CARDINAL INVARIANTS OF MONOTONE AND POROUS SETS

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ABSTRACT. A metric space (X, d) is *monotone* if there is a linear order $<$ on X and a constant c such that $d(x, y) \leqslant c\, d(x, z)$ for all $x < y < z$ in X. We investigate cardinal invariants of the σ-ideal Mon generated by monotone subsets of the plane. Since there is a strong connection between monotone sets in the plane and porous subsets of the line, plane and the Cantor set, cardinal invariants of these ideals are also investigated. In particular, we show that $\mathsf{non}(\mathsf{Mon}) \geqslant \mathfrak{m}_{\sigma\text{-linked}}$, but $\mathsf{non}(\mathsf{Mon}) < \mathfrak{m}_{\sigma\text{-centered}}$ is consistent. Also $\mathsf{cov}(\mathsf{Mon}) < \mathfrak{c}$ and $\mathsf{cof}(\mathcal{N}) < \mathsf{cov}(\mathsf{Mon})$ are consistent.

1. INTRODUCTION

Definition 1.1. Let (X, d) be a metric space.

- (X, d) is called *monotone* if there is $c > 0$ and a linear order $<$ on X such that $d(x, y) \leqslant c\, d(x, z)$ for all $x < y < z$ in X.
- (X, d) is called *σ-monotone* if it is a countable union of monotone subspaces (with possibly different witnessing constants).

The notions of monotone and σ-monotone space first occurred in [24], where they were used to prove existence of universal measure zero sets of large Hausdorff dimension. Their systematic investigation began in papers [10, 9]. In [23] the notions were used to prove that if a Borel set in $\mathbb{R}^n$ has Hausdorff dimension greater than m, then it maps onto the m-dimensional ball by a Lipschitz map.

The very basic fact established in [10] says that if X is a monotone metric space and $<$ is the witnessing order, then the metric topology is suborderable by $<$. In particular, the metric topology is finer than the order topology, i.e. every open interval $(x, y) = \{z \in X : x < z < y\}$ is open in the metric topology.

Of course, any subset of the line is monotone. So the ideal of σ-monotone subsets of the line is not interesting at all. On the other hand, the plane itself is clearly not σ-monotone. Thus the σ-ideal of σ-monotone subsets of the plane is nontrivial. The aim of this paper is to investigate this ideal and mainly its cardinal invariants.

Here is a brief account of what is known of monotone and σ-monotone sets in the plane. Any monotone set in the plane is homeomorphic to a subset of the line and any monotone connected set in the plane is homeomorphic to an interval, in particular, it is a curve ([10]), but there are homeomorphic copies of $[0, 1]$ in the

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plane that are not σ-monotone. Every σ-monotone set in the plane has topological dimension 0 or 1 ([10]). There is a zero dimensional compact set in the plane that is not σ-monotone ([9]). Every σ-monotone subset of the plane is contained in a countable union of compact monotone sets. This follows from the fact that every metric space with a dense monotone subset is monotone ([10]).

Definition 1.2. The ideal of all σ-monotone sets in a metric space X is denoted $\mathbf{Mon}(X)$. The ideal $\mathbf{Mon}(\mathbb{R}^2)$ of all σ-monotone sets in the plane is denoted $\mathbf{Mon}$.

Proposition 1.3 ([10])**.** *A closure of a monotone set in the plane is monotone. Hence* $\mathbf{Mon}$ *is generated by* F_σ*-sets.*

Cardinal invariants. Given an ideal $\mathcal{I}$ on a set X, the following are the usual cardinal invariants of $\mathcal{I}$:

$$\begin{aligned}
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\},\\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\},\\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq A)\},\\
\mathsf{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\}.
\end{aligned}$$

Denote by $\mathcal{M}, \mathcal{N}$, respectively, the ideals of meager and Lebesgue null subsets of 2^ω. For $f, g \in \omega^\omega$, the order by *eventual dominance* is defined by $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but finitely many $n \in \omega$. A family $F \subseteq \omega^\omega$ is *bounded* if there is $h \in 2^\omega$ such that $f \leqslant^* h$ for all $f \in F$; and F is *dominating* if for any $g \in \omega^\omega$ there is $f \in F$ such that $g \leqslant^* f$. The cardinal invariants associated with the eventual dominance are $\mathfrak{b}$, the minimal cardinality of an unbounded set, and $\mathfrak{d}$, the minimal cardinality of a dominating set.

We shall consider two *Martin numbers*, $\mathfrak{m}_{\sigma\text{-centered}}$ and $\mathfrak{m}_{\sigma\text{-linked}}$. Let $\mathbb{P}$ be a poset. A set $A \subseteq \mathbb{P}$ is *centered* (*linked*, respectively) if for any $p, q \in A$ there is $r \in A$ ($r \in \mathbb{P}$) such that $r \leqslant p$ and $r \leqslant q$. A poset $\mathbb{P}$ is called *σ-centered* or *σ-linked*, respectively, if there exists a cover $\{P_i : i \in \omega\}$ of $\mathbb{P}$ such that each P_i is centered or linked.

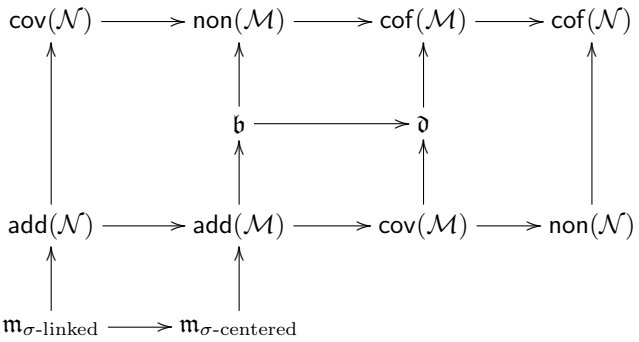
Given a cardinal κ, $\mathrm{MA}_{\sigma\text{-centered}}(\kappa)$ is the statement: For any σ-centered poset $\mathbb{P}$ and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$, with $|\mathcal{D}| \leqslant \kappa$, there is a filter that meets every member of $\mathcal{D}$, and $\mathrm{MA}_{\sigma\text{-linked}}(\kappa)$ is defined likewise. The corresponding Martin numbers are defined by

$$\begin{aligned}
\mathfrak{m}_{\sigma\text{-centered}} &= \min\{\kappa : \mathrm{MA}_{\sigma\text{-centered}}(\kappa) \text{ fails}\},\\
\mathfrak{m}_{\sigma\text{-linked}} &= \min\{\kappa : \mathrm{MA}_{\sigma\text{-linked}}(\kappa) \text{ fails}\}^1.
\end{aligned}$$

The provable inequalities between the listed cardinals are summarized in the following diagram[2].

[1] By Bell's theorem, $\mathfrak{m}_{\sigma\text{-centered}}$ is equal to the *pseudointersection number* $\mathfrak{p}$.

[2] As usual, the arrows in the diagram point from the smaller to the larger cardinal.



2^n means a set of $\{0,1\}$ sequences of length n if it makes sense; otherwise it denotes a number. $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ ordered by end extension. Given $p \in 2^{<\omega}$, the basic open set $\{x \in 2^\omega : p \subseteq x\}$ is denoted by $\langle p\rangle$. A set $T \subseteq 2^{<\omega}$ is a tree if it is closed under initial segments. Given a tree T the set $[T] = \{f \in 2^\omega : \forall n \in \omega\ f{\restriction}n \in T\}$ is the set of all branches of T. Given $s, t \in 2^{<\omega}$ the term $s^\frown t$ denotes concatenation of s followed by t.

A closed ball in a metric space, centered at x and of radius r, is denoted by $B(x, r)$.

2. Additivity, cofinality and cellularity of **Mon**

The values of additivity, cofinality and cellularity of **Mon** are easily derived from the following lemma.

Lemma 2.1. *Let $\mathscr{L}$ be a family of lines in $\mathbb{R}^2$. Then $\bigcup\mathscr{L}$ is σ-monotone if and only if $\mathscr{L}$ is countable.*

Proof. Suppose that $\mathscr{L} = \{L_\alpha : \alpha < \omega_1\}$ is an uncountable family of lines and aiming at contradiction suppose that $\bigcup_{\alpha<\omega_1} L_\alpha$ is σ-monotone. Then there is a countable family $\{C_n : n \in \omega\}$ of compact monotone sets such that $L_\alpha \subseteq \bigcup_{n\in\omega} C_n$ for all $\alpha < \omega_1$. By the Baire category argument, for each $\alpha < \omega_1$ there is $n \in \omega$ such that $L_\alpha \cap C_n$ contains an open straight segment. By the pigeonhole principle there is $n \in \omega$, an uncountable set $I \subseteq \omega_1$ and a family of open segments $\mathscr{S} = \{S_\alpha : \alpha \in I\}$ such that $S_\alpha \subseteq L_\alpha \cap C_n$ for each $\alpha \in I$. The union $X = \bigcup\mathscr{S}$ is a subset of C_n, hence it is a monotone set. Any distinct $S, S' \in \mathscr{S}$ that meet have exactly one common interior point. Hence the union $S \cup S'$ is not linearly ordered. It follows that the family $\mathscr{S}$ is pairwise disjoint. The segments S_α are open intervals in the linear order witnessing monotonicity of C_n and thus open sets in X. Since X is homeomorphic to a subset of the line, we arrived at a contradiction, as X is not *ccc*. □

Theorem 2.2. (i) $\mathsf{add}(\mathbf{Mon}) = \omega_1$,
(ii) $\mathsf{cof}(\mathbf{Mon}) = \mathfrak{c}$,
(iii) *the cellularity of* **Mon** *is* $\mathfrak{c}$.

Proof. (i) Since every line is monotone, any uncountable family of lines witnesses by the above lemma $\mathsf{add}(\mathbf{Mon}) \leqslant \omega_1$.

(ii) Now let $\mathscr{L} = \{\{x\} \times \mathbb{R} : x \in \mathbb{R}\}$. Suppose $\mathsf{cof}(\mathbf{Mon}) < \mathfrak{c}$. Then there is a family $\mathscr{B} \subseteq \mathbf{Mon}$ such that $|\mathscr{B}| < \mathfrak{c}$ and every element of $\mathscr{L}$ is covered by some

$B \in \mathscr{B}$. By the pigeonhole principle there is $B \in \mathscr{B}$ that covers uncountably many elements of $\mathscr{L}$, which is impossible by the above lemma. Thus $\mathsf{cof}(\mathbf{Mon}) = \mathfrak{c}$.

(iii) Split the line into $\mathfrak{c}$ many uncountable sets $\{A_\alpha : \alpha < \mathfrak{c}\}$. For each α the set $A_\alpha \times \mathbb{R}$ is not σ-monotone by the above lemma. Hence $\{A_\alpha \times \mathbb{R} : \alpha < \mathfrak{c}\}$ witnesses that cellularity of $\mathbf{Mon}$ is $\mathfrak{c}$. □

3. Porous sets

The other two cardinal invariants, $\mathsf{non}(\mathbf{Mon})$ and $\mathsf{cov}(\mathbf{Mon})$, are not so easy to evaluate. However there is a profound connection between monotone sets and porous sets, a notion from geometric measure theory, that can be used to approach them.

In this section we establish some relevant properties of σ-porous sets.

Definition 3.1 (see e.g. [5])**.** Let (X, d) be a metric space. A set $A \subseteq X$ is termed

- *porous at a point* $x \in X$ if there is $p > 0$ and $r_0 > 0$ such that for any $r \leqslant r_0$ there is $y \in X$ such that $B(y, pr) \subseteq B(x, r) \setminus A$,
- *porous* if it is porous at each point $x \in A$,[3]
- σ-porous if A is a countable union of porous sets.

Definition 3.2. Let X be a metric space. The ideal of all σ-porous sets in X is denoted $\mathbf{SP}(X)$.

We shall make use of a stronger form of porosity:

Definition 3.3. Let (X, d) be a metric space. A set $A \subseteq X$ is termed *strongly porous* if there is $p > 0$ such that for any $x \in X$ and any $r \in (0, \operatorname{diam} X)$, there is $y \in X$ such that $B(y, pr) \subseteq B(x, r) \setminus A$. The constant p will be called the *porosity constant of* A.

Lemma 3.4. *If A is strongly porous, then so is $\overline{A}$.*

Proof. Suppose $A \subseteq X$ is strongly porous and let p be its porosity constant. Denote by $B^\circ(x, r)$ the open ball with radius r centered at x. Clearly for each x and r there is y such that $B^\circ(y, pr) \subseteq B(x, r) \setminus A$ and since $B^\circ(y, pr)$ is open, it misses $\overline{A}$. Hence $\overline{A}$ is strongly porous with porosity constant any real number below p. □

Lemma 3.5. *Let X be a separable metric space. A set $A \subseteq X$ is σ-porous if and only if it is a countable union of strongly porous sets.*

Proof. It is obviously enough to prove that every porous set is a countable union of strongly porous sets. Splitting A into countably many pieces we may assume that

$$\exists r_0 > 0\ \exists p > 0\ \forall x \in A\ \forall r \leqslant r_0\ \exists y \in X\ B(y, pr) \subseteq B(x, r) \setminus A. \tag{1}$$

We now show that "$\forall x \in A$" can be replaced with "$\forall x \in X$" in (1). Let $x \in X$ and $r \leqslant r_0$. If $\operatorname{dist}(x, A) < \frac{r}{2}$, then there is $x' \in A$ such that $d(x, x') < \frac{r}{2}$. By (1) there is y such that $B(y, p\frac{r}{2}) \subseteq B(x', \frac{r}{2}) \setminus A \subseteq B(x, r) \setminus A$. If $\operatorname{dist}(x, A) \geqslant \frac{r}{2}$, then trivially $B(x, \frac{r}{3}) \subseteq B(x, r) \setminus A$. In either case, there is y such that $B(y, qr) \subseteq B(x, r) \setminus A$, where $q = \min\left(\frac{1}{3}, \frac{p}{2}\right)$. Overall

$$\exists r_0 > 0\ \exists q > 0\ \forall x \in X\ \forall r \leqslant r_0\ \exists y \in X\ B(y, qr) \subseteq B(x, r) \setminus A.$$

[3]There are many other notions of porosity, perhaps more than available names. This one is also called *lower porous*, *strongly porous* and *very porous*. We adhere to the simplest name in use for the scope of this paper.

The last step is to replace r_0 with $\operatorname{diam} X$. If $\operatorname{diam} X < \infty$, replace q with $q\frac{r_0}{\operatorname{diam} X}$. If $\operatorname{diam} X = \infty$, split A into countably many sets of diameter at most r_0. It is routine to show that each of these pieces is strongly porous. □

Proposition 3.6. *If X is separable, then $\mathbf{SP}(X)$ is generated by F_σ-sets.*

We now take a closer look at $\mathbf{SP}(2^\omega)$, the σ-porous sets on the Cantor set. The Cantor set 2^ω is equipped with a variant of the least difference metric defined by $d(x,y) = 2^{-n}$, where $n = \min\{i : x(i) \neq y(i)\}$ is the length of the maximal common initial segment.

Lemma 3.7. *A set $A \subseteq 2^\omega$ is strongly porous if and only if*

$$\exists n\ \forall p \in 2^{<\omega}\ \exists q \supseteq p\ |q| = |p| + n \wedge A \cap \langle q\rangle = \emptyset. \tag{2}$$

Proof. Note that $B(f, 2^{-n}) = \langle f{\restriction}n\rangle$ for all $x \in 2^\omega$ and $n \in \omega$. Thus if A is strongly porous and c is its porosity constant, then (2) obviously holds with any $n \geqslant \log_2 c$. On the other hand, if (2) holds, then A is strongly porous with porosity constant $c = 2^{-n}$. □

There are canonical strongly porous sets. For $n \in \omega$ and $\varphi : 2^{<\omega} \to 2^n$ set

$$X_\varphi = \{x \in 2^\omega : \forall k\ x \notin \langle x{\restriction}k{}^\frown\varphi(x{\restriction}k)\rangle\}$$

Proposition 3.8. (i) *X_φ is strongly porous for all $n \in \omega$ and each $\varphi : 2^{<\omega} \to 2^n$.*
(ii) *For every strongly porous set $A \subseteq 2^\omega$ there is $n \in \omega$ and $\varphi : 2^{<\omega} \to 2^n$ such that $A \subseteq X_\varphi$.*

Proof. Condition (2) can be obviously rephrased as follows:

$$\exists n\ \exists \varphi : 2^{<\omega} \to 2^n\ \forall p \in 2^{<\omega}\ A \cap \langle p{}^\frown\varphi(p)\rangle = \emptyset. \tag{3}$$

(i) Let $n \in \omega$ and $\varphi : 2^{<\omega} \to 2^n$. Let $p \in 2^{<\omega}$ and $x \in X_\varphi$. If $p \subseteq x$, then for $k = |p|$ the definition of X_φ yields $x \notin \langle p{}^\frown\varphi(p)\rangle$. If $p \not\subseteq x$, then $x \notin \langle p\rangle$ and *a fortiori* $x \notin \langle p{}^\frown\varphi(p)\rangle$. Hence (3) holds for X_φ.

(ii) Let A be strongly porous and n, φ be such that (3) holds. Let $x \in A$ and $k \in \omega$. Set $p = x{\restriction}k$ and use condition (3) to conclude that $A \cap \langle p{}^\frown\varphi(p)\rangle = \emptyset$ and in particular $x \notin \langle x{\restriction}k{}^\frown\varphi(x{\restriction}k)\rangle$. □

Let $T : 2^\omega \to [0,1]$ be the canonical mapping defined by $T(x) = \sum_{n\in\omega} 2^{-n-1}x(n)$. Let $\psi : 2^\omega \to 2^\omega{\times}2^\omega$ be the mapping that assigns to each $x \in 2^\omega$ the pair $x_1 : n \mapsto x(2n)$, $x_2 : n \mapsto x(2n+1)$.

Lemma 3.9. (i) *$A \subseteq 2^\omega$ is strongly porous if and only if $T[A] \subseteq [0,1]$ is strongly porous.*
(ii) *$A \subseteq 2^\omega{\times}2^\omega$ is strongly porous if and only if $(T{\times}T)[A] \subseteq [0,1]^2$ is strongly porous.*
(iii) *$A \subseteq 2^\omega$ is strongly porous if and only if $\psi[A] \subseteq 2^\omega{\times}2^\omega$ is strongly porous.*

Proof. (i) For each $k \in \omega$ let $\mathscr{D}_k$ be the family of binary closed intervals of length 2^{-k}, i.e. $\mathscr{D}_0 = \{[0,1]\}$, $\mathscr{D}_1 = \{[0,\frac12],[\frac12,1]\}$ etc., and $\mathscr{D}_k^\circ$ the family of binary open intervals of length 2^{-k}. It is easy to check that a set $A \subseteq [0,1]$ is strongly porous if and only if

$$\exists n\ \forall k\ \forall D \in \mathscr{D}_k^\circ\ \exists D' \in \mathscr{D}_{k+n}\ D' \subseteq D \setminus A.$$

Observe that if $p \in 2^{<\omega}$ then $T[\langle p\rangle] \in \mathscr{D}_{|p|}$ and that if $D \in \mathscr{D}_k^\circ$ then $T^{-1}(D) \subseteq \langle p\rangle$ for some $p \in 2^k$. Using (2) the proof is straightforward.

(ii) is proved in the same way, only the binary intervals are to be replaced by binary squares.

(iii) We show that if $A \subseteq 2^\omega$ is strongly porous, then so is $\psi[A]$, the proof of the other implication is similar. Assume that $A \subseteq 2^\omega$ is strongly porous and let $n \in \omega$ be as in (2). We may suppose n be even.

Let B be a closed ball in $2^\omega \times 2^\omega$ of radius r. There is k such that $2^{-k} \leqslant r < 2^{-k+1}$ and $p, q \in 2^k$ such that $B = \langle p\rangle \times \langle q\rangle$. The preimage of B is $\langle s\rangle$ for some $s \in 2^{2k}$. By (2) there is $t \supseteq s$, $|t| = |s| + n = 2k + n$ such that $A \cap \langle t\rangle = \emptyset$. Hence $\psi[A] \cap \psi[\langle t\rangle] = \emptyset$. It is straightforward that $\psi[\langle t\rangle]$ is a closed ball of radius 2^{-j}, where $j = \frac{1}{2}(2k+n) = k + \frac{n}{2}$, and that $\psi[\langle t\rangle] \subseteq B \setminus \psi[A]$. □

Theorem 3.10. $\mathsf{cov}(\mathbf{SP}(\mathbb{R})) = \mathsf{cov}(\mathbf{SP}(\mathbb{R}^2)) = \mathsf{cov}(\mathbf{SP}(2^\omega))$ *and likewise for* non, add *and* cof.

Proof. add, cov and cof are obviously preserved back and forth by any mapping that preserves strongly porous sets both ways; for non the mapping is moreover required to be countable-to-one. By the above lemma these conditions are met by all of the mappings T, $T \times T$ and ψ. Therefore

$$\mathsf{add}(\mathbf{SP}([0,1])) = \mathsf{add}(\mathbf{SP}(2^\omega)) = \mathsf{add}(\mathbf{SP}(2^\omega \times 2^\omega)) = \mathsf{add}(\mathbf{SP}([0,1]^2))$$

and it is clear that $\mathsf{add}(\mathbf{SP}([0,1])) = \mathsf{add}(\mathbf{SP}(\mathbb{R}))$ and $\mathsf{add}(\mathbf{SP}([0,1]^2)) = \mathsf{add}(\mathbf{SP}(\mathbb{R}^2))$. The same argument works for the other three invariants. □

So as to the cardinal invariants, it makes no difference which of the three $\mathbf{SP}$ ideals we investigate. Because of its simple combinatorial description we vote for $\mathbf{SP}(2^\omega)$ and from now on we abbreviate $\mathbf{SP}(2^\omega)$ by $\mathbf{SP}$.

4. Monotone vs. porous sets

We now show that monotone sets in $\mathbb{R}^2$ are strongly porous and that rectangles of porous sets are monotone.

The following combinatorial lemma is a simplified version of [9, Lemma 5.2]. We consider cyclic groups $\mathbb{Z}_m$. The corresponding subtraction modulo m is denoted $\ominus$.

Lemma 4.1. *Let $\mathbb{Z}_m$ be the cyclic group of an even order m. For any linear order $\prec$ on $\mathbb{Z}_m$ there are $x \prec y \prec z$ in $\mathbb{Z}_m$ such that $z \ominus x = 1$ and $z \ominus y = \frac{m}{2}$.*

Thinking of $\mathbb{Z}_m$ as a regular polygon, the lemma says that for any linear order there are $x \prec y \prec z$ such that x and z are neighboring vertices and y is opposite to x.

Theorem 4.2. *Every monotone set $X \subseteq \mathbb{R}^2$ is strongly porous. Consequently* $\mathbf{Mon} \subseteq \mathbf{SP}(\mathbb{R}^2)$.

Proof. Suppose X is monotone. Let $\prec$ be a linear order on X and $c > 0$ such that

$$x \prec y \prec z \implies |y - x| \leqslant c|z - x|. \tag{4}$$

We show that $p = \frac{1}{2c+3}$ is a porosity constant of X.

Aiming towards contradiction assume that there is $x \in \mathbb{R}^2$ and $r > 0$ such that

$$B(z, rp) \cap X \neq \emptyset \text{ for all } z \in B(x, r(1-p)).$$

Choose an even $m \in \mathbb{N}$ subject to

$$2\pi c \leqslant m < \pi(2c+1). \tag{5}$$

Let $\{x_1, \dots, x_m\}$ be the set of vertices of a regular m-polygon centered at x, with outer radius $r(1-p)$. We assume that the vertex x_{i+1} is next to x_i for each $i < m$.

By assumption, each of the balls $B(x_i, rp)$ intersects X. For each $i \leqslant m$ choose a point $z_i \in B(x_i, rp) \cap X$. The set $Z = \{z_1, \dots, z_m\}$ is thus a subset of X. We claim that Z is not c-monotone.

We first prove that $i \neq j$ implies $z_i \neq z_j$. Clearly $d(x_i, x_j) \geqslant 2r(1-p)\sin\frac{\pi}{m}$ and since $\sin\alpha > \frac{\alpha}{1+\alpha}$ holds for all $\alpha \in (0, \pi/2)$, (5) and the definition of p yield

$$d(x_i, x_j) > 2r(1-p)\frac{\pi/m}{1+\pi/m} = 2r\frac{2c+2}{2c+3}\frac{1}{1+m/\pi} > \frac{2rp(2c+2)}{1+2c+1} = 2rp.$$

Therefore the balls $B(x_i, rp)$ and $B(x_j, rp)$ are disjoint, and $z_i \neq z_j$ follows. Hence Z identifies with $\mathbb{Z}_m$.

Since $Z \subseteq X$, (4) holds for all $x, y, z \in Z$. By Lemma 4.1 there are $z_i \prec z_j \prec z_k$ such that $k \ominus i = 1$ and $j \ominus i = \frac{m}{2}$. Therefore

$$\begin{aligned} d(z_i, z_k) &\leqslant d(x_i, x_k) + 2rp = 2r(1-p)\sin\tfrac{\pi}{m} + 2rp, \\ d(z_i, z_j) &\geqslant d(x_i, x_j) - 2rp = 2r(1-2p). \end{aligned}$$

Using (5), the trivial estimate $\sin\alpha < \alpha$ and the definition of p yields

$$\begin{aligned} d(z_i, z_k) &< 2r\frac{1-p}{2c} + 2rp = 2rp\frac{2c+1}{c}, \\ d(z_i, z_j) &\geqslant 2rp(2c+1). \end{aligned}$$

Therefore $cd(z_i, z_k) < d(z_i, z_j)$, and since $z_i \prec z_j \prec z_k$, we arrived at a contradiction. □

Corollary 4.3. *Every* **SP** *set and, in particular, every σ-monotone set in $\mathbb{R}^2$ is contained in an F_σ-set of Lebesgue measure zero.*

Proof. P. Mattila [7] and A. Salli [15] (or see [8]) proved that every strongly porous set in the plane has Hausdorff dimension strictly below 2. In particular, every strongly porous set is Lebesgue null. Apply 3.6 and the previous theorem. □

Recall that $\mathcal{E}$, the *intersection ideal*, is the σ-ideal in 2^ω generated by closed sets of measure zero. It is obvious that $\mathsf{non}(\mathcal{E}) \leqslant \min\big(\mathsf{non}(\mathcal{M}), \mathsf{non}(\mathcal{N})\big)$. and it is consistent that $\mathsf{non}(\mathcal{E}) < \min\big(\mathsf{non}(\mathcal{M}), \mathsf{non}(\mathcal{N})\big)$. Also $\mathsf{cov}(\mathcal{E}) \geqslant \max\big(\mathsf{cov}(\mathcal{M}), \mathsf{cov}(\mathcal{N})\big)$, see [1].

Corollary 4.4. $\mathsf{non}(\mathbf{Mon}) \leqslant \mathsf{non}(\mathbf{SP}) \leqslant \mathsf{non}(\mathcal{E})$ *and* $\mathsf{cov}(\mathbf{Mon}) \geqslant \mathsf{cov}(\mathbf{SP}) \geqslant \mathsf{cov}(\mathcal{E})$

Lemma 4.5. *If $A \subseteq [0,1]$ is strongly porous, then it is Lipschitz equivalent to an ultrametric space.*

Proof. Let p be a porosity constant of A. Build a family $\{I_s : s \in 2^{<\omega}\}$ of closed intervals as follows: $I_\emptyset = [0,1]$. When I_s is constructed, remove from I_s an open interval of length $p \operatorname{diam} I_s$ disjoint with A. This is possible by porosity of A. Let $I_{s^\frown 0}$ and $I_{s^\frown 1}$ be, respectively, the left and right remaining closed intervals.

Let $C = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} I_s$. Clearly $A \subseteq C$.

If $s \in 2^n$, then obviously $\operatorname{diam} I_s \leqslant (1-p)^n$. Hence the set $\bigcap_{n\in\omega} I_{f\restriction n}$ consists of exactly one point for each $f \in 2^\omega$. Therefore if $x, y \in C$, $x \neq y$, then there is a unique $s \in 2^{<\omega}$ such that $x \in I_{s^\frown 0}$ and $y \in I_{s^\frown 1}$ (or the other way). It follows that $p \operatorname{diam} I_s \leqslant d(x,y) \leqslant \operatorname{diam} I_s$. Thus letting $\rho(x,y) = \operatorname{diam} I_s$ defines an ultrametric that is Lipschitz equivalent to the Euclidean metric. □

Proposition 4.6. *If $A, B \subseteq \mathbb{R}$ are σ-porous, then $A \times B \subseteq \mathbb{R}^2$ is σ-monotone.*

Proof. It is enough to show that if $A, B \subseteq \mathbb{R}$ are strongly porous, then $A \times B \subseteq \mathbb{R}^2$ is monotone. By the above lemma both A, B are Lipschitz equivalent to ultrametric spaces. Since the product (equipped with the maximum metric) of ultrametric spaces is ultrametric, $A \times B$ is Lipschitz equivalent to an ultrametric space. Since every ultrametric space is monotone ([10, 2.3]) and Lipschitz equivalence preserves monotonicity ([10, 2.2]), we are done. □

Corollary 4.7. $\mathsf{cov}(\mathbf{Mon}) = \mathsf{cov}(\mathbf{SP})$ *and* $\mathsf{non}(\mathbf{Mon}) = \mathsf{non}(\mathbf{SP})$.

Proof. Consider the σ-ideal in the plane generated by the family of rectangles $\mathcal{J} = \{A \times B : A, B \in \mathbf{SP}(\mathbb{R})\}$. It is easy to check that $\mathsf{non}(\mathcal{J}) = \mathsf{non}(\mathbf{SP}(\mathbb{R}))$ and $\mathsf{cov}(\mathcal{J}) = \mathsf{cov}(\mathbf{SP}(\mathbb{R}))$. Therefore theorem 3.10 yields $\mathsf{cov}(\mathcal{J}) = \mathsf{cov}(\mathbf{SP}(\mathbb{R}^2)) = \mathsf{cov}(\mathbf{SP})$ and $\mathsf{non}(\mathcal{J}) = \mathsf{non}(\mathbf{SP}(\mathbb{R}^2)) = \mathsf{non}(\mathbf{SP})$. Since $\mathcal{J} \subseteq \mathbf{Mon} \subseteq \mathbf{SP}(\mathbb{R}^2)$ by theorem 4.2 and proposition 4.6, we are done. □

5. Consistency results

In this section we present two forcing notions closely related to the ideal **SP** (and hence to the **Mon** ideal, too) and use them to prove some consistency results involving the cardinal invariants $\mathsf{non}(\mathbf{SP}) = \mathsf{non}(\mathbf{Mon})$ and $\mathsf{cov}(\mathbf{SP}) = \mathsf{cov}(\mathbf{Mon})$.

Recall that every strongly porous set in 2^ω is contained in a set of the form $X_\varphi = \{x \in 2^\omega : \forall k\ x \notin \langle x{\restriction}k^\frown\varphi(x{\restriction}k)\rangle\}$, where $\varphi : 2^{<\omega} \to 2^n$ for some fixed $n \in \omega$. There is a natural forcing making the ground-model reals σ-porous.

Given $n \geqslant 2$ define $\mathbb{P}^n$ as follows: $(s, F) \in \mathbb{P}^n$ if and only if

(i) s is a partial function from $2^{<\omega}$ to 2^n,
(ii) F is a finite subset of 2^ω,
(iii) $\forall \sigma \in \mathrm{dom}(s)\ F \cap \langle \sigma^\frown s(\sigma)\rangle = \emptyset$,
(iv) $\forall \sigma \in 2^{<\omega} \exists \rho \in 2^n\ F \cap \langle \sigma^\frown \rho\rangle = \emptyset$,

and order $\mathbb{P}^n$ by $(s, F) \leqslant (s', F')$ if $s \supseteq s'$ and $F \supseteq F'$.

Let $\mathbb{P} = \prod_{n \geqslant 2} \mathbb{P}^n$ be the finite support product of the forcing notions $\mathbb{P}^n$, ordered coordinatewise.

Lemma 5.1. *The partial order $\mathbb{P}$ is σ-linked.*

Proof. It suffices to show that each $\mathbb{P}^n$ is σ-linked, since a finite support product of a countable family of σ-linked partial orders is σ-linked.

Fix $n \geqslant 2$. For $k \in \omega$, $E \subseteq 2^{<k}$ and for a finite partial function s from $2^{<\omega}$ to 2^n such that $\mathrm{dom}(s) \subseteq 2^{k-n}$ let $\Delta_F = \min\{i : \forall x, y \in F (x \neq y \Rightarrow x{\restriction}i \neq y{\restriction}i\}$ and

$$\mathcal{F}_{k,E,s} = \big\{(s, F) \in \mathbb{P}^n : k \geqslant \Delta_F, \{x{\restriction}k : x \in F\} = E\big\}.$$

It is immediate that $\mathcal{F}_{k,E,s}$ is linked, as for any $(s, F), (s, F') \in \mathcal{F}_{k,E,s}$ the pair $(s, F \cup F') \in \mathbb{P}^n$. Since every element of $\mathbb{P}^n$ belongs to some $\mathcal{F}_{k,E,s}$, we are done: for the family of parameters k, E, s is countable. □

Theorem 5.2. $\mathfrak{m}_{\sigma\text{-linked}} \leqslant \mathsf{non}(\mathbf{SP})$.

Proof. Fix a set $X \subseteq 2^\omega$ od size $< \mathfrak{m}_{\sigma\text{-linked}}$. Consider the sets

$$H_x = \{p \in \mathbb{P} : \exists n \in \omega\ \exists (s, F)\ x \in F \wedge p(n) = (s, F)\}, \quad x \in X,$$
$$D_{\sigma,n} = \{p \in \mathbb{P} : \exists (s, F)\ p(n) = (s, F) \wedge \sigma \in \mathrm{dom}(s)\}, \quad n \in \omega, \sigma \in 2^{<\omega}.$$

Obviously, these sets are dense in $\mathbb{P}$. Let G be a filter intersecting all H_x and $D_{\sigma,n}$. Such a filter exists, for $|X| < \mathfrak{m}_{\sigma\text{-linked}}$. Define

$$\varphi_n(\sigma) = \rho \quad \iff \quad \exists p \in \mathbb{P}\ p(n) = (s, F) \wedge \sigma \in \mathrm{dom}(s) \wedge s(\sigma) = \rho.$$

Then

(a) $\varphi_n : 2^{<\omega} \to 2^n$ for each $n \geqslant 2$,
(b) $X \subseteq \bigcup_{n\geqslant 2} X_{\varphi_n}$.

To see (b), let $x \in X$. There is $p \in G$ and $n \geqslant 2$ such that $p(n) = (s, F)$ and $x \in F$. We claim that $x \in X_{\varphi_n}$. If not, then there is $q \in G$ with $q(n) = (s_q, F_q)$ and $k \in \omega$ such that $x{\restriction}k \in \mathrm{dom}(s_q)$ and $x \in \langle x{\restriction}k^\frown s_q(x{\restriction}k)\rangle$. Since G is a filter, there is a condition $r \in \mathbb{P}$ such that $r \leqslant p$ and $r \leqslant q$. In particular, $r(n) = (s_r, F_r) \geqslant (s, F)$ and $r(n) = (s_r, F_r) \geqslant (s_q, F_q)$. However, $x \in F \subseteq F_r$ and $x{\restriction}k \in \mathrm{dom}(s_q) \subseteq \mathrm{dom}(s_r)$, while $x \in \langle x{\restriction}k^\frown s_r(x{\restriction}k)\rangle$, which contradicts that r is a condition. (b) is proved.

It is clear that (a) and (b) show that X is σ-porous, as required. □

The same forcing notion lets us prove that $\mathsf{cov}(\mathbf{SP})$ need not be equal to $\mathfrak{c}$:

Theorem 5.3. *It is relatively consistent with* ZFC *that* $\mathsf{cov}(\mathbf{SP}) < \mathfrak{c}$.

Proof. Let $V \vDash \neg\mathrm{CH}$ and let $\mathbb{P}_{\omega_1}$ be a finite support iteration of the forcing $\mathbb{P}$. Let G be $\mathbb{P}_{\omega_1}$-generic over V. Then $V[G] \vDash \neg\mathrm{CH}$, since $\mathbb{P}_{\omega_1}$ is *ccc*, and $V[G] \vDash \mathsf{cov}(\mathbf{SP}) = \omega_1$, since $\mathbb{P}$ makes the set of ground-model reals σ-porous. □

Next we show that in theorem 5.2, the cardinal $\mathfrak{m}_{\sigma\text{-linked}}$ cannot be replaced with $\mathfrak{m}_{\sigma\text{-centered}}$. In other words, $\mathsf{non}(\mathbf{SP}) < \mathfrak{m}_{\sigma\text{-centered}}$ is consistent.

Theorem 5.4. *It is relatively consistent with* ZFC *that* $\mathfrak{m}_{\sigma\text{-centered}} = \mathfrak{c} > \omega_1$ *and* $\mathsf{non}(\mathbf{SP}) = \omega_1$.

Proof. Say that a partial order $\mathbb{P}$ *strongly preserves* $\mathsf{non}(\mathbf{SP})$ if for every $\mathbb{P}$-name $\dot{X}$ for a strongly porous set there is a σ-strongly porous set Y such that

$$\forall x \in 2^\omega (x \notin Y \Rightarrow \mathbb{1} \Vdash \text{“}x \notin \dot{X}\text{”}).$$

In other words, the set Y covers the ground-model part of $\dot{X}$. Equivalently, $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP})$ if for every $\mathbb{P}$-name φ such that

$$\mathbb{1} \Vdash \text{“}\dot{\varphi} : 2^{<\omega} \to 2^n\text{” for some fixed } n \geqslant 2$$

there are functions $\{\varphi_i : i \in \omega\}$, $\varphi_i : 2^{<\omega} \to 2^n$ such that for any $x \in 2^\omega$,

$$\text{if } \forall i\ \exists k\ x \in \langle x{\restriction}k^\frown \varphi_i(x{\restriction}k)\rangle, \text{ then } \mathbb{1} \Vdash \text{“}\exists k\ x \in \langle x{\restriction}k^\frown \dot{\varphi}(x{\restriction}k)\rangle\text{”}.$$

It should be obvious that if a forcing $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP})$, then for any $G \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over a model V, $V[G] \vDash 2^\omega \cap V \notin \mathbf{SP}$.

Claim. *Every σ-centered forcing strongly preserves* $\mathsf{non}(\mathbf{SP})$.

Proof of the claim. Let $\mathbb{B}$ be a σ-centered complete Boolean algebra and let $\{\mathcal{F}_i : i \in \omega\}$ be a family of ultrafilters on $\mathbb{B}$ such that $\mathbb{B}^+ = \bigcup_i \mathcal{F}_i$. Let $\dot{\varphi}$ be a $\mathbb{B}$-name such that for some $n \geqslant 2$, $\mathbb{1} \Vdash \text{“}\varphi : 2^{<\omega} \to 2^n\text{”}$. For $i \in \omega$ let

$$\varphi_i(\sigma) = \rho \quad \iff \quad [\![\dot{\varphi}(\sigma) = \rho]\!] \in \mathcal{F}_i.$$

The functions $\varphi_i : 2^{<\omega} \to 2^n$ are then well-defined. To finish the proof assume that $x \in 2^\omega$ is such that

$$\forall i \in \omega\ \exists k \in \omega\ x \in \langle x{\restriction}k^\frown \varphi_i(x{\restriction}k)\rangle. \tag{6}$$

We need to show that for every $p \in \mathbb{B}^+$ there are $q \leqslant p$ and $k \in \omega$ such that $q \Vdash$ "$x \in \langle x{\restriction}k^\frown \dot\varphi(x{\restriction}k)\rangle$".

Fix p and let $i \in \omega$ be such that $p \in \mathcal{F}_i$. By (6) there is $k \in \omega$ such that

- $x \in \langle x{\restriction}k^\frown \varphi_i(x{\restriction}k)\rangle$,
- $[\![\dot\varphi(x{\restriction}k) = \varphi_i(x{\restriction}k)]\!] \in \mathcal{F}_i$.

Since $\mathcal{F}_i$ is centered, there is $q \in \mathcal{F}_i$ such that $q \leqslant p$ and $q \leqslant [\![\dot\varphi(x{\restriction}k) = \varphi_i(x{\restriction}k)]\!]$, i.e. $q \Vdash$ "$x \in \langle x{\restriction}k^\frown \dot\varphi(x{\restriction}k)\rangle$". □

Claim. *Finite support iteration of ccc partial orders that strongly preserve* $\mathsf{non}(\mathbf{SP})$ *strongly preserves* $\mathsf{non}(\mathbf{SP})$ *as well.*

Proof of the claim. The claim clearly holds for iterations of finite and uncountable lengths. It thus suffices to show that if $\mathbb{P} = \langle \langle \mathbb{P}_n, \mathbb{Q}_n\rangle : n \in \omega\rangle$ is a finite support iteration such that $\mathbb{P}_0 = \{\mathbb{1}\}$ and $\mathbb{P}_n \Vdash$ "$\dot{\mathbb{Q}}_n$ strongly preserves $\mathsf{non}(\mathbf{SP})$", then $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP})$. In order to do this let $\dot\varphi$ be a $\mathbb{P}$-name such that for some $n \geqslant 2$ $\mathbb{1} \Vdash$ "$\dot\varphi : 2^{<\omega} \to 2^n$".

In each intermediate extension $V[G_j]$ find a function $\varphi_j : 2^{<\omega} \to 2^n$ and a decreasing sequence of conditions $p_{j,i} \in \mathbb{P}_{[j,\omega)}$ so that

$$p_{j,i} \Vdash_{\mathbb{P}_{[j,\omega)}} \text{"}\varphi_j{\restriction}2^{\leqslant i} = \dot\varphi{\restriction}2^{\leqslant i}\text{"}.$$

Since $\mathbb{P}_j$ strongly preserves $\mathsf{non}(\mathbf{SP})$, there are (in V) functions $\{\varphi_{j,i} : i \in \omega\}$ such that for any $x \in 2^\omega$

$$\text{if } \forall i\ \exists k\ x \in \langle x{\restriction}k^\frown \varphi_{i,j}(x{\restriction}k)\rangle, \text{ then } \mathbb{1}_{\mathbb{P}_j} \Vdash \text{"}\exists k\ x \in \langle x{\restriction}k^\frown \dot\varphi_j(x{\restriction}k)\rangle\text{"}.$$

In order to prove that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP})$ is suffices to show that for any $x \in 2^\omega$

$$\text{if } \forall i, j\ \exists k\ x \in \langle x{\restriction}k^\frown \varphi_{i,j}(x{\restriction}k)\rangle, \text{ then } \mathbb{1}_{\mathbb{P}} \Vdash \text{"}\exists k\ x \in \langle x{\restriction}k^\frown \dot\varphi(x{\restriction}k)\rangle\text{"}.$$

Aiming for a contradiction assume that there is $p \in \mathbb{P}$ such that $p \Vdash$ "$\forall k\ x \notin \langle x{\restriction}k^\frown \dot\varphi(x{\restriction}k)\rangle$". Then there is $j \in \omega$ such that $p \in \mathbb{P}_j$. Let G_j be $\mathbb{P}_j$-generic over V such that $p \in G_j$. Then (in $V[G_j]$) there is $k \in \omega$ such that

$$V[G_j] \vDash x \in \langle x{\restriction}k^\frown \varphi_j(x{\restriction}k)\rangle.$$

However, then $p^\frown p_{j,k} \Vdash$ "$x \in \langle x{\restriction}k^\frown \varphi(x{\restriction}k)\rangle$", which is a contradiction. □

We are now ready to prove the theorem. Start with a model V of GCH and let $\kappa > \omega$ be a regular cardinal. Using a standard bookkeeping argument construct a finite support iteration $\mathbb{P}$ of length κ of σ-centered partial orders of size less that κ, so that any such partial order which appears in an intermediate model is listed cofinally along the iteration. In this way (see e.g. [6] for the details of such bookkeeping) one constructs a model $V[G]$, where $\mathfrak{m}_{\sigma\text{-centered}} = \mathfrak{c} = \kappa$. On the other hand, the two claims entail that the set of ground-model elements of 2^ω is not in $\mathbf{SP}$, hence $\mathsf{non}(\mathbf{SP}) = \omega_1$. □

Remark 5.5. In fact, the cardinal characteristics $\mathsf{non}(\mathbf{SP})$ and $\mathfrak{m}_{\sigma\text{-centered}}$ are mutually incomparable. A model, where $\mathfrak{m}_{\sigma\text{-centered}} < \mathsf{non}(\mathbf{SP})$ can be described as follows: Start with a model V of $\mathrm{MA}_{\sigma\text{-linked}} + \neg\mathrm{CH}$ in which there is a Suslin tree $\mathbb{T}$. Force with the tree $\mathbb{T}$ (with reverse order). Let G be $\mathbb{T}$-generic over V. Then, in $V[G]$, $\mathfrak{m}_{\sigma\text{-centered}} = \omega_1$ (see e.g. [4]) and $\mathsf{non}(\mathbf{SP}) > \omega_1$, as $\mathbb{T}$ does not add reals and preserves cardinals.

We have described a natural forcing which increases $\mathsf{non}(\mathbf{SP})$. A natural forcing for increasing $\mathsf{cov}(\mathbf{SP})$ falls into the scope of the J. Zapletal's book [19]. The forcing $\mathbb{P}_{\mathbf{SP}} = \mathrm{Borel}(2^\omega)/\mathbf{SP}$ is proper, ω^ω-bounding and category preserving by a general theorem of Zapletal [19, 4.1.8], since the σ-ideal $\mathbf{SP}$ is σ-generated by a σ-compact collection of compact sets (a simple extension of our 3.6). We will show that the forcing, in fact, even has the Sacks property and preserves P-points. In order to do this we present an equivalent, combinatorial, version of the forcing.

Definition 5.6. A tree $T \subseteq 2^{<\omega}$ is *hyper-perfect* if

$$\forall s \in T\ \forall n\ \exists t \supseteq s\ \forall r \in 2^n\ t^\frown r \in T. \tag{7}$$

A set $P \subseteq 2^\omega$ is *hyper-perfect* if there is a hyper-perfect tree T such that $P = [T]$.

Zajíček and Zelený [22, 18] and Rojas-Rebolledo [14] proved that every $\mathbf{SP}$-positive Borel set contains a perfect $\mathbf{SP}$-positive subset. We will need a slight extension of their result.

Theorem 5.7. *Every Borel $\mathbf{SP}$-positive subset of 2^ω contains a hyper-perfect set.*

Proof. Let $A \subseteq 2^\omega$ be Borel, $A \notin \mathbf{SP}$. According to the mentioned theorem of Zajíček and Zelený [18, 3.4] we may assume that A is closed. *Mutatis mutandis* we may further assume that no nonempty relatively open subset of A is σ-porous. Writing $T = \{s \in 2^{<\omega} : \exists x \in A\ s \subseteq x\}$ and using condition (2) of Lemma 3.7 the latter reads

$$\forall s \in T\ \forall n\ \exists t \supseteq s\ \forall r \supseteq t\ (|r| = |t| + n \implies A \cap \langle r\rangle \neq \emptyset),$$

which is nothing but condition (7). Hence the tree T is hyper-perfect. Since A is closed, $[T] \subseteq A$, as required. □

Corollary 5.8. *The forcing notions $\mathbb{P}_{\mathbf{SP}}$ and $\mathbb{HP} = \{T \subseteq 2^{<\omega} : T$ is hyper-perfect$\}$ are forcing equivalent.*

Proof. By the above theorem 5.7 the function $\varphi : \mathbb{HP} \to \mathbb{P}_{\mathbf{SP}}$ defined by $\varphi(T) = [T]$ is a dense embedding. □

Theorem 5.9. *It is relatively consistent with* ZFC *that $\mathsf{cof}(\mathcal{N}) = \omega_1$ and $\mathsf{cov}(\mathbf{SP}) = \omega_2$.*

Proof. Start with a model of CH and iterate the forcing $\mathbb{HP}$ with countable support ω_2 times. It is immediate from the definition that $\mathbb{HP}$ adds a new real which is not contained in any element of $\mathbf{SP}$ coded in the ground-model. Hence $\mathbb{HP}_{\omega_2}$ forces $\mathsf{cov}(\mathbf{SP}) = \omega_2$. On the other hand, a standard fusion argument shows that the forcing $\mathbb{HP}$ has the *Sacks property*, which in turn is preserved under countable support iteration of proper partial orders (see [1, 6.3.F]) and hence $V^{\mathbb{HP}_{\omega_2}} \vDash \mathsf{cof}(\mathcal{N}) = \omega_1$. □

By [19, Theorem 4.1.8] of J. Zapletal the forcing $\mathbb{P}_{\mathbf{SP}}$ (and hence $\mathbb{HP}$) does not add independent (splitting) reals. By another theorem of J. Zapletal [20] a definable forcing which does not add an independent real and has the Sacks property (in fact, a lot less is needed) preserves P-points, which is preserved by a countable support iteration by a theorem of Shelah [2] (or see [1, 6.2.6]). So in our model there is an ultrafilter (P-point) of character ω_1.

6. Concluding remarks

Cardinal invariants of $\mathbf{SP}(X)$ obviously depend on the metric space X. To illustrate it, we calculate $\mathsf{non}(\mathbf{SP}(\omega^\omega))$ and $\mathsf{cov}(\mathbf{SP}(\omega^\omega))$. The metric we consider is $d(f,g) = 2^{-n}$, where $n = \min\{i : f(i) \neq g(i)\}$.

Theorem 6.1. $\mathsf{non}(\mathbf{SP}(\omega^\omega)) = \mathsf{non}(\mathcal{M})$ *and* $\mathsf{cov}(\mathbf{SP}(\omega^\omega)) = \mathsf{cov}(\mathcal{M})$.

Proof. For each $g \in \omega^\omega$ let $X_g = \{f \in \omega^\omega : \forall^\infty n\ g(n) \neq f(n)\}$ and let $\mathcal{J}$ be the ideal on ω^ω generated by $\{X_g : g \in \omega^\omega\}$.

Claim. $\mathcal{J} \subseteq \mathbf{SP}(\omega^\omega) \subseteq \mathcal{M}$.

Proof of the claim. The inclusion $\mathbf{SP}(\omega^\omega) \subseteq \mathcal{M}$ is obvious. Fix g and for every $k \in \omega$ consider the set $X_g^k = \{f \in \omega^\omega : \forall n \geqslant k\ g(n) \neq f(n)\}$. It is clear that $X_g = \bigcup_{k\in\omega} X_g^k$. Therefore it is enough to show that X_g^k is porous for each k. So suppose that $B(h,r)$ is a ball with $2^{-n+1} > r \geqslant 2^{-n} \geqslant 2^{-k}$. Then there is $p \in \omega^{<\omega}$ such that $\langle p\rangle \subseteq B(h, 2^{-n})$ and $|p| = n \geqslant k$. Let $q = p^\frown g(n)$ and consider $\langle q\rangle$, which is a ball of radius at least quarter r. If $f \in X_g^k$, then $f(n) \neq g(n)$. Therefore f does not extend q, i.e. $f \notin \langle q\rangle$. We showed that $X_g^k \cap \langle q\rangle = \emptyset$. Conclude that X_g^k is porous, with $r_0 = 2^{-k}$ and porosity constant $\frac{1}{4}$. Thus $\mathcal{J} \subseteq \mathbf{SP}(\omega^\omega)$. The claim is proved.

By [1, 2.4.1,2.4.7], $\mathsf{non}(\mathcal{J}) = \mathsf{non}(\mathcal{M})$ and $\mathsf{cov}(\mathcal{J}) = \mathsf{cov}(\mathcal{M})$. So the theorem follows at once from the claim. □

Note that since every bounded set in ω^ω is obviously σ-porous, theorem 5.7 fails for ω^ω. Also corollary 4.4 and theorem 5.9 fail for ω^ω.

A set in a metric space A is termed *upper porous at a point* x if there is a constant $p > 0$ and a sequence $r_n \to 0$ such that for every n there is y_n such that $B(y_n, pr_n) \subseteq B(x, r_n) \setminus A$. A set is *upper porous* if it is upper porous at each of its points. General references: [16, 17].

J. Brendle and M. Repický investigated cardinal characteristics of the ideal $\mathbf{UP}$ of σ-upper porous sets on the line. J. Brendle [3] proved that (a) $\mathsf{add}(\mathbf{UP}) = \omega_1$ and (b) $\mathsf{cof}(\mathbf{UP}) = \mathfrak{c}$. M. Repický [11, 12, 13] showed that (c) $\mathsf{cov}(\mathbf{UP}) \leqslant \mathsf{cof}(\mathcal{N})$, (d) $\mathsf{non}(\mathbf{UP}) \geqslant \mathfrak{m}_{\sigma\text{-centered}}$ and (e) $\mathsf{non}(\mathbf{UP}) \geqslant \mathsf{add}(\mathcal{N})$. As to (c) and (d), our theorems show that analogous inequalities $\mathsf{cov}(\mathbf{SP}) \leqslant \mathsf{cof}(\mathcal{N})$ and $\mathsf{non}(\mathbf{SP}) \geqslant \mathfrak{m}_{\sigma\text{-centered}}$ consistently fail. However, we do not know if the analogies of (a), (b) and (e) for $\mathbf{SP}$ hold.

Question 6.2. Is it true that

(i) $\mathsf{add}(\mathbf{SP}) = \omega_1$?
(ii) $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$?
(iii) $\mathsf{non}(\mathbf{SP}) \geqslant \mathsf{add}(\mathcal{N})$?

Cardinal invariants of the ideal $\mathbf{Mon}(X)$ of σ-monotone subsets of a metric space X probably also depend on the metric space X. Zelený [21] constructed an absolutely continuous function $f : [0,1] \to \mathbb{R}$, which graph X is not σ-porous and thus not σ-monotone. Since f is absolutely continuous, X is a rectifiable curve of finite length. Thus the 1-dimensional Hausdorff measure is a natural measure on X. It can be shown that $X = A \cup B$ with A σ-monotone and B of measure zero. This can be rephrased as follows.

Proposition 6.3. *There is a compatible metric ρ on $[0,1]$ such that*

(i) *$Z = ([0,1], \rho)$ is not σ-monotone,*
(ii) *there is a Lebesgue null set $A \subseteq Z$ such that $Z \setminus A$ is σ-monotone.*

In particular, in contrast with the situation in the plane, not every σ-monotone subset of Z is null. We wonder:

Question 6.4. What can one say about $\mathsf{add}(\mathbf{Mon}(X))$, $\mathsf{non}(\mathbf{Mon}(X))$, $\mathsf{cov}(\mathbf{Mon}(X))$ and $\mathsf{cof}(\mathbf{Mon}(X))$ when X is

(i) the space Z above,
(ii) the Hilbert cube,
(iii) the Urysohn universal space?

As to the minimal size of a metric space that is not σ-monotone, we know the following.

Theorem 6.5. *Every separable metric space of size $< \mathfrak{m}_{\sigma\text{-linked}}$ is σ-monotone.*

Proof. Fix a separable metric space X such that $|X| < \mathfrak{m}_{\sigma\text{-linked}}$ and a countable base $\mathcal{B}$. Given $n > 0$ define $\mathbb{P}^n$ as follows: $(\mathcal{U}, F, <) \in \mathbb{P}^n$ if and only if

(i) $\mathcal{U} \in [\mathcal{B}]^{<\omega}$,
(ii) $\max_{U\in\mathcal{U}} \operatorname{diam} U < \min_{V\neq W\in\mathcal{U}} \operatorname{dist}(V, W)$,
(iii) $<$ is a linear order on $\mathcal{U}$,
(iv) if $U < V < W$ and $x \in U, y \in V, z \in W$, then $d(x, y) \leqslant nd(x, z)$,
(v) $F \subseteq \bigcup \mathcal{U}$,
(vi) $\forall U \in \mathcal{U}\ |F \cap U| = 1$

and order $\mathbb{P}^n$ by $(\mathcal{U}, F, <) \leqslant (\mathcal{U}', F', <')$ if

(vii) $F \supseteq F'$,
(viii) $\forall U \in \mathcal{U}\ \exists! U' \in \mathcal{U}'\ (U \subseteq U')$,
(ix) $\forall U, V \in \mathcal{U}\ U < V \Rightarrow (U' = V' \vee U' <' V')$.

Claim. *$\mathbb{P}^n$ is σ-linked.*

Proof of the claim. Fix $\mathcal{U} = \{U_i : i < k\}$ and a linear order $<$ on $\mathcal{U}$ satisfying (i)–(iv). Let $F = \{x_i : i < k\}$ and $G = \{y_i : i < k\}$ be such that $x_i, y_i \in U_i$ for all $i < k$ and $(\mathcal{U}, F, <), (\mathcal{U}, G, <) \in \mathbb{P}^n$. For each $i < k$ choose from $\mathcal{B}$ neighborhoods $V_i, W_i \subseteq U_i$ of x_i, y_i, respectively, such that $\max(\operatorname{diam} V_i, \operatorname{diam} W_i) < \operatorname{dist}(V_i, W_i)$. Consider the open family $\mathcal{V} = \{V_i : i < k\} \cup \{W_i : i < k\}$ and order it as follows: $V_i \prec W_i$ for all i, $V_i \prec V_j$ iff $U_i < U_j$ and $W_i \prec W_j$ iff $U_i < U_j$. It is straightforward that $(\mathcal{V}, F \cup G, \prec) \in \mathbb{P}^n$ and that $(\mathcal{V}, F \cup G, \prec) \leqslant (\mathcal{U}, F, <)$ and $(\mathcal{V}, F \cup G, \prec) \leqslant (\mathcal{U}, G, <)$. Thus the family $(\mathcal{U}, F, <) \in \mathbb{P}^n : F \in [X]{<}\,\omega\}$ is linked for all $\mathcal{U}, <$ and since there are only countably many such pairs, we are done. □

Let $\mathbb{P} = \prod_{n>0} \mathbb{P}^n$ be the finite support product of the forcing notions $\mathbb{P}^n$, ordered coordinatewise. Since $\mathbb{P}^n$ are by the claim σ-linked, so is $\mathbb{P}$. For $p \in \mathbb{P}$ and $n \in \omega$ let $\mathcal{U}_{p(n)}$, $F_{p(n)}$ and $<_{p(n)}$ denote the coordinates of $p(n)$, i.e. $p(n) = \big(\mathcal{U}_{p(n)}, F_{p(n)}, <_{p(n)}\big)$. Define the following sets:

$$H_{k,n} = \big\{p \in \mathbb{P} : \max\{\operatorname{diam} U : U \in \mathcal{U}_{p(n)}\} < \tfrac{1}{k}\big\}, \quad n, k \in \omega,$$
$$D_x = \{p \in \mathbb{P} : \exists n\ (x \in F_{p(n)})\}, \quad x \in X.$$

It is easy to check that all of these sets are dense in $\mathbb{P}$. Since $|X| < \mathfrak{m}_{\sigma\text{-linked}}$, there is a filter $G \subseteq \mathbb{P}$ that meets all of them. Fix $n \in \omega$ for the moment and set $X_n = \bigcap_{p\in G} \bigcup \mathcal{U}_{p(n)}$. Order X_n as follows:

$$x \prec_n y \text{ if } \exists p \in G\ \exists U, V \in \mathcal{U}_{p(n)}\ x \in U \wedge y \in V \wedge U <_{p(n)} V.$$

Since G meets $H_{k,n}$ for all k, conditions (ii) and (iv) ensure that $\prec_n$ witnesses X_n to be monotone, the a monotonicity constant $c = n$. Since G meets all D_x, for any $x \in X$ there is n such that $x \in X_n$. Hence $X = \bigcup_n X_n$, i.e. X is σ-monotone. □

The above forcing $\mathbb{P}$ yields also a generalization of Theorem 5.3:

Theorem 6.6. *It is relatively consistent with* ZFC *that* $\mathsf{cov}(\mathbf{Mon}(X)) < \mathfrak{c}$ *for any Polish metric space* X.

Proof. Let $V \vDash \neg$CH and let $\mathbb{P}_{\omega_1}$ be a finite support iteration of the forcing $\mathbb{P}$. Let G be $\mathbb{P}_{\omega_1}$-generic over V. Then $V[G] \vDash \neg$CH, since $\mathbb{P}_{\omega_1}$ is *ccc*, and $V[G] \vDash \mathsf{cov}(\mathbf{Mon}(X)) = \omega_1$, since $\mathbb{P}$ makes the set of ground-model reals σ-monotone. □

However, we do not know if Theorem 6.5 remains true for nonseparable spaces:

Question 6.7. Is there a metric space of cardinality ω_1 that is not σ-monotone?

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