Modal Logic

Facts

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This text contains some basic facts about modal logic. For motivation, intuition and examples the reader should consult one of the standard textbooks in the field.

Contents

1 Expressive power

Many statements can be expressed in a meaningful way in propositional logic. Statements such as

1. John likes Mary and Susan ($L(John, Mary) \wedge L(John, Susan)$, where $L(x, y)$ is "x likes y"),

2. I do not like potatoes in summer". ($S \rightarrow \neg L(I, potatos)$, where S is "it is summer").

1.1 Predicate logic

Certain sentences do not have such a clear translation in the setting of propositional logic. A sentence like "John loves all women", cannot be expressed in propositional logic in a meaningful way: the only possibility is p, where p then denotes the statement "John likes all women", but this does not reflect any of the structure of the statement. To be able to capture the meaning of this kind of sentences one can improve the expressive power of the language by adding universal ($\forall$) and existentional ($\exists$) quantifiers which respectively denote "for all ..." and "there exists ...". In this setting "John likes all women" becomes $\forall x\big(W(x) \rightarrow L(John, x)\big)$, where $W(x)$ is "x is a woman". This logic is called *predicate logic*.

1.2 Propositional modal logic

It is not difficult to see that there are many other structures in sentences that still cannot be expressed in predicate logic. Sentences of the form "when it rains it is necessary I take a cab", or "there is the possibility that I graduate before I am 25". To capture the structure of these sentences one can extend propositional logic in a different way by adding two *modal operators*: $\Box$ and $\Diamond$. $\Box\varphi$ means "it is necessary that φ", and $\Diamond\varphi$ means "it is possible that φ". Thus the two sentences above become $R \rightarrow \Box C(I)$, where R is "it rains" and $C(x)$ "x takes a cab", and $\Diamond(G(I) \wedge T(I))$, where $G(x)$ is "x graduates" and $T(x)$ is "x is younger than 25".

Of course, to increase the expressive power of the system even more the modality operators could be added to predicate instead of propositional logic. These systems become considerably more complex than propositional modal logic and fall outside the scope of this course.

Important There are various names for the modal operators, $\Box$ is often denoted by K and $\Diamond$ by $\hat{K}$.

1.3 Brief overview of propositional logic

This section contains a brief summary of the necessary definitions of propositional logic needed to follow the rest of the exposition. Readers familiar with the subject will find nothing new here and can proceed with the next section.

Propositional logic is the logic of propositional formulas. Propositional formulas are build up in the usual way from propositional variables, also called atoms and often denoted by $p, q, p_1, p_2, \ldots$, the atoms $\bot$ (false) and $\top$ (true), and the connectives $\wedge$, $\vee$, $\to$ and $\neg$, that is, conjunction (and), disjunction (or), implication (if ..., then ...) and negation (not). Thus p is a formula, and so is $\neg p \wedge ((q \to p) \vee \neg r)$, but pp and $p \wedge q \to$ are not.

A *valuation* for a formula φ is a map from the propositional variables in φ to $\{0, 1\}$. A formula is *true* or satisfiable under a valuation v if, when we assign to atoms p the values $v(p)$ the formula becomes true. Here the evaluation of formulas in which all the atoms are replaced by 0's or 1's is as expected: 0 stands for false, and 1 stands for true, and

$$0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0 \quad 1 \wedge 1 = 1,$$

$$1 \vee 1 = 0 \vee 1 = 1 \vee 0 = 1 \quad 0 \vee 0 = 0,$$

$$(0 \to 0) = (0 \to 1) = (1 \to 1) = 1 \quad (1 \to 0) = 0,$$

$$\neg 0 = 1 \quad \neg 1 = 0.$$

A formula is *satisfiable* when there is at least one valuation under which the formula is true. It is a *tautology* when it is true under all valuations. It is *inconsistent* when it is not satisfiable.

Example 1 1. p is a satisfiable formula (take $v(p) = 1$).

2. $(p \to q) \wedge \neg p$ is a satisfiable formula (take $v(p) = 0$ and $v(q)$ is 0 or $v(q) = 1$).

3. $p \wedge \neg p$ is inconsistent.

4. $\neg\neg p \to p$ is a tautology.

5. $p \vee \neg p$ is a tautology.

2 Modal logic

Keeping the intuitive meaning of $\Box$ and $\Diamond$ in mind it is not difficult to write down some principles that we wish our modal operators to satisfy. Principles like

$$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi \qquad \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$$

and

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi),$$

and

$$\Box(p \vee \neg p) \qquad \neg\Diamond(p \wedge \neg p).$$

Also, it seems reasonable to say that whenever a propositional formula φ is a tautology, that then $\Box\varphi$. These considerations have lead to the formulation of the basic modal logic K. It consists, more of less, of the most obviously true principles about $\Box$ and $\Diamond$. Thus it is thereby the logic of necessity and possibility.

2.1 Kripke models

A *Kripke frame* is a pair (W, R) where W is a non-empty set and R is a relation on W, i.e. $R \subseteq W \times W$. A *Kripke model* is a triple (W, R, V), where (W, R) is a frame and V is a *valuation* that assigns sets of worlds to propositional variables, i.e. $V : \mathcal{P} \to P(W)$, where $\mathcal{P}$ is the set of propositional variables. $V(p)$ is interpretetd as the set of worlds where p is true. Of course, when we are only interested in formulas in, say, the variables $\{p_1 \ldots, p_n\}$ we do not have to define V for all variables, but only for the p_i. Frames are often denoted by F, models by M. The elements of W are often called *nodes* or *worlds*. R is called the *accessibility relation* of the frame or the model. The word Kripke is often omitted.

Modal formulas are evaluated at worlds in a model. We define what it is for a formula φ to be valid at a world w in a model M, denoted by $M, w \models \varphi$.

$$
\begin{array}{lcl}
M, w \models p & \Leftrightarrow & w \in V(p) \quad \text{(for propositional variables } p\text{)} \\
M, w \models \varphi \wedge \psi & \Leftrightarrow & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \varphi \vee \psi & \Leftrightarrow & M, w \models \varphi \text{ or } M, w \models \psi \\
M, w \models \varphi \to \psi & \Leftrightarrow & M, w \models \varphi \text{ implies } M, w \models \psi \\
M, w \models \neg\varphi & \Leftrightarrow & M, w \not\models \varphi \\
M, w \models \Box\varphi & \Leftrightarrow & \forall v(wRv \text{ implies } M, v \models \varphi) \\
M, w \models \Diamond\varphi & \Leftrightarrow & \exists v(wRv \text{ and } M, v \models \varphi).
\end{array}
$$

If M is clear from the context we omit it and write $w \models \varphi$. If $M, w \models \varphi$, we say that *w forces φ in M* or *that φ is valid at w in M* or *that φ holds at w in M*. Sometimes $M, w \models \varphi$ is denoted by $w \models_M \varphi$.

The definition of truth at a node in a model as given above naturally gives rise to three more global notions of truth. A formula φ is *valid in a model M*, denoted $M \models \varphi$, if for all worlds w in M, $M, w \models \varphi$. φ is *valid on a frame F*, denoted $F \models \varphi$, if for all valuations V on F, for all worlds w in F, $M, w \models \varphi$, where M is the model with frame F and valuation V, i.e. $M = (F, V)$. φ is *valid*, $\models \varphi$, if it holds on all frames. Thus we have three levels of valuation

$$
\begin{aligned}
M, w &\models \ldots \\
M &\models \ldots \\
F &\models \ldots \\
&\models \ldots
\end{aligned}
$$

Example 2 For models M, frames F and worlds w in it, the following holds.

- $M, w \models \varphi \vee \neg\varphi$, $M \models \varphi \vee \neg\varphi$, $F \models \varphi \vee \neg\varphi$, $\models \varphi \vee \neg\varphi$,
- $\models \varphi$ for all propositional tautologies φ,
- $F \models \Diamond\varphi \to \Diamond\top$,
- $F \models (\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$,
- $F \models (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$,
- $F \models \Box(\varphi \to \psi) \wedge \Box\varphi \to \Box\psi$,
- $(W, R, V) \models \varphi$ implies $(W, R, V), w \models \varphi$, for all worlds $w \in W$ and all formulas φ.
- $(W, R) \models \varphi$ implies $(W, R, V) \models \varphi$, for all valuations V on (W, R) and all formulas φ.

Given a model $M = (W, R, V)$ we write wR^*v if there exists nodes $u_1 \dots u_n$ such that $w = u_1$ and $v = u_n$ and $u_i R u_{i+1}$. Given a node w in M, M_w denotes the model which set of nodes is $\{v \in W \mid wR^*v\}$, and which relation and valuation are the restriction of R and V to this set.

Lemma 1 For all models M, all nodes w in M and all nodes v in M_w:

$$M, v \models \varphi \;\Leftrightarrow\; M_w, v \models \varphi.$$

Proof You will be asked to prove this in the exercises. ♡

2.2 Conventions

If R is the relation of the Kripke model we draw wRv as

$$w \longrightarrow v$$

We write the atoms that are forced at a node in brackets beside it. Thus in

$$(p, q)\ w \longrightarrow v\ (p)$$

$w \models p$, $w \models q$ and $v \models p$, and e.g. $v \models \neg q$.
In formulas $\Box$ and $\neg$ bind stronger than $\vee$ and $\wedge$, which bind stronger than $\to$. E.g. $p \wedge q \to r$ is short for $(p \wedge q) \to r$, and $\Box p \wedge q$ is short for $(\Box p) \wedge q$.

3 Interpretations of the modal operators

There are many interpretations of the modal operators $\Box$ and $\Diamond$. The particular interpretation we have in mind determines the principles (formulas) we are wishing to except, and the restrictions we wish to impose on the Kripke models. Here follow three examples.

3.1 Epistemic logic

In epistemic logic $w \models \Box\varphi$ is interpreted as (or is the formalization of) "being in world/state w I know φ". Thus the worlds are viewed as states of knowledge. Hence $w \models \Diamond\varphi$ means "being in w I consider it possible that φ holds", or "there is a knowledge state consistent with my knowledge in w where φ holds". We discuss some of the modal formulas that should hold in this context, and the properties the accessibility relation should satisfy.

Axioms Assuming that I can only know true things, the principle $\Box\varphi \to \varphi$ should be valid in this context (if I know φ, then φ is true). And so should $\Box\varphi \to \Box\Box\varphi$ be (if I know φ, then I know that I know φ). On the other hand, a principle like $\varphi \to \Box\varphi$ we would not be willing to accept (if φ is true, then I know φ).

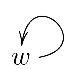
Models What is the corresponding meaning of the accessibility relation R in the Kripke models when interpreting the modal operators in this way? Here wRv should mean that v is a world that is consistent with the knowledge I have in w. In a picture,

$$w \longrightarrow v \quad \text{“}v\text{ is consistent with my knowledge in }w\text{”}$$

And indeed, by the definition of $\models \varphi$, we have

$$w \models \Box\varphi \Leftrightarrow \forall v(wRv \to v \models \varphi).$$

In words: it can never be the case that in v something (φ) holds which I know not to be the case ($\Box\neg\varphi$) when in w. What kind of properties should R have in this setting? For example, R should be reflexive, wRw, because the world w is consistent with the knowledge I have in w. Thus for all worlds w:



3.2 Tense logic

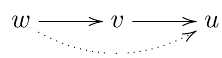
In tense logic $t \models \Box\varphi$ is interpreted as "from t on it is always going to be the case that φ", meaning that from time t on, φ will always hold. If one would wish to include point t in "from t on", then $t \models \Diamond\varphi$ means "there is a point in time, later than t or t itself, where φ holds". Some of the modal formulas that should hold and some of the properties that the relations in the frames should satisfy are the following.

Axioms Again $\Box\varphi \to \Box\Box\varphi$ seems naturally true in this setting. If you consider time as infinite, then $\Diamond\top$ should be valid: there is always a point later in time, at which $\top$ holds. If one wishes to include t in "from t on", then $\Box\varphi \to \varphi$ should hold too.

Models In this setting wRv should mean that v is a point later in time than w:

$$w \longrightarrow v \quad \text{“}v\text{ is a point later in time than }w\text{”}$$

In this setting R should be transitive: if wRv and vRu, then wRu:



Namely, if v is later than w and u later than v, then u is later than w.
If time is considered to be infinite, the Kripke models should look like

$$w_1 \longrightarrow w_2 \longrightarrow w_3 \longrightarrow \cdots$$

As mentioned above, under this interpretation, it is natural to require that R is transitive. But we should e.g. also require that for every w there is a v such that wRv and $w \neq v$: at every point in time there is a point later in time. Observe that this property of the accessibility relation is not one that one should require in the example above.

3.3 Agent logic

Here we interpret $w \models \Box\varphi$ as "all people w knows think that φ holds". Here the worlds can be viewed as persons (agents) and the formulas valid at a node (agent) w represent the things w thinks true. Thus $w \models \Diamond\varphi$ means "w knows a person that thinks φ is true".

Axioms Again, the formula $\Box\varphi \to \varphi$ should be valid, since all people know themselves: if all people w knows think φ is true ($w \models \Box\varphi$), then in particular w thinks φ is true ($w \models \varphi$).

On the other hand, the principle $\Box\varphi \to \Box\Box\varphi$ seems not plausible. If John knows Mary, and all people that John knows think φ is true (John$\models \Box\varphi$), then this does not imply that all acquaintances of Mary do so (Mary$\models \Box\varphi$), since Mary may know people that John does not know. Thus $\Box\varphi$ might not be valid for Mary. Hence $\Box\Box\varphi$ might be false for John.

Models What is the corresponding meaning of the accessibility relation R in this setting? wRv corresponds to person w knowing person v:

$$w \longrightarrow v \quad \text{"}w\text{ knows }v\text{"}$$

Under this interpretation we again should require that R is reflexive, as above. However, transitivity is not likely. The example above shows this: it might be the case that John knows Mary and Mary knows George, but John does not know George:

$$\text{John} \longrightarrow \text{Mary} \longrightarrow \text{George}$$

and whence there is no arrow from John to George.

4 Basic modal logic

All examples above share a collection of principles that holds for all of them. This is the *basic modal logic* K given by the following logic and rules:

Axioms Tautologies of propositional logic

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

Rules

$$\frac{\varphi \to \psi \qquad \varphi}{\psi} \text{ Modus Ponens}$$

$$\frac{\varphi}{\Box\varphi} \text{ Necessitation}$$

Axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is named after the logic and called the K-axiom. We say that φ is *derivable in K* and write $\vdash_{\mathsf{K}} \varphi$ if there is a derivation of φ in K.

The logic K is called the basic modal logic because it is the logic of all Kripke frames, see Theorem 6. The following theorem is a first step in that direction. It states that all formulas that are derivable in K are true on all frames.

Theorem 1 (Soundness theorem)

$$\vdash_{\mathsf{K}} \varphi \;\Rightarrow\; \varphi \text{ holds on all frames.}$$

Proof You will be asked to prove this in the exercises. ♡

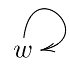
5 Frame properties

In Section 3 we saw how the interpretation of the modal operators determines the formulas which the operators should satisfy. Also, it naturally induces restrictions on the Kripke models. Note that in the examples above all these are restrictions on the asseccibility relation of the models. Thus they are independent of the paricular valuation of the model. Such properties are called *frame properties.*

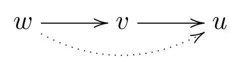
The relation between formulas and frame properties is a tight one, which is one of the reasons for the success of modal logic. The connection will be discussed in detail below. This section contains the definitions of certain frame properties that play an important role in modal logic. We first list the name, then the description of the property by a formula, and then the corresponding picture.

Definition 1

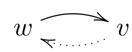
reflexive $\forall w(wRw)$



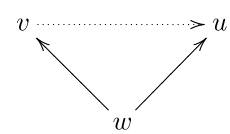
transitive $\forall w\forall v\forall u(wRvRu \rightarrow wRu)$



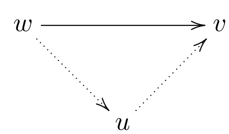
symmetric $\forall w\forall v(wRv \rightarrow vRw)$



euclidean $\forall w\forall v\forall u(wRv \wedge wRu \rightarrow vRu)$



dense $\forall w\forall v(wRv \rightarrow \exists u(wRuRv))$



The following properties are a bit harder to draw. Therefore only their description in terms of formulas is given.

Definition 2

irreflexive	$\forall x\neg(xRx)$
asymmetric	$\forall x\forall y(x \neq y \wedge xRy \rightarrow \neg yRx)$
antisymmetric	$\forall x\forall y(xRy \wedge yRx \rightarrow x = y)$
weakly ordered	$\forall x\forall y(xRy \vee yRx \vee x = y)$
partial order	reflexive, transitive and antisymmetric
equivalence relation	reflexive, transitive and symmetric
serial	$\forall x\exists y(xRy)$
completely disconnected	$\forall x\forall y\neg(xRy)$
well-founded	there is no infinite chain $\ldots x_3Rx_2Rx_1$

A frame $F = (W, R)$ is called *reflexive* if its accessibility relation R is reflexive, and similarly for the other properties. If wRv, then v is called a *successor* of w and w a *predeccessor* of v. We write $wRvRu$ for $(wRv \wedge vRu)$. Observe that irreflexive and antisymmetric are not the same as the properties of being not reflexive or not symmetric.

6 Important modal logics

There exist modal logics other than K that, like K, correspond to classes of frames. The following four logics are famous examples of such correspondences. They are extensions of K by the following axioms(s):

T	$\Box\varphi \to \varphi$
$\mathsf{K4}$	$\Box\varphi \to \Box\Box\varphi$
$\mathsf{S4}$	$\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$
$\mathsf{S5}$	$\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$ and $\Diamond\Box\varphi \to \varphi$.

Thus $\mathsf{S4}$ is the systems T and $\mathsf{K4}$ taken together, and $\mathsf{S5}$ is $\mathsf{S4}$ plus the axiom $\Diamond\Box\varphi \to \varphi$.
There exist beautiful connections between these logics and properties on frames, as will be explained in the next section. We first state the soundness theorems for these logics, which foreshadow the correspondence results.

Theorem 2 (Soundness theorem)

$\vdash_{\mathsf{T}} \varphi$	$\Rightarrow$	φ holds on all reflexive frames.
$\vdash_{\mathsf{K4}} \varphi$	$\Rightarrow$	φ holds on all transitive frames.
$\vdash_{\mathsf{S4}} \varphi$	$\Rightarrow$	φ holds on all reflexive and transitive frames.
$\vdash_{\mathsf{S5}} \varphi$	$\Rightarrow$	φ holds on all frames for which the relation is an equivalence relation.

Proof You will be asked to prove this in the exercises. ♡

7 Axioms and frame properties

Many frame properties are closely related to modal formulas. The possible interpretations of the operators as discussed above seem to imply such a correspondence. In this section this connection is made explicit via the *correspondence theorems.*

Theorem 3 (Correspondence theorem for reflexive frames)
For all frames F:

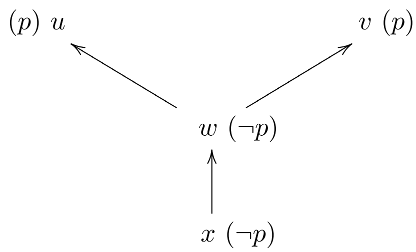
$$\forall\varphi\,(F \models \Box\varphi \to \varphi) \text{ if and only if } F \text{ is reflexive.} \tag{1}$$

Proof $\Leftarrow$: Suppose $F = (W, R)$ is reflexive. We have to show that $F \models \Box\varphi \to \varphi$, that is, that for all formulas φ, for all valuations V, for all $w \in W$, $w \models \Box\varphi \to \varphi$ in the model (W, R, V). Thus consider an arbitrary formula φ, an arbitrary valuation V and an arbitrary world w in W. Since R is reflexive wRw has to hold. Now suppose $w \models \Box\varphi$. This means that for all v, if wRv, then $v \models \varphi$. Since wRw, this implies that $w \models \varphi$. This proves that $w \models \Box\varphi \to \varphi$, and we are done.

$\Rightarrow$: This direction we show by contraposition. Thus we assume $F = (W, R)$ is not reflexive, and then show that $F \not\models \Box\varphi \to \varphi$ for some formula φ. In other words, we have to show that if F is not reflexive, then there is a formula φ and a valuation V and a world w in W such that $w \not\models \Box\varphi \to \varphi$ in the model (W, R, V). Note that $w \not\models \Box\varphi \to \varphi$ is the same as $w \models \Box\varphi \wedge \neg\varphi$. Thus suppose F is not reflexive. Then there is at least one world w such that not wRw. Now define the valuation V on F as follows. For all worlds v:

$$v \in V(p) \;\Leftrightarrow\; wRv.$$

Observe that in this definition the v are arbitrary, but w is the particular world such that not wRw that we fixed above. The definition implies that $v \models p$ if wRv, and for all other nodes x in W we put $x \not\models p$, i.e. $x \models \neg p$. E.g. as in this model:



Since not wRw, we have $w \models \neg p$. But the definition of V implies that all successors v of w, i.e. all nodes such that wRv, have $v \models p$. Thus $w \models \Box p$. Hence $w \models \Box p \wedge \neg p$. Hence $w \not\models \Box p \to p$. And thus there is a formula φ, namely the formula p, such that $w \not\models \Box\varphi \to \varphi$. This proves (1). ♡

Theorem 4 (Correspondence theorem for transitive frames)
For all frames F:

$$\forall\varphi\,(F \models \Box\varphi \to \Box\Box\varphi) \text{ if and only if } F \text{ is transitive.} \tag{2}$$

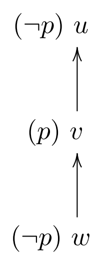
Proof $\Leftarrow$: Suppose $F = (W, R)$ is transitive. We have to show that $F \models \Box\varphi \to \Box\Box\varphi$, that is, that for all formulas φ, for all valuations V, for all $w \in W$, $w \models \Box\varphi \to \Box\Box\varphi$ in the model (W, R, V). Thus consider an arbitrary formula φ, an arbitrary valuation V and an arbitrary world w in W. Now suppose $w \models \Box\varphi$. We have to show that $w \models \Box\Box\varphi$, i.e. for all v such that wRv, $v \models \Box\varphi$. Thus consider a v such that wRv. To show $v \models \Box\varphi$, we have to show that for all u

with vRu, $u \models \varphi$. Thus consider a vRu. The transitivity of R now implies that wRu. Since $w \models \Box\varphi$, this means that all successors of w force φ. Since wRu, u is a successor of w. Hence $u \models \varphi$. Thus we have shown that for all u with vRu, $u \models \varphi$. Hence $v \models \Box\varphi$. And that is what we had to show, as it proves that $w \models \Box\varphi \to \Box\Box\varphi$.

$\Rightarrow$: This direction we show by contraposition. Thus we assume $F = (W, R)$ is not transitive, and then show that $F \not\models \Box\varphi \to \Box\Box\varphi$ for some φ. In other words, we have to show that if F is not transitive, then there is a formula φ and a valuation V and a world w in W such that $w \not\models \Box\varphi \to \Box\Box\varphi$ in the model (W, R, V). Note that $w \not\models \Box\varphi \to \Box\Box\varphi$ is the same as $w \models \Box\varphi \wedge \neg\Box\Box\varphi$. Thus suppose F is not transitive. Then there are at least three worlds w, v and u such that wRv and vRu and not wRu. Now define the valuation V on F as follows:

$$x \in V(p) \;\Leftrightarrow\; wRx.$$

Thus, we put $v \models p$ if wRv, and for all other nodes u in W we put $u \not\models p$, i.e. $u \models \neg p$. E.g. as in this model:



Since not wRu, we have $u \models \neg p$. This implies that $v \models \neg\Box p$. But this again implies that $w \models \neg\Box\Box p$. But the definition of V implies that all successors v of w, i.e. all nodes such that wRv, have $v \models p$. Thus $w \models \Box p$. Hence $w \models \Box p \wedge \neg\Box\Box p$. Thus $w \not\models \Box p \to \Box\Box p$. Thus there is a formula φ, namely p, such that $w \not\models \Box\varphi \to \Box\Box\varphi$. This proves (2). ♡

In a similar way one can prove several other correspondences between formulas and frame properties. You will be asked to prove the following correspondence theorems in the exercises.

Theorem 5 (Correspondence theorems)

$$F \models \Box\bot \text{ if and only if } F \text{ is completely disconnected.}$$

$$F \models \Diamond\top \text{ if and only if } F \text{ is serial.}$$

$$\forall\varphi\,(F \models \Diamond\Box\varphi \to \varphi) \text{ if and only if } F \text{ is symmetric.}$$

$$\forall\varphi\,(F \models \Box(\Box\varphi \to \varphi) \to \Box\varphi) \text{ if and only if } F \text{ is transitive and well-founded.}$$

8 Soundness and completeness

Soundness and completeness theorems link the syntax and semantics of modal logics, by providing a correspondence between derivability ($\vdash$) and validity ($\models$). The general outline of the proof of this main theorem in modal logic will be given below, in the section on canonical models.

Theorem 6 (Completeness theorem)

$\vdash_{\mathsf{K}} \varphi$	$\Leftrightarrow$	φ holds on all frames.
$\vdash_{\mathsf{T}} \varphi$	$\Leftrightarrow$	φ holds on all reflexive frames.
$\vdash_{\mathsf{K4}} \varphi$	$\Leftrightarrow$	φ holds on all transitive frames.
$\vdash_{\mathsf{S4}} \varphi$	$\Leftrightarrow$	φ holds on all reflexive and transitive frames.
$\vdash_{\mathsf{S5}} \varphi$	$\Leftrightarrow$	φ holds on all frames for which the relation is an equivalence relation.

Thus for these logics derivability is connected to a frame property in an elegant way. Because of the correspondence theorems we also know that these classes of frames can be characterized by one single formula, e.g. $\Box\varphi \to \varphi$ in case of the relexive frames, the formula that is the characteristic axiom of T.

8.1 The canonical model

Every modal logic has one special model that is in some sense as general as possible. It is close to the syntax of the logic because its worlds are sets of formulas. This model is called the *canonical model*. Its importance stems from the fact that from the existence of such a model one can in some cases easily prove the completeness of the logic in question. We will do so at the end of this section. We will consider the canonical model in detail for the logic K and later comment on its construction for other modal logics. Some definitions first.

Definition 3 A set of formulas is K-*consistent* if one cannot derive a contradiction from it, i.e. if it cannot derive $\phi \wedge \neg\phi$ in K for any ϕ. It is called *maximal* K-*consistent* if it is K-consistent and for every formula ϕ, either ϕ belongs to the set or $\neg\phi$ does. For other logics we define similar notions. E.g. T-consistent is defined as K-consistent but reading T for K: a set of formulas is T-*consistent* if one cannot derive a contradiction from it in T, i.e. if it cannot derive $\phi \wedge \neg\phi$ in T for any ϕ.

We will mainly work with K in this section, therefore the K-part is often omitted, so consistent means K-consistent, etc. A simple but important observation:

Proposition 1 If a set of formulas has a model, then it is consistent.

Proof For if not, it would derive $\phi \wedge \neg\phi$ for some ϕ. But then $\phi \wedge \neg\phi$ should hold in the model, which cannot be. ♡

Because of this, the set $\{p, \Box q\}$ clearly is consistent, as there are models in which both the formulas hold. The same argument applies to the set

$$\{p, \neg\Box p, \Box\Box p, \neg\Box\Box\Box p, \Box\Box\Box\Box p, \dots\}.$$

Obviously, the set $\{\phi, \neg\phi\}$ is not consistent, as it derives $\phi \wedge \neg\phi$. Also the set $\{\Box(\phi \to \psi), \Box(\top \to \phi), \Diamond\neg\psi\}$ is inconsistent, since $\Box\psi \wedge \neg\Box\psi$ follows from it (you will be asked to show all this in the exercises).

The set $\{p, \Box q\}$ is not maximal consistent since neither q nor $\neg q$ belongs to the set (and so do many other formulas). Examples of maximal consistent sets are a bit harder to desribe. The typical example is the following. Given a node w in a model, the set of formulas $L = \{\varphi \mid w \models \varphi\}$ is a maximal consistent set. That it is consistent is clear, as it has a model. That it is also maximal in this respect follows from the fact that for any formula ϕ, either $w \models \phi$ or $w \models \neg\phi$, and thus either $\phi \in L$ or $\neg\phi \in L$. Thus we see that nodes in a model naturally correspond to maximal consistent sets of formulas. This is the guiding idea behind the canonical model.

One more observation on the correspondence between nodes and maximal consistent sets of formulas. Given that wRv holds in a model, then for the sets

$$L_w = \{\phi \mid w \models \phi\} \qquad L_v = \{\phi \mid v \models \phi\},$$

it holds that $\Box\phi \in L_w$ implies $\phi \in L_v$, for all formulas ϕ. This immediately follows from the definition of forcing, and you will be asked to prove it in the exercises.

We are ready for the definition of a canonical model.

Definition 4 The K-*canonical model* is the Kripke model $M_{\mathsf{K}} = (W_{\mathsf{K}}, R_{\mathsf{K}}, V_{\mathsf{K}})$, where

1. $W_{\mathsf{K}} = \{\Gamma \mid \Gamma \text{ is a maximal } \mathsf{K}\text{-consistent set of formulas}\}$,
2. $\Gamma R_{\mathsf{K}} \Pi \;\Leftrightarrow\; \forall\phi\,(\Box\phi \in \Gamma \Rightarrow \phi \in \Pi)$,
3. $\Gamma \in V(p) \;\Leftrightarrow\; p \in \Gamma$.

Thus the canonical model consists of all maximal consistent sets, with arrows between them at the appropriate places (think of the remark on L_w and L_v above). As explained above, for every world w in a model, the set $\{\varphi \mid w \models \varphi\}$ is maximal K-consistent. Thus one could view the canonical model as containing all possible Kripke models together, and putting arrows between two sets $\{\varphi \mid w \models \varphi\}$ and $\{\varphi \mid v \models \varphi\}$ if for all $\Box\psi \in \{\varphi \mid w \models \varphi\}$ we have $\psi \in \{\varphi \mid v \models \varphi\}$.

Lemma 2 (Valuation lemma) For any maximal K-consistent set of formulas Γ (that is, for any node in the canonical model), for any formula φ:

$$M_{\mathsf{K}}, \Gamma \models \varphi \;\Leftrightarrow\; \varphi \in \Gamma.$$

(Note that here $M_{\mathsf{K}}, \Gamma \models \varphi$ means that Γ forces φ in the canonical model.)

Proof You will be asked to prove this in the exercises. ♡

Now we are ready to prove the completeness theorem. We only treat the case K, as the arguments for the other logics are more or less similar.

Theorem 7 $\vdash_{\mathsf{K}} \varphi \;\Leftrightarrow\; \models \varphi$ (φ holds on all frames).

Proof $\Rightarrow$: this is the soundness theorem, Theorem 1. You will be asked to prove this in the exercises.

$\Leftarrow$: this direction we prove by contraposition, showing that $\not\vdash_{\mathsf{K}} \varphi$ implies $\not\models \varphi$. If $\not\vdash_{\mathsf{K}} \varphi$, there is a maximal consistent set Γ containing $\neg\varphi$. We do not prove this here, but the argument is not difficult. We only remark in passing that if $\vdash_{\mathsf{K}} \varphi$, then there is no maximal consistent set containing $\neg\varphi$, as then the set would contain both φ and $\neg\varphi$ and whence be inconsistent.

To return to Γ, by the definition of canonical model it is a node in this model. By the Valuation lemma we have that $\Gamma \models \neg\varphi \;\Leftrightarrow\; \neg\varphi \in \Gamma$. And thus $\Gamma \models \neg\varphi$, since $\neg\varphi \in \Gamma$. Hence there is a Kripke model, namely M_{K}, and a node in it, namely Γ, that forces $\neg\varphi$. Therefore, $\not\models \varphi$, and that is what we had to show. ♡

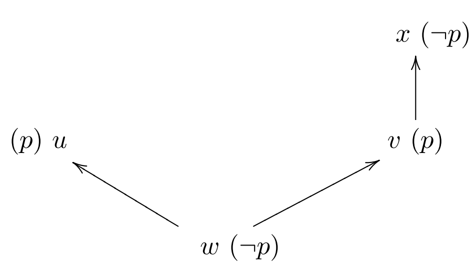
As remarked above, the proofs of Theorem 6 for the other logics follow the same pattern as the proof for K given above.

8.2 Small models

In view of the completeness theorem, to establish e.g. $\vdash_{\mathsf{K}} \varphi$ it suffices to show that φ holds on all frames. And to establish that $\not\vdash_{\mathsf{K}} \varphi$, it suffices to show that there is a frame F that refutes φ, i.e. such that $F \not\models \varphi$. Given that there are infinitely many frames, this might not be an easy task. However, we can restrict the frames that we have to consider in such a way that in order to check whether there is a frame that refutes φ, we only have to check a finite number of finite frames, which implies the decidability of the logic. This is the content of this section. We will see that the number of frames only depends on the size of the formula φ.

Intuitively, we establish "how far up" we have to inspect the frame in order to establish whether a certain node forces a formula. It turns out that the number of boxes decides this. First, consider the following example.

Example 3

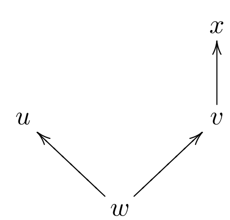


To see that $w \models \Box p$ it suffices to check for v and u whether they force p. In other words, the truth of $w \models \Box p$ does only depend on the forcing relation at the successors of w and not on the node x, which is not a successor of w. If x would force p, this would not change the truth of $w \models \Box p$, whereas a change in the forcing of u or v could. On the other hand, for a formula with two boxes, like $\Box\Box p$, whether $w \models \Box\Box p$ holds (it does not) depends on the valuation of p in x.

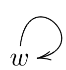
Before we continue we need a definition.

Definition 5 The depth of a frame F is the maximum length of a path from a root of the frame (a lowest node, a node that is no successor of another node) to the top. Formally: the depth of a frame F is the maximum number n for which there exists a chain $w_1 R w_2 R \ldots R w_n R w_{n+1}$ in the frame, where all w_i are distinct. Clearly, frames can have infinite depth.
The depth of a node v from a node w is the length of the shortest path from w to v. v is of depth 0 from w when it is equal to w or when it cannot be reached from w by travelling along the arrows.
Let $|\varphi|$ be the size of φ, i.e. the number of symbols in it, and let $b(\phi)$ denote the maximal nesting of boxes in ϕ. The size of a frame is the number of nodes in it.

Example 4 This frame has depth 2:



The node x has depth 2 from w and depth 1 from v and depth 0 from x and from u. And this frame has depth 0:

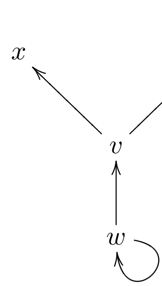


In this frame there are no nodes with depth > 0.
The maximal nesting of boxes in $(\Box\Box p \wedge \Box q)$ is 2, and in $\Box(\Box p \rightarrow \Box(\Box p \wedge q))$ it is 3 (coming from the box in front of p, and the box in front of the conjunction, and finally the box in front of the implication). Note that the nesting of boxes in $\Box\Diamond p$ is 2, not 1.

Returning to the first example, it seems to suggest that to evaluate a formula ϕ in a node w in a model M, we have to consider only the nodes in M that are of depth $\leq b(\phi)$ from w. Here follow two more examples to support this claim.
First we consider the case that the number of boxes in a formula ϕ is 0, i.e. $b(\phi) = 0$. This means that the formula does not contain boxes. Considering

the definition of $w \models \phi$, it is not difficult to see that to establish $w \models \phi$ for a formula without boxes, one only has to know which propositional variables are forced in w and which are not. Thus the truth of ϕ at w is indepedent of the model outside w.

In the following example,



the truth of $w \models \Box p$ does not depend x or y. In other words, $w \models \Box p$ holds if and only if $w \models p$ and $v \models p$, no matter whether x or y force p or not. However, the truth of $v \models \Box p$ depends on the forcing at x, since $v \models \Box p$ if and only if $x \models p$ and $y \models p$. On the other hand, to verify whether $w \models \Box p \to \Box\Box q$ all the nodes w, v, x, and y have to be taken into account.

This intuition is captured by the following theorem.

Theorem 8 (Finite depth theorem) For all numbers n, for all models M and all nodes w in M there exists a model N of depth n with root w' such that for all φ with $b(\varphi) \leq n$:

$$M, w \models \varphi \;\Leftrightarrow\; N, w' \models \varphi.$$

Proof We do not formally prove this statement, but only sketch the idea. Given a model M with node w, consider M_w. By Lemma 1 we have for all formulas φ that for all v in M_w,

$$M, v \models \varphi \;\Leftrightarrow\; M_w, v \models \varphi,$$

but this does not prove the lemma as M_w may still have depth $> n$. Therefore, in M_w we cut out all nodes that have depth $> n$ from w and call this model N. Observe that the root of N is w. The ideas explained above imply that for all formulas φ with $b(\varphi) \leq n$ we have $M, w \models \varphi$ if and only if $N, w \models \varphi$. ♡

Corollary 1

$$\vdash_K \varphi \;\Leftrightarrow\; F \models \varphi \text{ for all frames } F \text{ of depth} \leq b(\varphi).$$

Proof $\Rightarrow$: this direction is the soundness theorem.

$\Leftarrow$: this direction we show by contraposition. Thus assuming $\nvdash_K \varphi$ we show that there is a frame F of depth $\leq b(\varphi)$ such that $F \not\models \varphi$. Thus suppose $\nvdash_K \varphi$.

By the completeness theorem, there should be a frame G such that $G \not\models \varphi$. Thus there is a model M on this frame and a node w such that $M, w \models \neg\varphi$. By Theorem 8 there is a model N of depth $\leq b(\neg\varphi)$ and a node v such that $v \models_N \neg\varphi$. Since the number of boxes in φ and $\neg\varphi$ is the same, $b(\neg\varphi) = b(\varphi)$. Let F be the frame of N. This then shows that F has depth $\leq b(\varphi)$ and $F \not\models \varphi$, and we are done. ♡

8.3 The finite model property

Results similar to Corollary 1 hold for various modal logics. The result can also be improved in such a way that in the completeness theorem not only can we restrict ourselves to frames of finite depth, but even to frames that are finite. The precise formulation is as follows.

Theorem 9

$\vdash_{\mathsf{K}} \varphi$	$\Leftrightarrow$	φ holds on all frames of size $\leq 2^{\|\varphi\|}$.
$\vdash_{\mathsf{T}} \varphi$	$\Leftrightarrow$	φ holds on all reflexive frames of size $\leq 2^{\|\varphi\|}$.
$\vdash_{\mathsf{K4}} \varphi$	$\Leftrightarrow$	φ holds on all transitive frames of size $\leq 2^{\|\varphi\|}$.
$\vdash_{\mathsf{S4}} \varphi$	$\Leftrightarrow$	φ holds on all reflexive transitive frames of size $\leq 2^{\|\varphi\|}$.
$\vdash_{\mathsf{S5}} \varphi$	$\Leftrightarrow$	φ holds on all frames of size $\leq 2^{\|\varphi\|}$ for which the relation is an equivalence relation.

We say that a logic has the *finite model property* (FMP) if, whenever a formula φ is not derivable in the logic, there is a finite model of the logic (a model in which all formulas of the logic are forced) that contains a world in which φ is refuted.

Corollary 2 The logics K, $\mathsf{K4}$, T, $\mathsf{S4}$, $\mathsf{S5}$ have the finite model property.

Proof We prove it for T. Suppose $\not\vdash_{\mathsf{T}} \varphi$. Then by Theorem 9 there is a reflexive frame F of size $\leq 2^{|\varphi|}$ on which φ does not hold. Thus there is a model M on the frame and a node w such that $w \models \neg\varphi$. By the correspondence theorem $\Box\phi \to \phi$ holds on all reflexive frames. That is, T holds on all reflexive frames. Thus M is a finite model of T with a world that forces $\neg\varphi$. This proves that T has the finite model property. ♡

8.4 Decidability

Recall that a language is decidable if there is a Turing machine that decides it. We can define a similar notion for logics, by considering them as languages, namely as the set of all formulas that are derivable in the logic. We say that a formula *belongs* to a logic when it is derivable in it. E.g. with a logic L is associated the set $\{\varphi \mid \vdash_{\mathsf{L}} \varphi\}$. We call a Turing machine a decider for L when it decides $\{\varphi \mid \vdash_{\mathsf{L}} \varphi\}$. In general, we call a logic L *decidable* if there is a Turing machine that is a decider for L. The previous theorem implies the decidability of all modal logics mentioned there.

Corollary 3 The logics K, $\mathsf{K4}$, T, $\mathsf{S4}$, $\mathsf{S5}$ are decidable.

Proof We show that K is decidable and leave the other logics to the reader. Thus we have to construct a Turing machine that, given a formula φ, outputs "yes" if $\vdash_{\mathsf{K}} \varphi$ and "no" otherwise. By Theorem 9, $\vdash_{\mathsf{K}} \varphi$ is equivalent to φ being valid in all frames of size $\leq 2^{|\varphi|}$. Thus the TM has to do the following. Given φ it tests for all nodes w in all models M on all frames of size $\leq 2^{|\varphi|}$ whether $M, w \models \varphi$. If in all cases the answer is positive, it accepts, and otherwise it rejects. It is clear that this TM decides K. ♡

8.5 Complexity

In terms of complexity the TM constructed in the proof above might not do so well since there are at least exponentially many frames of size $\leq 2^{|\varphi|}$. The exponential factor is likely to be essential, as for many of these logics, including K, T, $\mathsf{K4}$ and $\mathsf{S4}$, one can show that the corresponding satisfiability problems are PSPACE-complete. That is, it can be solved in polynomial space whether a formula belongs to such a logic or not, and any problem in PSPACE can be reduced to such problems. (Recall that the satisfiability problem for propositional logic is NP-complete.) On the other hand, decidability is still nice. Recall that predicate logic is not decidable. Of course, propositional logic is, but since modal logics are extensions of propositional logic with much more expressive power, their decidability is not apparant, and indeed these facts have nontrivial proofs that, regrettably, fall outside the scope of this exposition.

9 Bisimulation

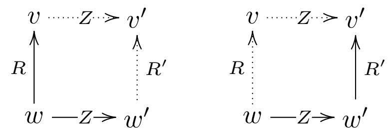
Bisimulation is a general method to establish whether two models are modally distinct, i.e. whether there is a modal formula that distinguishes the one from the other.

Given two models $M = (W, R, V)$ and $M' = (W', R', V')$, a *bisimulation* between M and M' is a relation Z on $W \times W'$ such that

1. wZw' implies that w and w' force the same propositional variables,
2. wZw' and wRv implies that there is a $v' \in W'$ such that $w'R'v'$ and vZv' (the forth condition),
3. wZw' and $w'R'v'$ implies that there is a $v \in W$ such that wRv and vZv' (the back condition).

If wZw' holds for some bisimulation Z, we say that w and w' are *bisimilar*.

There is neat way to visualize the back and forth conditions of bisimulations, the two *windows*:



Theorem 10 (Bisimulation theorem) If for two models $M = (W, R, V)$ and $M' = (W', R', V')$ there is a bisimulation Z such that wZw' for some $w \in W$ and $w' \in W'$, then w and w' force the same formulas: for all formulas φ

$$M, w \models \varphi \;\Leftrightarrow\; M', w' \models \varphi.$$

Proof By formula induction. You will be asked to prove this in the exercises. $\heartsuit$

10 P-morphisms

P-morphisms are functions between frames. They exist when there is a certain similarity between the frames. That is, given a p-morphism f one can define valuations on the frames such that a node and its image under the p-morphism cannot be distinguished modally: $w \models \varphi \;\Leftrightarrow\; f(w) \models \varphi$.

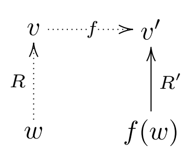
Given two frames $F = (W, R)$ and $F' = (W', R')$, a *p-morphism* $f : W \to W'$ between F and F' is a map such that

1. f is a surjection,
2. wRv implies $f(w)R'f(v)$,
3. $f(w)R'v'$ implies that there is a $v \in W$ such that wRv and $f(v) = v'$.

F' is called a *p-morphic image of* F.

Note the difference between p-morphisms and bisimulations: the former are *functions* between *frames*, while bisimulations are *relations* between *models*.

As for bisimulations, the third condition on p-morphisms can be depicted as follows:



Like the bisimulation theorem, there exists a theorem that states that the existence of a p-morphism implies that nodes that are connected via this function or relation are not modally indistinguishable:

Theorem 11 (P-morphism theorem) If $f : W \to W'$ is a p-morphism between $F = (W, R)$ and $F' = (W', R')$ and two valuations V and V' are such that for the models $M = (W, R, V)$ and $M' = (W', R', V')$, $M, w \models p \;\Leftrightarrow\; M', w' \models p$ holds, then

$$M, w \models \varphi \;\Leftrightarrow\; M', w' \models \varphi.$$

Proof By formula induction. The proof is similar to the proof of the bisimulation theorem, and you will be asked to provide such a proof in the exercises. ♡

From this theorem follows an interesting fact. If a frame F has a node w in which $\Box^n\bot$ is forced, and a frame G has a node v that does not force $\Box^n\bot$, then there is no p-morphism f from F to G such that $f(w) = v$. Namely, consider the valuations on the frames that do not force any propositional variable anywhere: $V(p) = \emptyset$ for all p. Then the conditions of the P-morphism theorem are met, and thus $w \models \phi$ iff $f(w) \models \phi$. And this implies that $f(w) \neq v$ since $w \models \Box^n\bot$ and $v \not\models \Box^n\bot$. Note that we use here the fact that the forcing of a formula $\Box^n\bot$ in a model does not depend on the valuation. Can you think of a more general statement than the above?

Observe that p-morphic images can be used to show that a certain frame property is not characterizable by a modal formula, in the following way.

Theorem 12 If a frame F is a p-morphic image of a frame G, then $G \models \varphi$ implies $F \models \varphi$.

Proof Assume F is a p-morphic image of G, and let f be the p-morphism and $G \models \varphi$. We have to show that $F \models \varphi$. Thus for an arbitrary valuation V' on F and an arbitrary world v in F, we have to show that $v \models \varphi$. Given V' we define a valuation V on G via

$$x \in V(p) \;\Leftrightarrow\; f(x) \in V'(p).$$

It is easy to see that in this case $w \models p \;\Leftrightarrow\; f(w) \models p$ for all propositional variables p, for all worlds w in G. Thus we can apply the P-morphism theorem, and conclude that $w \models \psi \;\Leftrightarrow\; f(w) \models \psi$ for all formulas ψ. Since $G \models \varphi$, it follows that $w \models \varphi$ for all nodes w in G. And thus $f(w) \models \varphi$. Observe that every node v in F is of the form $f(w)$ for some w, i.e. $v = f(w)$, because f is a surjection. Since also $f(w) \models \varphi$, it follows that every node v in F forces φ. And that is what we had to show. ♡

The converse of the previous theorem is in general not the case. You will be asked to prove this in the exercises.

Corollary 4 If for a certain property of frames there are frames F and G such that F is a p-morphic image of a frame G, and G has the property and F has it not, then this property is not characterizable by a modal formula.

Proof Suppose this property is characterizable by a formula φ. Then $G \models \varphi$, and by the previous theorem $F \models \varphi$. But this contradicts the fact that F does not have the property. ♡

By the previous corollary, to show that a certain frame property is not characterizable by a modal formula, it suffices to show that there are frames F and G as in the corollary.

The following theorem shows that bisimulations are generalizations of p-morphisms on the level of models. Historically, p-morphisms were first. Nowadays, bisimulations rule.

Theorem 13 If for a p-morphism $f : F \to F'$ we define valuations V and V' in such a way that

$$w \in V(p) \;\Leftrightarrow\; f(w) \in V'(p),$$

then the relation Z defined by $wRw' \;\Leftrightarrow w' = f(w)$ is a bisimulation between the models (F, V) and (F', V').

Proof We have to show that Z satisfies 1,2, and 3 in the definition of a bisimulation. Let $F = (W, R)$ and $F' = (W', R')$. Property 1. follows from the way in which the valuations and Z are defined. We prove 2. and leave 3., which is similar, to the reader. Thus assuming wZw' and wRv, we have to find a $v' \in W'$ such that vZv' and $w'R'v'$. Unwinding the definition of Z we find that wRv and $f(w) = w'$, and have to find a $v' \in W'$ such that $w'R'v'$ and $f(v) = v'$. But $f(w)R'f(v)$, that is $w'R'f(v)$, follows from the definition of p-morphism. ♡

11 Multimodal logics

Multimodal logics are logics with more modal operators than $\Box$ and $\Diamond$. To distinguish them from another these modalities are often indexed by letters or numbers: $\Box_a$, $\Box_1$, $\Diamond_a$, $\Diamond_1$. One also writes K_a for $\Box_a$.

On the syntactic side the formulas involve now the new operators, on the semantic side the Kripke models are now equiped with relations for every operator: R_a for $\Box_a$, R_1 for $\Box_1$, etc. Forcing is then defined for every modal operator $\Box_a$ seperately, in a way similar to the forcing of $\Box$:

$$w \models \Box_a\varphi \;\Leftrightarrow\; \forall v(wR_a v \;\Rightarrow\; v \models \varphi).$$

More modalities can model a lot more than one modality. For example, in epistemic logic, i.e. when interpreting $\Box_a\varphi$ as "person a knows φ", one can express statements like "a knows that b knows φ", via $\Box_a\Box_b\varphi$. Especially in this setting the extension to more modal operators is natural.

For the multimodal version of $\mathsf{S4}$, called $\mathsf{S4_m}$, we replace the axioms of $\mathsf{S4}$ by their multimodal versions. For all indices a we add axioms $\Box_a(\varphi \to \psi) \to$

$(\Box_a\varphi \to \Box_a\psi)$, $\Box_a\varphi \to \varphi$ and $\Box_a\varphi \to \Box_a\Box_a\varphi$. As in the unimodal case, we can prove frame characterizations and completeness theorems like the following.

Theorem 14 $\Box_a\varphi \to \varphi$ charaterizes the frames in which R_a is reflexive:

$$\forall\varphi\,(F \models \Box_a\varphi \to \varphi) \text{ if and only if } R_a \text{ is reflexive.}$$

Theorem 15 $\mathsf{S4_m}$ is sound and complete with respect to the frames in which the relations are reflexive and transitive.

There also exist theorems that relate different modalities and that therefore do not have a unimodal equivalent. The following theorem charecterizes the property that b knows all that a knows.

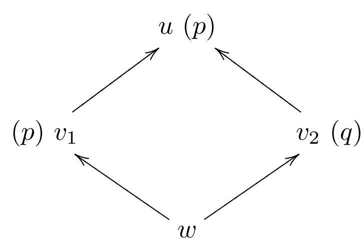
Theorem 16 $\Box_a\varphi \to \Box_b\varphi$ charaterizes the frames in which $R_b \subseteq R_a$:

$$F \models \Box_a\varphi \to \Box_b\varphi \;\Leftrightarrow\; \text{if } R_b \subseteq R_a.$$

Proof You will be asked to prove this theorem in the exercises. ♡

12 Exercises

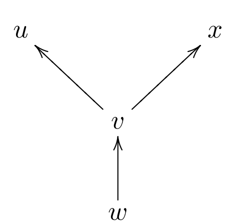
1. Given this Kripke model



Which of the following statements is true?

a.	$w \models \neg p$	*b.*	$v_1 \models \Box p$
c.	$u \models \Diamond\top$	*d.*	$u \models \Box\bot$
e.	$w \models \Box(p \vee q)$	*f.*	$w \models \Box p \vee \Box q$
g.	$v_2 \models \Box\Box\bot$	*h.*	$v_1 \models \Box q \to \neg p$
i.	$w \models \Diamond\bot$	*j.*	$w \models \Box\top$

2. Given the frame F



Define a valuation on the frame such that the following holds: $w \models \Diamond p$, $w \models \Box\Box p$, $v \models \Diamond q$, $u \models \neg q$. Does there exist more than one valuation that validates these constraints?

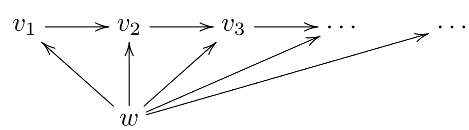
3. Show that $\Diamond(p \wedge \neg p)$ holds in no Kripke model.

4. Which of the following formulas is valid, that is, which formulas hold in all worlds of all models? If a formula is not valid, give a counter model.

a.	$\Box\top$	*b.*	$\Box\bot$
c.	$\Diamond\top$	*d.*	$\Diamond\bot$
e.	$\Box p \to \Diamond p$	*f.*	$\Box\varphi \wedge \Diamond\psi \to \Diamond\varphi$
g.	$\Box(p \vee q) \wedge \Diamond\neg p \to \Diamond q$	*h.*	$\Box(\varphi \to \varphi)$.

Here φ and ψ are arbitrary formulas and p and q are atoms.

5. Consider the following model M where w has infinitely many successors, $v_1, v_2, \ldots$:



Assume that p is only forced at the v_i for which i is odd. Which of the following statements is true?

a.	$w \models \Box p$	*b.*	$w \models \Box\Box p$
c.	$w \models \Diamond p$	*d.*	$w \models \Diamond\neg p$

For all of the following formulas, describe the i for which the formula holds at v_i.

a.	$\Box p$	*b.*	$\Box\Box p$
c.	$\Diamond p$	*d.*	$\Diamond\neg p$
e.	$\Diamond\Diamond p$	*f.*	$\Diamond\Diamond\Diamond\neg p$
g.	$\Diamond p \to \Box\Box p$	*h.*	$\Diamond\neg p \to \Diamond\Diamond p$

6. Explain why the following formulas hold in all worlds in all models:

   (a) $\varphi \vee \neg\varphi$,

   (b) φ, $\Box\varphi$, $\Box\Box\varphi$, ..., for all propositional tautologies φ,

   (c) $\Diamond\varphi \to \Diamond\top$,

   (d) $(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$,

   (e) $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$,

   (f) $\Box(\varphi \to \psi) \wedge \Box\varphi \to \Box\psi$,

7. Show that the other direction of the true formula $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$ given above, i.e. $\Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi$, is not generally valid. That is, give formulas φ and ψ and a model and world at which $\Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi$ does not hold.

8. Give instances of φ and ψ for which $\Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi$ does hold in all worlds in all models.

9. Which of the following formulas hold in the frame F given in Exercise 2?

a.	$\Box p$	*b.*	$\Box\Box\Box\bot$
c.	$\Diamond\top$	*d.*	$\Box p \vee \neg\Box p$
e.	$\Box\Box p \to (\Box\bot \to p)$	*f.*	$\Diamond q \to \neg\Box\bot$

10. Prove Theorems 1 and 2. For the former it suffices to show the following three things. First, show that all axioms of K hold on all frames. Second, for the modus ponens rule of K, show that if $\varphi \to \psi$ and φ hold on all frames, then so does ψ. Third, for the necessitation rule of K, show that if φ holds on all frames, then so does $\Box\varphi$. The proof of Theorem 2 has the same pattern.

11. Give a derivation of $\Box(p \vee \neg p)$ in K.

12. Why is $\Box p$ not derivable in K?

13. Prove that

$$\frac{\varphi \to \psi \qquad \varphi \to \chi}{\varphi \to \psi \wedge \chi}$$

is a derived rule in K, i.e. if K derives the premisses, then it derives the conclusion.

14. Prove that

$$\frac{\varphi \to \psi \qquad \psi \to \chi}{\varphi \to \chi}$$

is a derived rule in K.

15. Show that $\nvdash_{\mathsf{K}} p \to \Box p$, $\nvdash_{\mathsf{K}} \Box p \to p$, $\nvdash_{\mathsf{K}} \Box(p \vee q) \to \Box p \vee \Box q$.

16. Show that if $\vdash_{\mathsf{K}} \varphi \to \psi$, then $\vdash_{\mathsf{K}} \Box\varphi \to \Box\psi$.

17. Show that when we replace in K the axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ by the axioms $\Box\top$ and $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$, and we replace the necessitation rule by

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

we obtain a system that is equivalent to K in that it derives exactly the same formulas.

18. Show that if $\vdash_{\mathsf{K}} \varphi \leftrightarrow \psi$, then for any formula $\chi(p)$ in which variable p occurs, $\vdash_{\mathsf{K}} \chi(\varphi) \leftrightarrow \chi(\psi)$. Here $\chi(\varphi)$ is the result of substituting φ for p in χ. Use formula induction on χ.

19. Let A be a sequence of $\neg$ and $\Box$. Prove by induction on the length of A the following two statements. When $\neg$ occurs an even number of times in A, then if $\vdash_{\mathsf{K}} \varphi \to \psi$, then $\vdash_{\mathsf{K}} A\varphi \to A\psi$. When $\neg$ occurs an odd number of times in A, then if $\vdash_{\mathsf{K}} \varphi \to \psi$, then $\vdash_{\mathsf{K}} A\psi \to A\varphi$. Use the one statement in the induction of the other. First try the cases that $\neg$ occurs 0, 1 or 2 times in A.

20. Prove Lemma 1 by formula induction.

21. For all the frame properties discussed in Section 5, give a model with at least four nodes and four arrows that satisfies the property.

22. For all the frame properties discussed in Section 5, give a model that does not satisfy the property.

23. Show that for every euclidean relation it holds that $\forall x \forall y \forall z (xRy \wedge xRz \to yRz \wedge zRy)$.

24. Consider the Kripke model where the nodes are the integers and the accessibility relation R is defined as mRn iff $m \leq n$, i.e. the asseccibility relation is $\leq$. Is this frame well-founded, or reflexive, or euclidean?

25. Consider the Kripke frame where the nodes are the natural numbers and the relation is $<$. Is this frame dense? And is it dense if we replace the natural numbers by the rational numbers?

26. Is the Kripke model where the nodes are real numbers and the relation is $\leq$ a reasonable model for tense logic? And the same question for the rationals.

27. Prove that $\Box\bot$ characterizes the completely disconnected frames.

28. Which formula characterizes the frames where there are no three nodes w, v, u such that $wRvRu$?

29. Prove that $\Diamond\top$ characterizes the serial frames.

30. Prove that $\Diamond\Box\varphi \to \varphi$ characterizes the symmetric frames.

31. Which formula characterizes the frames where every chain of nodes $v_1, \ldots, v_m$ such that $v_1 R v_2 R \ldots R v_m$, has length at most n, thus $m \leq n$. (Hint: consider the exercise above on frames in which every node has at most one successor.)

32. Proof that $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ characterizes the class of transitive and well-founded frames.

33. Show that every reflexive frame is dense.

34. If a relation is well-founded, then it is irreflexive and asymmetric. Prove this fact.

35. Does $F \models \varphi \vee \neg\varphi$ hold for all frames F and for all formulas φ? And does ($F \models \varphi$ or $F \models \neg\varphi$) hold for all φ?

36. Show that $\{\Box(\phi \to \psi), \Box(\top \to \phi), \Diamond\neg\psi\}$ is inconsistent.

37. Show that this set is K consistent. Is it T-consistent?

$$\{p, \neg\Box p, \Box\Box p, \neg\Box\Box\Box p, \Box\Box\Box\Box p, \dots\}.$$

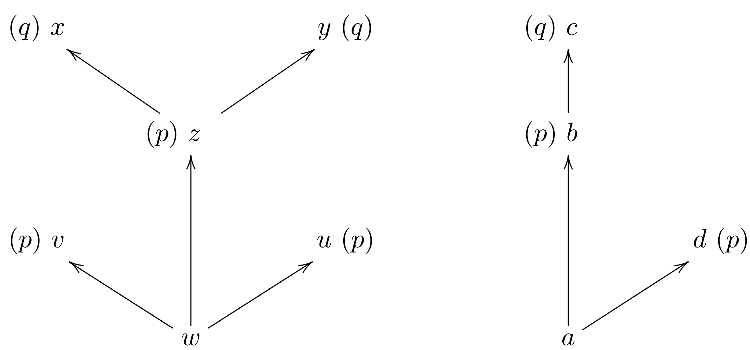
38. Given that wRv holds in a model, show that for the sets

$$L_w = \{\phi \mid w \models \phi\} \qquad L_v = \{\phi \mid v \models \phi\},$$

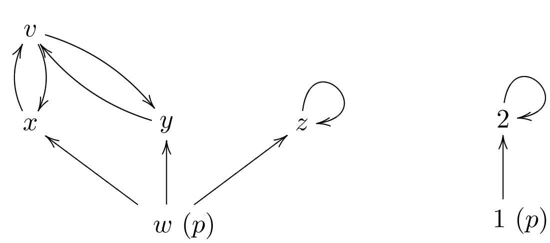
it holds that $\Box\phi \in L_w$ implies $\phi \in L_v$, for all formulas ϕ.

39. Prove the Valuation lemma, Lemma 2.

40. Give a bisimulation between the following two models such that w and a become bisimilar.

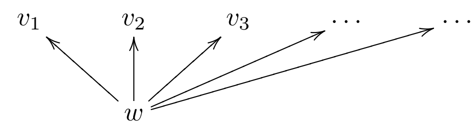


41. Give a bisimulation between the following two models such that w and 1 become bisimilar.



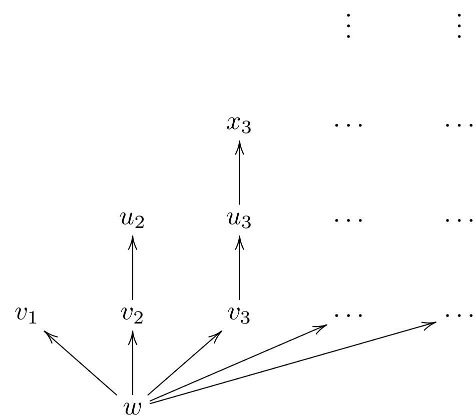
42. Is the model with frame the rational numbers at which no atoms are forced, bisimilar with the model with frame the real numbers at which no atom is forced? Explain your answer. The same question for the natural numbers instead of the reals.

43. Consider the following frame $F = (W, R)$ where w has infinitely many successors, $v_1, v_2, \ldots$.



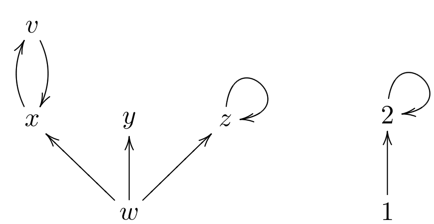
Does F validates the same formulas as F_{v_1}? Construct a finite frame that is a p-morphic image of F.

44. Consider the following infinite frame $F = (W, R)$



Does $\Box^n \bot$ hold in the frame for some n? And does $\Diamond \top$ hold? Show that no finite frame is a p-morphic image of F.

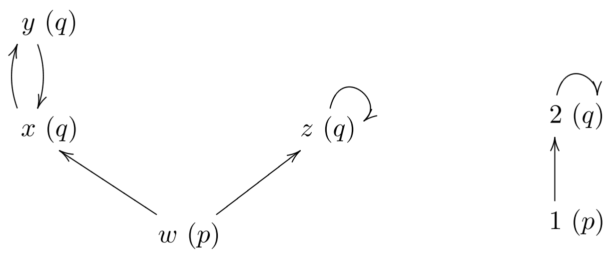
45. Show that there is no p-morphism between the following two frames.



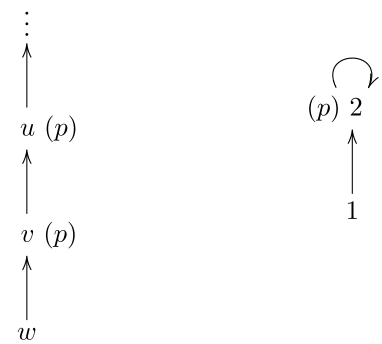
Are there models on the frames such that w and 1 force the same formulas? Is the generated subframe generated by x a p-morphic image of the frame generated by 2?

46. Show that the same formulas are forced in the following two models, using

the Bisimulation theorem.



47. Show that the same formulas are forced in the following two models, using the bisimulation theorem.



48. Prove that $\Box\phi \leftrightarrow \Box\Box\phi$ holds on all reflexive transitive frames.

49. Give a formula φ that characterizes the reflexive transitive frames, i.e.

$$F \models \varphi \;\Leftrightarrow\; F \text{ is a reflexive transitive frame,}$$

and prove this fact. Show that $\Box\varphi \leftrightarrow \Box\Box\varphi$ does not characterize this class of frames.

50. Prove the Valuation Theorem. Use formula induction. Some steps are explained in the syllabus, Lemma 18.

51. Prove the Bisimulation Theorem. Use formula induction.

52. Prove the P-morphism theorem. Use formula induction.

53. Show with an example that the converse of Theorem 12 does not hold.

54. The canonical model for T is defined in exactly the same way as for K, reading T everywhere for K. Thus the nodes of the T-canonical model are maximal T-consistent sets. Prove that the frame of the T-canonical model is reflexive, and that that of the $\mathsf{K4}$-canonical model is transitive.

55. Consider the multimodal logic with modal operators K_a, K_b, and K_c. When we read $K_a\varphi$ as "a knows φ", give formulas expressing the following statements.

    If a knows ϕ, then so does b (b knows everything that a knows).

    If c knows that a knows φ, then also b knows that a knows φ (b knows all about a that c knows).

    If a knows ψ it is possible that b knows it too.

    a knows that b knows that it, i.e. a, knows φ.

56. Consider the multimodal logic with two modal operators $\Box_a$ and $\Box_b$, and corresponding relations R_a and R_b in the Kripke models. Prove that $\Box_a\varphi \to \Box_b\varphi$ characterizes the class of frames in which $R_b \subseteq R_a$, i.e. wR_bv implies wR_av for all w and v.

57. Consider the multimodal logic with two modal operators $\Box_a$ and $\Box_b$, and corresponding relations R_a and R_b in the Kripke models. Give the class of frames on which $\Diamond_a\varphi \to \Diamond_b\varphi$ holds, and prove this fact.

58. Consider the multimodal logic with two modal operators K_1 and K_2 and corresponding relations R_1 and R_2 in the Kripke models. Prove that $\Diamond_1\varphi \to \Diamond_2\Diamond_1\varphi$ characterizes the class of frames in which wR_1v implies the existence of a u such that wR_2uR_1v.

59. Prove that the class of asymmetric frames is not characterizable by a modal formula.

60. Which formula characterizes the class of frames in which $wRvRuRs$ implies wRs?