INTERSECTION NUMBERS OF FAMILIES OF IDEALS

M. HRUŠÁK, C. A. MARTÍNEZ-RANERO, U. A. RAMOS-GARCÍA AND
O. A. TELLEZ-NIETO

Abstract. We study the intersection number of families of tall ideals. We show that the intersection number of the class of analytic P-ideals is equal to the bounding number $\mathfrak{b}$, the intersection number of the class of all meager ideals is equal to $\mathfrak{h}$ and the intersection number of the class of all F_σ ideals is between $\mathfrak{h}$ and $\mathfrak{b}$, consistently different from both.

1. Introduction

In [16] S. Plewik proved that the intersection of less that $\mathfrak{h}$ non-meager ideals is a non-meager ideal and he showed that there exists a family of size $\mathfrak{d}$ of non-meager ideals which has empty intersection. In [15] the same author proved that the intersection of less that $\mathfrak{c}$ ultrafilters is a non-meager filter. In [18] M. Talagrand proved that the intersection of countably many non-measurable filters is a non-measurable filter and in [3] T. Bartoszyński and S. Shelah proved that it is consistent with **ZFC** that the intesection of a family of less than $\mathfrak{c}$ ultrafilters has measure zero.

In this paper we investigate how many tall ideals from a given class Γ of ideals on ω are needed so that their intersection is not tall.

The first result of this sort is essentially due to Balcar, Pelant and Simon [1], who showed that there is a base tree of height $\mathfrak{h}$ in $\mathcal{P}(\omega)/\mathsf{fin}$ and, in effect, showed that $\mathfrak{h}$ is the minimal size of a family of tall ideals on ω whose intersection is fin (equivalently, not tall).

Definition 1. Let Γ be a class of tall ideals on ω such that $\bigcap \Gamma = \mathsf{fin}$ (that is, for all $A \in [\omega]^\omega$ there is $\mathscr{I} \in \Gamma$ such that $A \notin \mathscr{I}$). The *intersection number of* Γ is defined as $\mathfrak{h}_\Gamma = \min\{|\Omega| : \Omega \subseteq \Gamma\ (\bigcap \Omega \text{ is not tall })\}$.

We consider the intersection number for several naturally occurring classes of ideals. In particular, we show that the intersection number of the class of analytic P-ideals is equal to the bounding number $\mathfrak{b}$, the intersection number of the class of all meager ideals is equal to $\mathfrak{h}$ and the intersection number of the class of all F_σ ideals is between $\mathfrak{h}$ and $\mathfrak{b}$ and is consistently different from both of them.

We assume knowledge of the method of forcing as well as the basic theory of cardinal invariants of the continuum as covered in [2]. Our notation is standard

Date: June 29, 2011.

2000 *Mathematics Subject Classification.* Primary and Secondary: 03E15, 03E17, 03E05, 03E35.

Key words and phrases. Tall ideals, distributivity number, cardinal invariants of the continuum.

The authors gratefully acknowledge support from PAPIIT grant IN102311 and CONACyT grant 80355. The third author is also supported by CONACyT, scholarship 209499 and the fourth author by CONACyT, scholarship 189774.

and follows [2] and [11]. In particular, for a cardinal κ and a set A, $[A]^\kappa$ denotes $\{X \subseteq A : |X| = \kappa\}$. For any given function φ we denote by $\varphi'' A$ and $\varphi^{-1}[A]$ the sets $\{\varphi(x) : x \in A\}$ and $\{x : \varphi(x) \in A\}$, respectively. For any two sets A and B, we say that A is *almost contained* in B, in symbols $A \subseteq^* B$, if $A \setminus B$ is finite. For functions $f, g \in \omega^\omega$ we write $f \leq^* g$ to mean that there is $m \in \omega$ such that $f(n) \leq g(n)$ for all $n \geq m$. An *interval partition* is a partition of ω into finite intervals $\mathcal{I} = \{I_n = [i_n, i_{n+i}) : n \in \omega\}$. We say that the interval partition $\mathcal{I} = \{I_n : n \in \omega\}$ *dominates* another interval partition $\mathcal{J} = \{J_n : n \in \omega\}$ if there exists $m \in \omega$ such that for all $n > m$ there is $k \in \omega$ such that $J_k \subseteq I_n$. Recall that the *bounding number* $\mathfrak{b}$ is the least cardinal of a $\leq^*$-unbounded family of functions in ω^ω. Equivalently, it is the least cardinality of a family $\mathcal{F}$ of partitions of ω in intervals, such that there is no partition that dominates every element of $\mathcal{F}$ (see [4]). A family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a *splitting family* if for every infinite $A \subseteq \omega$ there is an $S \in \mathcal{S}$ such that both $S \cap A$ and $A \setminus S$ are infinite. The *splitting number* $\mathfrak{s}$ is the minimal size of a splitting family in $\mathcal{P}(\omega)$. We say that a family $\mathcal{D}$ of infinite subsets of ω is *dense* in $[\omega]^\omega$ if for all $A \in [\omega]^\omega$ there is $D \in \mathcal{D}$ almost contained in A. $\mathcal{D}$ is *open* if it is downward closed under $\subseteq^*$. The *distributivity number* $\mathfrak{h}$ of $\mathcal{P}(\omega)/\mathsf{fin}$ is the smallest size of a family of dense open sets with empty intersection.

An *ideal* on X is a family of subsets of X closed under finite unions and subsets. We assume throughout the paper that all ideals contain all singletons $\{x\}$ for all $x \in X$. An ideal $\mathscr{I}$ on ω is *tall* if for all $X \in [\omega]^\omega$ there is an $I \in \mathscr{I}$ such that $I \cap X$ is infinite. All the ideals that we consider are tall. A *filter* $\mathcal{F}$ on ω is a family of subsets of ω such that $\{X \subseteq \omega : \omega \setminus X \in \mathcal{F}\}$ is an ideal on ω and an *ultrafilter* is a maximal ultrafilter, that is, for all $X \subseteq \omega$, either $X \in \mathcal{F}$ or $\omega \setminus X \in \mathcal{F}$.

Ideals and filters on ω, as subsets of $\mathcal{P}(\omega)$ can be seen as subsets of the Cantor's set 2^ω (equipped with the product topology), by identifying each subset of ω with its characteristic function. When we speak about analytic complexity or some topological property of a filter or an ideal we refer to this topology. In particular, recall that a set is *meager* if it is the countable union of nowhere dense sets. Thus, an ideal $\mathscr{I}$ is meager if it is meager seen as subset of the Cantor's set.

The *uniformity of the null ideal* $\mathsf{non}(\mathcal{N})$ is the least cardinality of a subset of the real line which is not not of Lebesgue measure zero. The *additivity of the meager ideal* $\mathsf{add}(\mathcal{M})$ is the least κ such that the meager ideal is not κ-additive. The *covering number of the meager ideal* $\mathsf{cov}(\mathcal{M})$ is the smallest size of a family of meager sets wich cover the real line.

2. **ZFC** (in)equalities

Let Γ be a class of tall ideals. We say that Γ is *closed under restrictions and traslations* if given $\mathscr{I} \in \Gamma$, $X \notin \mathscr{I}$ and f a bijection between X and ω, the set $\mathscr{I} \restriction_f X = \{f[I \cap X] : I \in \mathscr{I}\}$ is an ideal of the class Γ. It is easy to see that all classes that we consider are closed under restrictions and translations.

Let Γ be a class of tall ideals closed under restrictions and translations. Suppose that $\Omega \subseteq \Gamma$ satisfies $\bigcap \Omega$ is not tall and $X \in [\omega]^\omega$ is a witness of that, then $\Omega' = \{\mathscr{I} \restriction_f X : \mathscr{I} \in \Omega\}$ is a subclass of Γ and $\bigcap \Omega' = \mathsf{fin}$. Therefore, the intesection number of Γ can be defined as $\min\{|\Omega| : \Omega \subseteq \Gamma \wedge \bigcap \Omega = \mathsf{fin}\}$.

We will use the following simple fact several times in the paper.

Lemma 2. *Let Γ, Δ be classes of tall ideals on ω. If for each $\mathscr{I} \in \Gamma$ there is $\mathscr{J} \in \Delta$ such that $\mathscr{J} \subseteq \mathscr{I}$, then $\mathfrak{h}_\Delta \leq \mathfrak{h}_\Gamma$. In particular, if $\Gamma \subseteq \Delta$, then $\mathfrak{h}_\Delta \leq \mathfrak{h}_\Gamma$.*

Proof. Let $\mathcal{H} = \{\mathscr{I}_\alpha : \alpha < \mathfrak{h}_\Gamma\} \subseteq \Gamma$ be a family such that $\bigcap \mathcal{H} = \mathsf{fin}$. For each α, let $\mathscr{J}_\alpha \subseteq \mathscr{I}_\alpha$ such that $\mathscr{J}_\alpha \in \Delta$. Then $\mathcal{H}' = \{\mathscr{J}_\alpha : \alpha < \mathfrak{h}_\Gamma\} \subseteq \Delta$ and $\bigcap \mathcal{H}' = \mathsf{fin}$, therefore $\mathfrak{h}_\Delta \leq \mathfrak{h}_\Gamma$. □

A family $\mathcal{A}$ of infinite subsets of ω is an *almost disjoint* family if for any $A, B \in \mathcal{A}$, $A \cap B$ is a finite set. A *maximal almost disjoint* (MAD) family is an infinite almost disjoint family of subsets of ω, maximal with respect to inclusion.

As it has already been mentioned, $\mathfrak{h}$ is the smallest possible value of the intersection number. The following theorem shows that the families of MAD and meager ideals realize the same intersection number $\mathfrak{h}$.

Proposition 3. $\mathfrak{h}_{\mathsf{MAD}} = \mathfrak{h}_{\mathsf{meager}} = \mathfrak{h}$.

Proof. It is sufficient to prove that $\mathfrak{h}_{\mathsf{MAD}} \leq \mathfrak{h}$ and $\mathfrak{h}_{\mathsf{meager}} \leq \mathfrak{h}_{\mathsf{MAD}}$. For the first inequality, let $\mathscr{I}$ be a tall ideal. Then $\mathscr{I}$ is a dense open family. It follows from proposition 6.18 of [4], that there is a MAD family $\mathcal{A}$ such that the ideal generated by $\mathcal{A}$, $\mathscr{I}(\mathcal{A}) = \{X \subseteq \omega : \exists \mathcal{B} \in [\mathcal{A}]^{<\omega}(X \subseteq^* \bigcup \mathcal{B})\}$ is a subset of $\mathscr{I}$. From lemma 2 we have the inequality. For the second inequality, recall that in [12] A. Mathias proved that the ideals based on MAD families are meager (that is, $\mathsf{MAD} \subseteq \mathsf{meager}$). The result follows from lemma 2 again. □

Let max denote the class of *maximal ideals.* Recall that an ideal $\mathscr{I}$ is *maximal* if its dual filter is an ultrafilter. The following proposition shows that the intersection number of the class of maximal ideals is the greatest possible.

Proposition 4. $\mathfrak{h}_{\mathsf{max}} = \mathfrak{c}$

Proof. It suffices to show that the intersection of less than $\mathfrak{c}$ maximal ideals is a tall ideal. Let $\kappa < \mathfrak{c}$ be given and let $\{\mathscr{I}_\alpha : \alpha < \kappa\}$ be a family of maximal ideals. Given an $A \in [\omega]^\omega$, let $\{A_\xi : \xi < \mathfrak{c}\}$ be an almost disjoint family of infinite subsets of A. First observe that for a fixed $\alpha < \kappa$, $|\{A_\xi : \xi < \mathfrak{c}\} \setminus \mathscr{I}_\alpha| \leq 1$. To see this, pick $\xi < \mathfrak{c}$ such that $A_\xi \notin \mathscr{I}_\alpha$. If $\chi \neq \xi$ then $A_\xi \cap A_\chi =^* \emptyset$, $A_\chi \subseteq^* \omega \setminus A_\xi$. We have $\omega \setminus A_\xi \in \mathscr{I}_\alpha$ since the ideal is maximal. Therefore, $A_\chi \in \mathscr{I}_\alpha$. It easily follows from the observation that there is a $\xi_0 < \mathfrak{c}$ such that $A_{\xi_0} \in \mathscr{I}_\alpha \cap [A]^\omega$ for all $\alpha < \kappa$. □

Recall that an ideal $\mathscr{I}_f$ is *summable* if there is $f : \omega \to (0, \infty)$ such that $\lim_{n\to\infty} f(n) = 0$, $\sum_{n\in\omega} f(n) = \infty$ and $\mathscr{I} = \{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\}$. An ideal $\mathscr{I}$ is a *P-ideal* if for any sequence $\langle I_n : n \in \omega\rangle \subseteq \mathscr{I}$ there is $I \in \mathscr{I}$ such that $I_n \subseteq^* I$ for all $n \in \omega$. An ideal $\mathscr{I}$ is ω*-hitting* if for any sequence $\langle A_n : n \in \omega\rangle \subseteq [\omega]^\omega$ there is $I \in \mathscr{I}$ such that $|A_n \cap I| = \aleph_0$ (equivalently, $A_n \cap I \neq \emptyset$ for all $n \in \omega$).

A *lower semicontinuous submeasure* on a set X is a function $\varphi : \mathcal{P}(X) \to [0, \infty]$ satisfying $\varphi(\emptyset) = 0$; $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$; $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ and $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for all $A, B \subseteq X$. If φ is a lower semicontinuous submeasure on ω then the ideals $\mathrm{Fin}(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ and $\mathrm{Exh}(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$ are F_σ and $F_{\sigma\delta}$ P-ideals, respectively.

In [17] S. Solecki showed that if $\mathscr{I}$ is an analytic P-ideal, then there is a lower semicontinuous submeasure φ on ω such that $\mathscr{I} = \mathrm{Exh}(\varphi)$ and K. Mazur in [13] proved that if an ideal $\mathscr{I}$ is F_σ then there is a lower semicontinuous submeasure φ such that $\mathscr{I} = Fin(\varphi)$.

An F_σ ideal $\mathscr{I}$ is *fragmented* [10] if there is partition $\langle I_n : n \in \omega\rangle$ of ω in finite sets and submeasures $\varphi_n : \mathcal{P}(I_n) \to [0, \infty)$ on I_n such that $\mathscr{I} = \{A \subseteq \omega : \exists k \forall n(\varphi_n(A \cap I_n) \leq k)\}$

Since $\mathsf{F}_\sigma \subseteq \cdots \subseteq \mathsf{Borel} \subseteq \mathsf{analytic}$ and $\mathsf{summable} \subseteq \mathsf{analytic\ P\text{-}ideal} \subseteq \mathsf{Borel}\ \omega\text{-}\mathsf{hitting} \subseteq \omega\text{-}\mathsf{hitting}$, we have from lemma 2 the following inequalities.

Proposition 5. *(1)* $\mathfrak{h} \leq \mathfrak{h}_{\mathsf{analytic}} \leq \mathfrak{h}_{\mathsf{Borel}} \leq \cdots \leq \mathfrak{h}_{F_\sigma}$.
(2) $\mathfrak{h} \leq \mathfrak{h}_{\omega\text{-hitting}} \leq \mathfrak{h}_{\mathsf{Borel}\ \omega\text{-hitting}} \leq \mathfrak{h}_{\mathsf{analytic\ P\text{-}ideal}} \leq \mathfrak{h}_{\mathsf{summable}}$.

Proof. (1) is obvious. For (2), if $\mathscr{I}_f$ is a summable ideal then the lower semicontinuous submeasure φ on ω defined by $\varphi(A) = \sum_{n\in A} f(n)$ shows that $\mathscr{I}_f$ is an analytic P-ideal as $\mathscr{I} = \mathrm{Exh}(\varphi)$.

Let $\mathscr{I}$ be an analytic P-ideal. Let us see that $\mathscr{I}$ is Borel ω-hitting. From Solecki's theorem we know that $\mathscr{I}$ is an $F_{\sigma\delta}$ ideal. Let $\langle A_n : n \in \omega\rangle \subseteq [\omega]^\omega$. Since $\mathscr{I}$ is tall, for every $n \in \omega$ there is $I_n \in \mathscr{I}$ such that $I_n \cap A_n$ is infinite. Let $I \in \mathscr{I}$ be such that $I_n \subseteq^* I$ for all $n \in \omega$. Then $I \cap A_n$ is infinite for all $n \in \omega$. □

The next class of ideals that we consider is the class of *eventually diferent* ideals. We consider this class for two reasons: the first one is because its intersection number admits a simple combinatorial characterization and the second one is, it allows us to relate the intersection number of the classes seen so far with the classical cardinal invariants $\mathfrak{b}$, $\mathfrak{s}$ and $\mathsf{non}(\mathcal{N})$.

Definition 6. Let $f \in \omega^\omega$ be such that $|\mathrm{ran}(f)| = \aleph_0$ and $\limsup_{n\to\infty} |f^{-1}(n)| = \infty$. We define the (tall F_σ) ideal $\mathcal{ED}_f = \{A \subseteq \omega : \exists m \forall l \geq m\, |A \cap f^{-1}(l)| \leq m\}$. The class of *ED-ideals* is defined as the class

$$\mathsf{ED} = \{\mathcal{ED}_f : f \in \omega^\omega \wedge |\mathrm{ran}(f)| = \aleph_0 \wedge \limsup_{n\to\infty} |f^{-1}(n)| = \infty\}.$$

The class of *ED_{fin}-ideals* is defined by

$$\mathsf{ED}_{\mathsf{fin}} = \{\mathcal{ED}_f : f \in \omega^\omega \wedge f \text{ is finite-to-one } \wedge \limsup_{n\to\infty} |f^{-1}(n)| = \infty\}.$$

Note that

$$\mathfrak{h}_{\mathsf{ED}} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall A \in [\omega]^\omega \exists f \in \mathcal{F}(\forall k\ \exists^\infty n(|f^{-1}(n) \cap A| > k))\} \text{ and}$$
$$\mathfrak{h}_{\mathsf{ED}_{\mathsf{fin}}} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall A \in [\omega]^\omega \exists f \in \mathcal{F} \text{ finite-to-one } (\forall k\ \exists^\infty n(|f^{-1}(n) \cap A| > k))\}.$$

From lemma 2 and $\mathsf{ED}_{\mathsf{fin}} \subseteq \mathsf{ED} \subseteq \mathsf{F}_\sigma$, we obtain the following inequalities.

Proposition 7. $\mathfrak{h}_{F_\sigma} \leq \mathfrak{h}_{\mathsf{ED}} \leq \mathfrak{h}_{\mathsf{ED}_{\mathsf{fin}}}$. □

We can estimate the values of $\mathfrak{h}_{\mathsf{ED}}$ and $\mathfrak{h}_{\mathsf{ED}_{\mathsf{fin}}}$.

Theorem 8. $\mathfrak{h}_{\mathsf{ED}_{\mathsf{fin}}} = \mathfrak{b}$.

Proof. First we prove that $\mathfrak{h}_{\mathsf{ED}_{\mathsf{fin}}} \leq \mathfrak{b}$. Let $\kappa < \mathfrak{h}_{\mathsf{ED}_{\mathsf{fin}}}$ and $\langle P_\alpha : \alpha < \kappa\rangle$ be a family of partitions of ω in intervals where $P_\alpha = \langle I^\alpha_n : n \in \omega\rangle$. Define $f_\alpha : \omega \to \omega$ by $f_\alpha(x) = n$ if $x \in I^\alpha_n$ (that means that $f_\alpha^{-1}(n) = I^\alpha_n$ for all $n \in \omega$). Since f_α is finite-to-one for all $\alpha < \kappa$ and $\kappa < \mathfrak{h}_{\mathsf{ED}_{\mathsf{fin}}}$, there is an $A \in [\omega]^\omega$ such that for each $\alpha < \kappa$ there are $k_\alpha, m_\alpha \in \omega$ such that $|f_\alpha^{-1}(n) \cap A| \leq k_\alpha$ for all $n > m_\alpha$. Let $e_A : \omega \to \omega$ be the enumerating function of A ($e_A(n)$ is the n-th element of A) and define the following partition of ω in intervals:

$$J_0 = [0, e_A(0));$$
$$J_{n+1} = [e_A(s_n), e_A(s_{n+1})),$$

where $s_n = \sum_{i=0}^{n} i$, for $n \geq 1$. Note that $|J_n \cap A| = n$.

We claim that $P = \langle J_n : n \in \omega \rangle$ dominates P_α for all $\alpha < \kappa$. Fix $\alpha < \kappa$, let $k_\alpha, m_\alpha \in \omega$ be such that $|f_\alpha^{-1}(n) \cap A| \leq k_\alpha$ for all $n > m_\alpha$. Let $N \in \omega$ such that $N > \max\{3k_\alpha, m_\alpha\}$ and if $s_{N-1} \in I_k^\alpha$. then $k \geq m_\alpha$. Let us see that for each $m \geq N$ there is an $r \in \omega$ such that $I_r^\alpha \subseteq J_m$.

For $m \geq N$, let $r_0 = \min\{n \in \omega : I_n^\alpha \cap J_m \neq \emptyset\}$. By the second condition on N, $r_0 \geq m_\alpha$. If $I_{r_0} \subseteq J_m$ we are done. If not, we claim that $I_{r_0+1} \subseteq J_m$. Suppose not, then $J_m \subseteq I_{r_0}^\alpha \cup I_{r_0+1}^\alpha$ and therefore $A \cap J_m \subseteq A \cap (I_{r_0}^\alpha \cup I_{r_0+1}^\alpha)$ wich implies that $|A \cap J_m| \leq |A \cap (I_{r_0}^\alpha \cup I_{r_0+1}^\alpha)|$ but $|A \cap J_m| = m \geq 3k_\alpha$ while $|A \cap (I_{r_0} \cup I_{r_0+1})| \leq 2k_\alpha$, which is a contradiction.

On the other hand, let $\kappa < \mathfrak{b}$ and $\langle f_\alpha : \alpha < \kappa \rangle \subseteq \omega^\omega$ be a family of finite-to-one functions. For each $\alpha < \kappa$ we define a partition of ω in intervals as follows:

$$I_0^\alpha = [0, k_0),$$

where $k_0 = \min\{m \in \omega : f_\alpha^{-1}(0) \subseteq [0, m)\}$, and

$$I_{n+1}^\alpha = [k_n, k_{n+1}),$$

where $k_{n+1} = \min\{m \in \omega : \forall x \in I_n (f_\alpha^{-1}[f_\alpha(x)] < m)\}$ for $n \geq 1$. Put $P_\alpha = \langle I_n^\alpha : n \in \omega \rangle$. Observe that $f_\alpha^{-1}(m)$ is contained in at most two consecutive intervals of P_α, for all $m \in \operatorname{ran}(f_\alpha)$. Let $P = \langle J_n : n \in \omega \rangle$ be a partition dominating the family $\langle P_\alpha : \alpha < \kappa \rangle$ (that is, for each $\alpha < \kappa$ there is $N_\alpha \in \omega$ such that for all $n \geq N_\alpha$ exists $r \in \omega$ such that $I_r^\alpha \subseteq J_n$) and let A be a selector of P. For $\alpha < \kappa$, consider N_α and r such that $I_r^\alpha \subseteq I_{N_\alpha}$. If $m_\alpha = \max\{f_\alpha(x) : x \in I_r^\alpha\}$, then for each $n \geq m_\alpha$, $|f_\alpha^{-1}(n) \cap A| < 3$, because $f_\alpha^{-1}(n)$ is contained in at most two intervals of P_α. This proves that $\langle f_\alpha : \alpha < \kappa \rangle$ is not a wintness for $\mathfrak{h}_{\mathsf{ED_{fin}}}$, and therefore $\mathfrak{b} \leq \mathfrak{h}_{\mathsf{ED_{fin}}}$. □

Recall that an ideal $\mathscr{I}$ is a *Q-ideal* if for every partition $\langle F_n : n < \omega \rangle$ of ω into finite sets, there is an $\mathscr{I}$-positive set X such that $|X \cap F_n| \leq 1$ for all $n < \omega$.

We use the following theorem [9].

Theorem 9. *For each Borel ideal $\mathscr{I}$, the following are equivalent:*

(1) $\mathscr{I}$ is not a Q-ideal,
(2) $\mathscr{I}$ is an ω-hitting ideal. □

Lemma 10. *(1)* $\mathfrak{h}_{\mathsf{summable}} \leq \mathfrak{b}$.
(2) $\mathfrak{h}_{\mathsf{ED_{fin}}} \leq \mathfrak{h}_{\mathsf{Borel}\ \omega\text{-}\mathsf{hitting}}$.
(3) $\mathfrak{h}_{\mathsf{Borel}\ \omega\text{-}\mathsf{hitting}} \leq \mathfrak{h}_{\mathsf{fragmented}}$.

Proof. For (1), we use the caracterization of $\mathfrak{b}$ from [7], that is

$$\mathfrak{b} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq c_0 \wedge \forall X \in [\omega]^\omega \exists s \in \mathcal{S} (s \restriction X \notin \ell_1)\},$$

where c_0 and ℓ_1 denote the standard Banach spaces of sequences of reals.

Let $\kappa < \mathfrak{h}_{\mathsf{summable}}$ and $\mathcal{S} = \{s_\alpha : \alpha < \kappa\} \subseteq c_0$, without loss of generality we can suppose that $\sum_{n \in \omega} s_\alpha(n) = \infty$. For each $\alpha < \kappa$, we define the summable ideal $\mathscr{I}_\alpha = \{A \subseteq \omega : \sum_{n \in A} s_\alpha(n) < \infty\}$. Since $\kappa < \mathfrak{h}_{\mathsf{summable}}$, there is an $A \in \bigcap_{\alpha < \kappa} \mathcal{I}_\alpha \cap [\omega]^\omega$. Then $s_\alpha \restriction A \in \ell_1$, and therefore $\kappa < \mathfrak{b}$.

Let us see (2). By the theorem 9, if $\mathscr{I}$ is a Borel ω-hitting ideal, then $\mathscr{I}$ is not a Q-ideal, that means that there is a partition $\langle F_n : n \in \omega \rangle$ of ω into finite sets such that every selector of the partition belongs to $\mathscr{I}$. Consider the function $f : \omega \to \omega$ given by $f(x) = n$, where $x \in F_n$, it is easy to see that f is finite-to-one and $\mathcal{ED}_f \subseteq \mathscr{I}$ (each element of $\mathcal{ED}_f$ is a finite union of selectors of $\langle F_n : n \in \omega\rangle$). Thus, lemma 2 gives the desired conclusion.

In order to show (3), let $\mathscr{I}$ be a fragmented ideal, with $\langle I_n : n \in \omega\rangle$ and $\langle \varphi_n : n \in \omega\rangle$ witnessing it. Let us see that $\mathscr{I}$ is an ω-hitting ideal. Suppose not. Then by theorem 9, $\mathscr{I}$ is a Q-ideal. That means that for $\langle I_n : n \in \omega\rangle$ there is an $\mathscr{I}$-positive set X such that $|X \cap I_n| \leq 1$ for all $n \in \omega$. If $X \cap I_n \neq \emptyset$, let $x_n \in X \cap I_n$. Since X is an $\mathscr{I}$-positive set, then $\sup\{\varphi_n(X \cap I_n) : n \in \omega\} = \sup\{\varphi_n(x_n) : X \cap I_n \neq \emptyset\} = \infty$. Then there is a $Y \subseteq X$ infinite such that the n-th element of Y has submeasure at least n. Therefore, $\mathscr{I}$ is not a tall ideal, a contradiction. Again lemma 2 gives the conclusion. □

Theorem 11. $\mathfrak{b} = \mathfrak{h}_{\mathsf{ED_{fin}}} = \mathfrak{h}_{\mathsf{Borel}\ \omega\text{-}\mathsf{hitting}} = \mathfrak{h}_{\mathsf{analytic\ P\text{-}ideal}} = \mathfrak{h}_{\mathsf{summable}} = \mathfrak{h}_{\mathsf{fragmented}}$.

Proof. Follows directly from 2 of proposition 5, theorem 8 and lemma 10. □

In order to simplify the calculations for $\mathfrak{h}_{\mathsf{ED}}$ we introduce the following cardinal

$$\nu = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall A \in [\omega]^\omega \exists f \in \mathcal{F}(\forall n \in \omega(|A \cap f^{-1}(n)| = \aleph_0))\}.$$

Obviously $\mathfrak{h}_{\mathsf{ED}} \leq \nu$.

It is not known if $\mathfrak{h}_{\mathsf{ED}}$ is equal to any of the known cardinal invariants, as in the case of $\mathfrak{h}_{\mathsf{ED_{fin}}}$. However, we can bound it from both sides.

Theorem 12. $\min\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{h}_{\mathsf{ED}} \leq \min\{\mathfrak{b}, \mathsf{non}(\mathcal{N})\}$.

Proof. Obviously $\mathfrak{h}_{\mathsf{ED}} \leq \mathfrak{h}_{\mathsf{ED_{fin}}} = \mathfrak{b}$. Now we will prove that $\nu \leq \mathsf{non}(\mathcal{N})$. Consider the measure μ_0 on ω given by $\mu_0(n) = \frac{1}{2^{n+1}}$ and let μ be the product measure on ω^ω. Let $N_A = \{f \in \omega^\omega : \exists n \in \omega(|A \cap f^{-1}(n)| < \aleph_0)\}$.

We show that $\mu(N_A) = 0$. Observe that

$$N_A = \bigcup_{n \in \omega, F \in [A]^{<\omega}} N_A(n, F),$$

where $N_A(n, F) = \{f \in \omega^\omega : f^{-1}(n) \cap A = F\}$.

Fix $n \in \omega$ and $F = \{a_0, \ldots, a_r\} \in [A]^{<\omega}$ (in increasing order). Let us see that $\mu(N_A(n, F)) = 0$, and therefore $\mu(N_A) = 0$.

$$\begin{aligned}\mu(N_A(n,F)) &= \lim_{m\to\infty} (\mu_0(\omega \setminus \{n\}))^{a_0}(\mu_0(\{n\}))(\mu_0(\omega \setminus \{n\}))^{a_1 - a_0 - 1}(\mu_0(\{n\})) \\ &\qquad \ldots (\mu_0(\omega \setminus \{n\}))^{a_r - a_{r-1} - 1}(\mu_0(\{n\}))(\mu_0(\omega \setminus \{n\}))^m \\ &= \lim_{m\to\infty} (\mu_0(\omega \setminus \{n\}))^{a_r - r - 1}(\mu_0(\{n\}))^{r-1}(\mu_0(\omega \setminus \{n\}))^m \\ &= \lim_{m\to\infty} \left(1 - \frac{1}{2^{n+1}}\right)^{a_r - r - 1}\left(\frac{1}{2^{n+1}}\right)^{r-1}\left(1 - \frac{1}{2^{n+1}}\right)^m = 0\end{aligned}$$

Take $\kappa < \nu$ and $\mathcal{F} \subseteq \omega^\omega$ where $|\mathcal{F}| = \kappa$. Then, there is $A \in \omega^\omega$ such that for all $f \in \mathcal{F}$ there is $n \in \omega$ such that $|f^{-1}(n) \cap A| < \aleph_0$. Then $\mathcal{F} \subseteq N_A$, and therefore $\mathcal{F}$ is a null set.

Recall that

$$\begin{aligned}\min\{\mathfrak{b}, \mathfrak{s}\} = \min\{|\mathcal{X}| : \forall \varphi \in \mathcal{X}(\varphi : [\omega]^2 \to 2) \wedge \forall A \in [\omega]^\omega \exists \varphi \in \mathcal{X} \\ \forall n \in \omega(\varphi''[A \setminus n]^2 = 2)\}\end{aligned}$$

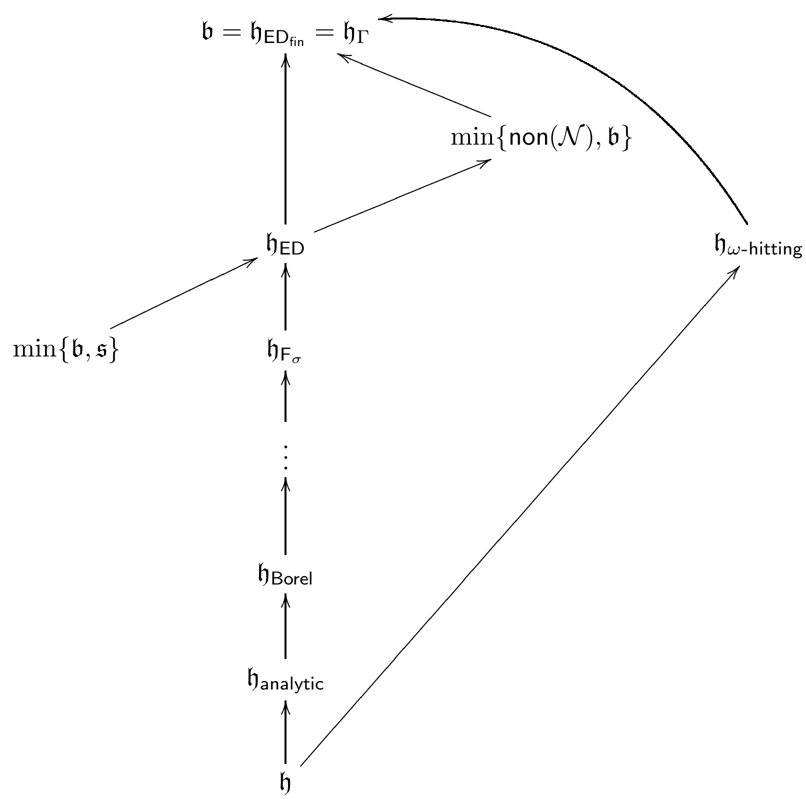
(see [4], theorem 3.5).

Let $\kappa < \min\{\mathfrak{b}, \mathfrak{s}\}$ and $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$. For each $\alpha < \kappa$ we define $\varphi_\alpha : [\omega]^2 \to 2$ as follows: $\varphi_\alpha(\{m, n\}) = 0$ if and only if $f_\alpha(n) = f_\alpha(m)$. Since $\kappa < \min\{\mathfrak{b}, \mathfrak{s}\}$, there is $A \in [\omega]^\omega$ such that for every $\alpha < \kappa$ exists $n_\alpha \in \omega$ such that $|\varphi''[A \setminus n_\alpha]^2| = 1$. Now the proof proceed by cases.

If $\varphi''[A \setminus n_\alpha]^2 = \{0\}$ and $m_\alpha = \max\{f_\alpha(k) : k \leq n_\alpha + 1\}$, then $f^{-1}(n) \cap A = \emptyset$ for all $n > m_\alpha$.

If $\varphi''[A \setminus n_\alpha]^2 = \{1\}$ and $m_\alpha = \max\{f_\alpha(k) : k \leq n_\alpha + 1\}$, then $|f^{-1}(n) \cap A| = 1$ for all $n > m_\alpha$. Therefore, $\mathfrak{h}_{\mathsf{ED}} \geq \min\{\mathfrak{b}, \mathfrak{s}\}$. □

The relations that we have seen so far can be summarized in the following diagram (Γ is any of the following classes of tall ideals: Borel ω-hitting, analytic P-ideal, summable or fragmented).



3. Consistency results

In this section we will show that each of the following statements is consistent with **ZFC**:

(1) $\mathfrak{h} < \mathfrak{h}_{\mathsf{analytic}}$.
(2) $\mathfrak{h}_{F_\sigma} = \mathfrak{h}_{\mathsf{ED}} < \mathfrak{h}_{\omega\text{-hitting}}$.
(3) $\mathfrak{h}_{\mathsf{ED}} = \mathfrak{h}_{\omega\text{-hitting}} < \mathfrak{b}$
(4) $\mathfrak{h}_{\mathsf{ED}} < \mathsf{add}(\mathcal{M})$.

3.1. **The consistency of $\mathfrak{h} < \mathfrak{h}_{\mathsf{analytic}}$.** We use the following forcing notions (see [2]).

The Laver forcing $\mathbb{L}$: $T \in \mathbb{L}$ if and only if $T \subseteq \omega^{<\omega}$ is a tree, there is $s_T \in T$ (called *stem of* T) such that for all $t \in T$ either $t \subseteq s_T$ or $s_T \subseteq t$ and for all $t \in T$ if $t \supseteq s_T$, then $\mathrm{succ}_T(t) = \{n \in \omega : t^\frown n \in T\}$ is infinite. For $T, T' \in \mathbb{L}$, define $T' \leq T$ if $T' \subseteq T$.

The Mathias forcing $\mathbb{M}$: $\langle s, A\rangle \in \mathbb{M}$ if and only if $s \in [\omega]^{<\omega}$, $A \in [\omega]^{\omega}$ and $\max(s) < \min(A)$. If $\langle s, A\rangle, \langle s^{'}, A^{'}\rangle \in \mathbb{M}$ define $\langle s, A\rangle \leq \langle s', A'\rangle \in \mathbb{M}$ if and only if $s' \subseteq s$, $A \subseteq A'$ and $s \setminus s' \subseteq A'$.

The Mathias forcing asociated to an ultrafilter $\mathcal{U}$, $\mathbb{M}_{\mathcal{U}}$: $\langle s, A\rangle \in \mathbb{M}$ if and only if $s \in [\omega]^{<\omega}$, $A \in \mathcal{U}$ and $\max(s) < \min(A)$. The same order as $\mathbb{M}$.

$\mathbb{LM}$ denotes the two step iteration $\mathbb{L} * \mathbb{M}$, and for a forcing notion $\mathbb{P}$, $\mathbb{P}_{\omega_2}$ denotes the countable support iteration of $\mathbb{P}$ of lenght ω_2.

We recall the following theorem due to A. Mathias [12].

Theorem 13 (Mathias, [12])**.** *Let $\mathcal{U}$ be an ultrafilter on ω. Then $\mathcal{U}$ is selective if and only if $\mathcal{U} \cap \mathscr{I} \neq \emptyset$ for each tall analytic ideal $\mathscr{I}$.* □

Lemma 14 (Folklore)**.** $\mathbb{M} \simeq \mathcal{P}(\omega)/\mathsf{fin} * \mathbb{M}_{\dot{\mathcal{U}}}$, *where $\dot{\mathcal{U}}$ is the selective ultrafilter added by $\mathcal{P}(\omega)/\mathsf{fin}$.*

Proof. Consider the mapping $\iota : \mathbb{M} \to \mathcal{P}(\omega)/\mathsf{fin} * \mathbb{M}_{\mathcal{U}}$ given by $\iota(\langle a, A\rangle) = \langle A, (a, A)\rangle$. It is easy to see that ι is a dense embedding. □

As $\mathcal{P}(\omega)/\mathsf{fin}$ adds a selective ultrafilter $\mathcal{U}$ and $\mathbb{M}_{\mathcal{U}}$ adds a pseudo-intersection of $\mathcal{U}$, if G is $\mathbb{M}$-generic over V, then $V[G] = V[\mathcal{U}][A]$ where A is the pseudo-intersection added by $\mathbb{M}_{\mathcal{U}}$. If G is a $\mathbb{L}$-generic over V and f_G is the Laver real added by G, then we write $V[f_G]$ instead of $V[G]$.

Theorem 15. *It is consistent with* **ZFC** *that* $\mathfrak{h} = \omega_1$ *and* $\mathfrak{h}_{\mathsf{analytic}} = \omega_2$

Proof. It is shown in [6] that if $V \models$ **CH** and G a $\mathbb{LM}_{\omega_2}$-generic over V, then $V[G] \models \mathfrak{h} = \omega_1$.

Let us show that $V[G] \models \mathfrak{h}_{\mathsf{analytic}} = \omega_2$. Let $\langle \mathscr{I}_\alpha : \alpha < \omega_1\rangle \in V[G]$ be a family of analytic ideals on ω.

Claim 1. *There exists $\beta < \omega_2$ such that $\mathscr{I}_\alpha \in V[G_\beta]$ for all $\alpha < \omega_1$.*

Fix $\alpha < \omega_1$, since the ideal $\mathscr{I}_\alpha$ is analytic, it is the countinuous image of a Polish space. As a continuous function from a Polish space is determined by the values in a countable dense subset there is $\beta_\alpha < \omega_2$ such that $\mathscr{I}_\alpha \in V[G_{\beta_\alpha}]$. Let $\beta = \sup\{\beta_\alpha : \alpha < \omega_1\}$. Then $\beta < \omega_2$ and $\mathscr{I}_\alpha \in V[G_\beta]$ for all $\alpha < \omega_1$.

Having fixed such β note that, by Schoenfield's absoluteness, $V[G_\beta]$ as well as any larger model thinks that $\mathscr{I}_\alpha$ is tall for every $\alpha < \omega_1$, Now, from the previous remark, $V[G_{\beta+1}] \simeq V[G_\beta][f][\mathcal{U}][A]$, where f is the Laver real, $\mathcal{U}$ is the selective ultrafilter added by $\mathcal{P}(\omega)/\mathsf{fin}$ and A is the pseudo-intersection of $\mathcal{U}$. By the theorem of Mathias, there is $I_\alpha \in \mathcal{U} \cap \mathscr{I}_\alpha$ for each $\alpha < \omega_1$. Since $A \subseteq^* I_\alpha$, we have that $A \in \mathscr{I}_\alpha$ for all $\alpha < \omega_1$. Thus, $A \in \bigcap_{\alpha \in \omega_1} \mathscr{I}_\alpha$. □

3.2. **The consistency of $\mathfrak{h}_{\mathsf{F}_\sigma} = \mathfrak{h}_{\mathsf{ED}} < \mathfrak{h}_{\omega\text{-}\mathsf{hitting}}$.** The forcing notion that we use is the Laver forcing. First we show that the range of a Laver real belongs to any ω-hitting ideal.

Lemma 16. *Let $\mathscr{I} \in V$ be an ω-hitting ideal. If G is $\mathbb{L}$-generic over V, f_G is the Laver real added by G and $A = \operatorname{ran}(f_G)$, then $V[G] \models A \in \mathscr{I}$.*

Proof. We show that the set $\{S \in \mathbb{L} : S \Vdash \text{“}A \in \mathscr{I}\text{”}\}$ is dense. Let $T \in \mathbb{L}$ be a Laver condition. Since $\mathscr{I}$ is ω-hitting, there is $I \in \mathscr{I}$ such that $|I \cap \operatorname{succ}_T(t)| = \omega$ for all $t \in T$ with $s_T \subseteq t$. Define $T' \leq T$ by recursion as follows: $s_{T'} = s_T$, $\operatorname{succ}_{T'}(s_{T'}) = \operatorname{succ}_T(s_T) \cap I$. Suppose defined $\operatorname{succ}_{T'}(t)$ for $|t| = |s_{T'}| + n$. For $t \in T'$ with $|t| = |s_{T'}| + n + 1$, let $\operatorname{succ}_{T'}(t) = \operatorname{succ}_T(t) \cap I$. Hence, T' is a Laver condition which for each $t \in T'$, $\operatorname{succ}_{T'}(t) \subseteq I$. That means that $T' \Vdash \text{“}f_G(n) \in I\text{”}$ for all $n \geq |s_{T'}|$, then $T' \Vdash \text{“}\operatorname{ran}(f_G) = A \subseteq^* I\text{”}$ and therefore $T' \Vdash \text{“}A \in \mathscr{I}\text{”}$. □

Now we prove the consistency of the statement.

Theorem 17. *It is consistent with* **ZFC** *that $\mathfrak{h}_{\mathsf{ED}} = \omega_1$ and $\mathfrak{h}_{\omega\text{-hitting}} = \omega_2$.*

Proof. Let $V \models$ **CH** and G be a $\mathbb{L}_{\omega_2}$-generic over V. In [14] it shows that $V[G] \models \mathsf{non}(\mathcal{N}) = \omega_1$, and from theorem 12, we have that $V[G] \models \mathfrak{h}_{\mathsf{ED}} = \omega_1$.

It remains to verify that $V[G] \models \mathfrak{h}_{\omega\text{-hitting}} > \omega_1$. Let $\langle \mathscr{I}_\alpha : \alpha < \omega_1 \rangle \in V[G]$ be a family of ω-hitting ideals. The following claim will be necessary to finish the proof.

Claim 2. *There is $\gamma < \omega_2$ such that $V[G_\gamma] \models \mathscr{I}_\alpha \cap V[G_\gamma]$ is ω-hitting for all $\alpha < \omega_1$.*

Proof of the claim. Start with $\alpha_0 < \omega_2$. In $V[G_{\alpha_0}]$, enumerate all sequences of infinite subsets of ω, $\langle \langle A_n^\xi : n \in \omega \rangle : \xi < \omega_1 \rangle$ (by **CH**, there are only ω_1). For each $\xi < \omega_1$, let $I_\alpha^\xi \in \mathscr{I}_\alpha$ be such that (in $V[G]$) $I_\alpha^\xi \cap A_n^\xi \neq \emptyset$ for all $n \in \omega$. The set $\{I_\alpha^\xi : \alpha, \xi < \omega_1\}$ has cardinality ω_1, then there is $\alpha_1 < \omega_2$ such that $I_\alpha^\xi \in V[G_{\alpha_1}]$. Iterating this proccess ω_1 times, we find α_{ω_1}. Then $\gamma = \alpha_{\omega_1}$ works. If $\langle A_n : n \in \omega \rangle \in M[V_\gamma]$ there is $\xi < \omega_1$ such that $\langle A_n : n \in \omega \rangle \in M[V_{\alpha_\xi}]$. In $M[V_{\alpha_{\xi+1}}]$, we know that for each $\alpha < \omega_1$, there is $I_\alpha \in \mathscr{I}_\alpha$ such that $I_\alpha \cap A_n \neq \emptyset$. But $M[V_{\alpha_{\xi+1}}] \subseteq M[V_\gamma]$. ■

Let $\gamma < \omega_2$ obtained from the claim. Now, in $V[G_{\gamma+1}]$ we have by the lemma 16 that the range A of the $(\gamma+1)$-st Laver real is an infinite set that belongs to each $\mathscr{I}_\alpha$, that is $A \in \bigcap_{\alpha<\omega_1} \mathscr{I}_\alpha$. □

3.3. **The consistency of $\mathfrak{h}_{\mathsf{ED}} = \mathfrak{h}_{\omega\text{-hitting}} < \mathfrak{b}$.** For this consistency proof we use the random forcing $\mathbb{B}(\omega_1)$. Let μ be the standard product measure on 2^{ω_1} and $\mathcal{N}_{\omega_1} = \{X \subseteq 2^{\omega_1} : \mu(X) = 0\}$. For $A, B \in \mathbf{Borel}(2^{\omega_1})$ let $A \simeq B$ if and only if $A \triangle B \in \mathcal{N}_{\omega_1}$ and denote $[A]_{\mathcal{N}}$ the equivalence class of the set A with respect this equivalence relation. Define $\mathbb{B}(\omega_1) = \{[A]_{\mathcal{N}} : A \in \mathbf{Borel}(2^{\omega_1})\}$ with the order $[A]_{\mathcal{N}} \leq [B]_{\mathcal{N}}$ if $A \setminus B \in \mathcal{N}_{\omega_1}$.

The random forcing $\mathbb{B}(\omega_1)$ preserves $\mathfrak{b}$ and $\mathsf{non}(\mathcal{N})$ (see [2]) and adds ω_1 reals in the following way: if G is $\mathbb{B}(\omega_1)$-generic filter and $r_G \in 2^{\omega_1}$ is the generic function, then the α-th real is defined by $r_\alpha(n) = r_G(\alpha \cdot \omega + n)$ for $\alpha < \omega_1$.

We can see $V[G]$ as $V[r_\alpha : \alpha < \omega_1]$, where $r_\alpha : \omega \to 2$ is the α-th random real added by G.

Theorem 18. *It is consistent with* **ZFC** *that $\mathfrak{h}_{\mathsf{ED}} = \mathfrak{h}_{\omega\text{-hitting}} = \omega_1$ and $\mathfrak{b} = \omega_2$.*

Proof. Start with a model V such that $V \models \mathsf{non}(\mathcal{N}) = \omega_1 < \mathfrak{b} = \omega_2 = \mathfrak{c}$ (for example, the model obtained in theorem 17 works) and let G be a $\mathbb{B}(\omega_1)$-generic over V. Then, $V[G] \models \mathsf{non}(\mathcal{F}) = \omega_1 < \mathfrak{b} = \omega_2$, because, as already mentioned, random forcing preserve it.

Let us show that $V[G] \models \mathfrak{h}_{\omega\text{-hitting}} = \omega_1$. Let $\{r_\alpha : \alpha < \omega_1\}$ the ω_1 random reals added by G. For $\alpha, \beta < \omega_1$, let $J_\beta = r_\beta^{-1}(1)$ and $\mathscr{I}_\alpha = \langle J_\beta : \beta > \alpha \rangle$.

We claim that $\mathscr{I}_\alpha$ is an ω-hitting ideal (and therefore, tall) for each $\alpha < \omega_1$, and $\bigcap_{\alpha<\omega_1} \mathscr{I}_\alpha = \mathsf{fin}$.

To see that $\mathscr{I}_\alpha$ is ω-hitting, let us see that if $\langle A_n : n \in \omega \rangle \in V[r_\gamma : \gamma < \alpha]$, then $J_\beta \cap A_n \neq \emptyset$ for all $\beta > \alpha$ and for all $n \in \omega$. For this, note that $\mu[\![J_\beta \cap A_n = \emptyset]\!] = \mu(\{f \in 2^{\omega_1} : \forall k \in A_n(f(\beta + k) = 0)\}) = 0$, for each $n \in \omega$.

To check that $\bigcap_{\alpha<\omega_1} \mathscr{I}_\alpha = \mathsf{fin}$, note that if $V[G] \models A \in [\omega]^\omega$, then there is $\alpha < \omega_1$ such that $A \in V[G_\alpha]$. Now, $\mu[\![A \subseteq J_\beta]\!] = \mu(\{f \in 2^{\omega_1} : \forall k \in A(f(\beta+k) = 1)\}) = 0$, which implies $A \not\subseteq J_\beta$ for all $\beta > \alpha$. □

Remark. It follows that the cardinal invariant $\mathfrak{h}_{\omega\text{-hitting}}$ is not *tame*.[1] As a consequence of theorem 6.1.11 of [19] (under an appropriate large cardinal assumption), we have that for every tame cardinal invariant $\mathfrak{j}$, if $\mathfrak{j} < \mathfrak{b}$ holds in some forcing extension, then it holds in $V^{\mathbb{L}_{\omega_2}}$. Theorem 17 shows that $V^{\mathbb{L}_{\omega_2}} \models \mathfrak{h}_{\omega\text{-}hitting} = \mathfrak{b}$. On the other hand, theorem 18 shows that $V^{\mathbb{B}(\omega_1)} \models \mathfrak{h}_{\omega\text{-hitting}} = \omega_1 < \mathfrak{b} = \omega_2$.

3.4. **The consistency of $\mathfrak{h}_{\mathsf{ED}} < \mathsf{add}(\mathcal{M})$.** We consider $\mathbb{L}_{\mathsf{Fr}}$, *Laver forcing* associated to the Fréchet filter Fr (the filter of co-finite sets of ω). It is defined as the set of those trees $T \subseteq \omega^{<\omega}$ for which there is $s_T \in T$ (the *stem of* T) such that for all $t \in T$, $t \subseteq s_T$ or $s_T \subseteq t$ and such that for all $t \in T$, wich $t \supseteq s_T$ the set $\mathrm{succ}_T(t) = \{n \in \omega \colon t^\frown n \in T\} \in \mathsf{Fr}$. It is ordered by inclusion. It is well-known that the forcing $\mathbb{L}_{\mathsf{Fr}}$ is σ-centered.

Similar to the definition of an ω-hitting family of sets, we say that a family $\mathcal{F} \subseteq \omega^\omega$ is *ω-hitting* if given $\langle A_n \colon n \in \omega \rangle \subseteq [\omega]^\omega$ there is a $f \in \mathcal{F}$ such that $f^{-1}(m) \cap A_n$ is infinite for all m and n. An important property of ω-hitting families of functions, which will be used several times in what follows, is that if an ω-hitting family is partitioned into countably many pieces, then at least one of the pieces is ω-hitting.

We now turn to the preservation of ω-hitting for functions in iterations. The argument is based on [5]. In order to do that, we introduce a stronger property: We say that a forcing notion $\mathbb{P}$ *strongly preserves ω-hitting for functions* if for every $\mathbb{P}$-name $\dot{A}$ for infinite subset of ω there is a $\langle A_n \colon n \in \omega \rangle \subseteq [\omega]^\omega$ such that for any $f \in \omega^\omega$, $f^{-1}(m) \cap A_n$ is infinite for all m and n then $\Vdash_{\mathbb{P}}$ "$f^{-1}(m) \cap \dot{A}$ is infinite for all m". Clearly, every forcing notion that strongly preserves ω-hitting for functions preserves ω-hitting for functions.

Lemma 19. *$\mathbb{L}_{\mathsf{Fr}}$ strongly preserves ω-hitting for functions.*

Proof. Let $\dot{A}$ be an $\mathbb{L}_{\mathsf{Fr}}$-name for countable subset of ω. Aiming towards a contradiction, assume that for each $\langle A_n \colon n \in \omega \rangle \subseteq [\omega]^\omega$ there is $f \in \omega^\omega$ such that $f^{-1}(m) \cap A_n$ is infinite for all m and n, yet there are a condition T_f and natural numbers n_f, m_f such that

$$T_f \Vdash \text{“}f^{-1}(m_f) \cap \dot{A} \subseteq n_f\text{”}. \qquad (\star)$$

[1] Recall that a cardinal invariant $\mathfrak{j}$ is *tame* (see [19]) if it is the minimum size of a set $A \subseteq X$, where X is a Polish space, with properties $\phi(A)$, and $\forall x \in X \exists y \in A\ \theta(x, y)$ where ϕ quantifies over natural numbers and elements of A only and θ is a projective formula not mentioning the set A.

Let $\mathcal{F}$ be the family of all such $f \in \omega^\omega$, that is, the family of all $f \in \omega^\omega$ such that there is a condition T_f and natural numbers n_f, m_f such that $T_f \Vdash$ "$f^{-1}(m_f) \cap \dot{A} \subseteq n_f$". By our assumption $\mathcal{F}$ is ω-hitting.

Recall the standard rank analysis for Laver forcing. For $s \in [\omega]^{<\omega}$, say *s favors $k \in \dot{A}$* if there is no condition $T \in \mathbb{L}_{\mathsf{Fr}}$ with stem s such that $T \Vdash$ "$k \notin \dot{A}$", or equivalently, every condition $T \in \mathbb{L}_{\mathsf{Fr}}$ with stem s has an extension T' such that $T' \Vdash$ "$k \in \dot{A}$". Define the *rank* $\mathrm{rk}(s)$ by recursion on the ordinals by

$$\mathrm{rk}(s) = 0 \Leftrightarrow \begin{cases} \text{either} & \exists K \in [\omega]^\omega \, \forall k \in K (s \text{ favors } k \in \dot{A}) \\ \text{or} & \exists X \in [\omega]^\omega,\ f\colon X \to \omega \text{ finite-to-one} \\ & \forall l \in X (s^\frown l \text{ favors } f(l) \in \dot{A}) \end{cases}$$

and $\mathrm{rk}(s) \leqslant \alpha$ if and only if there is a $X \in [\omega]^\omega$ such that $\mathrm{rk}(s^\frown l) < \alpha$ for all $l \in X$, when $\alpha > 0$.

Claim 3. $\mathrm{rk}(s) < \infty$ *for all* s.

Proof of the claim. Assume $\mathrm{rk}(s) = \infty$. So $K = \{k\colon s \text{ favors } k \in \dot{A}\}$ is finite. Recursively build $T \in \mathbb{L}_{\mathsf{Fr}}$ with stem s such that for all $t \in T$ extending s,

- $\mathrm{rk}(t) = \infty$, and
- $\{k\colon t \text{ favors } k \in \dot{A}\} \subseteq K$.

Let such t be given. First, there is $X_0 \in \mathsf{Fr}$ such that $\mathrm{rk}(t^\frown l) = \infty$ for all $l \in X_0$. Let $X_1 = \{l \in X_0\colon \exists k \notin K (t^\frown l \text{ favors } k \in \dot{A})\}$. If X_1 is infinite, then we can define a function as in the definition of rk, and so $\mathrm{rk}(t) = 0$, a contradiction. Thus X_1 is finite and $X_0 \setminus X_1 \in \mathsf{Fr}$. For $t^\frown l$ with $l \in X_0 \setminus X_1$, both clauses above are satisfied, and the construction proceeds.

Now find $T' \leqslant T$ and $k \notin K$ such that $T' \Vdash$ "$k \in \dot{A}$". Then the stem of T' in particular favors $k \in \dot{A}$, a contradiction. ■

Let s_f be the stem of T_f. By strengthening T_f, if necessary, we may assume that $\mathrm{rk}(s_f) = 0$ for all $f \in \mathcal{F}$. Since $\mathcal{F}$ is ω-hitting, there are s and natural numbers n, m such that the family $\mathcal{F}_{s,n,m} = \{f \in \mathcal{F}\colon s = s_f,\ n = n_f \text{ and } m = m_f\}$ is ω-hitting. Fix such s, n and m.

We consider two cases, according to the definition of rk.

Case 1. $\exists K \in [\omega]^\omega \, \forall k \in K (s$ *favors* $k \in \dot{A})$

Let $f \in \mathcal{F}_{s,n,m}$ be such that $f^{-1}(m) \cap K$ is infinite. So there is $k > n$ such that $k \in f^{-1}(m) \cap K$. Thus there is $T' \leqslant T_f$ with $T' \Vdash$ "$k \in \dot{A}$", a contradiction to the initial assumption $(\star)$.

Case 2. $\exists X \in [\omega]^\omega$, $f\colon X \to \omega$ *finite-to-one* $\forall l \in X (s^\frown l$ *favors* $f(l) \in \dot{A})$.

Let $g \in \mathcal{F}_{s,n,m}$ be such that $g^{-1}(m) \cap \mathrm{ran}(f)$ is infinite. Since $X \subseteq^* \mathrm{succ}_{T_g}(s)$, there is a $k \in g^{-1}(m) \cap \mathrm{ran}(f)$ with $k > n$ such that $f^{-1}(k) \cap \mathrm{succ}_{T_g}(s) \neq \emptyset$. Let $l \in f^{-1}(k) \cap \mathrm{succ}_{T_g}(s)$. Thus $s^\frown l$ favors $k \in \dot{A}$. Hence there is $T \leqslant T_g$ whose stem extends $s^\frown l$ such that $T \Vdash$ "$k \in \dot{A}$", again a contradiction. □

Lemma 20. *Finite support iteration of forcings strongly preserving ω-hitting for functions strongly preserves ω-hitting for functions.*

Proof. This is a standard argument. We provide the details for the sake of completeness. Obviously, it suffices to consider limit stages of cofinality ω.

Let $\langle \mathbb{P}_k, \dot{\mathbb{Q}}_k \colon k \in \omega \rangle$ be a finite support iteration of ccc forcing such that each $\mathbb{P}_k$ strongly preserves ω-hitting for functions.

Let $\dot{A}$ be a $\mathbb{P}_\omega$-name for an infinite subset of ω. In the intermediate extension $V[G_k]$ find a decreasing sequence of conditions $\langle p_{n,k} \colon n \in \omega \rangle$ and infinite subsets $A_{n,k}$ of ω such that

$$p_{n,k} \Vdash_{\mathbb{P}_{[k,\omega)}} \text{“the first } n \text{ elements of } A_{n,k} \text{ and } \dot{A} \text{ agree”}$$

The $A_{n,k}$ are approximations to $\dot{A}$.

Now, as each $\mathbb{P}_k$ strongly preserves ω-hitting for functions, there is a $\langle A^m_{n,k} \colon m \in \omega \rangle \subseteq [\omega]^\omega$ such that for every $f \in \omega^\omega$, if $f^{-1}(i) \cap A^m_{n,k}$ is infinite for all i and m then

$$\Vdash_{\mathbb{P}_k} \text{“} f^{-1}(i) \cap \dot{A}_{n,k} \text{ is infinte for all } i\text{”}$$

Consider $\langle A^m_{n,k} \colon n, k, m \in \omega \rangle$ and let $f \in \omega^\omega$ be such that $f^{-1}(i) \cap A^m_{n,k}$ is infinite for all n, k and m. To finish the proof, it suffices to show that

$$\Vdash_{\mathbb{P}_\omega} \text{“} f^{-1}(i) \cap \dot{A} \text{ is infinite for all } i\text{”}.$$

If not, then there are a $q \in \mathbb{P}_\omega$, $i \in \omega$ and $m \in \omega$ such that $q \Vdash_{\mathbb{P}_\omega}$ “$f^{-1}(i) \cap \dot{A} \subseteq m$”. Let k be such that $q \in \mathbb{P}_k$.

Let G_k be a $\mathbb{P}_k$-generic such that $q \in G_k$. As $f^{-1}(i) \cap A_{n,k}$ is infinite, let $l \geqslant m$ with $l \in f^{-1}(i) \cap A_{n,k}$. For large enough n,

$$p_{n,k} \Vdash_{\mathbb{P}_{[k,\omega)}} \text{“} l \in \dot{A}\text{”}.$$

Since $q \in G_k$, this contradicts the initial assumption about q. □

Combining the previous two lemmas, we obtain the following consistency result.

Theorem 21. *It is consistent with* **ZFC** *that* $\mathfrak{h}_{\mathsf{ED}} = \omega_1$ *and* $\mathsf{add}(\mathcal{M}) = \omega_2$.

Proof. Start with a model of **CH** and iterate the forcing $\mathbb{L}_{\mathsf{Fr}}$ with finite support ω_2 times. To establish the first assertion, let $\kappa = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \omega^\omega \text{ is } \omega\text{-hitting}\}$. Then $\mathfrak{h}_{\mathsf{ED}} \leqslant \kappa$. By lemmas 19 and 20, the cardinal κ is preserved along the iteration, and hence $V^{\mathbb{L}_{\mathsf{Fr}}^{\omega_2}} \models \mathfrak{h}_{\mathsf{ED}} = \omega_1$. On the other hand, it is well known that $\mathbb{L}_{\mathsf{Fr}}$ adds a Cohen real and also adds an unbounded real, and since $\mathsf{add}(\mathcal{M}) = \min\{\mathsf{cov}(\mathcal{M}), \mathfrak{b}\}$ (see [2]), it follows that $V^{\mathbb{L}_{\mathsf{Fr}}^{\omega_2}} \models \mathsf{add}(\mathcal{M}) = \omega_2$. □

4. Final remarks and questions

In [8] the author asked which of the following inequalities can be consistently strict: $\mathfrak{h} \leq \mathfrak{h}_{\mathsf{analytic}} \leq \mathfrak{h}_{\mathsf{Borel}} \leq \cdots \leq \mathfrak{h}_{\mathsf{F}_\sigma} \leq \mathfrak{b}$. There is no known consistency result that distinguishes between the intersection numbers of analytic ideals and that of Borel (or even F_σ) ideals. Note that a positive answer to the following question provides an answer to the previous question.

Question 1. [8] *Let $\mathscr{I}$ be a tall Borel (analytic) ideal. Is there a tall F_σ ideal $\mathscr{J}$ such that $\mathscr{J} \subseteq \mathscr{I}$?*

The author of [8] (the first listed author of this note) also claimed that “obviously $\mathfrak{h}_{\mathsf{F}_\sigma} \leq \min\{\mathfrak{b}, \mathfrak{s}\}$”. We do not whether this is true, but definitely, it does not seem obvious.

Question 2. *Is $\mathfrak{h}_{\mathsf{F}_\sigma} \leq \mathfrak{s}$?*

Question 3. *Is $\mathfrak{h}_{\mathsf{ED}} = \min\{\mathfrak{b}, \mathfrak{s}\}$?*

References

[1] B. Balcar, J. Pelant, and P. Simon, The space of ultrafilters on **N** covered by nowhere dense sets. *Fund. Math.*, 110(1):11-24,1980.

[2] T. Bartoszynski and H. Judah. *Set theory: On the structure of the Real Line.* A. K. Peters, Wellesley, MA, 1995.

[3] T. Bartoszyński, and S. Shelah. Intersection of $< 2^{\aleph_0}$ ultrafilters may have measure zero. *Archive for Mathematical Logic*, 31(4):221-226, 1992.

[4] A. Blass.*Combinatorial Cardinal Characteristics of the Continnum.* In Handbook of Set Theory, vol. 1, 395-489. M. Foreman, A. Kanamori (eds). New York, Springer Verlag 2010.

[5] J. Brendle, and M. Hrušák. Countable Fréchet Boolean groups: an independence result. *Journal of Symbolic Logic* 74(3): 1061-1068, 2009.

[6] A. Dow. Tree π-bases for β $\mathbf{N}-\mathbf{N}$ in various models. *Topology and its Applications*, 33: 3-19, 1989.

[7] F. Hernández-Hernández and M. Hrušák. Cardinal invariants of analytic P-ideals. *Canadian Journal of Mathematics*, 59(3): 575-595, 2007.

[8] M. Hrušák. Combinatorics of filters and ideals. In *Set theory and its applications (Boise, ID, 1995-2010)*, volume 533 of *Contemp. Math.*, Amer. Math. Soc. (2011), 29-69.

[9] M. Hrušák, D. Meza-Alcántara, and H. Minami. Pair-splitting, pair-reaping and cardinal invariants of F_σ ideals. *Jornal of Symbolic Logic*, 75(2): 661-677, 2010.

[10] M. Hrušák, D. Rojas and J. Zapletal. Cofinalities of Borel ideals. Preprint.

[11] K. Kunen. *Set Theory. An introduction to Independence Proofs.* North Holland, Amsterdam, 1980.

[12] A. R. D. Mathias. Happy families. *Ann. Math. Logic,* 12(1): 59-111, 1977.

[13] K. Mazur. F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $\mathcal{P}(\omega)/I$. *Fundamenta Mathematicae*, 138(2):103-111, 1991.

[14] J. Pawlikowski. Laver's forcing and outer measure. In *Proceeding of BEST Conference 1991-1994, 1995.*

[15] S. Plewik. Intersection and unions of ultrafilters without the Baire property. *Bulletin of the Polish Academy of Sciences Mathematics*, 35(11-12):805-808, 1987.

[16] S. Plewik. Ideals of second category. *Fundamenta Mathematicae*, 138(1):23-26, 1991.

[17] S. Solecki. Filters and sequences. *Fundamenta Mathematicae*, 163:215-228, 2000.

[18] M. Talagrand. Compacts de fonctions mesurables et filtres non mesurables. *Studia Mathematica*, 67(1):13-43, 1980.

[19] J. Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2008.

Centro de Ciencias Matemáticas, UNAM, Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, México.

E-mail address: michael@matmor.unam.mx, ariet@matmor.unam.mx, otellez@matmor.unam.mx

Department of Mathematics, University of Toronto, Room 6290, 40 St. george street, Toronto, Ontario, Canada

E-mail address: azarel@math.toronto.edu