

Hamiltonian Privilege

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May 28, 2023

Abstract

We argue that Hamiltonian mechanics is more fundamental than Lagrangian mechanics. Our argument provides a non-metaphysical strategy for privileging one formulation of a theory over another: *ceteris paribus*, a more general formulation is more fundamental. We illustrate this criterion through a novel interpretation of classical mechanics, based on three physical conditions. Two of these conditions suffice for recovering Hamiltonian mechanics. A third condition is necessary for Lagrangian mechanics. Hence, Lagrangian systems are a proper subset of Hamiltonian systems. Finally, we provide a geometric interpretation of the principle of stationary action and rebut arguments for privileging Lagrangian mechanics.

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1 Introduction

The intellectual richness of classical mechanics manifests itself in an abundance of formulations, with Newtonian, Hamiltonian, Lagrangian, and Hamilton–Jacobi being among the most prominent. Butterfield (2004) has aptly referred to these formulations as different *schemes* for treating classical systems. Recent work in philosophy of science has focused largely on how these schemes relate to one another, with the aim of arriving at a deeper understanding of classical mechanics. By considering the conceptual differences between these formulations, we can better interpret the physical significance of our representations.

In the context of comparing Hamiltonian and Lagrangian mechanics, three interpretive stances have emerged. North (2009) has argued for privileging Hamiltonian mechanics on the grounds that it requires less mathematical structure to formulate. In contrast, Curiel (2014) recommends privileging Lagrangian mechanics, arguing that it isolates physically-significant kinematical constraints that all classical systems must satisfy. Finally, we can interpret Lagrangian and Hamiltonian mechanics as theoretically equivalent in a precise mathematical sense. Within a subclass of systems known as the *hyper-regular domain*, the two formalisms are categorically equivalent (Teh and Tsementzis 2017; Barrett 2018). Categorical equivalence provides a natural interpretive criterion for theoretical equivalence because it yields mutual inter-translatability between theory formulations. This equivalence motivates the view that there are no physically significant differences between these two formalisms, at least within the hyper-regular domain.

Here, we will argue that there are indeed physical grounds for privileging Hamiltonian mechanics over Lagrangian mechanics, their categorical equivalence notwithstanding.¹ Whereas North's argument relies on a problematic criterion for comparing structure based on the size of symmetry groups (Swanson and Halvorson 2012), our argument avoids this pitfall. We show that a classical system needs to satisfy strictly fewer physical conditions in order to be Hamiltonian. To be Lagrangian, systems must satisfy an additional physical condition, making Lagrangian mechanics a proper subset of Hamiltonian mechanics. Hamiltonian mechanics thereby captures a wider range of systems than Lagrangian mechanics. Unlike Lagrangian mechanics, Hamiltonian mechanics provides a unique solution to the equations of motion even outside the hyper-regular domain. Hence, Lagrangian mechanics is restricted to the hyper-regular regime. The construction at the heart of our argument provides a novel interpretation of classical mechanical systems in terms of their invariant phase space density.

Section 2 previews our argument for privileging Hamiltonian mechanics. We introduce three physical conditions that define classical systems while distinguishing Hamiltonian from

1. Barrett shows that in the hyper-regular domain, Lagrangian and Hamiltonian mechanics are categorically equivalent provided that we define categories of models over the tangent and cotangent bundle formulations, respectively (2018, 1181–82). However, he also shows that if we consider a more general formulation of Hamiltonian mechanics on a symplectic manifold (which does not require a cotangent bundle), then the two formulations are *not* categorically equivalent (2018, 1182–83). Our construction of Hamiltonian mechanics in Section 4 matches this more general symplectic manifold formulation and illuminates why it is more general.

Lagrangian mechanics. Section 3 explains how our position differs from North's. Specifically, our argument does not require metaphysical commitments to fundamental structure or perfectly natural properties. Section 4 describes in detail the derivation of Hamiltonian mechanics for systems that satisfy two physical conditions: infinitesimal reducibility and deterministic/reversible evolution. Our derivation shows that even non-hyper-regular Hamiltonian systems are physically interpretable. We then show that Lagrangian systems must satisfy an additional physical condition, which we call motion equivalence. This completes our primary argument for privileging Hamiltonian mechanics. Section 5 considers the status of Lagrangian mechanics within our framework. We provide a geometric interpretation of the action principle that applies even outside the hyper-regular domain. We end by rebutting objections that arise from Curiel's (2014) privileging of Lagrangian mechanics.

2 Privileging Hamiltonian mechanics

In claiming that Hamiltonian mechanics is more fundamental than Lagrangian mechanics, we must provide a criterion for assessing whether one framework is more fundamental than another. Of various notions of relative fundamentality, we focus on an entirely non-mysterious one. Given two formalisms that capture a class of systems, if one is logically more general, then it is more fundamental.² This aspect of fundamentality is familiar from mathematics. A more fundamental framework encompasses a less fundamental one as a special case. It is in this sense that the generalized Stokes' theorem is more fundamental than special cases such as the fundamental theorem of calculus and Green's theorem.³ This aspect of fundamentality is

2. Hunt (2022, Ch. 6) provides an expressivist conception of fundamentality, based on an attitude of *being for privileging*. In terms of this conception, adopting our specific criterion here expresses an attitude of being for privileging frameworks that are logically more general. Hunt (2021, 816) considers a separate reason for privileging Hamiltonian mechanics: in systems with an ignorable coordinate, Hamiltonian mechanics makes manifest conservation of momentum.

3. One might worry that our criterion is subject to elementary counterexamples. For instance, can't we make any theory more general by adjoining arbitrary disjunctions? Importantly, we require that the theory remains wholly *about* classical mechanics. Hamiltonian mechanics is more general than Lagrangian mechanics while still being about classical systems, where "aboutness" can be understood using Yablo's framework (2014). Having an account of *aboutness* or *subject matter* is a general philosophical problem that any account of fundamentality must address. It is not a special problem facing our criterion.

distinct from aspects associated with differences in energy scale, such as how the Standard Model of particle physics is more fundamental than classical mechanics. We do not mean that Hamiltonian mechanics is more fundamental than Lagrangian mechanics in this sense. As Gilton notes, “many philosophers of science have moved away from thinking of fundamentality in science in terms of the very small toward thinking of it in terms of comprehensiveness, or breadth of applicability” (2021, 637).

To apply our criterion for relative fundamentality, we must show that Lagrangian mechanics is less general than Hamiltonian mechanics. We do this in Section 4 by recovering the mathematical formalisms of both Hamiltonian and Lagrangian mechanics from three physical conditions. Our derivation illuminates the relationship between Hamiltonian and Lagrangian mechanics. As North (2009) argues, there is a sense in which Hamiltonian mechanics is more general than Lagrangian mechanics. Barrett (2018) makes this sense more precise by showing that the symplectic manifold formulation of Hamiltonian mechanics is categorically inequivalent to Lagrangian mechanics formulated on a tangent bundle. Still, one might wonder why it is that Hamiltonian mechanics can be formulated in a more general framework than Lagrangian mechanics. Additionally, one might wonder whether Lagrangian mechanics could be meaningfully formulated in an analogously general manner. Our derivation helps answer these questions: the physical principles that undergird the symplectic manifold formulation of Hamiltonian mechanics are simply more general than the principles required for Lagrangian mechanics.⁴ In this section, we describe how our argument works in principle. Section 3 demonstrates some philosophical advantages of our criterion for relative fundamentality.

Shorn of many details, our argument goes as follows. First, we show that the formalism of Hamiltonian mechanics applies to systems that are both i) infinitesimally reducible and ii) follow a deterministic and reversible evolution. A system is *infinitesimally reducible* provided that specifying the state of the whole system is equivalent to specifying the states of its parts (and vice versa). Colloquially, we can say that the whole state equals the sum of its parts, where we take these infinitesimal parts to be classical particles. Infinitesimal reducibility distinguishes

4. We thank a reviewer for suggesting this helpful way of framing the significance of our derivation.

classical from quantum systems, where the latter are irreducible (i.e. we cannot specify the whole state by specifying the states of its subsystems).⁵ We show in Section 4.1 that if one chooses to model continuous quantities using a manifold, then this manifold must be symplectic. To recover Hamiltonian mechanics, it then suffices to identify when trajectories on phase space obey Hamiltonian evolution (i.e. evolve according to a symplectomorphism). Section 4.2 shows that this requires the system to follow a *deterministic and reversible evolution*, i.e. the present state of the system uniquely determines both its future and past states. We thereby characterize Hamiltonian mechanics as the mathematical framework for infinitesimally reducible systems with deterministic and reversible evolution. Importantly, our derivation encompasses Hamiltonian mechanics in the non-hyper-regular regime as well.

Next, we note that some classical systems satisfy a third, further condition, which we call *motion equivalence*. Motion equivalence holds when a system's trajectory in physical space determines—and is determined by—its evolution in phase space. In other words, knowledge of the trajectory in physical space (e.g. Euclidean three-space \mathbb{R}^3) suffices for knowledge of the trajectory in phase space, and vice versa. Physically, knowledge of the trajectory in physical space corresponds to knowledge of position and velocity, while knowledge of the trajectory in phase space requires knowing the system's momentum. Hence, motion equivalence entails a correspondence between velocity and momentum, which formally leads to the framework of Lagrangian mechanics (Section 4.3). Since a system satisfies motion equivalence if and only if it is hyper-regular, this condition illuminates the physical significance of hyper-regularity. Moreover, since Lagrangians lack unique solutions to the equations of motion outside the hyper-regular regime, we take ourselves to have recovered the physically significant Lagrangian systems.⁶ As a result, Lagrangian mechanics requires that a further physical condition is satisfied, and hence it is less general than Hamiltonian mechanics.

To show that motion equivalence is a further condition, it suffices to show that some Hamil-

5. The conditions of infinitesimal (ir)reducibility are similar to various notions of (non-)separability in the literature on physical holism (Healey 2007, 46, 124). We adopt our preferred terminology to avoid controversies surrounding how best to characterize separability.

6. See Gelfand and Fomin (1963, 119) for discussion of the fact that Lagrangians yield unique solutions only in the hyper-regular domain.

tonian systems violate it. This follows from the fact that knowing a system’s position and velocity does not always suffice for knowing its momentum. Hence, we cannot always determine the evolution of a system in phase space from its trajectory in physical space. For instance, if we consider a photon as a classical particle, then we cannot determine its momentum from its trajectory through space. Since photons travel at constant speed, we can recover the direction of momentum—but not the magnitude—from the trajectory. Two photons with different momenta can follow the same trajectory in physical space. Hence, knowledge of the physical trajectory of a photon is insufficient for knowledge of its phase space evolution. In general, different Lagrangians—differing even by more than a constant—can correspond to the same particle trajectories while not agreeing on the total energy, even up to a constant.⁷ Hence, the set of classical systems that satisfy motion equivalence is a proper subset of the classical systems that satisfy the conditions for Hamiltonian mechanics. For at least this reason, Hamiltonian mechanics is more fundamental than Lagrangian mechanics.

Of course, different definitions of “Hamiltonian” and “Lagrangian” systems can result in different interpretive relationships. On our account, Hamiltonian systems are infinitesimally reducible and follow deterministic and reversible evolution.⁸ We thereby set aside well-known cases of non-deterministic classical systems.⁹ Additionally, we set aside dissipative systems, i.e. those that fail to conserve energy. Possessing a deterministic and reversible evolution is equivalent to being non-dissipative. Our derivation of Lagrangian mechanics in Section 4.3 takes place within the context of conservative systems, so we do not consider systems that satisfy inhomogenous Euler–Lagrange equations. Although both the Lagrangian and Hamiltonian formulations apply to dissipative systems, we exclude these.¹⁰ This restricts our claim about the fundamentality of Hamiltonian mechanics to the non-dissipative regime. Finally, our treat-

7. Barrett (2018, 1185–86) discusses this in the context of the failure of equivalence between the vector field formulations of Lagrangian and Hamiltonian mechanics. In contrast, the trajectories in phase space (encoded by the Hamiltonian vector field) uniquely define the Hamiltonian up to a constant.

8. Although not sufficient for a definition, Newtonian systems also satisfy the motion equivalence condition. Since Lagrangian systems satisfy all three conditions (infinitesimal reducibility, deterministic/reversible evolution, and motion equivalence), they lie in the intersection of Newtonian and Hamiltonian systems.

9. For non-deterministic classical systems, see Baez (unpublished, 3–4), Earman (1986), and Norton (2008).

10. For details, see Cline (2021, §10.4). Curie (2014, 311) acknowledges that the Hamiltonian framework can be extended to treat dissipative systems. For a Lagrangian approach to dissipative systems, see Smith (2008).

ment is not restricted to n -particle mechanics but applies as well to rigid-body and continuum mechanics. This supports our claim that the property of infinitesimal reducibility fruitfully characterizes classical systems. Before presenting the details of our argument in Section 4, we describe how we avoid problems that arise in North's privileging of Hamiltonian mechanics.

3 Fundamentality, *sans metaphysics*

As indicated at the start of Section 2, we are interested in a non-metaphysical aspect of fundamentality. At least some differences in fundamentality are differences in generality: other things equal, a theoretical framework is more fundamental if it requires fewer conditions to hold. This aspect of fundamentality leads to a correspondingly non-metaphysical notion of structure. Hamiltonian mechanics employs less structure than Lagrangian mechanics because the former requires fewer physical conditions. Compared with North's (2009) account of comparative structure, our criterion has two advantages. First, it avoids metaphysical disputes surrounding perfectly natural or joint-carving properties, which scientific anti-realists, pragmatists, and even some scientific realists view with suspicion. Secondly, it avoids a technical objection that Swanson and Halvorson (2012) levy against North's more precise mathematical criterion for comparing structure.

Despite our different commitments, our approach has methodological parallels to North's. We agree with North that it is fruitful to identify both invariants and "the mathematical structure needed to formulate the theory in an invariant, frame-independent way" (2009, 65). Additionally, we agree that it is worthwhile to identify what particular mathematical structure is required by a formulation (78). Indeed, we undertake precisely this sort of investigation in Section 4.¹¹ However, North advocates a metaphysical account of structure, about which we remain agnostic. According to North, "structure comprises the objective, fundamental, intrinsic features, the ones that remain the same regardless of arbitrary or conventional choices and

11. Curiel also aims to identify the necessary and sufficient mathematical structures for formulating classical mechanics. He sets out to establish the "geometrical structure necessary (and manifestly sufficient) for the formulation of the Euler–Lagrange equation" relevant for all kinematically-possible solutions (2014, 292). See Section 5.2 for discussion.

description” (2009, 66). These notions of fundamentality and intrinsicity correspond to differences in joint-carving, where more fundamental or intrinsic features are more joint-carving (or in Lewis’s (1983) terminology, more natural).

Owing to the epistemic difficulty of accessing these putative joints, we do not endorse these metaphysical dimensions of structure or fundamentality.¹² While Hamiltonian mechanics is significant partly because it elucidates invariant densities, this does not entail that additional mathematical structure is physically insignificant or surplus. Additional structure may correspond to an additional physically-significant condition that applies to the system under consideration. Precisely this scenario occurs in the context of Lagrangian mechanics, which requires the additional condition of motion equivalence to hold. Whereas North views Lagrangian mechanics as containing “excess, superfluous structure” (2009, 75), we take this additional condition to still be physically significant. Thus, while it is illuminating to identify the minimum mathematical structures necessary for a formulation, additional structure is not automatically rendered physically insignificant.

Additionally, whereas North (2009, 76) seeks to determine the “real” fundamental state space structure of the theory—and correspondingly, of a classical world—we remain neutral on this underlying metaphysical question.¹³ Whether Hamiltonian phase space structure is metaphysically fundamental in a classical world does not matter for the questions we seek to answer. Regardless of its metaphysical status, Hamiltonian mechanics is conceptually more fundamental. Indeed, our argument does not privilege any one of the various mathematical formalizations of Hamiltonian mechanics. These include formulating Hamiltonian mechanics on a $2n + 1$ -dimensional contact manifold, on an extended phase space, on a Poisson manifold, and the standard formulation on a symplectic manifold that we recover in Section 4.2.¹⁴

12. For a detailed discussion of difficulties facing the notion of metaphysical naturalness, see Dorr and Hawthorne (2014).

13. Indeed, Swanson and Halvorson (2012) note that many physicists and philosophers would not interpret state-space structure ontologically, but rather as a formal device for encoding physical information or a calculational tool. As Wilson illustrates, we can interpret classical mechanics while avoiding—and even rejecting—the search for what is “fundamental in the bottom-layer sense” (2013, 53).

14. Tulczyjew’s reformulation of Hamiltonian mechanics further supports viewing symplectic geometry—and hence Hamiltonian mechanics—as conceptually fundamental. According to Teh and Tsementzis, this reformulation shows that “the very concept of a classical mechanical system (and not merely the phase space of that system)

North also introduces a technical notion of comparative structure, based on the symmetry group of statespace. According to this criterion, statespace A has more structure than statespace B if the symmetry group of A is a proper subgroup of B's symmetry group (87–88).¹⁵ Being a proper subgroup entails that strictly fewer transformations preserve the space's structure. This indicates that statespace A has more mathematical structure to be preserved. These symmetry groups are defined in terms of the coordinate transformations that preserve the space's geometric structure, amounting to re-coordinatizations (redescriptions) of that structure. For Lagrangian systems, North takes the relevant symmetry group to be that of the point transformations, and for Hamiltonian systems, she assumes that canonical transformations play this role. Since point transformations are a proper subgroup of canonical transformations, North's criterion entails that the statespace of Lagrangian mechanics has more structure than the statespace of Hamiltonian mechanics—in virtue of the former having a smaller symmetry group.

However, Swanson and Halvorson (2012) develop a series of problems for North's argument. Most compellingly, they argue that we cannot in general assume that point transformations and canonical transformations are the proper symmetry groups for Lagrangian and Hamiltonian mechanics, respectively. As they note, Galilean boosts are often symmetry transformations of Lagrangian systems (in the sense of preserving the equations of motion), but they are not point transformations. Additionally, the two-body problem admits canonical transformations that map physically distinct solutions to each other (specifically, solutions whose orbits possess the same energy but different eccentricity).¹⁶ Hence, we cannot straightforwardly compare the structure of Lagrangian and Hamiltonian mechanics in terms of the symmetry groups of their statespaces.

Advantageously, our criterion for comparing Hamiltonian and Lagrangian mechanics avoids these problems. Hamiltonian mechanics has less structure than Lagrangian mechanics simply in the sense of logical strength: a system must satisfy strictly fewer physical conditions in order to

should be described in terms of symplectic geometry" (2017, 46).

15. This criterion follows from the most charitable reconstruction of North's argument, given by Swanson and Halvorson (2012). North sometimes speaks in terms of comparing the dimensions of symmetry groups, which they note leads to counterexamples.

16. For more details on the classical two-body problem, see Belot (2013).

be Hamiltonian. Whereas to be Lagrangian, the system must also satisfy the motion equivalence condition. Motion equivalence places an additional constraint on the Hamiltonian, a constraint that need not be satisfied in general.

4 Deriving Hamiltonian and Lagrangian mechanics

Our derivation has three main steps. Section 4.1 shows that if one represents continuous quantities using a manifold, then this manifold must be symplectic. For systems with deterministic and reversible evolution, Section 4.2 shows they must evolve according to a symplectomorphism. Hence, we derive the formalism of Hamiltonian mechanics from the conditions of infinitesimal reducibility and deterministic/reversible evolution. Section 4.3 shows how systems that satisfy the further condition of motion equivalence can be treated within the formalism of Lagrangian mechanics. In fact, when motion equivalence fails, Lagrangian mechanics does not yield unique solutions to the equations of motion.

These three physical conditions place constraints on both physical systems and, in turn, the mathematical objects that can be used to represent them. Each condition leads to additional constraints, introduced gradually within each section. Some of these constraints are regulative in nature: they provide requirements for the math and therefore cannot be derived from mathematics alone. Our goal is to show how these constraints follow logically from the physical conditions, how they are encoded in the math, and how they recover well-known structures.

Figure 1 summarizes the results of this derivation. The left column lists the three central physical conditions. Each condition entails physically significant properties, described in the middle column. Each physical property corresponds to a formal/mathematical structure depicted in the right column, including phase space, Hamiltonian mechanics, and Lagrangian mechanics. In this way, our framework develops a correspondence between physical conditions and their associated mathematical formalisms. We thereby elucidate the physical significance of these formalisms.

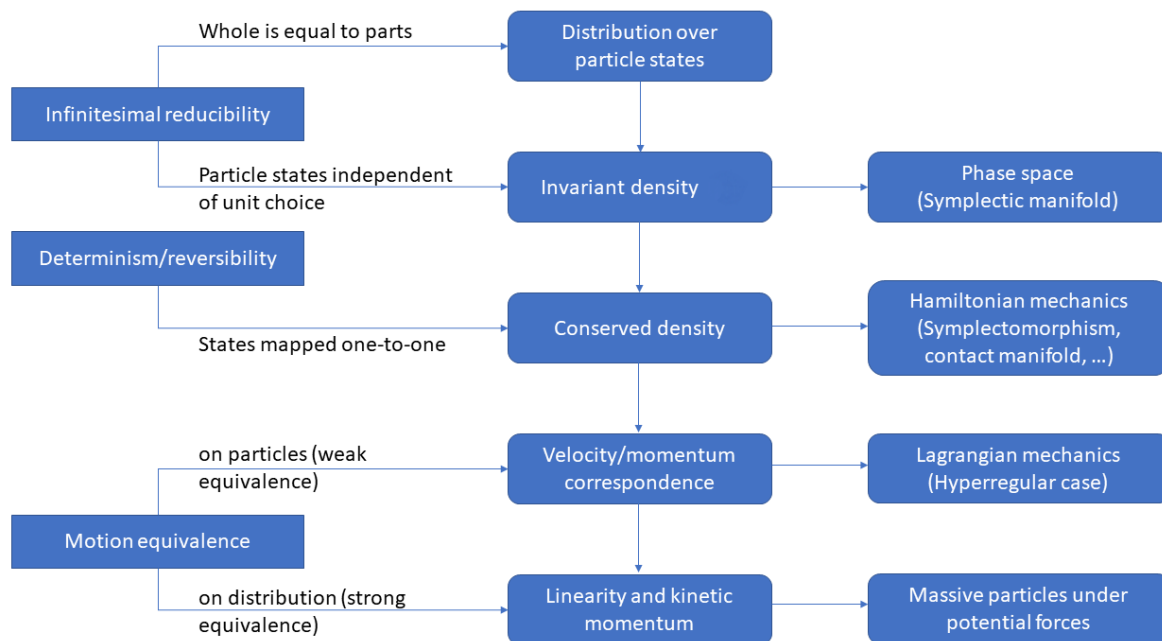


Figure 1: Physical conditions and corresponding properties and formalisms

4.1 Infinitesimal reducibility

First, we will show that a classical phase space arises if a system satisfies the following physical condition:

Infinitesimal reducibility: the state of the system is reducible to the state of its infinitesimal parts. That is, specifying the state of the whole system is equivalent to specifying the state of its parts.

This assumption characterizes classical systems as those whose infinitesimal parts completely specify the state of the system. We define a *classical particle* as one of these infinitesimal parts, namely the limit of recursive subdivision.¹⁷ Ultimately, this condition entails that we can describe the state of the whole system as a statistical distribution over the state space of its parts. If we model the state space of the parts using a manifold, the distribution must be characterized by a density that depends on the state and not on the coordinates. From this condition, we

¹⁷ This definition of a classical particle differs from viewing point-particles as extensionless (Butterfield 2006) or zero-dimensional (Wilson 2013).

show that the manifold must be symplectic. Readers satisfied with this overview can proceed to Section 4.2.

Let \mathcal{C} be the state space for the whole system, so that any $c \in \mathcal{C}$ represents a particular state of the whole system. Given infinitesimal reducibility, a classical system consists of a collection of classical particles. Let \mathcal{S} be the state space for the particles, so that each point $s \in \mathcal{S}$ describes a possible state of a classical particle. Then for each system state $c \in \mathcal{C}$, there exists a unique distribution $\rho_c : \mathcal{S} \rightarrow \mathbb{R}$ that describes the state of c 's parts. That is, the state of the whole system is a distribution over its infinitesimal parts. This distribution describes the proportion of classical particles in any particular state from \mathcal{S} .¹⁸

When the system is characterized by continuous quantities, \mathcal{S} can be taken to be a manifold.¹⁹ For $U \subseteq \mathcal{S}$, we define $\mu(U) := \int_U \rho_c(s) d\mathcal{S}$ to measure the number of particles in region U . *State variables* ξ^a are any set of quantities that fully identify states, so that $\xi : \mathcal{S} \rightarrow \mathbb{R}^n$ is an invertible function between each state and the numeric values of the variables.²⁰ The map ξ represents a state $s \in \mathcal{S}$ by a set of state variables $\xi^a \in \mathbb{R}^n$, such that $\xi(s) = \xi^a$. To represent the distribution ρ_c in terms of state variables, we define a density $\rho_{c,\xi} : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\rho_{c,\xi} \equiv (\rho_c \circ \xi^{-1})$. This density plays a crucial role throughout our argument. It allows us to compare and track the amount of matter in different possible states.

Since $\rho_c : \mathcal{S} \rightarrow \mathbb{R}$ is a function of the particle state s , we assume that its value is independent of the choice of state variables ξ^a (i.e. the choice of state variable map ξ). If this assumption did not hold, then we could not model the system in the first place using state variables, and this would preclude many standard representational tools. Hence, the density $\rho_{c,\xi}$ must transform as a scalar under state variable changes.²¹ Likewise, we assume that the integral $\int_U \rho_c(s) d\mathcal{S}$ is in-

18. For further details, see Carcassi et al. (2018) and Carcassi and Aidala (2020).

19. Note that the use of manifolds constitutes a choice of how to represent the physics. One could make another choice, for example, by representing the state variables not as local coordinates but as an algebra of operators that act on the state. In this case, we conjecture that a similar argument would demonstrate an analogue of a symplectic structure for this alternative mathematical representation.

20. In the language of manifolds, these ξ^a are called ‘‘coordinates’’ and ξ is a coordinate map. To avoid confusion, we will call the coordinates of the state space \mathcal{S} ‘‘state variables.’’ As usual with coordinate charts, ξ is properly defined on some $U \subseteq \mathcal{S}$, and ξ^{-1} is defined on $\xi(U) \subseteq \mathbb{R}^n$. We will neglect these subtleties in what follows.

21. For two state variable maps ξ and $\hat{\xi}$, we have $\rho_c(s) = (\rho_c \circ \xi^{-1} \circ \xi)(s) = (\rho_c \circ \hat{\xi}^{-1} \circ \hat{\xi})(s)$, which entails that $\rho_c(s) = \rho_{c,\xi}(\xi^a) = \rho_{c,\hat{\xi}}(\hat{\xi}^b)$. Here, $\xi(s) \equiv \xi^a$ and $\hat{\xi}(s) \equiv \hat{\xi}^b$.

dependent of the choice of state variables, so that $\int_{\xi(U)} \rho_{c,\xi}(\xi^a) d\xi^a = \int_{\hat{\xi}(U)} \rho_{c,\hat{\xi}}(\hat{\xi}^b) d\hat{\xi}^b$.²² Applying the change of variables formula, we know that $\int_{\hat{\xi}(U)} \rho_{c,\hat{\xi}}(\hat{\xi}^b) d\hat{\xi}^b = \int_{\xi(U)} \rho_{c,\xi}(\hat{\xi}^b) \left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right| d\xi^a$.²³ Here, $\left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right|$ is the determinant of the Jacobian for this change of state variables. Matching integrands from the preceding two equations, we have $\rho_{c,\xi}(\xi^a) = \rho_{c,\hat{\xi}}(\hat{\xi}^b) \left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right|$. From the scalar transformation property, $\rho_{c,\xi}(\xi^a) = \rho_{c,\hat{\xi}}(\hat{\xi}^b)$. Hence, we see that $\rho_{c,\hat{\xi}}(\hat{\xi}^b) = \rho_{c,\hat{\xi}}(\hat{\xi}^b) \left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right|$, entailing that the Jacobian determinant must equal one. By dimensional homogeneity, it must also be dimensionless.

We will now show that in order for the density $\rho_{c,\xi}$ to have these transformation properties, the particle phase space \mathcal{S} must be at least a symplectic manifold. In other words, given that i) we assume we can model a classical system by a density over its infinitesimal states and ii) these transformation properties are required for the density to play this role, it follows that the phase space must have a symplectic manifold structure. Our next assumption involves the relationship between state variables and measurement units.

4.1.1 The particle state space is even-dimensional

We assume that a proper subset of the state variables suffices to define a unit system for all of the state variables. Again, this assumption is regulative in the sense that without it, we could not quantify properties of the system using measured values. Using this assumption, we will show that the state space \mathcal{S} must be even-dimensional and hence a candidate for having symplectic structure.

We define a *unit variable* $q \in \xi^a$ as a particular state variable that defines a unit. For instance, q might be position, defined in units of meters. Two unit variables are *independent* if one unit variable can be changed without changing the other. For example, one can measure vertical

22. Here, we assume that ξ and $\hat{\xi}$ map the background measure on \mathcal{S} to the standard Lebesgue measure on \mathbb{R}^n . In order to match integrands below, we also assume that ξ and $\hat{\xi}$ agree on their orientation. These are regulative assumptions so that we can model the system using coordinates on \mathbb{R}^n . They amount to assuming that the particle state space \mathcal{S} has a volume form.

23. The change of variables transformation from ξ^a to $\hat{\xi}^b$ is given by $\phi \equiv \hat{\xi} \circ \xi^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$. For convenience, we denote $\rho_{c,\hat{\xi}}(\phi(\xi^a))$ by $\rho_{c,\hat{\xi}}(\hat{\xi}^b)$. Likewise, in the numerator of the Jacobian, we denote $\partial\phi$ by $\partial\hat{\xi}^b$, where $\hat{\xi}^b$ is here understood as a function of ξ^a .

position in meters while measuring horizontal position in miles. The *unit variables* q^i are a privileged subset of the state variables that define the unit system. These q^i determine the units not only for themselves, but for all other state variables. A *change of units* is therefore a map $\hat{q}^j = \hat{q}^j(q^i)$ that defines new unit variables in terms of old unit variables. Since the unit variables define the units for all state variables, this change must induce a unique change of state variables $\hat{\xi}^b = \hat{\xi}^b(\xi^a)$ for all variables.

Consider first the simplest case: suppose that a single unit variable q suffices to define the unit system for the particle state space. Then, the state variables are a set $\xi^a = \{q, k^\alpha\}$, where *prima facie* the index α could be any non-negative integer. We will show that, in fact, there must be exactly one k^α variable, conjugate to the single q variable.

As a scalar, the units of the density $\rho_{c,\xi}$ are invariant under unit changes of the state variables. Suppose we change units by transforming q to a new unit variable $\hat{q} = \hat{q}(q)$. Call the new state variables $\hat{\xi}^b = \{\hat{q}, \hat{k}^\beta\}$. As shown above, the Jacobian determinant $\left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right|$ must equal one. Assume for contradiction that there is only one state variable, so that there are no k^α variables. Then we would have $\left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right| = \left| \frac{\partial \hat{q}}{\partial q} \right| = 1$. But this would mean the unit change cannot be arbitrary. Therefore we must have two or more state variables, requiring at least one k^α variable.

Next, we show that in fact there must be exactly one k variable, conjugate to the single q variable. Using the fact that the Jacobian determinant equals one, we determine how the state variables k^α must change in order to compensate the change from q to \hat{q} . Since \hat{q} depends only on q , $\frac{\partial \hat{q}}{\partial k^\alpha} = 0$. Hence, the Jacobian is a lower triangular block matrix:

$$|J| = \left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right| = \left| \begin{array}{cc} \frac{\partial \hat{q}}{\partial q} & \frac{\partial \hat{q}}{\partial k^\alpha} \\ \frac{\partial \hat{k}^\beta}{\partial q} & \frac{\partial \hat{k}^\beta}{\partial k^\alpha} \end{array} \right| = \left| \begin{array}{cc} \frac{\partial \hat{q}}{\partial q} & 0 \\ \frac{\partial \hat{k}^\beta}{\partial q} & \frac{\partial \hat{k}^\beta}{\partial k^\alpha} \end{array} \right| \quad (1)$$

The determinant of a triangular block matrix equals the product of determinants of the block matrices along the diagonal: $\left| \frac{\partial \hat{\xi}^b}{\partial \xi^a} \right| = \left| \frac{\partial \hat{q}}{\partial q} \right| \left| \frac{\partial \hat{k}^\beta}{\partial k^\alpha} \right|$. For the Jacobian determinant to equal one, we must therefore have $\left| \frac{\partial \hat{k}^\beta}{\partial k^\alpha} \right| = \left| \frac{\partial \hat{q}}{\partial q} \right|^{-1}$. Since \hat{q} is a function only of q , $\frac{\partial \hat{q}}{\partial q}$ is a total derivative, so we obtain the inverse by swapping the numerator and denominator. Hence, we see that $\left| \frac{\partial \hat{k}^\beta}{\partial k^\alpha} \right| = \left| \frac{\partial q}{\partial \hat{q}} \right|$.

This equation provides only a single constraint, placed on the determinant of the transformation from k^α to \hat{k}^β . Hence, if there were three or more ξ^a variables, this constraint would be insufficient to recover the transformation of the k^α uniquely. In this case, \hat{q} would not fully define the units for all other state variables, contrary to assumption. Consequently, there must be exactly two state variables: q and a single k . So in the case of a single q variable, defining an invariant density $\rho_{c,\xi}$ requires a two-dimensional manifold.

We extend this construction to the general case with multiple unit variables q^i . Recall that two unit-variables are independent if changing the units for one does not change the units for the other. Suppose our particle state space \mathcal{S} is such that its units are fully defined by n independent unit-variables q^i . If we change the first unit-variable q^1 without changing the others, then the variable k_1 must change as before, so that the density remains invariant. Likewise, transforming the second unit-variable q^2 in the same way (while fixing k_1) requires a corresponding k_2 . Continuing, we find that \mathcal{S} is $2n$ -dimensional, with state variables $\xi^a = \{q^i, k_i\}$. We define an *independent degree of freedom* as the space charted by a pair of such variables. Each independent degree of freedom thereby defines a two-dimensional surface in phase space. These play a crucial role in showing that \mathcal{S} must have a non-degenerate, closed two-form ω , i.e. a symplectic form.

4.1.2 The particle state space has a symplectic form

We now show that \mathcal{S} must have a well-defined two-form ω . Recall that a density is defined in terms of an amount per unit volume. Representing the system state c in terms of its infinitesimal parts requires that the density $\rho_{c,\xi}$ is defined on infinitesimal volumes $d\mathcal{S}$ of the phase space. Since each independent degree of freedom comprises a pair of state variables, each infinitesimal volume decomposes into independent infinitesimal areas. To define the density $\rho_{c,\xi}$, we must be able to measure each of these independent infinitesimal areas. This is precisely the role performed by a two-form ω , which assigns an infinitesimal area to each infinitesimal surface. We can then define integrals of the form $\int_\Sigma \rho_c \omega(d\Sigma)$, for $\Sigma \subset \mathcal{S}$ (i.e. the fraction of the system whose parts are in the region Σ of state space \mathcal{S}). Because the degrees of freedom are indepen-

dent, the total number of states in a volume is the product of the possible configurations of each degree of freedom. This means that the volume form is given by ω^n .

Since each volume of phase space contains some possible states, the volume form cannot be degenerate. If the volume form were degenerate, then there would be a “volume” in state space that did not contain states. But then this region of state space could not be a volume. Since the volume form ω^n is non-degenerate, the two-form ω must also be non-degenerate. Otherwise, there would be infinitesimal surfaces in phase space assigned zero area, i.e. there would be states of phase space that ω does not count. But, ω must count all possible states in order for the density $\rho_{c,\xi}$ to be well-defined.

Next, we show that ω must be closed, using the fact that the degrees of freedom are independent. If we translate a surface along an independent degree of freedom, this does not change the area of the surface, meaning that the number of states on that surface is invariant. We can therefore construct a parallelepiped P , whose opposite sides have equal areas. Integrating ω over the closed surface ∂P of this parallelepiped, opposite sides provide equal and opposite contributions. Hence, the integral over this closed surface must equal zero: $\int_{\partial P} \omega = 0$. By Stokes’ theorem, the integral of $d\omega$ over this volume P must equal zero as well: $\int_P d\omega = \int_{\partial P} \omega = 0$. Since, we can approximate any volume V by such parallelepipeds, $\int_V d\omega = 0$ for arbitrary V . Hence, the integrand $d\omega$ must vanish, entailing that ω is closed. So \mathcal{S} must come equipped with a two-form ω that is closed and not degenerate, and therefore \mathcal{S} is a symplectic manifold.

Moreover, we choose the same units for ω as the units we choose to count states over an independent degree of freedom. This matches common practice in statistical mechanics. When normalized, the density $\rho_{c,\xi}$ is given in terms of the inverse of the units for ω . Following standard conventions in quantum mechanics and statistical mechanics, the k_i are given in terms of inverse units of the respective q^i , such that $dq^i \wedge dk_i$ is a pure number. We then use \hbar to define the units for state counting and therefore set $\omega = \hbar dq^i \wedge dk_i = dq^i \wedge dp_i$ where $p_i = \hbar k_i$.

We can thereby understand canonical coordinates as a set of state variables that express the density in the correct units. A canonical transformation is one that preserves the unit and the numerical value of the density. A change of unit is a special type of canonical transformation:

one where we have an arbitrary change in the unit variables and an induced (covariant) change in the conjugate variables.

In sum, under the infinitesimal reducibility assumption, a state c of the system is represented by a coordinate-invariant density over the state space \mathcal{S} of its infinitesimal parts. We have shown that the state space of the particles must be a symplectic manifold. Additionally, any symplectic manifold structure supports the definition of coordinate-invariant densities: any scalar function can be integrated using the symplectic form in a coordinate-invariant way. Therefore symplectic structure is exactly the structure needed to define an invariant density.

4.2 Deterministic and reversible evolution

Next, we focus on those classical systems that are deterministic and reversible. This amounts to satisfying the following joint condition:

Deterministic and reversible evolution: given the present state of the system, all future (determinism) and past (reversibility) states are uniquely identified.

In our case, this applies to both the state of the whole system and the state of its parts. Under this assumption, we show that the symplectic structure must be preserved over time, which gives us Hamiltonian evolution.

First, we will show that deterministic and reversible evolution entails that the density $\rho_{c,\xi}$ is constant in time. We consider a particle state s_0 at time t_0 . Given deterministic and reversible evolution, there exists a unique evolution $\lambda : \mathbb{R} \rightarrow \mathcal{S}$ such that $\lambda(t_0) = s_0$. Hence, the initial state s_0 at time t_0 uniquely identifies the state of this particle at all times. Let $\rho(s, t)$ be the density associated to a state s at a time t .²⁴ Then $\rho(\lambda(t_0), t_0)$ represents the part of the composite system in particle state s at that time. Likewise, for $t_1 > t_0$, $\rho(\lambda(t_1), t_1)$ represents the part of the system associated to the evolved state at this future time. Given deterministic and reversible evolution, all particles that start in a given initial state must end up in the same final state, and vice-versa. Hence, $\rho(\lambda(t_0), t_0) = \rho(\lambda(t_1), t_1)$ for all t_0, t_1 . In other words, the density remains the same throughout the evolution.

24. For convenience, we sometimes drop the composite state c and coordinate-choice ξ indices from the invariant density $\rho_{c,\xi}(\xi^a)$. We reserve the word “distribution” for the coordinate-free distribution $\rho_c(s)$.

Second, we show that the two-form ω is invariant under this evolution, and hence that the system follows Hamiltonian evolution. Consider a surface $\Sigma_0 \subset \mathcal{S}$ of particle states at some time t_0 . If two-dimensional, this surface defines an independent degree of freedom. The integral $\int_{\Sigma_0} \rho_c(s, t_0) \omega(d\Sigma_0)$ represents the fraction of the system in states defined on Σ_0 . Since each particle state evolves by λ , Σ_0 evolves to a surface Σ_1 at time t_1 . This leads to a corresponding $\int_{\Sigma_1} \rho_c(s, t_1) \omega(d\Sigma_1)$, representing the fraction of the system on this evolved surface. Since the evolution is deterministic and reversible, all and only the particle states that began on surface Σ_0 evolve to Σ_1 . Consequently, the fraction of the system on these two surfaces at these two times must be the same: $\int_{\Sigma_0} \rho_c(s, t_0) \omega(d\Sigma_0) = \int_{\Sigma_1} \rho_c(s, t_1) \omega(d\Sigma_1)$. Since the density is constant, we know that $\rho_c(s, t_0) = \rho_c(s, t_1)$. Hence, the two-form ω must remain the same. That is, the areas in phase space must be mapped to areas of equal size, and independent degrees of freedom must be mapped to independent degrees of freedom (otherwise, volumes would not be mapped to equal volumes, violating determinism). The evolution is therefore a symplectomorphism and hence corresponds to a Hamiltonian evolution.

Intuitively, our argument is the inverse of Liouville's theorem: instead of positing Hamiltonian evolution and finding conservation of areas and volumes, deterministic and reversible evolution imposes the conservation of areas and volumes (i.e. sets of states are mapped to sets of states of equal size). This leads to Hamiltonian evolution. In summary, we have shown that any system that satisfies the conditions of both infinitesimal reducibility and deterministic/reversible evolution is a classical Hamiltonian system. Note in particular that our derivation is not restricted to only hyper-regular Hamiltonians. As we will see, this is a key difference between Hamiltonian and Lagrangian mechanics.

4.3 Motion equivalence

Having recovered classical Hamiltonian mechanics, the laws of evolution are further restricted when the following condition holds:

Motion equivalence: the motion of the system (i.e. trajectories in physical space)

suffices to recover its dynamics (i.e. evolutions in state space) and vice-versa.²⁵

If we focus on the parts of the system, this means that for every evolution in phase space there should be one and only one trajectory in physical space. This assumption means that momentum can be expressed as a function of position and velocity, which mathematically corresponds to the case in which the Hamiltonian is hyper-regular. In this regime we find Lagrangian mechanics. Moreover, to be able to express the symplectic form (and therefore the density) over position and velocity, we find that momentum must be a linear function in velocity, which constrains the Hamiltonian to be that of a massive particle under scalar and vector potential forces.

Note that each space variable x^i is a unit-variable, i.e. a state variable that defines a unit. In fact, the trajectories can be fully described only by these unit-variables. For, to specify a trajectory, we need epistemic access to a unit system, where the number of independent units is determined by the dimensionality of physical space. It follows that (i) $x^i = q^i$, (ii) each q^i is paired with a conjugate p_i , and (iii) each state $\{q^i, p_i\}$ corresponds to one and only one trajectory in physical space.

At each point x^i , then, there must be infinitely many trajectories, one for each possible combination of $\{p_i\}$. Since these trajectories are differentiable in x^i , we can define a velocity $v^i = d_t x^i$. If the equations of motion contained the velocity as a function of only the unit-variables (i.e. $v^i = v^i(q^i)$), then motion equivalence would fail. For if both x^i and v^i were functions of only q^i , this relationship would not be invertible due to dimensional considerations (a surjection from an n -dimensional manifold to a $2n$ -dimensional manifold cannot be invertible). Hence, we must have $v^i = v^i(q^j, p_k)$, so that knowledge of the state in phase space suffices for knowledge of the trajectory. If motion equivalence holds, then this relationship must be invertible, meaning that knowledge of the trajectory suffices for knowledge of the phase space state. For systems that satisfy the motion equivalence condition, the following relationships

25. Abraham and Marsden (1978, 218) show that in the hyper-regular domain, Lagrangian and Hamiltonian mechanics describe the same evolution of particle positions over time (Theorem 3.6.2). See Barrett (2018, 1180–81) for discussion. Motion equivalence can be understood as the converse of this theorem: assuming that Lagrangian and Hamiltonian mechanics agree on particle evolutions, we derive that the Legendre transformation is well-defined in the hyper-regular domain.

thus hold at any given time:

$$\begin{aligned} x^i &= q^i \\ v^i &= d_t x^i = v^i(q^j, p_k) \end{aligned} \tag{2}$$

In this case—which we call *weak equivalence*— $v^i(q^j, p_k)$ is invertible at every q^j . Therefore, for fixed q , the Jacobian matrix $\frac{\partial v^i}{\partial p_k}$ must be invertible. Let H denote the Hamiltonian. By Hamilton's equations, $v^i = d_t q^i = \frac{\partial H}{\partial p_i}$, so the Hessian $\frac{\partial^2 H}{\partial p_k \partial p_i} = \frac{\partial}{\partial p_k} \left(\frac{\partial H}{\partial p_i} \right)$ equals the Jacobian $\frac{\partial v^i}{\partial p_k}$. Since the Jacobian is invertible, the Hessian must be non-zero everywhere, and therefore it must have the same sign everywhere—which we take to be positive. This, by definition, means that the Hamiltonian is hyper-regular if and only if weak equivalence is satisfied. Weak equivalence allows us to express particle states in terms of either position/momentum or position/velocity variables. These are exactly the cases where a Lagrangian can be constructed, and where the Lagrangian leads to a unique solution (via the stationary action principle). Hence, weak equivalence suffices for the existence of a hyper-regular Lagrangian. As discussed in Section 5.2, Lagrangian mechanics only yields a unique evolution in the hyper-regular domain.²⁶

Whereas weak equivalence involves motion equivalence at the level of particle states, *full equivalence* extends this to the entire distribution, i.e. the composite state of the system. Full equivalence holds when we can express the distribution—represented by the density—in terms of position and velocity variables. Formally, this requires that $\rho(q^i, p_j) = |J| \rho(x^i, v^j) = \left| \frac{\partial v^i}{\partial p_j} \right| \rho(x^i, v^j)$ since

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial x^i}{\partial q^j} & \frac{\partial x^i}{\partial p_j} \\ \frac{\partial v^i}{\partial q^j} & \frac{\partial v^i}{\partial p_j} \end{vmatrix} = \begin{vmatrix} \delta_j^i & 0 \\ \frac{\partial v^i}{\partial q^j} & \frac{\partial v^i}{\partial p_j} \end{vmatrix} \\ &= |\delta_j^i| \left| \frac{\partial v^i}{\partial p_j} \right| = \left| \frac{\partial v^i}{\partial p_j} \right|. \end{aligned} \tag{3}$$

Note that while the value given by $\rho(q^i, p_j)$ is invariant under unit changes, the value given by $\rho(x^i, v^j)$ depends on the choice of units through $\left| \frac{\partial v^i}{\partial p_j} \right|$. If x^i truly sets the unit system by itself, then $\left| \frac{\partial v^i}{\partial p_j} \right|$ must be a function of position only, since the position variables define the units.

26. Weak equivalence corresponds to a general treatment of Lagrangian mechanics that does not presuppose Riemannian geometry. As we discuss in Section 5.2, Equation 2 corresponds to the kinematical constraints central to Curiel's (2014) argument for privileging Lagrangian mechanics.

Similar considerations hold for marginal distributions (i.e. distributions on a subset of the unit variables), entailing that all components of $\frac{\partial v^i}{\partial p_j}$ must be a function of position only. We set:

$$\begin{aligned}\frac{\partial v^i}{\partial p_j} &= \frac{1}{m} g^{ij} \\ \frac{\partial p_j}{\partial v^i} &= m g_{ji}\end{aligned}\tag{4}$$

where m is the unit conversion constant between velocity and conjugate momentum, while g_{ij} represents the linear dependency of the velocity on the momentum.

If we integrate, we have:

$$\begin{aligned}v^i &= \frac{1}{m} g^{ij} (p_j - A_j) \\ p_j &= m g_{ji} v^i + A_j\end{aligned}\tag{5}$$

where A_j are arbitrary functions. We recognize $m g_{ji} v^i$ as kinetic momentum. Note that:

$$v^i = d_t q^i = \frac{\partial H}{\partial p_i} = \frac{1}{m} g^{ij} (p_j - A_j)\tag{6}$$

We integrate yet again and find:

$$H = \frac{1}{2m} (p_i - A_i) g^{ij} (p_j - A_j) + V\tag{7}$$

where V is another arbitrary function. We recognize this as the Hamiltonian for massive particles under potential forces, relevant for classical electrodynamics and Newtonian gravitation. For example, A can play the role of the magnetic vector potential, with V playing the role of the electric potential. Whereas Curiel (2014) argues that this Hamiltonian requires the imposition of *ad hoc* conditions, we take ourselves to have given an eminently natural construction of it. We return to this point in Section 5.2, using it to defuse Curiel's argument for the primacy of Lagrangian mechanics.

This completes our primary argument for privileging Hamiltonian mechanics over Lagrangian mechanics. We have seen that Lagrangian systems are a proper subset of Hamiltonian systems,

specifically those Hamiltonian systems that also satisfy the motion equivalence condition. In the next section, we provide a geometric interpretation of the principle of stationary action within Hamiltonian mechanics. Our construction makes manifest the gauge-dependency of the Lagrangian, providing a further reason to privilege the Hamiltonian. Finally, we respond to Curiel’s arguments against the fundamentality of Hamiltonian mechanics.

5 Interpreting Lagrangian mechanics

By viewing classical systems through their invariant phase space density, we gain new insights into Lagrangian mechanics. Section 5.1 uses the invariant phase space density to give a geometric interpretation the action principle. We show that this principle holds even when the action is not defined as a function of position and velocity, since it can always be expressed as a function of position and conjugate momentum. Our construction makes the gauge-dependence of the Lagrangian manifest. Section 5.2 considers Curiel’s (2014) argument for privileging Lagrangian mechanics, which he alleges encodes the necessary and sufficient structure for capturing the physically significant kinematical constraints of classical systems. This raises a number of objections to our own position, which we rebut. In short, we argue that we have given a natural construction of Hamiltonian mechanics.

5.1 A geometric interpretation of the action principle

Hamilton’s principle of stationary action serves as a foundational principle for Lagrangian mechanics. According to this principle, infinitesimal variations of the action \mathcal{A} along the system’s trajectory γ equal zero: $\delta\mathcal{A}[\gamma] = 0$. Although central to Lagrangian mechanics, we can actually derive the stationary action principle using Hamiltonian mechanics. Our derivation simply assumes that the system’s trajectory follows a deterministic and reversible evolution. We present the derivation for the one-dimensional case—where it is most easily visualized—but it can be generalized using differential geometry.²⁷

27. For a related argument, see Arnold (1989, 236–37, 243–45).

The derivation uses the extended phase space (q, p, t) , i.e. the standard phase space extended by a time variable. Knowing that the system is deterministic and reversible, it follows from Section 4.2 that the density of states is invariant. This means that possible trajectories of the system in the extended phase space are never pushed toward each other or pulled apart over time. In other words, the flux of trajectories through any closed surface in the extended phase space is always zero. Formally, the vector field $\vec{S} := (\frac{dq}{dt}, \frac{dp}{dt}, \frac{dt}{dt})$ represents the displacement of states along the trajectories. This field is divergenceless, $\nabla \cdot \vec{S} = 0$, because for any region U , the same number of states will enter the region as exit. Moreover, note that the displacement vector \vec{S} is tangent to the trajectory $\gamma[q, p, t](t)$ in extended phase space, where the variable t functions as both a coordinate and the evolution parameter.

Since \vec{S} is divergenceless, there exists another vector field $-\vec{\theta}$ such that $\vec{S} = \text{curl}(-\vec{\theta})$.²⁸ As a vector field in the extended phase space, $\vec{\theta}$ can be written in terms of its components $\vec{\theta} = (\theta_q, \theta_p, \theta_t)$. Since the curl of the gradient of a scalar function is always zero, the displacement vector \vec{S} is invariant under the gauge transformation $\vec{\theta}' = \vec{\theta} + \text{grad}(f)$, where f is an arbitrary scalar function. Hence, we can choose f such that $\partial f / \partial p = -\theta_p$. Without loss of generality, we can thereby redefine $\vec{\theta}$ as $(\theta_q, 0, \theta_t)$.

Treating time t as both a coordinate and the evolution parameter, the time component of the displacement $S_t = dt/dt = 1$.²⁹ This places another constraint on $\vec{\theta}$, since $\vec{S} = -\text{curl}(\vec{\theta})$. In components, this means that $S_t = -(\frac{\partial}{\partial q} \cdot \theta_p - \frac{\partial}{\partial p} \cdot \theta_q) = -(\frac{\partial}{\partial q} \cdot 0 - \frac{\partial}{\partial p} \cdot \theta_q) = \frac{\partial \theta_q}{\partial p}$. Hence, $S_t = 1$ entails that $\frac{\partial \theta_q}{\partial p} = 1$. Integrating this expression, we find that $\theta_q = p$ up to addition by an arbitrary constant function $g(q)$. Again using the gauge freedom of \vec{S} , we can set $g(q) = 0$, so that $\theta_q = p$. Thus far, we have shown that $\vec{\theta} = (p, 0, \theta_t)$.

Next, we introduce a function H and let $\theta_t \equiv -H$. Hence, we write $-\vec{\theta} = (-p, 0, H)$ as the vector potential for the displacement vector \vec{S} . By matching components of \vec{S} and $-\text{curl}(\vec{\theta})$, we show that H is the Hamiltonian for the system. First, by definition, $S_q \equiv \frac{dq}{dt}$ and $S_p \equiv \frac{dp}{dt}$. Secondly, since $\vec{S} = -\text{curl}(\vec{\theta})$, we calculate that $S_q = -\frac{\partial \theta_t}{\partial p} = \frac{\partial H}{\partial p}$ and $S_p = \frac{\partial \theta_t}{\partial q} = -\frac{\partial H}{\partial q}$.

28. For reasons that will become clear, we choose $-\vec{\theta}$ rather than $\vec{\theta}$.

29. In the expression “ dt/dt ,” the time variable in the numerator functions as a coordinate, while the time variable in the denominator functions as a parameter. We could instead use a parameter τ , and the derivation would remain valid.

Matching terms, we see that $\frac{\partial H}{\partial p} = \frac{dq}{dt}$ and $-\frac{\partial H}{\partial q} = \frac{dp}{dt}$. These simply are Hamilton's equations, which justifies interpreting H as the Hamiltonian.

At this point, we return to the action \mathcal{A} . If the motion equivalence condition holds, then \mathcal{A} is defined as the integral of the Lagrangian over a trajectory beginning at time t_0 and ending at time t_1 : $\int_{t_0}^{t_1} L(q, \dot{q}, t) dt$. If motion equivalence does not hold, we can define the action more generally over the function $L_p \equiv p\dot{q} - H(q, p, t)$, considering paths in the extended phase space. By deriving the stationary action principle in this more general setting, we simultaneously derive its usual version within Lagrangian mechanics. We have then that $\mathcal{A}[\gamma] = \int_{t_0}^{t_1} L_p(q, p, t) dt = \int_{t_0}^{t_1} [p \frac{dq}{dt} - H(q, p, t)] dt$, where γ is an actual trajectory followed by the system. Multiplying the Hamiltonian by $dt/dt = 1$ and adding a vanishing $0 \cdot \frac{dp}{dt}$, we obtain $\mathcal{A}[\gamma] = \int_{t_0}^{t_1} [p \frac{dq}{dt} + 0 \cdot \frac{dp}{dt} - H(q, p, t) \frac{dt}{dt}] dt$. We notice that the integrand simply equals the inner product of $\vec{\theta}$ with \vec{S} , entailing that $L_p = \vec{\theta} \cdot \vec{S}$. Consequently, the action $\mathcal{A}[\gamma] = \int_{t_0}^{t_1} \vec{\theta} \cdot \vec{S} dt$. Note that $\vec{S} dt = (\frac{dq}{dt} dt, \frac{dp}{dt} dt, \frac{dt}{dt} dt) = (dq, dp, dt) = d\vec{\gamma}$ represents the infinitesimal displacement along the trajectory. Therefore, the action equals the line integral of $\vec{\theta}$ along the trajectory γ : $\mathcal{A}[\gamma] = \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma}$. To derive Hamilton's principle of stationary action, it suffices to show that $\delta \mathcal{A}[\gamma] = \delta \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma} = 0$.

By definition, a variation over the integral $\int_{\gamma} \vec{\theta} \cdot d\vec{\gamma}$ consists in taking the difference between the integrand and an integral over a nearby trajectory γ' : $\delta \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma} = \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma} - \int_{\gamma'} \vec{\theta} \cdot d\vec{\gamma}'$. Since the trajectories γ and γ' have the same endpoints, they define a closed-curve C , enclosing a region Σ . Hence, by Stokes' theorem $\delta \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma} = \oint_C \vec{\theta} \cdot d\vec{C} = \iint_{\Sigma} \text{curl}(\vec{\theta}) \cdot d\vec{\Sigma}$. Recalling that i) the $\text{curl}(\vec{\theta})$ equals the negative of the displacement vector \vec{S} , and ii) \vec{S} is tangent to the trajectory γ , we see that $\text{curl}(\vec{\theta})$ is tangent to the surface Σ .³⁰ Hence, the surface integral must vanish, showing that $\delta \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma}$ does indeed equal zero. This establishes a more general version of Hamilton's principle of stationary action, which we have derived simply by assuming that the displacement vector is divergenceless. In the special case that motion equivalence holds, the Legendre transformation is well-defined and $L_p = p\dot{q} - H(q, p, t)$ equals the usual Lagrangian $L(q, \dot{q}, t)$, thereby recovering the principle of stationary action for Lagrangian mechanics.

30. Since \vec{S} is tangent to γ , and γ' defines an infinitesimal variation, \vec{S} must be tangent to the entire surface Σ .

As a corollary, this construction shows that the Lagrangian $L = \vec{\theta} \cdot \vec{S}$ is manifestly gauge-dependent. The vector potential $\vec{\theta}$ is gauge-dependent because we can transform it by adding the gradient of an arbitrary scalar function. The displacement vector $\vec{S} = \text{curl}(-\vec{\theta})$ is invariant under this gauge transformation. Hence, L must be gauge-dependent, which complicates the physical interpretation of any particular Lagrangian. In contrast, the Hamiltonian is defined up to a constant, endowing it with greater physical significance.

5.2 Curiel's privileging of Lagrangian mechanics

Curiel argues that Lagrangian mechanics represents classical mechanical systems in a *natural* manner whereas Hamiltonian mechanics involves an essentially artificial construction (2014, 270).³¹ According to Curiel, all physically-meaningful classical systems satisfy a certain set of kinematic constraints. In the simplest case, these constraints characterize velocity as the first temporal derivative of position: $v^i = d_t x^i = v^i(q^j, p_k)$ (Equation 2).³² Curiel argues that Lagrangian mechanics encodes the necessary and sufficient conditions for a system to satisfy these kinematic constraints, both within and outside the hyper-regular domain (2014, 307–8, 311). In contrast, Hamiltonian mechanics does not automatically encode these constraints. Since Curiel views these constraints as an essential feature of physically-meaningful classical systems, he argues that Lagrangian mechanics picks out the proper class of classical systems.

Imposing these kinematic constraints within the Hamiltonian formalism requires focusing on a proper subset of all possible Hamiltonians. Within this subset, conjugate momentum is a linear function of velocity: $p_i = \lambda_i^j \dot{q}_j$ (2014, 304). This relationship between momentum and velocity entails that the Hamiltonian takes the following quadratic form—where the functions U and α^{mn} are arbitrary functions of the generalized position coordinates q_j :

$$H = \alpha^{mn} p_m p_n + U(q_j) \quad (8)$$

31. Curiel characterizes “natural” in terms of natural coordinate systems, defined using the almost-tangent structure of the tangent bundle (2014, 290–91). He also uses the almost-tangent structure to give a general characterization of what he takes to be the fundamental kinematical constraints.

32. Curiel treats this constraint as “kinematical” in the sense that it must hold independently of the interactions between a system and its environment. He interprets it as a substantive physical condition, rather than an analytic truth (2014, 282).

Curiel contends that this form is the “necessary and sufficient condition [a Hamiltonian must have] for it to model an abstract classical system” (2014, 305). Since many possible Hamiltonians do not take this form,³³ Curiel believes we must artificially restrict Hamiltonian mechanics to those systems that satisfy Equation 8.

To rebut Curiel’s argument, it suffices to provide a natural interpretation of (8). We have in fact already provided one in Section 4.3. There, we saw how both the linear relationship between momentum and velocity and the quadratic form (8) follow from imposing the motion equivalence condition. In Equation 5, we recovered the linear relationship between conjugate momentum and velocity, up to the addition of an arbitrary function. Just as Curiel does, we then used this relationship to derive a quadratic form for the Hamiltonian (Equation 7). Imposing motion equivalence thereby allows us to interpret this quadratic form as the Hamiltonian for massive particles under potential forces. It is not an *ad hoc* condition imposed on Hamiltonian systems. Instead, it arises in all contexts where motion equivalence holds, i.e. where knowledge of the trajectory of the system in physical space is necessary and sufficient for knowledge of its evolution in state space. Thus, we take ourselves to have responded to Curiel’s main criticism of Hamiltonian mechanics, namely that it must impose *ad hoc* the condition (8).

Furthermore, Curiel’s definition of “classical systems” leads to interpretive challenges that our framework avoids. His prohibition against Hamiltonians that violate (8) excludes Hamiltonians for non-hyper-regular systems. However, we believe that there are physically-meaningful non-regular Hamiltonian systems. These include classical models of a massless particle moving at constant speed, such as a classical photon. In general, Hamiltonian systems outside the hyper-regular domain correspond to cases where we cannot express the density in terms of position and velocity. These are cases where the weak equivalence condition (2) fails. Nevertheless, we can still define a distribution over the phase space points and consider its evolution. We thereby view the broader scope of Hamiltonian mechanics as an advantage, rather than a disadvantage. Moreover, our interpretation remains compatible with the claim that many—perhaps even most—non-regular Hamiltonian systems are physically uninteresting.

33. As an example, Curiel considers the Hamiltonian $H = \frac{1}{2}p_i p^i + \sum_j p_j$ (2014, 305).

Whereas we can easily interpret non-hyper-regular Hamiltonians, non-hyper-regular Lagrangians pose interpretative challenges. For example, consider the case where the Lagrangian equals zero. For this “system,” all paths have the same action, so all paths are possible. Given an initial probability distribution over states, the Lagrangian fails to specify how this distribution will evolve in time. In contrast, a Hamiltonian still defines a unique solution—a unique evolution in phase space—even when outside the hyper-regular domain, i.e. when the Hessian is zero. For instance, a zero Hamiltonian describes a static system (by Hamilton’s equations, the derivatives of the position and momenta are zero). Indeed, Curiel acknowledges that only hyper-regular Lagrangians have unique solutions whereas Hamiltonian systems have unique solutions even outside the hyper-regular domain (2014, 291, 308). Although Curiel argues this favors Lagrangian mechanics, it generates challenges for understanding non-hyper-regular Lagrangians.

Of course, if one insists on defining a “classical system” as Curiel does, one has grounds for privileging Lagrangian mechanics. Nevertheless, our preferred notion of classical systems—focusing on systems that are infinitesimally reducible—has substantive advantages over Curiel’s. The general geometrical understanding of the principle of stationary action illustrates one of these advantages (Section 5.1). Additionally, our definition accommodates models of classical massless particles by maintaining the uniqueness of solutions, even outside the hyper-regular domain. Moving beyond classical mechanics, our framework has the advantage of providing an elegant relationship with both classical statistical mechanics and quantum mechanics. First, the interpretation of the symplectic form is exactly the one used by the fundamental postulate of statistical mechanics, which uses phase space volumes—i.e. the volume form generated by the symplectic form—to count states. Second, by changing the requirement of infinitesimal reducibility to a corresponding notion of *irreducibility*, we recover quantum mechanical systems. Hence, our framework perspicuously distinguishes quantum from classical systems. We take these considerations to favor our preferred notion of a classical system.³⁴

34. For more details on the quantum and statistical mechanical cases, see Carcassi et al. (2018) and Carcassi and Aidala (2020). Additionally, one can show that Hamiltonian mechanics provides both the proper setting for the invariant density and a simple understanding of the gauge-dependency of conjugate momentum.

6 Conclusion

By focusing on three basic physical conditions, we have derived the mathematical structures that underpin Hamiltonian and Lagrangian mechanics. These three physical conditions are (i) infinitesimal reducibility, (ii) deterministic and reversible evolution, and (iii) motion equivalence. Each condition gives rise to corresponding mathematical structure, allowing us to interpret this structure in a physically well-motivated fashion. By building up the formalism in this way, we have seen that Lagrangian mechanics requires one more condition to hold than Hamiltonian mechanics, namely motion equivalence.

We have used this additional constraint to argue that Hamiltonian mechanics is more general—and hence more fundamental—than Lagrangian mechanics. Our argument for privileging Hamiltonian mechanics avoids both the technical problems facing North’s (2009) argument and her more substantive metaphysical commitments. Likewise, our construction provides a principled response to Curiel’s (2014) criticism that Hamiltonian mechanics requires an *ad hoc* condition. Although both frameworks adequately treat hyper-regular systems, there are compelling reasons to interpret Hamiltonian mechanics as more fundamental than Lagrangian mechanics. Our formalism emphasizes the invariant phase space density and the role of the symplectic form in counting states for each degree of freedom. Finally, we have provided a geometrical interpretation of the action principle that makes the gauge-dependency of the Lagrangian manifest. Overall, our approach provides a better understanding of not just the relationship between Hamiltonian and Lagrangian mechanics, but of classical systems in general.

Acknowledgements

This paper forms part of the ongoing *Assumptions of Physics* project (Carcassi and Aidala 2021). This project aims to both (i) isolate the physical assumptions required for different frameworks in mathematical physics and (ii) consider their philosophical implications. The authors would like to thank Gordon Belot for comments, along with two reviewers for this journal and four reviewers at two other journals, all of whom provided extremely helpful and construc-

tive comments. Funding for this work was provided in part by the MCubed program of the University of Michigan.

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