How to Learn Concepts, Consequences, and Conditionals

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This is the penultimate version of a paper that is forthcoming in *analytica*.

Abstract

In this brief note I show how to model conceptual change, logical learning, and revision of one's beliefs in response to conditional information such as indicative conditionals that do not express propositions.

1 Introduction

Stanford (2006) illustrates the importance of the capacity to learn new concepts with case studies from the history of science. Hintikka (1970: 135) calls formal philosophy's inability to model logical learning "a scandal of deduction." Van Fraassen (1981)'s "Judy Benjamin problem" highlights how difficult it is to model the learning of conditional information such as indicative conditionals that, unlike material conditionals, do not express propositions. This brief note shows that these three problems are easily dealt with in Spohn (1988; 2012)'s theory of ranking functions. I will sketch the latter theory in section 2, deal with conceptual change and logical learning in section 3, and conditional information in section 4.

2 Ranking functions

Ranking functions have been introduced by Spohn (1988; 2012) in order to model qualitative conditional belief. The theory is quantitative or numerical in the sense that ranking functions assign numbers to propositions, which are the objects of belief in this theory. These numbers are needed for the definition of conditional ranking functions representing conditional beliefs. As we will see, though, once conditional ranking functions are defined we can interpret everything in purely qualitative, but conditional terms.

Consider a set of possible worlds W and an algebra of propositions $\mathcal{A}$ over W. A function $\varrho : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ from $\mathcal{A}$ into the set of natural numbers $\mathbb{N}$ extended by ∞, $\mathbb{N} \cup \{\infty\}$, is a *finitely / countably / completely minimitive ranking function* on $\mathcal{A}$ just in case for all finite / countable / arbitrary sets of propositions $\mathcal{B} \subseteq \mathcal{A}$:

$$\varrho(W) = 0 \tag{1}$$

$$\varrho(\emptyset) = \infty \tag{2}$$

$$\varrho\left(\bigcup \mathcal{B}\right) = \min\{\varrho(A) : A \in \mathcal{B}\} \tag{3}$$

For a non-empty or consistent proposition $A \neq \emptyset$ from $\mathcal{A}$ the conditional ranking function $\varrho(\cdot \mid A) : \mathcal{A} \backslash \{\emptyset\} \to \mathbb{N} \cup \{\infty\}$ based on the unconditional ranking function $\varrho(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is defined as

$$\varrho(\cdot \mid A) = \begin{cases} \varrho(\cdot \cap A) - \varrho(A), & \text{if} \quad \varrho(A) < \infty, \\ \infty \text{ or } 0, & \text{if} \quad \varrho(A) = \infty. \end{cases}$$

Goldszmidt & Pearl (1996: 63) suggest ∞. Huber (2006: 464) suggests 0 and stipulates $\varrho(\emptyset \mid A) = \infty$ to ensure that every conditional ranking function is a ranking function on $\mathcal{A}$.

A ranking function ϱ is *regular* if and only if

$$\varrho(A) < \varrho(\emptyset) \tag{4}$$

for all non-empty or consistent propositions A from $\mathcal{A}$. In contrast to probability theory it is always possible to define a regular ranking function, no matter how rich or fine-grained the underlying algebra of propositions.

Doxastically ranks are interpreted as grades of disbelief. A proposition A is *disbelieved* just in case A is assigned a positive rank, $\varrho(A) > 0$. A is *believed* just in case its negation is disbelieved, $\varrho\left(\overline{A}\right) > 0$.

A proposition A is *disbelieved conditional on* a proposition C just in case A is assigned a positive rank conditional on C, $\varrho(A \mid C) > 0$. A is *believed conditional on* C just in case its negation is disbelieved conditional on C, $\varrho\left(\overline{A} \mid C\right) > 0$.

It takes getting used to reading positive numbers in this "negative" way, but mathematically this is the simplest formulation of ranking theory. Note that a proposition A is believed just in case A is believed conditional on the tautological proposition W. This is so, because $\varrho\left(\overline{A}\right) = \varrho\left(\overline{A} \mid W\right)$.

It follows from the definition of conditional ranks that the ideal doxastic agent should not disbelieve a non-empty or consistent proposition A conditional on itself: $\varrho(A \mid A) = \varrho(A \cap A) - \varrho(A) = 0$. I'll refer to this consequence below. In doxastic terms the first axiom says that the ideal doxastic agent should not disbelieve the tautological proposition W. The second axiom says that she should disbelieve the empty or contradictory proposition $\emptyset$ with maximal strength ∞.

Given the definition of conditional ranks, the second axiom can be read in purely qualitative, but conditional terms. Read this way it says that the ideal doxastic agent should disbelieve the empty or contradictory proposition conditional on any proposition with a finite rank. This implies that she should believe the tautological proposition with maximal strength, or conditional on any proposition with a finite rank.

Finite minimitivity is the weakest version of the third axiom. It states that $\varrho(A \cup B) = \min\{\varrho(A), \varrho(B)\}$ for any two propositions A and B. Part of what finite minimitivity says is that the ideal doxastic agent should disbelieve a disjunction $A \cup B$ just in case she disbelieves both its disjuncts A and B. Given the definition of conditional ranks, finite minimitivity extends this requirement to conditional beliefs. As noted above the definition of conditional ranks implies that the ideal doxastic agent should not disbelieve a proposition conditional on itself. Given this consequence, finite minimitivity says – in purely qualitative, but conditional terms – the following: the ideal doxastic agent should conditionally disbelieve a disjunction $A \cup B$ just in case she conditionally disbelieves both its disjuncts A and B. Countable and complete minimitivity extend this requirement to disjunctions of countably and arbitrarily many disjuncts, respectively.

Interpreted doxastically these axioms are synchronic norms for organizing the ideal doxastic agent's beliefs and conditional beliefs at a given moment in time. They are supplemented by diachronic norms for updating her beliefs over time if new information of various formats is received. The first update rule is defined for the case where the new information comes in form a "certainty", a proposition that the ideal doxastic agent comes to believe with maximal strength.

Update Rule 1 (Plain Conditionalization, Spohn 1988) *If $\varrho(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is the ideal doxastic agent's ranking function at time t, and between t and t' she becomes certain of $E \in \mathcal{A}$ and no logically stronger proposition (in the sense that E is the logically strongest proposition whose negation is assigned ∞ as new rank at t'), and her ranks are not directly affected in any other way such as forgetting etc., then her ranking function at time t' should be $\varrho_E(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$, $\varrho_E(\cdot) = \varrho(\cdot \mid E)$, where for all non-empty or consistent $A \in \mathcal{A}$:*

$$\varrho_E(A) = \varrho(A \cap E) - \varrho(E) \quad \textit{and} \quad \varrho(\emptyset \mid E) = \infty.$$

Plain conditionalization mirrors the update rule of strict conditionalization from probability theory (Vineberg 2000). The second update rule is defined for the case where the new information comes in form of new ranks for the elements of an "evidential partition." It mirrors the update rule of Jeffrey conditionalization from probability theory (Jeffrey 1983).

Update Rule 2 (Spohn Conditionalization, Spohn 1988) *If $\varrho(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is the ideal doxastic agent's ranking function at time t, and between t and t' her ranks on the evidential partition $\{E_i \in \mathcal{A} : i \in I\}$ change* ***to*** *$n_i \in \mathbb{N} \cup \{\infty\}$ with $\min\{n_i : i \in I\} = 0$, and $n_i = \infty$ if $\varrho(E_i) = \infty$, and her finite ranks change on no finer partition, and her ranks are not directly affected in any other way such as forgetting etc., then her ranking function at time t' should be $\varrho_{E_i \to n_i}(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$,*

$$\varrho_{E_i \to n_i}(\cdot) = \min\{\varrho(\cdot \mid E_i) + n_i : i \in I\}.$$

Here and below I is an arbitrary index set. The third update rule is defined for the case where the new information reports the differences between the old and the new ranks for the elements of an evidential partition. It mirrors the update rule of Field conditionalization from probability theory (Field 1978).

Update Rule 3 (Shenoy Conditionalization, Shenoy 1991) *If $\varrho(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is the ideal doxastic agent's ranking function at time t, and between t and t' her ranks on the evidential partition $\{E_i \in \mathcal{A} : i \in I\}$ change* ***by*** *$z_i \in \mathbb{N}$, where $\min\{z_i : i \in I\} = 0$, and her finite ranks change on no finer partition, and her ranks are not directly affected in any other way such as forgetting etc., then her ranking function at time t' should be $\varrho_{E_i \uparrow z_i}(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$,*

$$\varrho_{E_i \uparrow z_i}(\cdot) = \min\{\varrho(\cdot \cap E_i) + z_i - m : i \in I\}, \quad m = \min\{z_i + \varrho(E_i) : i \in I\}.$$

Spohn conditionalizing E and $\overline{E}$ to 0 and n, respectively, keeps the relative positions of all possible worlds in E and all possible worlds in $\overline{E}$ fixed. It improves the rank of E to 0 (remember that low numbers represent low grades of *dis*belief), and it changes the rank of $\overline{E}$ to n. Shenoy conditionalizing E and $\overline{E}$ by 0 and n, respectively, improves the possibilities within E by n, as compared to the possibilities in $\overline{E}$. m is a normalization parameter. It ensures that at least one possible world is assigned rank 0 so that the result is a ranking function.

In the case of Spohn and Shenoy conditionalization the new information consists of a partition of "evidential propositions" together with a list of numbers for these evidential propositions. The evidential propositions are those which are directly affected by experience. They are paired with numbers, which reflects the fact that the quality of new information varies with the reliability of its source: it makes a difference if the weather forecast predicts that it will rain, if a friend the ideal doxastic agent trusts tells her so, or if she sees herself that it is raining. In each case the evidential proposition the ideal doxastic agent learns is that it is raining, and its negation is the only other cell or element of the evidential partition. However, the effect the new information should have on her beliefs will be a different one in each case. The difference in the reliability of the sources of the new information – the weather forecast, a friend, her vision – is reflected in the numbers that are paired with the evidential propositions. The effect the new information should have on the ideal doxastic agent's beliefs depends on these numbers.

The package consisting of the synchronic norms (1-3) and the diachronic norms (Update Rules 1-3) can be justified by the consistency argument (Huber 2007) in much the same way that probability theory can be justified by the Dutch book argument. The consistency argument shows that obeying the synchronic and diachronic rules of ranking theory is a necessary and sufficient means to attaining the cognitive end of always holding beliefs that are jointly consistent and deductively closed. To the extent that the ideal doxastic agent has this goal, she should obey the norms of ranking theory. It is not that we are telling her what and how to believe. *She* is the one who has this goal. We merely point out the objectively obtaining means-end relationships. Of course, if the ideal doxastic agent does not aim at always holding beliefs that are jointly consistent and deductively closed, our response will cut no ice. But that is besides the point: it is mistaking a hypothetical imperative for a categorical imperative. Alternatively one may use the representation result by Hild & Spohn (2008), or the rank-theoretic decision theory by Giang & Shenoy (2000), to obtain a justification of ranking theory that is deontological in spirit.

3 Conceptual change and logical learning

Plain, Spohn, and Shenoy conditionalization handle belief revision when the new information the ideal doxastic agent receives takes the form of propositions together with numbers. In the case of plain conditionalization this number is ∞, indicating that the proposition is learned with certainty. In the case of Spohn conditionalization the new information comes in the form of new grades of disbelief for the propositions in the evidential partition. In the case of Shenoy conditionalization the new information comes in the form of differences between the old and new grades of disbelief for the propositions in the evidential partition.

In addition there are at least three other forms in which an ideal doxastic agent can receive new information: she can learn a new concept without learning any factual information; she can learn about the logical relations between various concepts, and I will treat such logical learning as a special case of conceptual learning; and she can learn an indicative conditional that, unlike a material conditional, does not express a (conditional or other) proposition.

In the case of a conceptual change the ideal doxastic agent learns that her language or algebra was too poor or coarse-grained. For instance, Sophia may start out with a language that allows her to distinguish between red wine and white wine, and then may acquire the concept of rosé. Or she may learn that among these wines one can distinguish between barriques and non-barriques. When the ideal doxastic agent receives such *conceptual information* she should perform a *conceptual change*. As we will see below, *logical learning* can be viewed as a prominent special case of a conceptual change.

In many instances when we ordinary doxastic agents learn a new concept, we do not merely learn the concept without any factual information, but we learn the concept together with a host of other things that are not purely conceptual. For instance, someone who learns that one can distinguish between barriques and non-barriques usually also learns that barriques tend to be red wines. The update rule I will propose below only deals with the clinically clean case where the new information is purely conceptual. This is no restriction, though, as the additional factual information that often accompanies a conceptual change can be dealt with in a separate step by plain, Spohn, or Shenoy conditionalization. Phenomenologically the two changes may appear to be one, but for the purpose of constructive theorizing it is best to separate them.

As a preparatory step, note that in probability theory there is no such thing as an unbiased assignment of probabilities, an *ur*- or *tabula rasa* prior, as we may call it. This is so even if we consider just a finite set of (more than two) possibilities.

For instance, it is sometimes said that assigning a probability of $1/6$ to each of the six outcomes of a throw of a die is such an unbiased assignment. To see that this is not so it suffices to note that it follows from this assignment that the proposition that the number of spots the die will show after being thrown is greater than one is five times the probability of its negation. More generally, for every probability measure Pr on the power-set of $\{1, \ldots, 6\}$ there exists a contingent proposition A such that $\Pr(A) > \Pr\left(\overline{A}\right)$, where a proposition is contingent just in case both it and its complement are non-empty. This is the sense in which there is no genuinely unbiased ur- or tabula rasa prior probability measure. It highlights the fact that, in probability theory, the tendency that is represented by a probability measure is inseparably tied to the underlying space of possibilities, a fact employed by Betrand (1889) in his famous paradoxes. The meaning of probability depends on what the other options are, so to speak. To speak of the probability of something without relativizing, or contrasting, it to all the other options is, strictly speaking, meaningless.

In ranking theory the tabula rasa prior is that function $R : \mathcal{A} \rightarrow N \cup \{\infty\}$ such that $R(A) = 0$ for all non-empty propositions A in $\mathcal{A}$, no matter how rich the field or algebra of propositions $\mathcal{A}$. R suspends judgment with respect to every contingent proposition and only believes the tautology, and disbelieves the contradiction. This tabula rasa prior ranking function turns out to be very useful.

In probability theory we cannot adequately model conceptual changes, especially those that are due the ideal doxastic agent's not being logically omniscient. Prior to learning a new concept Sophia's friend Bay is equipped with a probability measure Pr on some algebra of propositions $\mathcal{A}$ over some set of possibilities W. When Bay learns a new concept, the possibilities w in W become more fine grained. For instance, Bay's set of oenological possibilities with regard to a particular bottle of wine prior to learning the concept of barrique may be $W_1 = \{\text{red}, \text{white}\}$. After learning this concept her set of possibilities could[1] be $W_2 = \{\text{red \& barr.}, \text{red \&}\neg\text{ barr.}, \text{white \& barr.}, \text{white \&}\neg\text{ barr.}\}$. To model this conceptual change adequately, the new algebra of propositions over W_2 will contain a unique counterpart-proposition for each proposition in the old algebra of propositions over W_1. In our example the algebras are the power-sets. The unique counterpart-proposition of the old proposition that the bottle of wine is red, $\{\text{red}\} \subseteq W_1$, is $\{\text{red \& barr.}, \text{red \&}\neg\text{ barr.}\} \subseteq W_2$.

[1]Here I assume Bay to be logically omniscient. Suppose she is not, and is unaware that barr. &¬ barr. is inconsistent. Then her set of possibilities may include red & barr. &¬ barr. Her new doxastic attitude towards the latter will be her old doxastic attitude towards red.

The important feature of the purely conceptual part of this learning episode is that Bay does not learn anything about which of the possibilities is the actual one. If Sophia is the one who undergoes this conceptual change, and R_1 is her ranking function on the power-set of W_1, we want her R_2 to be such that $R_1(A) = R_2(A')$ for each old proposition A in the power-set of W_1 and its counterpart proposition A' in the power-set of W_2. We also want her R_2 to be such that $R_2(B) = R_2\left(\overline{B}\right)$ for each contingent new proposition B. This is easily achieved by letting R_2 copy R_1 on the counterpart-propositions of the old propositions, and by letting it copy the tabula rasa prior on all new propositions. For Bay there is no way to obtain probability measures Pr_1 on the old algebra and Pr_2 on the new algebra that are related in this way.

The same is true for the different conceptual change that occurs when Sophia learns the concept of rosé, and thus that her old set of possibilities was not exhaustive. If R_1 is Sophia's ranking function on the power-set of W_1, her R_3 on the power-set of $W_3 = \{\text{red}, \text{rosé}, \text{white}\}$ is that function R_3 such that $R_1(\{w\}) = R_3(\{w\})$ for each old singleton-proposition $\{w\}$, and $R_3(\{w'\}) = 0$ for each new singleton-proposition $\{w'\}$. For Sophia's friend Bay there is no way to undergo this conceptual change, since the only new probability measure that, in this sense, *conservatively* extends the old one assigns 0 to the union of all new possibilities.[2] Arntzenius (2003) relies on just this inability of Bay to cope with changes of the underlying set of possibilities when he uses "spreading" to argue against strict conditionalization and van Fraassen (1984; 1995)'s principle of "reflection."

Before turning to the special case of logical learning let me officially state[3]

Update Rule 4 (Conceptual Conditionalization) *Suppose $\varrho(\cdot) : \wp(W) \to \mathbb{N} \cup \{\infty\}$ is the ideal doxastic agent's ranking function at time t, and between t and t' her algebra of propositions $\wp(W)$ over W explodes in the sense that W expands to W^*, where for each $w \in W$ there is at least one $w^* \in W^*$ such that w^* is at least as specific as w. Each such $w^* \in W^*$ is called a* refinement *of $w \in W$. This includes explosions where W expands to some superset W^*.*

Suppose further that the ideal doxastic agent's set of possibilities W and her ranks on the algebra of propositions $\wp(W)$ over W are not directly affected in any other way such as forgetting etc.

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[2]Readers will have noticed that one sentence may pick out different propositions with respect to the two sets of possibilities. For instance, with respect to W_1 the sentence 'It is not a bottle of red wine' picks out the proposition that it is a bottle of white wine, $\{\text{white}\}$, while with respect to W_2 this sentence picks out the proposition that it is a bottle of rosé or white wine, $\{\text{rosé}, \text{white}\}$.

[3]To avoid complications discussed in Huber (2006) I assume the algebra of propositions to be the power-set $\wp(W)$ over the set of possibilities W so that $\varrho(\{w\})$ is defined for each $w \in W$.

Then her algebra of propositions at time t' should be $\wp(W^*)$, and her ranking function at time t' should be $\varrho'(\{w^*\}) = \varrho(\{w\})$ for each refinement $w^* \in W^*$ of some $w \in W$, and $\varrho'(\{w^*\}) = 0$ for each $w^* \in W^*$ that is not a refinement of some $w \in W$.

If the ideal doxastic agent's algebra of propositions $\wp(W)$ over W implodes in the sense that W shrinks to some subset W^-, then her algebra of propositions at time t' should be $\wp(W)$, and her ranking function at time t' should be $\varrho'(\{w\}) = \varrho(\{w\})$ for each $w \in W^-$ and $\varrho'(\{w^-\}) = \infty$ for each $w^- \in W \setminus W^-$.

Implosions are dealt with in the same way as factual information is handled by the update rules from the previous section. In particular, implosions do not have to come in the form of certainties, where the possibilities $w^- \in W \setminus W^-$ are sent to ∞, but can come in the gradual form that Spohn and Shenoy conditionalization deal with. In this case they are sent to some finite rank $n \in \mathbb{N}$.

One form of logical learning is to learn that, contrary to what the ideal doxastic agent had thought initially, some hypothesis H logically implies some evidence E. This amounts to learning that $H \wedge \neg E$ is not logically possible after all, and can be thought of as learning with certainty that $\neg H \vee E$. Logical learning of this sort corresponds to an implosion of the ideal doxastic agent's algebra of propositions. Another form of logical learning is to learn that, contrary to what she had thought initially, H does not logically imply E. This amounts to learning that $H \wedge \neg E$ is logically possible after all. It corresponds to an explosion of the ideal doxastic agent's algebra of propositions. Mixed changes of explosions and implosions can be dealt with in a stepwise fashion, as the order does not matter.

So far I have assumed the sets of possible worlds W to be given, and the notion of refinement, or specificity, to be a primitive[4]. To deal with logical learning in a bit more detail I will now present one way to think of them. A formal language is generally defined recursively as follows. We start with a set of propositional letters $\{p, q, r, \ldots\}$ and say that each of them is a sentence. Then we say that $\neg A$, $A \wedge B$, $A \vee B$, and $A \supset B$ are sentences, if A and B are. Finally we add that nothing else is a sentence. This is how we obtain a formal language or set of sentences $\mathcal{L}$. In the case of a predicate or modal language things are slightly more complicated, but otherwise proceed in the same way.

The powerset of such a formal language, $\wp(\mathcal{L})$, is one way to generate a set of possible worlds. A sentence A from $\mathcal{L}$ is said to be true in a possible world $w \in \wp(\mathcal{L})$ just in case $A \in w$. One possible world w is a refinement of another possible world w' just in case $w \subseteq w'$. Among these possible worlds there are, of

[4]The latter can take several forms, e.g. by using Cartesian products of sets of possibilities.

course, many that are contradictory in the sense that the rules of inference for the language $\mathcal{L}$ – classical, as I will assume below, or non-classical – disprove them. But this is logical knowledge the ideal doxastic agent may not yet possess. All possible worlds w that have non-contradictory refinements other than themselves are redundant or non-maximal, but this is again logical knowledge the ideal doxastic agent may not yet posses. An ideal doxastic agent is logically omniscient only if she has eliminated all possible worlds except those that are maximal and non-contradictory. If the ideal doxastic agent comes across a new word, she just throws it into her bag of propositional letters or predicates and lets the recursive machinery sketched above generate a new language containing the old one. Conceptual conditionalization does the rest.

To illustrate suppose the set of propositional variables is $\{p, q\}$ which generates the formal language $\mathcal{L}$, and Sophia initially has no logical knowledge. Then her set of possible worlds is $\wp(\mathcal{L})$. Sophia takes a course in propositional logic and learns with certainty that p logically implies $p \vee q$. Then Sophia eliminates all sets of sentences $w \in \wp(\mathcal{L})$ that contain p, but that do not contain $p \vee q$. Next Sophia learns with certainty that the rules of inference for $\mathcal{L}$ disprove the schema $A \wedge \neg A$. Then she eliminates all sets of sentences that contain any sentence of the form $A \wedge \neg A$. In the third weak Sophia learns, again with certainty, that the set of sentences $\{p, q\}$ disproves $\neg(p \wedge q)$. How she learns this piece of logical knowledge depends on the way her instructor sets things up: one way of disproving works by showing that $\{p, q\} \cup \{\neg(p \wedge q)\}$ proves $p \wedge \neg p$, which Sophia already knows to be disprovable (from the empty set). Another way of disproving works by showing that $\{p, q\}$ logically implies $p \wedge q$. In this latter case Sophia does not only learn that all sets of sentences including both p and q, but excluding $p \wedge q$, are redundant or inconsistent, but also that all sets of sentences including p and q and $\neg(p \wedge q)$ are inconsistent. All these changes correspond to implosions. If Sophia does not learn with certainty, then the parameter ∞ has to be lowered to some finite number.

Finally, if Sophia initially thinks that q and $p \supset q$ logically implies p, then her set of possibilities W excludes all sets of sentences containing both q and $p \supset q$, but not p. If Sophia then learns that she was mistaken about this alleged logical implication, she expands W by adding all these sets of sentences. Some of these newly added ones will be refinements of sets of sentences Sophia already deemed possible initially. Sophia's ranks for the latter determine her ranks for the former. All the others of these newly added sets of sentences receive rank 0.

4 Conditional information

It's Sophia's birthday and her friend Bay treats Sophia to a nice bottle of red or white wine. The bottle of wine is wrapped in paper, and Sophia is curious to learn if it is a bottle of barrique or not. Bay tells Sophia: it is a barrique, given that it is a bottle of red wine. Sophia deems Bay reliable to a very high, but finite, grade n. How should Sophia revise her beliefs and conditional beliefs in response to this piece of conditional information? The answer is given by

Update Rule 5 (Conditional Conditionalization) *If $\varrho(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is the ideal doxastic agent's ranking function at time t, and between t and t' her conditional rank for A given C improves by $n \in \mathbb{N}$, and her ranks are not directly affected in any other way such as forgetting etc., then her ranking function at time t' should be $\varrho_{(A|C)\uparrow n}(\cdot) : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$, which results from ϱ by n **consecutive** Shenoy shifts on the evidential partition*

$$\left\{A \cap C, \overline{A} \cap C, A \cap \overline{C}, \overline{A} \cap \overline{C}\right\}$$

with input parameters

$$z_{A\cap C} = 0, z_{A\cap\overline{C}} = x, z_{\overline{A}\cap C} = 1, z_{\overline{A}\cap\overline{C}} = x,$$

where $x = 1$ if $\varrho(A \cap C) > \varrho\left(\overline{A} \cap C\right)$, and $x = 0$ otherwise.

This looks awfully complicated, but that is because of a technical detail. The idea itself is very simple: we improve the rank of $A \cap C$ compared to the rank of $\overline{A} \cap C$. This can happen in more than one way, though: by decreasing the rank of $A \cap C$ and by holding fixed the rank of the other three cells of the partition, or by increasing the rank of $\overline{A} \cap C$ and by holding fixed the rank of the other three cells of the partition. Which of these two ways of improving should happen depends on the ideal doxastic agent's initial beliefs. In many cases one first has to start improving in the first manner and then, mid-way, switch to the second manner of improving. This is why I have chosen to formulate the update rule in terms of n consecutive Shenoy shifts, which makes it look complicated. I will present a different, and perhaps more perspicuous, formulation below.

A is a proposition in Sophia's algebra, and so is the condition C. However, the conditional information A given C is not itself a proposition. If it were, we would use plain, Spohn, or Shenoy conditionalization. To stress that we are dealing with a conditional belief in A given C rather than the belief in a conditional proposition "If A, then C" I continue to use 'A' for the target proposition, and I use

'C' for the condition (and *not* for the *a*ntecedent and *c*onsequent of a conditional proposition).

Receiving the conditional information that it is a barrique given that it is red wine from her friend Bay, who Sophia deems reliable to grade n, tells Sophia to improve her rank for the proposition that it is a red barrique compared to her rank for the proposition that it is a red non-barrique by n ranks. Everything else depends on Sophia's initial grading of disbelief R.

Suppose Sophia initially has no clue about what wine Bay will bring, except that it is red wine or white wine and that it might be a barrique or not:

$$\begin{aligned} R(\{\text{red} \,\&\, \text{barrique}\}) &= 0 \\ R(\{\text{red} \,\&\neg\, \text{barrique}\}) &= 0 \\ R(\{\text{white} \,\&\, \text{barrique}\}) &= 0 \\ R(\{\text{white} \,\&\neg\, \text{barrique}\}) &= 0 \end{aligned}$$

Then Sophia's new grading of disbelief R^* is such that she holds the conditional belief that it is a barrique given that it is red wine, but she continues to suspend judgment with respect to whether it is red wine or white wine, and whether it is a barrique or not. The only contingent belief Sophia holds is that it is not a red non-barrique:

$$\begin{aligned} R^*(\{\text{red} \,\&\, \text{barrique}\}) &= 0 \\ R^*(\{\text{red} \,\&\neg\, \text{barrique}\}) &= n \\ R^*(\{\text{white} \,\&\, \text{barrique}\}) &= 0 \\ R^*(\{\text{white} \,\&\neg\, \text{barrique}\}) &= 0 \end{aligned}$$

Suppose next that Sophia initially believes that Bay will bring red wine, but she suspends judgment with respect to whether it is a barrique or not. She also believes that it is a barrique given that it is white wine.

$$\begin{aligned} R(\{\text{red} \,\&\, \text{barrique}\}) &= 0 \\ R(\{\text{red} \,\&\neg\, \text{barrique}\}) &= 0 \\ R(\{\text{white} \,\&\, \text{barrique}\}) &= 7 \\ R(\{\text{white} \,\&\neg\, \text{barrique}\}) &= 11 \end{aligned}$$

Then Sophia's new grading of disbelief R^* is such that she holds the conditional belief that it is a barrique given that it is red wine. In addition Sophia continues to believe that it is red wine, but now also believes that it is a barrique. Sophia also continues to believe that it is a barrique given that it is white wine.

$$\begin{aligned} R^*(\{\text{red \& barrique}\}) &= 0 \\ R^*(\{\text{red \&}\neg\text{ barrique}\}) &= n \\ R^*(\{\text{white \& barrique}\}) &= 7 \\ R^*(\{\text{white \&}\neg\text{ barrique}\}) &= 11 \end{aligned}$$

Now suppose Sophia initially believes that Bay will bring white wine, but she suspends judgment with respect to whether it is a barrique or not. Initially Sophia also holds the conditional belief that it is a non-barrique given that it is red wine.

$$\begin{aligned} R(\{\text{red \& barrique}\}) &= 9 \\ R(\{\text{red \&}\neg\text{ barrique}\}) &= 7 \\ R(\{\text{white \& barrique}\}) &= 0 \\ R(\{\text{white \&}\neg\text{ barrique}\}) &= 0 \end{aligned}$$

If $n > 2$ so that Sophia deems Bay more reliable than the strength with which she holds her conditional belief, then Sophia's new grading of disbelief R^* will be such that she holds the opposite conditional belief that it is a barrique given that it is red wine. If $n \geq 9$ Sophia even gives up her belief that it is white wine, although she won't believe that it is red wine. In either case she continues to suspend judgment with respect to whether it is a barrique or not, unconditionally as well as conditional on it being white wine.

$$\begin{aligned} R^*(\{\text{red \& barrique}\}) &= 9 - x \quad \text{where} \quad x = \min\{n, 9\} \\ R^*(\{\text{red \&}\neg\text{ barrique}\}) &= 7 + \max\{n - x, 0\} \\ R^*(\{\text{white \& barrique}\}) &= 0 \\ R^*(\{\text{white \&}\neg\text{ barrique}\}) &= 0 \end{aligned}$$

This time suppose Sophia initially already believes that Bay will bring a barrique, unconditionally and conditional on it being red wine and conditional on it being white wine, but suspends judgment with respect to whether it is red wine or white wine. In addition Sophia holds the conditional belief that it is red wine given that it is a non-barrique.

$$\begin{aligned} R(\{\text{red \& barrique}\}) &= 0 \\ R(\{\text{red \&}\neg\text{ barrique}\}) &= 5 \\ R(\{\text{white \& barrique}\}) &= 0 \\ R(\{\text{white \&}\neg\text{ barrique}\}) &= 7 \end{aligned}$$

Then Sophia's new grading of disbelief R^* is such that she continues to hold the belief that it is a barrique, and the conditional belief that it is a barrique given that it is red wine, and the conditional belief that it is a barrique given that it is white wine. In addition she continues to suspend judgment with respect to whether it is red or white wine. Depending on how reliable Sophia deems Bay to be she may give up her initial belief that it is red wine given that it is a non-barrique ($n \geq 2$), and she may even adopt the belief that it is white wine given that it is a non-barrique ($n > 2$).

$$\begin{aligned} R^* (\{\text{red \& barrique}\}) &= 0 \\ R^* (\{\text{red \&} \neg \text{ barrique}\}) &= 5 + n \\ R^* (\{\text{white \& barrique}\}) &= 0 \\ R^* (\{\text{white \&} \neg \text{ barrique}\}) &= 7 \end{aligned}$$

Yet another possibility is that Sophia initially believes that Bay will bring a non-barrique, unconditionally and conditional on it being white wine and conditional on it being red wine, but she suspends judgment with respect to whether it is red or white wine, unconditionally and conditional on it being a barrique and conditional on it being a non-barrique.

$$\begin{aligned} R (\{\text{red \& barrique}\}) &= 5 \\ R (\{\text{red \&} \neg \text{ barrique}\}) &= 0 \\ R (\{\text{white \& barrique}\}) &= 5 \\ R (\{\text{white \&} \neg \text{ barrique}\}) &= 0 \end{aligned}$$

If Sophia deems Bay sufficiently reliable for her to give up the conditional belief that it is a non-barrique given that it is red wine ($n \geq 5$), then Sophia continues to suspend judgment with respect to whether it is red or white wine. If $0 < n < 5$ so that Sophia holds her initial beliefs more firmly than she deems Bay reliable, but she still deems Bay reliable to some degree, then she adopts the belief that it is red wine given that it is a barrique, but continues to suspend judgment with respect to whether it is red or white wine given that it is a non-barrique. In all these cases Sophia retains her belief that it is a non-barrique given that it is white wine.

$$\begin{aligned} R^* (\{\text{red \& barrique}\}) &= 5 - x \quad \text{where} \quad x = \min\{n, 5\} \\ R^* (\{\text{red \&} \neg \text{ barrique}\}) &= 0 + \max\{n - x, 0\} \\ R^* (\{\text{white \& barrique}\}) &= 5 \\ R^* (\{\text{white \&} \neg \text{ barrique}\}) &= 0 \end{aligned}$$

We thus see that it is difficult to say what happens to the ideal doxastic agent's beliefs and conditional beliefs when she receives conditional information such as an indicative conditional that does not express a proposition. It is difficult to say in the precise sense that there is not a single contingent proposition that she is guaranteed to believe after the update. Nor is there a single contingent proposition that she is guaranteed to conditionally believe after the update. This includes the very piece of conditional information that the ideal doxastic agent learns, as she may deem the source of information somewhat, but insufficiently reliable.

If R disbelieves A given C, then the rank of $A \cap C$ is lowered and the remaining three cells are held fixed until R does not disbelieve A given C anymore. At this point the parameter x in conditional conditionalization changes to from $x = 1$ to $x = 0$. If R does not disbelieve A given C, then the cell $\overline{A} \cap C$ is moved upwards and the remaining three cells are held fixed. If R initially disbelieves the condition C, then she gives up her disbelief in C only if she gives up her disbelief in what the source claims was the wrong thing to disbelieve: $A \cap C$. If R initially does not disbelieve the condition C, then R continues to do so. However, the reason for doing so may change. Initially it may be because R assigns rank 0 to what the source claims is the wrong thing to not disbelieve: $\overline{A} \cap C$. In the new grading of disbelief R^* it may be because $A \cap C$, but not $\overline{A} \cap C$, receives rank 0.

What can be said in general is relative to the ideal doxastic agent's initial grading of disbelief R. Receiving the conditional information that A given C from a source she deems reliable to grade n improves the rank of A by n compared to the rank of $\overline{A}$ within the condition C, but not within the condition $\overline{C}$. Within the latter condition everything is kept as it was initially. In other words, conditional conditionalization transforms a given grading of disbelief R into a new grading of disbelief R^* such that: $R^*\left(\overline{A} \mid C\right) - R^*(A \mid C) = R\left(\overline{A} \mid C\right) - R(A \mid C) + n$. Furthermore, conditional conditionalization is purely conditional in the sense that $R^*\left(\cdot \mid \overline{C}\right) = R\left(\cdot \mid \overline{C}\right)$ and $R^*\left(\overline{C}\right) = R\left(\overline{C}\right)$. The latter feature uniquely characterizes conditional conditionalization among all consecutive Shenoy-shifts on the evidential partition $\left\{A \cap C, \overline{A} \cap C, A \cap \overline{C}, \overline{A} \cap \overline{C}\right\}$ that possess the former feature. It is important to note that the rank of the condition C may increase or decrease, whereas the rank of the proposition A can decrease, but cannot increase, and the rank of its negation $\overline{A}$ can increase, but cannot decrease.

The feature that the rank of the condition C may increase or decrease distinguishes conditional conditionalization from Bradley (2005)'s "Adams conditionalization." The latter transforms a given probability measure Pr into a new probability measure $\Pr^*$ in response to input of the form $\Pr^*(A \mid C) = p$ such that:

P1 $\Pr^*(\cdot \mid A \cap C) = \Pr(\cdot \mid A \cap C)$

P2 $\Pr^*\left(\cdot \mid \overline{A} \cap C\right) = \Pr\left(\cdot \mid \overline{A} \cap C\right)$

P3 $\Pr^*\left(\cdot \mid \overline{C}\right) = \Pr\left(\cdot \mid \overline{C}\right)$

P4 $\Pr^*\left(\overline{C}\right) = \Pr\left(\overline{C}\right)$

Bradley (2005) shows that these four conditions and input of the form $\Pr^*(A \mid C) = p$ transform a given old Pr into a unique new $\Pr^*$. In the exact same way $R_{(A|C)\uparrow n} = R^*$ is determined uniquely by input of the form $R^*\left(\overline{A} \mid C\right) - R^*(A \mid C) = n$ and the following four conditions:

R1 $R^*(\cdot \mid A \cap C) = R(\cdot \mid A \cap C)$

R2 $R^*\left(\cdot \mid \overline{A} \cap C\right) = R\left(\cdot \mid \overline{A} \cap C\right)$

R3 $R^*\left(\cdot \mid \overline{C}\right) = R\left(\cdot \mid \overline{C}\right)$

R4 $R^*\left(\overline{C}\right) = R\left(\overline{C}\right)$

However, in the probabilistic case the condition P4 implies that $\Pr^*(C) = \Pr(C)$, and hence that Adams conditionalization can never change the probability of the condition C. Douven & Romeijn (2011) conclude that this very feature prevents Adams conditioning from being an adequate rule to respond to new information of conditional form. Douven (2012) and Hartmann & Rad (ms) then propose to remedy this situation by making external information available to the ideal doxastic agent – such as information about explanatory relationships (Douven 2012), or information about independence relationships that are suggested by some causal structure (Hartmann & Rad ms). In doing so these authors go beyond the resources provided by the probability calculus, and thus abandon the Bayesian idea that the probability calculus is all there is to scientific reasoning. In the present framework of ranking functions no such maneuvers are necessary: the rank of the condition C is not determined by the rank of its complement $\overline{C}$, and can go up or down or stay the same, depending on the conditional information received and the initial grading of disbelief.

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