PAIR-SPLITTING, PAIR-REAPING AND CARDINAL INVARIANTS OF F_σ-IDEALS

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ABSTRACT. We investigate the pair-splitting number $\mathfrak{s}_{pair}$ which is a variation of splitting number, pair-reaping number $\mathfrak{r}_{pair}$ which is a variation of reaping number and cardinal invariants of ideals on ω. We also study cardinal invariants of F_σ ideals and their upper bounds and lower bounds. As an application, we answer a question of S. Solecki by showing that the ideal of finitely chromatic graphs is not locally Katětov-minimal among ideals not satisfying Fatou's lemma.

INTRODUCTION

The splitting number $\mathfrak{s}$ and the reaping number $\mathfrak{r}$ are cardinal invariants which play important role when we study $\mathcal{P}(\omega)/fin$.

For $X, Y \in [\omega]^\omega$ we say X *splits* Y if both $X \cap Y$ and $Y \setminus X$ are infinite. We call $\mathcal{S} \subset [\omega]^\omega$ a *splitting family* if for each $Y \in [\omega]^\omega$, there exists $X \in \mathcal{S}$ such that X splits Y. The *splitting number* $\mathfrak{s}$ is the least size of a splitting family.

We call $\mathcal{R}$ a *reaping family* if for each $X \in [\omega]^\omega$, there exists $Y \in \mathcal{R}$ such that Y is not split by X, that is, $X \cap Y$ is finite or $Y \setminus X$ is finite. The *reaping number* $\mathfrak{r}$ is the least size of a reaping family.

We shall study variations of splitting number and of reaping number and study cardinal invariants of ideals of ω.

The pair-reaping number $\mathfrak{r}_{pair}$ and the pair-splitting number $\mathfrak{s}_{pair}$ are introduced in two different contexts with the same definition independently.

One is motivated by the investigation of the dual-reaping number $\mathfrak{r}_d$ and the dual-splitting number $\mathfrak{s}_d$ which are reaping number and

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splitting number for the structure of all infinite partitions of ω ordered by "almost coarser" $((\omega)^\omega, \leq^*)$ respectively.

We call $A \subset [\omega]^2$ *unbounded* if for $k \in \omega$, there exists $a \in A$ such that $a \cap k = \emptyset$. For $X \in [\omega]^\omega$ and unbounded $A \subset [\omega]^2$, X *pair-splits* A if there exist infinitely many $a \in A$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. We call $\mathcal{S} \subset [\omega]^\omega$ a *pair-splitting family* if for each unbounded $A \subset [\omega]^2$, there exists $X \in \mathcal{S}$ such that X pair-splits A. The *pair-splitting number* $\mathfrak{s}_{pair}$ is the least size of a pair-splitting family.

We call $\mathcal{R} \subset \mathcal{P}([\omega]^2)$ a *pair-reaping family* if for each $A \in \mathcal{R}$, A is unbounded and for $X \in [\omega]^\omega$, there exists $A \in \mathcal{R}$ such that X does not pair-split A, that is, for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. The *pair-reaping number* $\mathfrak{r}_{pair}$ is the least size of a pair-reaping family.

In [13] it is proved that there is the following relationship between $\mathfrak{r}_{pair}$, $\mathfrak{s}_{pair}$ and other cardinal invariants.

Proposition 0.1. *(1)* $\mathfrak{s}_{pair} \leq \mathsf{non}(\mathcal{M}), \mathsf{non}(\mathcal{N})$.
(2) $\mathfrak{r}_{pair} \geq \mathsf{cov}(\mathcal{M}), \mathsf{cov}(\mathcal{N})$.
(3) $\mathfrak{s}_{pair} \geq \mathfrak{s}$.
(4) $\mathfrak{r}_{pair} \leq \mathfrak{r}, \mathfrak{s}_d$.

It is not known whether $\mathfrak{r}_d \leq \mathfrak{s}_{pair}$ or not.

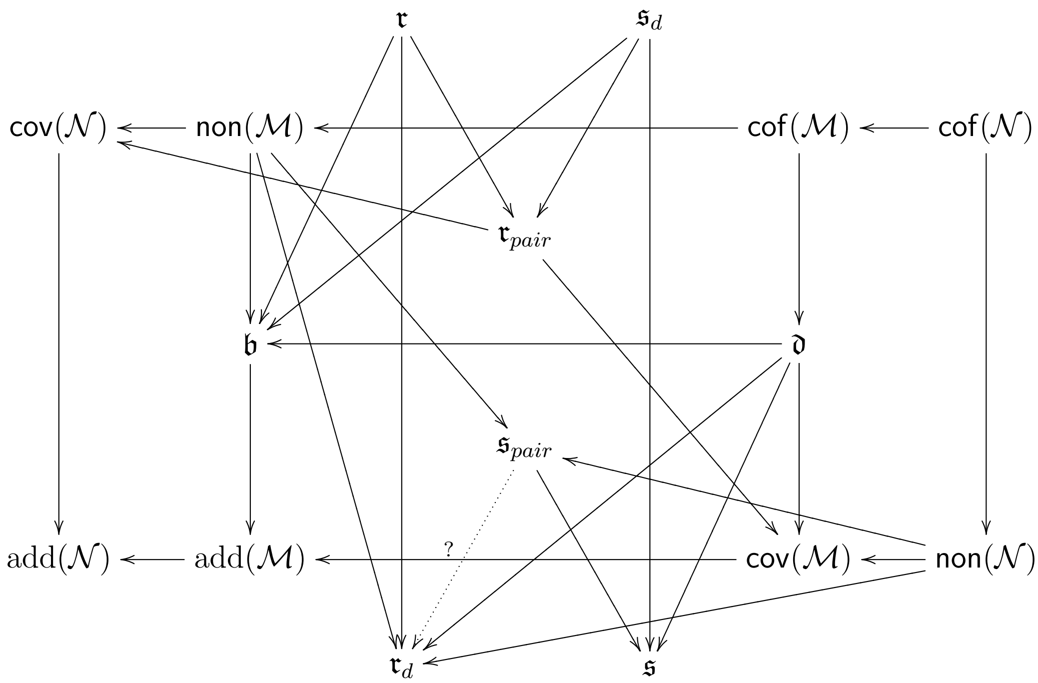
For $\mathcal{G} \subset \omega^\omega$, we call $\mathcal{G}$ a *dominating family* if for each $f \in \omega^\omega$, there exists $g \in \mathcal{G}$ such that for all but finitely many $n \in \omega$, $f(n) \leq g(n)$, denoted by $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the least size of a dominating family.

For $\mathcal{G} \subset \omega^\omega$, we call $\mathcal{G}$ an *unbounded family* if for each $f \in \omega^\omega$, there exists $g \in \mathcal{G}$ such that $g \not\leq^* f$, that is, there exist infinitely many $n \in \omega$ such that $g(n) > f(n)$. The *unbounded number* $\mathfrak{b}$ is the least size of an unbounded family.

$\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{r} \geq \mathfrak{b}$ hold (see in [3]). Kamo proved the following statement in [13]:

Theorem 0.2. $\mathfrak{r}_d \leq \mathfrak{d}$ *and* $\mathfrak{s}_d \geq \mathfrak{b}$.

So we have the following diagram:



An arrow $\kappa \to \lambda$ denotes the inequality $\kappa \geq \lambda$.

In [13] by using finite support iteration of Hechler forcing, the following consistency results are proved.

Theorem 0.3. *It is consistent that $\mathfrak{s}_{pair} <$ add$(\mathcal{M})$. Dually it is consistent that $\mathfrak{r}_{pair} > \mathsf{cof}(\mathcal{M})$.*

$\mathfrak{r}_{pair}$ is a lower bound of $\mathfrak{r}$ and $\mathfrak{s}_d$, and $\mathfrak{s}_{pair}$ is an upper bound of $\mathfrak{s}$ (and maybe of $\mathfrak{r}_d$). So it is natural to ask the question whether $\mathfrak{s}_{pair} \leq \mathfrak{d}$ or not and whether $\mathfrak{r}_{pair} \geq \mathfrak{b}$ or not. In [14] the consistency of $\mathfrak{s}_{pair} > \mathfrak{d}$ and of $\mathfrak{r}_{pair} < \mathfrak{b}$ are shown and an upper bound of $\mathfrak{s}_{pair}$ and a lower bound of $\mathfrak{r}_{pair}$ are given.

The other source of motivation stems from the study of Borel ideals on ω.

For a set X, we call $\mathcal{I} \subset \mathcal{P}(X)$ an *ideal on* X if $\mathcal{I}$ satisfies the following:

(1) for $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$,
(2) for $A, B \subset X$, $A \subset B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$ and
(3) $X \notin \mathcal{I}$.

In this paper we assume that all ideals on X contain all finite subsets of X. We say an ideal $\mathcal{I}$ on ω is *tall* if for each $X \in [\omega]^\omega$ there exists $I \in \mathcal{I}$ such that $I \cap X$ is infinite.

If $\mathcal{I}$ is an ideal on ω and $Y \in [\omega]^\omega$, we denote by $\mathcal{I} \restriction Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$ on Y.

The topology of $\mathcal{P}(\omega)$ is induced by identifying each subset of ω with its characteristic function, where 2^ω is equipped with the product topology of the discrete topology of $2 = \{0,1\}$. We call $\mathcal{I}$ a Borel ideal on ω if $\mathcal{I}$ is an ideal on ω and $\mathcal{I}$ is Borel in this topology.

Let $\mathcal{I}$ be a tall ideal on ω. Then the *uniformity number* of $\mathcal{I}$, denoted by $\mathsf{non}^*(\mathcal{I})$ and the *covering number* of $\mathcal{I}$, denoted by $\mathsf{cov}^*(\mathcal{I})$ are given by

$$\begin{aligned}
\mathsf{non}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(|A \cap I| < \aleph_0)\}, \\
\mathsf{cov}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge (\forall X \in [\omega]^\omega)(\exists A \in \mathcal{A})(|X \cap A| = \aleph_0)\}.
\end{aligned}$$

The (pre)orderings on the family of ideals are crucial in describing some properties of ideals on ω. For example, Cohen-destructibility of an ideal $\mathcal{I}$ on ω is equivalent to the statement $\mathcal{I}$ is smaller than the nowhere dense ideal in the Katětov order ([8, 6]).

Suppose $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω. Then $\mathcal{I} \leq_K \mathcal{J}$ if there exists a function $f : \omega \to \omega$ such that for each $I \in \mathcal{I}$, $f^{-1}[I] \in \mathcal{J}$. We call this ordering *Katětov order*.

When we investigate the Katětov order, the uniformity number of ideals and the covering number of ideals are significant.

Proposition 0.4. *If* $\mathcal{I} \leq_K \mathcal{J}$*, then* $\mathsf{non}^*(\mathcal{I}) \leq \mathsf{non}^*(\mathcal{J})$ *and* $\mathsf{cov}^*(\mathcal{I}) \geq \mathsf{cov}^*(\mathcal{J})$.

In the study, the Katětov order between the finite chromatic ideal on $[\omega]^2$, denoted by $\mathcal{G}_{FC}$, which is an F_σ-ideal, and other Borel ideals is investigated. The pair-reaping number and the pair-splitting number are introduced as other descriptions of the uniformity number of $\mathcal{G}_{FC}$ and the covering number of $\mathcal{G}_{FC}$.

The encounter of these two different studies produces more general results.

In the present paper we shall investigate the relationship between $\mathfrak{r}_{pair}$, $\mathfrak{s}_{pair}$, cardinal invariants of ideals on ω and other classical cardinal invariants.

In Section 1 we shall show $\mathfrak{r}_{pair} = \mathfrak{r}_n$ for $n \geq 3$ and $\mathfrak{s}_{pair} = \mathfrak{s}_n$ for $n \geq 3$. In Section 2 we shall investigate the relation between $\mathfrak{s}_{pair}$, $\mathfrak{r}_{pair}$ and cardinal invariants of the ideal of finitely chromatic graphs. In Section 3 we shall show the consistency of $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$ for F_σ-ideals on ω. In Section 4 we shall answer a question by Solecki from [15].

1. n-SPLITTING NUMBER AND n-REAPING NUMBER

In this section we shall show $\mathfrak{s}_{pair} = \mathfrak{s}_n$ and $\mathfrak{r}_{pair} = \mathfrak{r}_n$ for $n \geq 2$.

We call $A \subset [\omega]^n$ unbounded if for $k \in \omega$ there exists $a \in A$ such that $a \cap k = \emptyset$.

For $X \in [\omega]^\omega$ and unbounded $A \subset [\omega]^n$, X n-splits A if there exist infinitely many $a \in A$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. We call $\mathcal{S} \subset [\omega]^\omega$ an n-splitting family if for each unbounded $A \subset [\omega]^n$ there exists $X \in \mathcal{S}$ such that X n-splits A. The n-splitting number $\mathfrak{s}_n$ is the least size of an n-splitting family.

We call $\mathcal{R} \subset \mathcal{P}([\omega]^n)$ an n-reaping family if for each $A \in \mathcal{R}$, A is unbounded and for $X \in [\omega]^\omega$, there exists $A \in \mathcal{R}$ such that X does not n-split A, that is, for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. The n-reaping number $\mathfrak{r}_n$ is the least size of an n-reaping family. So $\mathfrak{s}_{pair} = \mathfrak{s}_2$ and $\mathfrak{r}_{pair} = \mathfrak{r}_2$.

The following relations hold between $\mathfrak{s}_n$ for $n \geq 2$ and between $\mathfrak{r}_n$ for $n \geq 2$.

Proposition 1.1. *(1) $\mathfrak{s}_{pair} = \mathfrak{s}_2 \geq \mathfrak{s}_3 \geq \ldots \geq \mathfrak{s}_n \geq \ldots$ and $\mathfrak{s}_n \geq \mathfrak{s}$ for $n \geq 2$.*

(2) $\mathfrak{r}_{pair} = \mathfrak{r}_2 \leq \mathfrak{r}_3 \leq \ldots \leq \mathfrak{r}_n \leq \ldots$ and $\mathfrak{r} \geq \mathfrak{r}_n$ for $n \geq 2$.

Proof. Fix $n \geq 2$. Let $\mathcal{S}$ be an n-splitting family of cardinality $\mathfrak{s}_n$. For an unbounded $A \subset [\omega]^{n+1}$, let $A^* \subset [\omega]^n$ be the collection of the initial n-many elements of an element of A. Then there exists $X \in \mathcal{S}$ which n-splits A^*. So there exist infinitely many $a \in A^*$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. Since for each $a \in A^*$, there exists $a^* \in A$ such that $a \subset a^*$, there exist infinitely many $a^* \in A$ such that $a^* \cap X \neq \emptyset$ and $a^* \setminus X \neq \emptyset$. So $\mathcal{S}$ is an $(n+1)$-splitting family. Hence $\mathfrak{s}_{n+1} \leq \mathfrak{s}_n$.

We shall show $\mathfrak{s}_n \geq \mathfrak{s}$. Let $\mathcal{S}$ be an n-splitting family of cardinality $\mathfrak{s}_n$. For $Y \in [\omega]^\omega$, fix an infinite subset A_Y of $[Y]^n$ whose elements are pairwise disjoint. Then A_Y is unbounded. Pick $X \in \mathcal{S}$ which n-splits A_Y. So there exist infinitely many $a \in A_Y$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. Hence $|X \cap Y| = |Y \setminus X| = \omega$. So X splits Y. Therefore $\mathcal{S}$ is a splitting family. So $\mathfrak{s}_n \geq \mathfrak{s}$.

We shall show $\mathfrak{r}_n \leq \mathfrak{r}_{n+1}$. Let $\mathcal{R}$ be an $(n+1)$-reaping family of cardinality $\mathfrak{r}_{n+1}$. Put $\mathcal{R}^*$ the set of the initial n-many elements of an element of $\mathcal{R}$. Given $X \in [\omega]^\omega$, pick $A \in \mathcal{R}$ such that for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. Put A^* the set of initial segments of size n of elements of A. Then for all but finite many $a^* \in A^*$, $a^* \cap X = \emptyset$ or $a^* \subset X$. So $\mathcal{R}^*$ is an n-reaping family of cardinality $\mathfrak{r}_{n+1}$. Hence $\mathfrak{r}_n \leq \mathfrak{r}_{n+1}$.

We shall prove $\mathfrak{r} \geq \mathfrak{r}_n$. Let $\mathcal{R}$ be a reaping family of cardinality $\mathfrak{r}$. For each $Y \in \mathcal{R}$, fix an infinite subset A_Y of $[Y]^n$ whose elements are pairwise disjoint. $\mathcal{R}^*$ is the collection of A_Y with $Y \in \mathcal{R}$.

For $X \in [\omega]^\omega$, pick $Y \in \mathcal{R}$ such that $Y \setminus X$ is finite or $X \cap Y$ is finite. Then for all but finitely many $a \in A_Y$, $a \subset X$ or for all but finitely many $a \in A_Y$, a does not meet X. So $\mathcal{R}^*$ is an n-reaping family of cardinality $\mathfrak{r}$. Therefore $\mathfrak{r}_n \leq \mathfrak{r}$. □

Proposition 1.1 was proved as early as $\mathfrak{r}_n$ and $\mathfrak{s}_n$ were defined. However, it was not known whether $\mathfrak{s}_{pair} = \mathfrak{s}_n$ for $n \geq 3$ or not.

Between $\mathfrak{r}_{pair}$ and $\mathfrak{r}_n$, we can prove the following statement.

Proposition 1.2. *$\mathfrak{r}_{pair} = \mathfrak{r}_n$ for $n \geq 3$.*

Proof. We shall prove that $\mathfrak{r}_{pair} \geq \mathfrak{r}_4$. Let $\mathcal{R}$ be a pair-reaping family of cardinality $\mathfrak{r}_{pair}$. Without loss of generality we can assume $\mathcal{R}$ is closed under finite changes, i.e, if $C \in \mathcal{R}$ and $|D \triangle C| < \aleph_0$ then $D \in \mathcal{R}$; and A is pairwise disjoint for each $A \in \mathcal{R}$. Let e_A be a bijection from A to ω. Put

$$\mathcal{R}^* = \{C : \exists A, B \in \mathcal{R} \left(C = \{\bigcup e_A^{-1}[b] : b \in B\} \right)\}.$$

We shall prove such $\mathcal{R}^*$ is a 4-reaping family. Let $X \in [\omega]^\omega$. Then we can find $A \in \mathcal{R}$ such that for all $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. Then define $Y_{A,X} \subset \omega$ so that

$$n \in Y_{A,X} \text{ if } \begin{cases} e_A^{-1}(n) \subset X & \text{if } \exists^\infty m \in \omega \left(e_A^{-1}(m) \subset X \right), \\ e_A^{-1}(n) \cap X = \emptyset & \text{otherwise} \end{cases}$$

Pick $B \in \mathcal{R}$ such that for all $b \in B$, $b \cap Y_{A,X} = \emptyset$ or $b \subset Y_{A,X}$. Let $C_{A,B} = \{\bigcup e_A^{-1}[b] : b \in B\} \in \mathcal{R}^*$. Let $b \in B$. Since for $a \in A$, $a \cap X = \emptyset$ or $a \subset X$, $e_A^{-1}(i) \cap X = \emptyset$ or $e_A^{-1}(i) \subset X$ for $i \in b$. Since $b \cap Y_{A,X} = \emptyset$ or $b \subset Y_{A,X}$ for $b \in B$, $\bigcup e_A^{-1}[b] \cap X = \emptyset$ or $\bigcup e_A^{-1}[b] \subset X$. So X does not 4-split $C_{A,B}$. Since $|\mathcal{R}^*| = \mathfrak{r}_{pair}$, $\mathfrak{r}_4 \leq \mathfrak{r}_{pair}$. By Proposition 1.2 $\mathfrak{r}_{pair} = \mathfrak{r}_3 = \mathfrak{r}_4$.

Similarly we can prove $\mathfrak{r}_{pair} = \mathfrak{r}_{2n}$ for $n \geq 2$. □

David Asperó conjectured that $\mathfrak{s}_{pair} = \mathfrak{s}_3$. Shizuo Kamo gave the proof. The proofs for the splitting numbers are not dual to the proofs for the reaping numbers. It might simplify in terms of Galois-Tukey connections as in [16]. However it might be difficult. In [11] and [12], Mildenberger introduced another variation of reaping numbers $\mathfrak{r}_n$ and $\mathfrak{r}_n = \mathfrak{r}_m (= \mathfrak{r})$ holds for $n, m \in \omega$ but it is proved that there are no nice Galois-Tukey connections between Mildenberger's reaping numbers.

Theorem 1.3. *(Kamo) $\mathfrak{s}_{pair} = \mathfrak{s}_n$ for $n \geq 3$.*

Proof. We shall prove $\mathfrak{s}_{pair} = \mathfrak{s}_4$. Let ZFC$^-$ be a large enough fragment of ZFC. Suppose $\mathfrak{s}_4$, $\mathfrak{s}_3 < \mathfrak{s}_{pair}$ holds. Let M_0 be a model of ZFC$^-$ such that the cardinality is $\mathfrak{s}_3$ and $M_0 \cap [\omega]^\omega$ is a 3-splitting family and 4-splitting family.

Pick an infinite subset A of $[\omega]^2$ which is not 2-split by $M_0 \cap [\omega]^\omega$. Without loss of generality we can assume this A is pairwise disjoint.

Let M_1 be a model of ZFC$^-$ of cardinality $\mathfrak{s}_3$ which contains A and all elements of M_0. Pick B in M_1 such that B is an infinite subset of $[A]^2$ and B is not 2-split by any elements in $M_1 \cap [A]^\omega$. We can also assume this B is pairwise disjoint.

Let $C = \{a \cup b : \{a, b\} \in B\}$. Since $M_0 \cap [\omega]^\omega$ is a 4-splitting family, there exists $X \in M_0 \cap [\omega]^\omega$ such that X 4-splits C. Since A is not 2-split by X, there exist infinitely many $a \in A$ such that $a \subset X$ or $X \cap a = \emptyset$. So there exist infinitely many $\{a, b\} \in B$ such that $a \subset X$ and b does not meet X. Put $Y = \{a \in A : a \subset X\}$. Then $Y \in M_1$ and Y 2-splits B. However, this is a contradiction to the fact B is not split by any infinite subset of A in M_1.

Similarly we can prove that $\mathfrak{s}_{pair} = \mathfrak{s}_{2n}$ for $n \geq 2$. Therefore $\mathfrak{s}_{pair} = \mathfrak{s}_n$ for $n \in \omega$. $\square$

2. The ideal of finitely chromatic graphs

In this section we shall investigate the relation between the finite chromatic ideal, pair-splitting number and pair-reaping number.

The *finite chromatic ideal* on $[\omega]^2$ is defined by

$$\mathcal{G}_{FC} = \{A \subset [\omega]^2 : \chi(\omega, A) < \aleph_0\}$$

where $\chi(\omega, A) = \min\{k \in \omega : (\exists f \in k^\omega)(\forall a \in A)(|f[a]| = 2)\}$.

Theorem 2.1. *The following conditions hold.*

(1) $\mathfrak{s}_{pair} = \mathsf{cov}^*(\mathcal{G}_{FC})$,
(2) $\mathsf{non}^*(\mathcal{G}_{FC})$ *is the minimal cardinality of a family* $\mathcal{A} \subseteq [[\omega]^2]^\omega$ *such that for any finite partition* $\mathcal{P}$ *of* ω *there is an element* A *of* $\mathcal{A}$ *such that for every* $r \in A$ *there is* $P \in \mathcal{P}$ *such that* $r \subseteq P$ *and*
(3) $\mathfrak{r}_{pair} \leq \mathsf{non}^*(\mathcal{G}_{FC})$.

Proof. First we shall prove $\mathfrak{s}_{pair} \leq \mathsf{cov}^*(\mathcal{G}_{FC})$. Let $\mathcal{A}$ be a subset of $\mathcal{G}_{FC}$ such that $|\mathcal{A}| = \mathsf{cov}^*(\mathcal{G}_{FC})$ and

$$(1) \qquad (\forall X \subset [\omega]^2)(\exists A \in \mathcal{A})\,(|X| = \aleph_0 \to |A \cap X| = \aleph_0)\,.$$

Claim 2.2. *If* $A \in \mathcal{G}_{FC}$*, then there exist* $n \in \omega$ *and* $A_i \subset A$ *for* $i < n$ *such that* $A = \bigcup_{i<n} A_i$ *and* $\chi(A_i) = 2$ *for* $i < n$.

Proof of Claim. Suppose $A \in \mathcal{G}_{FC}$, $k \in \omega$ and $f : \omega \to k$ such that for all $a \in A$ $|f[a]| = 2$. For $i, j < k$ with $i < j$, put $A_{i,j} = \{a \in A : f[a] = \{i, j\}\}$. Then $\chi(\omega, A_{i,j}) = 2$ and $A = \bigcup_{i,j<k, i<j} A_{i,j}$. □

By this claim, we can assume $\chi(\omega, A) = 2$ for $A \in \mathcal{A}$. For each $A \in \mathcal{A}$, fix $f : \omega \to 2$ so that f witnesses $\chi(\omega, A) = 2$. Put $A_0 = f^{-1}(0) \cap \bigcup A$ and $\mathcal{A}_0 = \{A_0 : A \in \mathcal{A}\}$.

Then $\mathcal{A}_0$ is a pair-splitting family. Let $B \subset [\omega]^2$ be infinite. Since $\mathcal{A}$ satisfies (1), there is $A \in \mathcal{A}$ such that $|A \cap B| = \aleph_0$. So there exist infinitely many $b \in B$ such that $b \in A$. So there exist infinitely many $b \in B$ such that $b \cap A_0 \neq \emptyset$ and $b \setminus A_0 \neq \emptyset$. Therefore $\mathfrak{s}_{pair} \leq \mathsf{cov}^*(\mathcal{G}_{FC})$.

We shall prove $\mathfrak{s}_{pair} \geq \mathsf{cov}^*(\mathcal{G}_{FC})$. Let $\mathcal{S} \subset [\omega]^\omega$ be a pair-splitting family. For each $S \in \mathcal{S}$, put $A_S = \{a \in [\omega]^2 : a \cap S \neq \emptyset \wedge a \cap \omega \setminus S \neq \emptyset\}$ and $\mathcal{A}(\mathcal{S}) = \{A_S : S \in \mathcal{S}\}$.

Then $\mathcal{A}(\mathcal{S})$ satisfies that for each infinite $X \in [\omega]^2$, there exists an $A_S \in \mathcal{A}(\mathcal{S})$ such that $|X \cap A_S| = \aleph_0$. Let $X \subset [\omega]^2$ be infinite. Since $\mathcal{A}$ is a pair-splitting family, there exists an $S \in \mathcal{S}$ such that S pair-splits X. So there exist infinitely many $a \in X$ such that $a \cap S \neq \emptyset$ and $a \setminus S \neq \emptyset$. Hence $|X \cap A_S| = \aleph_0$. Therefore $\mathsf{cov}^*(\mathcal{G}_{FC}) \leq \mathfrak{s}_{pair}$.

In order to prove (2), note that if P is a finite partition of ω then $G_P = \{\{n, m\} : (\exists a \neq b \in P)(n \in a \wedge m \in b)\} \in \mathcal{G}_{FC}$, and moreover, $\{G_P : P \text{ is a finite partition of } \omega\}$ is a base of $\mathcal{G}_{FC}$. Then, if $\mathcal{A}$ is a family as in (2) then $\mathcal{A}$ itself witnesses $\mathsf{non}^*(\mathcal{G}_{FC})$; and if $\mathcal{B}$ is a witness of $\mathsf{non}^*(\mathcal{G}_{FC})$ then defining $\mathcal{A}$ as the family of finite changes of elements of $\mathcal{B}$ we are done. (3) follows directly from (2). □

It can be easily seen that $\mathcal{G}_{FC}$ is an F_σ-ideal. In particular, $\mathfrak{s}_{pair}$ is equal to the covering number of an F_σ-ideal and $\mathfrak{r}_{pair}$ is bounded by the uniformity number of an F_σ-ideal.

Concerning to the covering number of F_σ-ideals and $\mathfrak{b}$, we can construct a proper forcing notion which destroys tallness of an F_σ-ideal and preserves the unbounded number.

Theorem 2.3. [9] *For each F_σ-ideal $\mathcal{I}$, there exists a proper forcing notion $\mathbb{P}_\mathcal{I}$ which is ω^ω-bounding and adds a new element X in the extension such that $|X \cap I| < \aleph_0$ for $I \in \mathcal{I} \cap V$.*

By using ω_2-stage countable support iteration of $\mathbb{P}_\mathcal{I}$, we can show the following statement.

Corollary 2.4. *Suppose $\mathcal{I}$ is an F_σ-ideal on ω. Then it is consistent that $\mathsf{cov}^*(\mathcal{I}) > \mathfrak{d}$.*

Corollary 2.5. *It is consistent that $\mathfrak{s}_{pair} = \mathsf{cov}^*(\mathcal{G}_{FC}) > \mathfrak{d}$.*

3. The uniformities of F_σ-ideals

The *eventually different ideal* is defined by

$$\mathcal{ED} = \{A \subset \omega\times\omega : (\exists m, n \in \omega)(\forall k > n)\,(|\{l : \langle k, l\rangle \in A\}| \leq m)\}.$$

Define $\mathcal{ED}_{fin} = \mathcal{ED} \restriction \triangle$, where $\triangle = \{\langle m, n\rangle : n \leq m\}$.

On the $\mathsf{cov}^*(\mathcal{ED})$ we have the following result.

Lemma 3.1. $\mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$.

Proof. We will use the following lemma, due to Bartoszyński and Miller.

Lemma 3.2 ([1], Lemma 2.4.8)**.** *For any cardinal κ the following are equivalent:*

(a) $\kappa < \mathsf{non}(\mathcal{M})$,

(b) $(\forall F \in [\omega^\omega]^\kappa)(\exists g \in \omega^\omega)(\exists X \in [\omega]^\omega)(\forall f \in F)(\forall^\infty n \in X)(f(n) \neq g(n))$ *and*

(c) $(\forall F \in [\mathcal{C}]^\kappa)(\exists g \in \omega^\omega)(\forall S \in F)(\forall^\infty n)(g(n) \notin S(n))$

Let $\mathcal{F}$ be a subset of ω^ω of minimal cardinality such that

$$(\forall g \in \omega^\omega)(\forall X \in [\omega]^\omega)(\exists f \in \mathcal{F})(\exists^\infty n \in X)(f(n) = g(n))$$

(We are identifying every function $f \in \omega^\omega$ with its graph $\{(n, f(n)) : n < \omega\}$.) Define $\mathcal{A} = \mathcal{F} \cup \{\{n\}\times\omega : n < \omega\}$. Obviously $\mathcal{A} \subseteq \mathcal{ED}$. We claim that $\mathcal{A}$ is a covering family. Let X be an infinite subset of $\omega\times\omega$. If there exists $n < \omega$ such that $X_n = X \cap (\{n\}\times\omega)$ is infinite, then X_n is an infinite subset of an element of $\mathcal{A}$. If the set $A = \{n < \omega : X_n \neq \emptyset\}$ is infinite then there exists $f \in \mathcal{F}$ such that $f(n) = \min(X_n)$ for infinitely many $n \in A$. Hence, $f \cap X$ is infinite.

On the other hand, let $\mathcal{A}$ be a subset of $\mathcal{ED}$ with $|\mathcal{A}| < \mathsf{non}(\mathcal{M})$. For every $A \in \mathcal{A}$, let $n_A < \omega$ such that $|A_k| \leq n_A$ for all $k \geq n_A$, and define a slalom S_A by

$$S_A(n) = \begin{cases} \emptyset & \text{if } n < n_A \\ A_n & \text{if } n \geq n_A \end{cases}$$

Note that $|\{S_A : A \in \mathcal{A}\}| \leq |\mathcal{A}|$, and by the lemma above, there exists $g \in \omega^\omega$ such that for every $A \in \mathcal{A}$, $g(n) \notin S_A(n)$, for almost all $n < \omega$. Hence, $g \cap A$ is finite for all $A \in \mathcal{A}$, and so, $\mathcal{A}$ is not a covering family. □

Theorem 3.3. *If $\mathcal{I}$ is a Borel ideal on ω, then* $\mathsf{non}^*(\mathcal{I}) = \omega$ *or* $\mathcal{ED}_{fin} \leq_K \mathcal{I}$. *So* $\mathsf{non}^*(\mathcal{I}) = \omega$ *or* $\mathsf{non}^*(\mathcal{ED}_{fin}) \leq \mathsf{non}^*(\mathcal{I})$.

Proof. For a Borel ideal $\mathcal{I}$, let us consider the following two-player game: In stage k, Player I chooses a finite subset F_k of ω and then, Player II chooses a natural number $n_k \notin F_k$.

I	$F_0 \in [\omega]^{<\omega}$		$F_1 \in [\omega]^{<\omega}$		$\dots$
II		$n_0 \notin F_0$		$n_1 \notin F_1$	$\dots$

Player I wins if $\{n_i : i \in \omega\} \in \mathcal{I}$ and Player II wins $\{n_i : i \in \omega\} \in \mathcal{I}^+$.

Claim 3.4. *If* Player I *has a winning strategy then* $\mathcal{ED}_{fin} \leq_K \mathcal{I}$.

Proof of Claim. If Player I has a winning strategy then there is a cofinite-branching tree $T \subset \omega^{<\omega}$ such that every $t \in T$ is an increasing sequence and $rng(f) \in \mathcal{I}$ for all $f \in [T]$. Choose $g : \omega \to \omega$ a strictly increasing function such that if $n \in \omega$ and $t \in T$ with $rng(t) \subset g(n)$ then $[g(n+1), \infty) \subseteq Succ_T(t)$. Then every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is the range of a branch of T. Therefore every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is in $\mathcal{I}$.

Choose $f : \omega \to \triangle$ an injection so that for each $n \in \omega$, there exists $k \in \omega$ such that $f[[g(n), g(n+1))] \subset \{\langle k, l\rangle : l \leq k\}$.

We shall show this f witnesses $\mathcal{ED}_{fin} \leq_K \mathcal{I}$. Let $I \in \mathcal{ED}_{fin}$ and $m \in \omega$ be such that for all but finitely many k, $|\{\langle k, l\rangle : l \leq k \wedge \langle k, l\rangle \in I\}| \leq m$. So $f^{-1}[I]$ is a union of m-many selectors of $\{[g(n), g(n+1)) : n \in \omega\}$. Since every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is in $\mathcal{I}$, $f^{-1}[I] \in \mathcal{I}$ i.e., $\mathcal{ED}_{fin} \leq_K \mathcal{I}$. □

Claim 3.5. *If* Player II *has a winning strategy, then* $\mathsf{non}^*(\mathcal{I}) = \omega$.

Proof of Claim. Player II has a winning strategy if and only if there exists an infinitely-branching tree $T \subset \omega^{<\omega}$ such that $rng(f) \in \mathcal{I}^+$ for all $f \in [T]$.

We shall show $\{succ_T(t) : t \in T\}$ is a witness of $\mathsf{non}^*(\mathcal{I})$. Assume to the contrary that there exists $I \in \mathcal{I}$ such that $|I \cap succ_T(t)| = \omega$ for all $t \in T$. Then there exists $b \in [T]$ such that $rng(b) \subset I \in \mathcal{I}$. This is a contradiction. Therefore $\mathsf{non}^*(\mathcal{I}) = \omega$. □

By Borel determinacy this game is determined i.e., either Player I or Player II has a winning strategy. So $\mathcal{ED}_{fin} \leq_K \mathcal{I}$ or $\mathsf{non}^*(\mathcal{I}) = \omega$. □

Concerning to the cardinal invariants of $\mathcal{ED}_{fin}$, we have proved the following.

Proposition 3.6. *The following relations hold:*
(1) $\mathsf{non}^*(\mathcal{ED}_{fin}) \leq \mathfrak{r}$*,*
(2) $\mathsf{cov}(\mathcal{M}) = \min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\}$ *and*
(3) $\mathsf{non}(\mathcal{M}) = \max\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{ED}_{fin})\}$.

Proof. For any $A \subseteq \Delta$ we will denote by $A_n = \{m \leq n : \langle n, m\rangle \in A\}$. Let us prove (1). We will say that a family $\mathcal{R}$ of infinite subsets of ω is *hereditarily reaping* if for every $X \in \mathcal{R}$ and every infinite subset Y of X there is R in $\mathcal{R}$ such that $R \subseteq Y$ or $R \subseteq X \setminus Y$.

Lemma 3.7. $\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R}$ *is hereditarily reaping*$\}$

Proof. It will be enough to prove that there is a hereditarily reaping family with cardinality $\mathfrak{r}$. Let $\mathcal{Q}$ be a reaping family with cardinality $\mathfrak{r}$. Define $\mathcal{Q}_n$ by recursion on $n < \omega$. Let $\mathcal{Q}_0 = \mathcal{Q}$. Given $\mathcal{Q}_n$ and $A \in \mathcal{Q}_n$, let $\mathcal{Q}_{n+1} \restriction A$ be a reaping family on A with cardinality $\mathfrak{r}$. Put $\mathcal{Q}_{n+1} = \bigcup_{A\in\mathcal{Q}_n} \mathcal{Q}_{n+1} \restriction A$. So, $\mathcal{R} = \bigcup_{n<\omega} \mathcal{Q}_n$ is a hereditarily reaping family. $\square$

Let $\mathcal{R}$ be a hereditarily reaping family, and for every $R \in \mathcal{R}$ and $n < \omega$ define $X_{R,n} = \{(m,n) : m \geq n \wedge m \in R\}$. We will see that $\mathcal{A} = \{X_{R,n} : R \in \mathcal{R} \wedge n < \omega\}$ witnesses $\mathsf{non}^*(\mathcal{ED}_{fin})$. Let I be in $\mathcal{ED}_{fin}$, and choose $\{f_i : i \leq n\}$ functions such that $I \subseteq \bigcup_{i\leq n} f_i$. Define $A_j = \{k : (\exists i \leq n)(f_i(k) = j)\}$, for $j \leq n$. If A_j is finite for some $j \leq n$, then $I \cap X_{R,j}$ is finite for every $R \in \mathcal{R}$. So we can assume A_j is infinite for $j \leq n$. Let R_0 be in $\mathcal{R}$ such that $R_0 \cap A_0 = \emptyset$ or $R_0 \subseteq A_0$. In general, for $1 \leq j \leq n$ we can find $R_j \in \mathcal{R}$ such that $R_j \cap (R_{j-1} \cap A_j) = \emptyset$ or $R_j \subseteq R_{j-1} \cap A_j$. If the first case is true for a $j \leq n$ we are done, because for j minimal, we have that $X_{R_j,j} \cap I = \emptyset$. Suppose that $R_j \subseteq R_{j-1} \cap A_j$ for all $j \leq n$. Then, for any $k \in R_n$, $I \cap (\{k\} \times \omega) = n + 1$, and so, $X_{R_n,n+1} \cap I = \emptyset$.

In order to prove (2) we will need the following lemma, due to Bartoszyński and Miller.

Lemma 3.8 ([1], Lemma 2.4.2)**.** *For any cardinal κ the following conditions are equivalent:*

(i) $\kappa < \mathsf{cov}(\mathcal{M})$ *and*
(ii) $(\forall F \in [\omega^\omega]^\kappa)(\forall G \in [[\omega]^\omega]^\kappa)(\exists g \in \omega^\omega)(\forall f \in F)(\forall X \in G)(\exists^\infty n \in X)(f(n) = g(n))$. $\square$

Let $\mathcal{X}$ be a subset of $[\Delta]^{\aleph_0}$ with $|\mathcal{X}| < \mathsf{cov}(\mathcal{M})$. For every $X \in \mathcal{X}$ define $G_X = \{n < \omega : X \cap (\{n\} \times \omega) \neq \emptyset\}$ and let $f_X \in \omega^\omega$ be a function such that $f_X(n) \in X \cap (\{n\} \times \omega)$. By the previous lemma, there is a function $g \in \omega^\omega$ such that $f_X(n) = g(n)$ for infinitely many elements n of G_X, for all $X \in \mathcal{X}$. Then, $\Delta \cap g$ is an element of $\mathcal{ED}_{fin}$ having an infinite intersection with every element of $\mathcal{X}$, proving $|\mathcal{X}| < \mathsf{non}^*(\mathcal{ED}_{fin})$. So $\mathsf{cov}(\mathcal{M}) \leq \mathsf{non}^*(\mathcal{ED}_{fin})$. In addition, it is a well known fact that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$. Therefore $\mathsf{cov}(\mathcal{M}) \leq \min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\}$.

We shall show $\min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\} \leq \mathsf{cov}(\mathcal{M})$. Let κ be a cardinal lower than $\mathfrak{d}$ and $\mathsf{non}^*(\mathcal{ED}_{fin})$. We will prove and use the following lemma.

Lemma 3.9. *Let κ be an infinite cardinal. The following conditions are equivalent.*

(a) $\kappa < \mathsf{non}^*(\mathcal{ED}_{fin})$ *and*

(b) for every bounded family $\mathcal{F}$ *of* κ *functions in* ω^ω *and every family* $\mathcal{A}$ *of* κ *infinite subsets of* ω *there exists a function* $g \in \omega^\omega$ *such that for all* $f \in \mathcal{F}$ *and* $A \in \mathcal{A}$, $f(n) = g(n)$ *for infinitely many* $n \in A$.

Proof. Suppose that κ satisfies (b) and let $\mathcal{B}$ be a family of κ infinite subsets of Δ. For every $B \in \mathcal{B}$, let $X_B = \{n : B_n \neq \emptyset\}$ and $f_B : \omega \to \omega$ such that $(n, f_B(n)) \in B$ if $n \in X_B$, and $f_B(n) = 0$ if not. The families $\mathcal{F} = \{f_B : B \in \mathcal{B}\}$ and $\mathcal{A} = \{X_B : B \in \mathcal{B}\}$ have cardinality κ, and so, there exists a function $g \in \omega^\omega$ such that for all $B \in \mathcal{B}$ there are infinitely many $n \in X_B$ such that $g(n) = f_B(n)$, showing that g has an infinite intersection with B.

On the other hand assume that $\kappa < \mathsf{non}^*(\mathcal{ED}_{fin})$, $\mathcal{F} \subseteq \omega^\omega$ and $\mathcal{A} \subseteq [\omega]^\omega$ have cardinality κ, and $\mathcal{F}$ is bounded by an increasing function $h \in \omega^\omega$. We will identify every $f \in \mathcal{F}$ with a subset of an $\mathcal{ED}_{fin}$-positive subset Δ' of Δ, as follows: Define $X = h[\omega]$, $\Delta' = \prod_{n \in X} n$, $A' = h[A]$ if $A \in \mathcal{A}$, and for $f \in \mathcal{F}$, define $f' : X \to \omega$ by $f'(n) = f(h^{-1}(n))$. So, $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is a family of infinite subsets of Δ'. Let $\mathcal{B} = \{f' \restriction A' : f \in \mathcal{F} \wedge A \in \mathcal{A}\}$. Since $|\mathcal{B}| = \kappa$, there exists $I \in \mathcal{ED}_{fin}$ such that $I \cap B$ is infinite for all $B \in \mathcal{B}$. Let $\{g_i : i \leq N\}$ be a set of functions in ω^ω such that $I \subseteq \bigcup_{i \leq N} g_i$. Define $B_{f,A} = \{n \in A' : f'(n) = g_i(n)\}$, for some $i \leq N$ such that $|(f' \restriction A') \cap g_i| = \aleph_0$, and define $\mathcal{C} = \{B_{f,A} : f \in \mathcal{F} \wedge A \in \mathcal{A}\}$. By Proposition 3.6 (1) $|\mathcal{C}| \leq \kappa < \mathfrak{r}$, and so, there exists $Y \in [\omega]^\omega$ such that $|Y \cap B_{f,A}| = \omega = |B_{f,A} \setminus Y|$. We can find a partition $\{Y_0, Y_1\}$ of Y such that $|Y_0 \cap B_{f,A}| = \aleph_0 = |Y_1 \cap B_{f,A}|$, for all $f \in \mathcal{F}$ and for all $A \in \mathcal{A}$, and inductively, we can find a partition $\{Y_0, Y_1, \ldots, Y_n\}$ of Y such that for every $i \leq n$, $|B_{f,A} \cap Y_i| = \aleph_0$. Now, we define $g(n) = g_i(n)$ if $n \in Y_i$ and $g(n) = 0$ if $n \notin Y$. Given f and A, if $i \leq n$ is such that $B_{f,A} = \{n \in A' : f'(n) = g_i(n)\}$ then $f'(n) = g(n)$ for infinitely many $n \in Y_i \cap A'$, and so, $f(n) = g(h(n))$ for infinitely many $n \in h^{-1}[Y_i] \cap A$. □

Let us prove that $\kappa < \mathsf{cov}(\mathcal{M})$ when $\kappa < \min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\}$, by using Lemma 3.8. Let F and G be families such that $F \in [\omega^\omega]^\kappa$ and $G \in [[\omega]^\omega]^\kappa$.

Claim 3.10. *There exists* $h \in \omega^\omega$ *such that for all* $X \in G$ *and for all* $f \in F$, $f(n) < h(n)$ *for infinitely many* $n \in X$.

Proof of the Claim. For all $f \in F, X \in G$, let e_X be the enumeration of X and let $h_{f,X} \in \omega^\omega$ be such that $h_{f,X}(n) \geq f(e_X(i))$ for all $i \leq n$. Since $\kappa < \mathfrak{d}$, there is a function h which is not dominated by $\{h_{f,X} : X \in G \wedge f \in F\}$. This h does the work. □

Now, for every $f \in F$ define $f' \in \omega^\omega$ such that $f'(n) = f(n)$ if $f(n) < h(n)$ and $f'(n) = 0$ otherwise; for every $f \in F$ and for every $X \in G$ define $C_{f,X} = \{n \in X : f(n) < h(n)\}$, $\mathcal{A} = \{C_{f,X} : f \in F \wedge X \in G\}$ and $\mathcal{F} = \{f' : f \in F\}$. $\mathcal{F}$ is bounded and so, by Lemma 3.9, there is $g \in \omega^\omega$ such that for all $f \in \mathcal{F}$ and for all $A \in \mathcal{A}$, $g(n) = f'(n)$ for infinitely many $n \in A$ and in consequence, $g(n) = f(n)$ for infinitely many $n \in C_{f,X} \subset X$ for every $X \in G$. Therefore $\kappa < \mathsf{cov}(\mathcal{M})$ by Lemma 3.9.

We shall prove (3). It is well known that $\mathfrak{b} \leq \mathsf{non}(\mathcal{M})$ and note that $\mathcal{ED} \leq_K \mathcal{ED}_{fin}$ and so, $\mathsf{cov}^*(\mathcal{ED}_{fin}) \leq \mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$. So $\max\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{ED}_{fin})\} \leq \mathsf{non}(\mathcal{M})$.

To show $\max\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{ED}_{fin})\} \geq \mathsf{non}(\mathcal{M})$, we are going to use the following lemma.

Lemma 3.11 ([1], Theorem 2.4.7)**.** $\mathsf{non}(\mathcal{M})$ *is the size of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is an element f of $\mathcal{F}$ such that $f(n) = g(n)$ for infinitely many $n \in \omega$.* □

Let κ be a cardinal greater than $\mathsf{cov}^*(\mathcal{ED}_{fin})$ and greater than $\mathfrak{b}$. Let $\mathcal{G} = \{f_\alpha : \alpha < \kappa\}$ be an unbounded family of functions in ω^ω, and let G_α a witness of $\mathsf{cov}^*(\mathcal{ED}_{fin})$ in $\Delta_\alpha = \{\langle n, m\rangle : m \leq f_\alpha(n)\}$, for all $\alpha < \kappa$. Without loss of generality we can assume that every element of I of G_α is the graph of a function in ω^ω. We will prove that $\mathcal{F} = \bigcup_{\alpha<\kappa} G_\alpha$ is such that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ such that $f(n) = g(n)$ for infinitely many $n \in \omega$. Given $g \in \omega^\omega$, let $\alpha < \kappa$ be such that $f_\alpha \not\leq^* g$. Then, $g \cap \Delta_\alpha$ is infinite and so, there is $I \in G_\alpha$ such that $I \cap (g \cap \Delta_\alpha)$ is infinite. Since I is the graph of a function in $\mathcal{F}$, we are done. □

By Proposition 3.6, it is consistent that $\mathsf{non}^*(\mathcal{ED}_{fin}) < \mathfrak{b}$. For example if the ground model satisfies Martin axiom, then the random forcing corresponding to the product space 2^{ω_1} forces $\mathsf{non}^*(\mathcal{ED}_{fin}) = \mathsf{cov}(\mathcal{M}) = \omega_1 < \mathfrak{b} = \mathfrak{c}$. However, we cannot use this argument to show the consistency of $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$ for every F_σ-ideal $\mathcal{I}$ because $\mathsf{cov}(\mathcal{N}) \leq \mathsf{non}^*(\mathcal{G}_{FC})$ and the random forcing corresponding to the product space 2^{ω_1} forces $\mathsf{cov}(\mathcal{N}) = \mathfrak{c}$ whenever the ground model satisfies Martin axiom.

However, F_σ-ideals on ω have the following good property.

Theorem 3.12. [10] *$\mathcal{I}$ is an F_σ-ideal on ω if and only if $\mathcal{I} = Fin(\varphi)$ for some lower semi-continuous submeasure φ, where $Fin(\varphi) = \{A \subset \omega : \varphi(A) < \infty\}$. Here $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a lower semi-continuous submeasure if*

(1) $\varphi(\emptyset) = 0$,

(2) whenever $X, Y \subset \omega$ and $X \subset Y$, $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$,

(3) $\varphi(\{n\}) < \infty$ *for* $n \in \omega$ *and*
(4) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ *for every* $A \subset \omega$.

To show the consistency of $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$, we shall use the Laver forcing $\mathbb{L}$. $\mathbb{L}$ is defined by $T \in \mathbb{L}$ if $T \subset \omega^{<\omega}$ is a tree and for $s \in T$ with $stem(T) \subset s$, $|succ_T(s)| = \aleph_0$. $\mathbb{L}$ is ordered by inclusion. Then $\mathbb{L}$ adds an unbounded real.

Proposition 3.13. *Let* G *be a* $\mathbb{L}$*-generic over* V *and* $f_G = \bigcup\{stem(T) : T \in G\}$. *Then* $f_G \in \omega^\omega$ *and* f_G *dominates for all* $g \in \omega^\omega \cap V$.

Therefore, if $\mathbb{L}_{\omega_2}$ *is an* ω_2*-stage countable support iteration of Laver forcing, then* $V^{\mathbb{L}_{\omega_2}} \models \mathfrak{b} = \mathfrak{c}$.

By Proposition 3.13 it is enough to show that $\mathbb{L}_{\omega_2}$ preserves $\mathsf{non}^*(\mathcal{I})$ for each F_σ-ideal $\mathcal{I}$ on ω. We shall use the Laver property.

Definition 4. [4] A forcing notion $\mathbb{P}$ have the Laver property if for every $H : \omega \to \omega \in V$

$$\Vdash \left(\forall f \in \prod_{n\in\omega} H(n) \cap V[\dot{G}]\right) (\exists A : \omega \to \omega^{<\omega} \in V)$$
$$(\forall n \in \omega)\left(f(n) \in A(n) \wedge |A(n)| \le 2^n\right).$$

The Laver property has the following good property.

Theorem 4.1. [4] *The Laver property is preserved under countable support iteration of proper forcing notions.*

Theorem 4.2. [1, p353] *The Laver forcing* $\mathbb{L}$ *has the Laver property.*

So $\mathbb{L}_{\omega_2}$ has the Laver property.

Theorem 4.3. *If* $\mathcal{I}$ *is an* F_σ*-ideal on* ω*, then it is consistent that* $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$.

Proof. Let $\mathcal{I}$ be an F_σ-ideal and let φ be a lower semi-continuous submeasure such that $\mathcal{I} = Fin(\varphi)$.

If a forcing notion $\mathbb{P}$ has the Laver property, then $\mathbb{P}$ has the following good property:

Lemma 4.4. *If* $\mathbb{P}$ *has the Laver property, then*

$$\Vdash_{\mathbb{P}} \text{“}(\forall X \in \mathcal{I} \cap V[\dot{G}])(\exists A \in [\omega]^\omega \cap V)\left(|X \cap A| < \aleph_0\right)\text{”}.$$

Proof of Lemma. Let $p \in \mathbb{P}$ and let $\dot{X}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ “$\dot{X} \in \mathcal{I}$”. Without loss of generality we can assume that there exists $n \in \omega$ such that $p \Vdash_{\mathbb{P}}$ “$\varphi(\dot{X}) < n$”.

Claim 4.5. *Let $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ be a lower semi-continuous submeasure such that $Fin(\varphi) = \mathcal{I}$ for some F_σ-ideal on ω. For each $k \in \omega$ and $l \in \omega$, there exists $m \in \omega$ such that $\varphi([l,m]) > k$.*

Proof of Claim. Since $[l,\infty) \notin \mathcal{I}$, $\varphi([l,\infty)) = \infty$. Because φ has the lower semi-continuous, there exists $m > l$ such that $\varphi([l,m]) > k$. $\square$

Let $\Pi = \langle I_j : j \in \omega\rangle$ be an interval partition of ω such that $\varphi(I_j) > 2^j \cdot n$. By the Laver property, there exist $q \leq p$ and $A : \omega \to \bigcup_{j\in\omega} \mathcal{P}(2^{I_j}) \in V$ such that for $j \in \omega$, $A(j) \subset 2^{I_j}$ and $|A(j)| \leq 2^j$ and $q \Vdash_{\mathbb{P}}$ "$\forall j \in \omega \left(\dot{X} \upharpoonright I_j \in A(j)\right)$". Without loss of generality we can assume $\varphi(J) \leq n$ for $J \in A(j)$ and for $j \in \omega$. By the finite subadditivity of φ, $\varphi(\bigcup A(j)) \leq \sum_{J\in A(j)} \varphi(J) \leq 2^j \cdot n$. So $I_j \setminus A_j \neq \emptyset$ for $j \in \omega$. Choose $y_j \in I_j \setminus \bigcup A(j)$ for $j \in \omega$. Put $Y = \{y_j : j \in \omega\}$. Then $q \Vdash_{\mathbb{P}}$ "$\dot{X} \cap Y = \emptyset$". Therefore $\Vdash_{\mathbb{P}}$ "$\forall X \in \mathcal{I} \exists Y \in [\omega]^\omega \cap V \,(|X \cap Y| < \aleph_0)$". $\square$

So if the ground model satisfies CH, then $V^{\mathbb{L}_{\omega_2}} \models [\omega]^\omega \cap V$ witnesses $\mathsf{non}^*(\mathcal{I})$. Therefore it is consistent $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$. $\square$

In [7] Masaru Kada introduced a cardinal invariant associated with the Laver property.

We call a function from ω to $[\omega]^{<\omega}$ a slalom. Let $\mathcal{S}$ be the collection of slaloms such that $\forall \phi \in \mathcal{S} \forall n \in \omega (|\phi(n)| \leq 2^n)$. $\mathfrak{l}$ is the smallest cardinal κ such that for every $h \in \omega^\omega$ there is a set $\Phi \subset \mathcal{S}$ with cardinality κ such that, for every $f \in \omega$ with $f(n) < h(n)$ for all $n < \omega$, there is $\phi \in \Phi$ such that for all but finitely many $n \in \omega$, we have $f(n) \in \phi(n)$.

Pawlikowski showed that the dual notion to the definition of $\mathfrak{l}$ characterizes $\mathsf{trans\text{-}add}(\mathcal{N})$, transitive additivity of the null ideal (see [1, p.91]). That is, $\mathsf{trans\text{-}add}(\mathcal{N})$ is the smallest size of $\leq^*$-bounded family $F \subset \omega^\omega$ such that for every $\phi \in \mathcal{S}$ there is $f \in F$ such that for infinitely many $n \in \omega$, $f(n) \notin \phi(n)$.

As the proof of Theorem 4.3 we can prove the following statement.

Corollary 4.6. *If $\mathcal{I}$ is an F_σ-ideal, then*

(1) $\mathsf{non}^*(\mathcal{I}) \leq \mathfrak{l}$ *and*
(2) $\mathsf{cov}^*(\mathcal{I}) \geq \mathsf{trans\text{-}add}(\mathcal{N})$.

Proof of Corollary. 1. Let $\mathcal{I}$ be an F_σ-ideal on ω and let φ be a lower semi-continuous submeasure such that $Fin(\varphi) = \mathcal{I}$. Choose $\Pi = \langle I_j : j \in \omega\rangle$ an interval partition of ω such that $\varphi(I_j) > 2^j \cdot j$. Choose Φ a family of functions from ω to $\bigcup_{j\in\omega} \mathcal{P}(2^{I_j})$ such that

i. $|\Phi| \leq \mathfrak{l}$,
ii. for each $j \in \omega$ and $\phi \in \Phi$, $\phi(j) \in 2^{I_j}$ and $|\phi(j)| \leq 2^j$ and

iii. for each $X \in [\omega]^\omega$, there exists $\phi \in \Phi$ such that for all but finitely many $j \in \omega$, $X \cap I_j \in \phi(j)$,

Without loss of generality we can assume that for each $\phi \in \Phi$ and each $j \in \omega$, $J \in \phi(j)$ implies $\varphi(J) \leq j$. For each $j \in \omega$ and $\phi \in \Phi$, $\varphi(\bigcup \phi(j)) \leq \sum_{J \in \phi(j)} \varphi(J) \leq 2^j \cdot j$. So for each $j \in J$, $I_j \setminus \bigcup \phi(j) \neq \emptyset$.

For each $\phi \in \Phi$, choose $X_\phi \in [\omega]^\omega$ such that $X_\phi \cap I_j \setminus \bigcup \phi(j) \neq \emptyset$. Put $\mathcal{A} = \{X_\phi : \phi \in \Phi\}$. We shall show for each $I \in \mathcal{I}$, there exists $X \in \mathcal{A}$ such that $|A \cap I| < \aleph_0$.

Let $I \in \mathcal{I}$ and let $n \in \omega$ such that $\varphi(I) < n$. Choose $m \in \omega$ and $\phi \in \Phi$ so that for $j \geq m$, $I \cap I_j \in \phi(j)$. Then for $j \geq \max n, m$ $X_\phi \cap I_j \cap I = \emptyset$. So $|X_\phi \cap I| < \aleph_0$. Hence $\mathsf{non}^*(\mathcal{I}) \leq \mathfrak{l}$.

2. Let $\mathcal{I}$ be an F_σ-ideal. Let $\mathcal{A} \subset \mathcal{I}$ such that $|\mathcal{A}| < \mathsf{trans\text{-}add}(\mathcal{N})$. Let $\Pi = \langle I_j : j \in \omega \rangle$ be an interval partition of ω such that $\varphi(I_j) > 2^j \cdot j$.

Since $|\mathcal{A}| < \mathsf{trans\text{-}add}(\mathcal{N})$, there exists $\phi : \omega \to \bigcup_{j \in \omega} \mathcal{P}(2^{I_j})$ such that

i. for each $j \in \omega$, $\phi(j) \subset \mathcal{P}(2^{I_j})$,
ii. for each $j \in \omega$, $|\phi(j)| \leq 2^j$ and
iii. for each $I \in \mathcal{A}$ for all but finitely many $j \in \omega$, $I \cap I_j \in \phi(j)$.

Without loss of generality we can assume that for each $j \in \omega$ and $J \in \phi(j)$, $\varphi(J) < j$. By the finite subadditivity of φ, $\varphi(\bigcup \phi(j)) \leq \sum_{J \in \phi(j)} \varphi(J) \leq 2^j \cdot j$ for each $j \in \omega$. So $I_j \setminus \bigcup \phi(j) \neq \emptyset$ for $j \in \omega$.

Choose $X_\phi \in [\omega]^\omega$ such that $X_\phi \cap I_j \setminus \bigcup \phi(j)$ for $j \in \omega$. For each $I \in \mathcal{A}$, there exists $m \in \omega$ such that $j \geq m$ implies $I \cap I_j \in \phi(j)$. Then $j \geq m$ implies $I \cap I_j \cap X_\phi = \emptyset$. So $|I \cap X_\phi| < \aleph_0$. Therefore $\mathsf{trans\text{-}add}(\mathcal{N}) \leq \mathsf{cov}^*(\mathcal{I})$. $\square$

$$\begin{tikzcd}
\mathsf{cov}(\mathcal{N}) \arrow[dddddd] & \mathsf{cov}^*(\mathcal{ED}_{fin}) \arrow[l] \arrow[d] & \mathsf{non}(\mathcal{M}) \arrow[l] \arrow[ddd] & \mathsf{cof}(\mathcal{M}) \arrow[l] \arrow[ddd] & & \mathsf{cof}(\mathcal{N}) \arrow[ll] \arrow[dddddd] \\
 & \mathsf{cov}^*(\mathcal{I}) \arrow[d] & & & & \\
 & \mathsf{trans\text{-}add}(\mathcal{N}) & & & & \\
 & & \mathfrak{b} \arrow[ddd] & \mathfrak{d} \arrow[l] \arrow[ddd] & & \\
 & & & & \mathfrak{l} \arrow[d] & \\
 & & & & \mathsf{non}^*\mathcal{I} \arrow[d] & \\
\mathsf{add}(\mathcal{N}) & & \mathsf{add}(\mathcal{M}) \arrow[ll] & \mathsf{cov}(\mathcal{M}) \arrow[l] & \mathsf{non}^*(\mathcal{ED}_{fin}) \arrow[l] & \mathsf{non}(\mathcal{N}) \arrow[l]
\end{tikzcd},$$

where $\mathcal{I}$ is an F_σ-ideal on ω and $\mathsf{non}^*(\mathcal{I}) \neq \omega$.

Corollary 4.7. *(1) It is consistent* $\mathfrak{r}_{pair} < \mathfrak{b}$.
(2) $\mathfrak{r}_{pair} \leq \mathfrak{l}$ *and* $\mathfrak{s}_{pair} \geq \mathsf{trans\text{-}add}(\mathcal{N})$.

Question 4.8. *(1)* $\mathfrak{r}_d \leq \mathfrak{s}_{pair}$*?*
(2) If $\mathcal{I}$ *is a Borel ideal, then* $\mathsf{non}^*(\mathcal{I}) \leq \mathsf{cof}(\mathcal{N})$*?*

5. Fatou's lemma and a question of Solecki

In this section we answer a question of S. Solecki related to the Katětov order by using cardinal invariants of Borel ideals.

For a sequence of $(a_n)_{n\in\omega}$ of real numbers and an ideal $\mathcal{I}$ on ω, $\lim_{\mathcal{I}} \inf a_n = \sup\{r \in \mathbb{R} : \{n \in \omega : a_n < r\} \in \mathcal{I}\}$.

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with μ defined on σ-algebra $\mathcal{B}$. Let $f_n : X \to [0, \infty]$ be a sequence of μ-measurable functions and let $\mathcal{I}$ be an ideal on ω. We say that Fatou's lemma holds on $\langle f_n : n \in \omega\rangle$ with respect to $\mathcal{I}$ if

$$\underline{\int} \lim_{\mathcal{I}} \inf f_n d\mu \leq \lim_{\mathcal{I}} \inf \int f_n d\mu$$

where $\underline{\int}$ is the lower integral, i.e., if $g \geq 0$, then

$$\underline{\int} g d\mu = \sup\left\{\int f d\mu : f \leq g \text{ and } f \text{ is } \mu\text{-measurable}\right\}.$$

Let $\mathcal{I}$ be an ideal on ω. We say that Fatou's lemma holds for $\mathcal{I}$ if Fatou's lemma holds with respect to $\mathcal{I}$ for any sequence $\langle f_n : n \in \omega \rangle$ of measurable functions from X to $[0, \infty)$ on any σ-finite measure space.

The ideal $\mathcal{S}$ is a critical (locally minimal in the Katětov order) among the ideals which satisfy Fatou's lemma. Let $\Omega = \{U \in \mathrm{Clopen}(2^\omega) : \mu(U) = \frac{1}{2}\}$. $\mathcal{S}$ is an ideal on Ω generated by the set $\{I_x : x \in 2^\omega\}$ where $I_x = \{U \in \Omega : x \in U\}$.

Theorem 5.1. [15] *Let $\mathcal{I}$ be a Borel ideal on ω.*
$\mathcal{I}$ does not satisfy Fatou's lemma if and only if there exists $X \in \mathcal{I}^+$ such that $\mathcal{S} \leq_K \mathcal{I} \restriction X$.

Concerning this theorem, Solecki asked the following question.

Question 5.2. [15] *Can $\mathcal{S}$ be replaced by $\mathcal{G}_{FC}$?*

When we think about question related to the Katětov order, cardinal invariants of ideals are significant.

Theorem 5.3. $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$.

To prove this theorem, we will use the following lemmas.

Lemma 5.4. [5] *For any $\{U_n : n \in \omega\} \subset \Omega$,*

$$\mu(\{x \in 2^\omega : \exists^\infty n \, (x \in U_n)\}) \geq \frac{1}{2}.$$

Proof of Lemma. Assume to the contrary that there exists $\{U_n : n \in \omega\} \in [\Omega]^\omega$ with $\mu(\{x \in 2^\omega : \exists^\infty n \in \omega(x \in U_n)\}) < \frac{1}{2}$. Then there exists a compact set $K \subset 2^\omega$ such that $\mu(K) > \frac{1}{2}$ and K is disjoint with $\{x \in 2^\omega : \exists^\infty n \in \omega(x \in U_n)\}$. Let $\delta = \mu(K) - \frac{1}{2} > 0$. Then $\mu(K \cap U_n) \geq \frac{1}{2}$ for each $n \in \omega$.

For each $k \in \omega$, define $A_k \subset K$ by

$$A_k = \{x \in K : |\{n \in \omega : x \in U_n\}| = k\}.$$

Then $\mu(K) = \sum_{k \in \omega} \mu(A_k)$. So there exists $m \in \omega$ such that $\sum_{k \geq m} \mu(A_k) < \frac{\delta}{2}$. For each $n < m$, choose a compact subset C_n of A_n so that $\mu(A_n \setminus C_n) \leq \frac{\delta}{2m}$.

Put $C = \bigcup_{n<m} C_n$. Then $\mu(\bigcup_{n<m} A_n \setminus C) \leq \frac{\delta}{2}$. Since

$$\mu(K \setminus C) = \sum_{n \geq m} \mu(A_n) + \mu(\bigcup_{n<m} A_n \setminus C) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

$\mu(C \cap U_n) \geq \mu(C) + \frac{1}{2} - 1 > 0$ for $n \in \omega$. However, $\sum_{n \in \omega} \mu(C \cap U_n) \leq m \cdot \mu(C) < \infty$ by $C_n \subset A_n$ for $n < m$. This is a contradiction. Therefore $\mu(\{x \in 2^\omega : \exists^\infty n(x \in U_n)\}) \geq \frac{1}{2}$.

$\square$

Lemma 5.5. *Given $X \subset 2^\omega$.*

(1) If $\mu^*(X) < \frac{1}{2}$, then $\{I_x : x \in X\}$ does not witness to $\mathsf{cov}^*(\mathcal{S})$.
(2) If $\{I_x : x \in X\}$ does not witness to $\mathsf{cov}^*(\mathcal{S})$, then $\mu^*(X) \leq \frac{1}{2}$.

Proof of Lemma. (1). Assume $\mu^*(X) < \frac{1}{2}$. By the definition of the outer measure, there exists a compact subset K of 2^ω such that $\mu(K) = \frac{1}{2}$ and $K \cap X = \emptyset$.

Let $\{U_n : n \in \omega\}$ be a strictly decreasing sequence of open sets such that $K = \bigcap_{n\in\omega} U_n$. Choose $V_n \in \Omega$ such that $V_n \notin \{V_i : i < n\}$ and $V_n \subset U_n$. Let $Y = \{V_n : n \in \omega\}$.

Since $K \cap X = \emptyset$, for each $x \in X$, there exists $n \in \omega$ such that $x \notin U_n$. So $|Y \cap I_x| < \omega$ for every $x \in X$.

(2). Suppose $\{I_x : x \in X\}$ does not witness to $\mathsf{cov}^*(\mathcal{S})$. Choose $Y = \{U_n : n \in \omega\} \in [\Omega]^\omega$ such that $|I_x \cap Y| < \omega$. By Lemma 5.4, $\mu(\{x \in 2^\omega : |I_x \cap Y| = \omega\}) = \frac{1}{2}$. So

$$\mu^*(X) \leq \mu(\{x \in 2^\omega : |I_x \cap Y| < \omega\}) \leq \frac{1}{2}.$$

$\square$

Proof of Theorem 5.3. Firstly we shall show $\mathsf{cov}^*(\mathcal{S}) \leq \mathsf{non}(\mathcal{N})$.

Let X be a non-null set with $\mu^*(X) > 0$.

Claim 5.6. *There exists $Y \subset 2^\omega$ such that $|Y| = |X|$ and $\mu^*(Y) = 1$.*

Then $\{I_x : x \in Y\}$ is a witness to $\mathsf{cov}^*(\mathcal{S})$ by Lemma 5.5.
Next we shall show $\mathsf{cov}^*(\mathcal{S}) \geq \mathsf{non}(\mathcal{N})$. Let $\kappa < \mathsf{non}(\mathcal{N})$ and let $X \subset 2^\omega$ with $|X| = \kappa$. Then $\mu^*(X) = 0$. By Lemma 5.5, $\{I_x : x \in X\}$ does not witness to $\mathsf{cov}^*(\mathcal{S})$. So $\kappa < \mathsf{cov}^*(\mathcal{S})$. Therefore $\mathsf{non}(\mathcal{N}) \leq \mathsf{cov}^*(\mathcal{S})$. $\square$

Corollary 5.7. *$\mathcal{G}_{FC} \geq_K \mathcal{S}$ but $\mathcal{G}_{FC} \not\leq_K \mathcal{S}$.*

Proof. $\mathcal{G}_{FC} \geq_K \mathcal{S}$ is proved in [15]. We shall only show $\mathcal{G}_{FC} \not\leq_K \mathcal{S}$.

In the Cohen model, $\mathsf{cov}^*(\mathcal{G}_{FC}) = \mathfrak{s}_{pair} < \mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$ since $\mathfrak{s}_{pair} \leq \mathsf{non}(\mathcal{M})$ [13]. By Proposition 0.4, $\mathcal{G}_{FC} \not\leq_K \mathcal{S}$ in the Cohen model. By absoluteness of the Katětov order on Borel ideals, $ZFC \vdash \mathcal{G}_{FC} \not\leq_K \mathcal{S}$. $\square$

We need to find a Borel ideal $\mathcal{I}$ such that $\mathcal{I} \geq_K \mathcal{S}$ but for every $X \in \mathcal{I}^+$, $\mathcal{I} \restriction X \not\geq_K \mathcal{G}_{FC}$.

nwd denotes the ideal of nowhere dense subsets of $\mathbb{Q}$.

By the Sierpiński's characterization of $\mathbb{Q}$ we have the following.

Theorem 5.8. [2] $\mathsf{nwd} \simeq_K \mathsf{nwd} \restriction X$ *for every $X \in \mathsf{nwd}^+$.*

Given a forcing notion $\mathbb{P}$, we say an ideal $\mathcal{I}$ on ω is $\mathbb{P}$-indestructible if $\mathbb{P}$ does not add an infinite subset of ω which is almost disjoint from every element of $\mathcal{I}$. We say an ideal $\mathcal{I}$ is $\mathbb{P}$-destructible if $\mathcal{I}$ is not $\mathbb{P}$-indestructible. The ideal nwd is important when we think which ideals on ω are Cohen-destructible.

Theorem 5.9. [8, 6] *$\mathcal{I}$ is Cohen-destructible if and only if $\mathcal{I} \leq_K \mathsf{nwd}$.*

Using this theorem, we can decide the Katětov order between $\mathcal{G}_{FC}$ and nwd and between $\mathcal{S}$ and nwd

Theorem 5.10. *(1) $\mathcal{S} \leq_K \mathsf{nwd}$.*
(2) $\mathcal{G}_{FC} \not\leq_K \mathsf{nwd}$.

Proof. Since adding $\mathfrak{c}^+$-many Cohen reals enlarges $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N}) \geq \mathsf{cov}(\mathcal{M})$, Cohen forcing destroys $\mathcal{S}$. By Theorem 5.9, $\mathcal{S} \leq_K \mathsf{nwd}$.

However, adding ω_2-many Cohen reals implies that $\mathsf{cov}^*(\mathcal{G}_{FC}) = \mathfrak{s}_{pair} \leq \mathsf{non}(\mathcal{M}) = \omega_1$, while $\mathsf{cov}^*(\mathsf{nwd}) \geq \omega_2$. Hence $\mathcal{G}_{FC}$ is Cohen-indestructible. So $\mathcal{G}_{FC} \not\leq_K \mathsf{nwd}$. □

By Theorem 5.8 and 5.10, $\mathcal{S}$ can not be replaced by $\mathcal{G}_{FC}$ in Theorem 5.1. So the answer of Question 5.2 is in the negative.

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