x, y \rangle : x \leq_B y$ and $x, y \in A\}$; moreover the meet, join, and complement operations of the A-algebra are restrictions of those of the B-algebra (e.g., for $x \in A$, the complement of x in $\langle A, \leq_A \rangle$ is the complement of x in $\langle B, \leq_B \rangle$).

Now call an element x of a Boolean algebra a "bit of gunk" if $x \neq 0$ and no atoms are $\leq x$. Let $\langle B, \leq \rangle$ be a Boolean algebra that has infinitely many atoms and also at least one bit of gunk. Let P be some infinite set of pairwise disjoint bits of gunk of B. (x and y are disjoint if their meet is 0; there must be such a set if there is at least one bit of gunk in B.) Let G be $\{x \in B : \exists y \in P\, x \leq y\}$. Let T be the set of atoms of B and let A be the closure of $T \cup G$ under (binary) join, (binary) meet and complement. Then $\langle A, \leq \restriction A \rangle$ is a Boolean algebra in which the set of atoms has no least upper bound. (To show this, it helps to put each member of A into a "normal form" analogous to Conjunctive Normal Form in propositional logic. Since members of A are "generated out of $T \cup G$ by inductive closure on the three operations," given the Boolean laws governing these operations, one can deduce that every member of A can be represented as a finite join of terms, each of which is a finite meet of terms, each of which (in turn) is a member of T or of G or a complement

N2F (with Zero) and sBA

We are now in a position to start linking mereology with Boolean algebra. A weakening of **N2** naturally provides an alternate set of axioms equivalent with **sBA**. We simply add Zero and reduce MubE to its finite counterpart, which effectively asserts a minimal upper bound for any x and y:

Binary Join $\forall x \forall y \exists z (x \leq z \wedge y \leq z \wedge \forall w (x \leq w \wedge y \leq w \rightarrow z \leq w))$

(In the presence of Reflexivity and Transitivity, Binary Join is equivalent with BLUB above.) Let **N2F** (N2 Finite) be the conjunction of Anti-Symmetry, Transitivity, Binary Join, and Strong Complement, and let **BA** be **N2F** conjoined with the axiom Zero. We can derive from **BA** the standard relational axioms for Boolean Algebra, **sBA**.

First, we can derive in **BA** that there are unique objects that satisfy $0(x)$ and $1(x)$, where these are defined as above, or equivalently, $0(x) \leftrightarrow \forall y\, x \leq y$ and $1(x) \leftrightarrow \forall y\, y \leq x$. (Zero yields the 0; apply Strong Complement to it to yield the 1.) Each is the strong complement of the other (and this holds even if there is exactly one thing). Thus we have

Exceptionless Strong Complement
$\forall x \exists z (z \wr x \wedge \forall y ((y \wr x \rightarrow y \leq z) \wedge (y \wr z \rightarrow y \leq x)))$

and so we may speak of "the strong complement" of x (notated again x^*) for any x. The Easy Lemmas above go through. Applying anti-symmetry, binary joins are unique, so we may use '$x + y$' as a term for the object asserted to exist by Binary Join. Then it can fairly easily be shown that $(x^* + y^*)^*$ is a meet for x and y. ($x^* \leq x^* + y^*$, so $(x^* + y^*)^* \leq x$; similarly $(x^* + y^*)^* \leq y$. And if $a \leq x$ and $a \leq y$, then: $x^* \leq a^*$, and $y^* \leq a^*$, so $x^* + y^* \leq a^*$, so $a \leq (x^* + y^*)^*$.) Notate the meet of x and y with '$x \cdot y$'.

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thereof: i.e., notating the meet, join, and complementation operations of $\langle B, \leq \rangle$ as '$\cdot$', '$+$', and '$'$' respectively, each member of A is of the form

$((a_1 \cdot \ldots \cdot a_n \cdot b'_1 \cdot \ldots \cdot b'_m \cdot g_1 \cdot \ldots \cdot g_i \cdot h'_1 \cdot \ldots \cdot h'_j) + (\ldots) + \ldots + (\ldots))$

(where $n, m, i, j \geq 0$ and each a and each $b \in T$ and each g and each $h \in G$). A key observation is then that for each x that is a complement of a member of $T \cup G$, there are at most finitely many members of $T \cup G$ that are not $\leq x$.)

The Stong Overlap Lemma (SOL) goes through as above. To derive Complement, we need to show that for any x, $x + x^* = 1$ and $x \cdot x^* = 0$. The latter is easy since $x \wr x^*$; to get the former, note that for any $a \neq 0$, if $a \wr x$, then $a \leq x^*$, so $a \bullet x + x^*$, while if not, $a \bullet x$, so again, $a \bullet x + x^*$. So for any a with $a \bullet 1$, $a \bullet x + x^*$; apply SOL and conclude $1 \leq x + x^*$, hence $1 = x + x^*$.

To derive Distributivity, we will show both

(i) $x + (y \cdot z) \leq (x + y) \cdot (x + z)$ and
(ii) $(x + y) \cdot (x + z) \leq x + (y \cdot z)$

The first can be shown to follow basically from the definitions of join and meet. To derive the second, we derive

(iia) $a \bullet [(x + y) \cdot (x + z)] \rightarrow a \bullet (x + (y \cdot z))$

(using $\bullet$ as defined above) and apply SOL.

The key lemmas for deriving *(iia)* are

(iia)L1 $a \wr x \wedge a \wr y \rightarrow a \wr x + y$ and
(iia)L2 $a \wr x \wedge a \leq x + y \rightarrow a \leq y$

To show L1, note that the antecedent implies that $x \leq a^*$ and similarly for y, so $x + y \leq a^*$; so $a \leq (x + y)^*$, so $a \wr x + y$. (The only way $b \leq z$ and $b \leq z^*$ is if $b = 0$.) To show L2, assume the antecedent, and that $a \neq 0$ (otherwise we are done). Now consider any b with $b \bullet a$: get $c \neq 0$ with $c \leq a$ and $c \leq b$. Get that $c \wr x$ while $c \bullet x + y$. Apply (the contrapositive of) L1 to deduce $c \bullet y$ and infer $b \bullet y$. So for any b, if $b \bullet a$, $b \bullet y$; apply SOL and conclude $a \leq y$.

Now to derive *(iia)*, suppose $a \bullet [(x + y) \cdot (x + z)]$. Get $b \neq 0$ with $b \leq a$ and $b \leq (x + y) \cdot (x + z)$; get that $b \leq x + y$ and $b \leq x + z$. Now, if $b \bullet x$, then we have a non-zero c with $c \leq b \leq a$ and $c \leq x \leq x + (y \cdot z)$. Otherwise $b \wr x$; apply *(iia)L2* twice to get $b \leq y$ and $b \leq z$ and thus $b \leq y \cdot z$. Thus, in any case, $a \bullet x + (y \cdot z)$.

To show that **sBA** yields **BA**, we need only show that the complements postulated by Complement are in fact strong complements, in the presence of the other axioms. Use $\bar{x}$ to denote the (weak) complement of x, and x^* for the strong complement; we want $\forall x\, x^* = \bar{x}$. To show this, we need to show $x \wr \bar{x}$ and $\forall y\, (y \wr x \rightarrow y \leq \bar{x})$ and $\forall y\, (y \wr \bar{x} \rightarrow y \leq x)$. One can easily show that $x \wr y \leftrightarrow x \cdot y = 0$. Hence $x \wr \bar{x}$. Now suppose $y \wr x$. Then $y \cdot x = 0$, and so:

$$\bar{x} = \bar{x} + 0 = \bar{x} + (y \cdot x) = (\bar{x} + y) \cdot (\bar{x} + x) = (\bar{x} + y) \cdot 1 = \bar{x} + y$$

Hence $y \leq \bar{x}$. The other needed fact is derived similarly.

Main results of Part Four

Thus **BA**, which is **N2F** (the finite version of **N2**) plus Zero, is equivalent with **sBA**. With this shown, it is clear that **N2** plus Zero is equivalent with the "infinite" or complete version of Boolean algebra, **sCBA** (provided that the mechanism for generality represented by our ϕ_x's is the same on both sides). To see their equivalence, we need only the above arguments and the observation that Zero and MubE together are equivalent with Supremum.

As we noted earlier, in the presence of Zero, we can derive in **N2F** that there are unique elements 0 and 1 with the expected properties and that each is the strong complement of the other. Putting all this together, we get that we may axiomatize Boolean algebra with the conjunction of Anti-Symmetry, Transitivity, Binary Join, and Exceptionless Strong Complement. To get complete Boolean algebra, replace Binary Join with Supremum.

Summing up, **N2** is an axiomatic "middle ground" between Classical Mereology and Boolean algebra. We have that **CLM** is equivalent with **N** plus NoZero, which, in turn, is equivalent with **N2** plus NoZero. And **sCBA** is equivalent with **N2** plus Zero, which, in turn, is equivalent with **N** plus Zero. This brings out the small difference between Classical Mereology and complete Boolean algebra. Further, we have seen how the defined notion of strong complement helps to bring all of these theories together. In the presence of partial ordering and mubs (finite or not), the effect of the strong complement axiom is basically the same as the combination of WeakSup and Filtration, and, again, basically the same as the combination of Complement and Distributivity.

Summary of axiom sets

Classical mereology is the core notion of "mereology" in the philosophical literature. *Within* a system that yields it, the two main notions of fusion and the notion of least upper bound are equivalent; but in axiomatizing, one must be more careful. The stronger, second type of fusion seems to be the more natural notion, and the notion of minimal upper bound is perhaps more natural still.

Here are five ways to axiomatize classical mereology. (See Part One above for the definitions of other terms and the use of ϕ_x.)

TYPE-1 FUSION $Fu_1(t, \phi_x)$ abbreviates
$\forall y(y \circ t \leftrightarrow \exists x(\phi_x \wedge y \circ x))$

TYPE-2 FUSION $Fu_2(t, \phi_x)$ abbreviates
$\forall x(\phi_x \rightarrow x \leq t) \wedge \forall y(y \leq t \rightarrow \exists x(\phi_x \wedge y \circ x))$

MIN UPPER BOUND $Mub(t, \phi_x)$ abbreviates
$\forall x(\phi_x \rightarrow x \leq t) \wedge \forall w(\forall x(\phi_x \rightarrow x \leq w) \rightarrow t \leq w)$

One may not replace type-2 fusion with type-1 fusion in any of the axiom sets.

First way:

Reflexivity $\forall x\, x \leq x$
Anti-symmetry $\forall x \forall y((x \leq y \wedge y \leq x) \rightarrow x = y)$
Transitivity $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$
StrongSup $\forall z \forall y(\forall x(x \leq y \rightarrow x \circ z) \rightarrow y \leq z)$
Fusion1E $\exists x\, \phi_x \rightarrow \exists z\, Fu_1(z, \phi_x)$

Second way:

Transitivity $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$
WeakSup $\forall x \forall y(x \ll y \rightarrow \exists z(z \leq y \wedge x \wr z))$
Fusion2E $\exists x\, \phi_x \rightarrow \exists z\, Fu_2(z, \phi_x)$

Third way:

Transitivity $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$
WeakSup $\forall x \forall y(x \ll y \rightarrow \exists z(z \leq y \wedge x \wr z))$
Filtration $\forall y \forall z((y \leq z \wedge Mub(z, \phi_x)) \rightarrow \exists x(\phi_x \wedge y \circ x))$
MubE $\exists x\, \phi_x \rightarrow \exists z\, Mub(z, \phi_x)$

Fourth way:

Transitivity $\forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z)$
Fusion2UE $\exists x\, \phi_x \rightarrow \exists! z\, Fu_2(z, \phi_x)$

Fifth way:

Anti-symmetry $\forall x \forall y ((x \leq y \wedge y \leq x) \rightarrow x = y)$
Transitivity $\forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z)$
MubE $\exists x\, \phi_x \rightarrow \exists z\, Mub(z, \phi_x)$
Strong Complement $\forall x (\exists y\, y \not\leq x \rightarrow \exists z (z \wr x \wedge \forall y ((y \wr x \rightarrow y \leq z) \wedge (y \wr z \rightarrow y \leq x))))$
NoZero $\exists x \exists y\, x \neq y \rightarrow \neg \exists x \forall y\, x \leq y$

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