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On Rules

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Abstract This paper contains a brief overview of the area of admissible rules with an emphasis on results about intermediate and modal propositional logics. No proofs are given but many references to the literature are provided.

Keywords Admissible rules · Modal logic · Intermediate logic · Consequence relations

1 Introduction

In the inferential or proof-theoretic approach to logic one describes a theory via a set of inference rules. This in contrast to most semantic approaches where a logic is a set of theorems, characterized as those formulas that hold in certain models, or algebras, or categories, or some other semantics. In the inferential approach one wishes to describe how one formula follows from another and thereby provide a procedure to generate all theorems of the theory. There are, of course, numerous ways to do this, as illustrated by the abundance of proof systems developed over the years. But even if one restricts oneself to one sort of proof system, such as sequent calculi, or natural deduction systems, many possibilities remain. We all know that a single theorem can have many different proofs, and that a theory can have all kinds of axiomatizations. This paper is about the variety of axiomatizations that a theory can have, and in particular about the role of rules in this setting.

The phenomenon we wish to study can best be illustrated by the role of the cut rule in sequent calculi. As is well-known, one can show that in the standard cut-free

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calculus for classical logic the cut rule is admissible. This means that one can add it to the calculus and use it in derivations but in doing so one does not obtain new theorems, just new proofs. Thus the resulting system represents the same theory (the same set of theorems) as the original system. The merit of having different calculi is that each system comes with its own benefits. Proof in calculi with cut are in general shorter (in a precise sense) than cut-free proofs. On the other hand, the latter satisfy the subformula property and are easy to generate.

The admissible rules of a theory are those rules that can be used in derivations without obtaining theorems that cannot be obtained without them. Thus they can facilitate reasoning or shorten proofs or have some other effect, but they cannot prove what is not true in the original theory. The collection of all admissible rules of a theory therefore is the set of all possible inference steps one can use to prove the theorems of the theory.

Over the last thirty years the study of admissible rules has flourished. Starting with Rybakov's book [42] many results on the decidability and characterization of admissible rules have appeared. Most of these results concern propositional logics: intermediate, modal and substructural ones. This is the reason that we restrict this overview to those logics and leave the discussion of admissible rules in predicate logic for another occasion.

One of the beautiful aspects of the study of rules is its relation with unification theory, as discovered by Ghilardi and developed by him and many others after him. The connection to substitutions stems from the fact that a rule A/B is admissible in a propositional logic, be it modal, intermediate or substructural, if and only if whenever a substitution instance σA of A is derivable, so is σB. The exact connection with unification theory will be explained below.

What follows in the next sections is a brief overview of the study of admissible rules. I have not included any proofs but provide references where needed. Because of lack of space I cannot discuss all results on rules and therefore a slightly biased summary is what the reader can expect. But I hope I have done justice to this area in logic that has grown large and wide over the last twenty years and that combines philosophical relevance with deep mathematics.

2 Consequence

The study of admissible rules does not depend on a particular proof system but only on the logic itself. We therefore chose to let logics be given by consequence relations [56], rather than by a specific proof system such as natural deduction or sequent calculi. In this section we define consequence relations and thereby provide the framework for the discussion of rules in the rest of the paper.

To maintain a certain level of generality we do not yet specify a particular logic, but just assume we have a *language* $\mathcal{L}$ that contains propositional variables $p, q, r, \dots$, certain logical constants such as connectives and possibly modal or other operators. *Formulas* in this language are defined as usual. When we speak of formulas, we will always mean formulas in $\mathcal{L}$. Sometimes sequents instead of formulas are used to simplify notation. They are of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas, and interpreted as usual: $I(\Gamma \Rightarrow \Delta)$ denotes $\bigwedge \Gamma \to \bigvee \Delta$ (in case implication, conjunction and disjunction are in the language).

A *substitution* is a map that assigns to each propositional variable a formula. Every subsitution can be uniquely extended to a map from formulas to formulas that commutes with the logical constants (algebraically, that is a homomorphism). Ambiguously, both maps will be called substitutions from now on. σ and τ range over substitutions, and $\sigma\Gamma$ denotes $\{\sigma A \mid A \in \Gamma\}$.

Multi-conclusion consequence relations are relations $\vdash$ on sets of formulas. We write $\Gamma \vdash \Delta$ if the pair (Γ, Δ) belongs to the relation. We also write Γ/Δ for the pair (Γ, Δ), and $A, \Gamma \vdash \Delta, B$ for $\{A\} \cup \Gamma \vdash \{B\} \cup \Delta$.

A *finitary structural multiple-conclusion consequence relation* or *m-logic* is a relation $\vdash$ on finite sets of formulas that satisfies for all finite sets of formulas $\Gamma, \Gamma', \Delta, \Delta'$ and formulas A:

reflexivity	$A \vdash A$,
weakening	if $\Gamma \vdash \Delta$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$,
transitivity	if $\Gamma \vdash \Delta, A$ and $\Gamma', A \vdash \Delta'$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$,
structurality	if $\Gamma \vdash_{\mathsf{L}} \Delta$, then $\sigma\Gamma \vdash_{\mathsf{L}} \sigma\Delta$.

A *finitary structural consequence relation* or *logic* is a relation satisfying the single-conclusion variants of the four properties above, that is, where the right side of $\vdash$ consists of exactly one formula (identifying $\{A\}$ with A). Although our primary interest is the single-conclusion consequence relation, the multi-conclusion analogue allows us to express certain properties more naturally, such as the disjunction property discussed below. We omit the words "finitary" and "structural" in what follows and just speak of "consequence relations".

Any single-conclusion consequence relation $\vdash$ has a natural multi-conclusion analogue:

$$\Gamma \vdash^m \Delta \equiv_{def} \exists A \in \Delta\ \Gamma \vdash A.$$

The single-conclusion part of a multi-conclusion consequence relation $\vdash$ is

$$\Gamma \vdash^s A \equiv_{def} \Gamma \vdash A.$$

A is a *theorem* if $\emptyset \vdash A$, which we write as $\vdash A$. The set of all theorems of a consequence relation $\vdash$ is denoted by $\mathsf{Th}(\vdash)$. Δ is a *multi-conclusion theorem* if $\vdash \Delta$, and the set of all multi-conclusion theorems is denoted by $\mathsf{Thm}(\vdash)$.

3 Rules

The first part of this section contains the necessary definitions regarding rules, and it is followed by general remarks on what has been defined.

A *(multi-conclusion) rule* is an ordered pair of finite sets of formulas, written Γ/Δ or $\dfrac{\Gamma}{\Delta}$. It is *single-conclusion* if $|\Delta| \leq 1$. For $R = \Gamma/\Delta$, σR is short for $\sigma\Gamma/\sigma\Delta$, and similarly for sets of rules. If disjunction belongs to the language, the *single-conclusion version* of a multi-conclusion rule $R = \Gamma/\Delta$ is the rule $A \vee \bigwedge \Gamma / A \vee \bigvee \Delta$. The addition of "$A\vee$" is a technicality that will not be explained, given the survey character of this text. Rules Γ/Δ such that $\Gamma \vdash \Delta$ are the *rules of the consequence relation* $\vdash$ and $\mathsf{Ru}(\vdash)$ is the set of all such rules. If Γ is empty, the rule is also called an *axiom*, and a *proper rule* otherwise.

Given a multi-conclusion consequence relation $\vdash$ and a set of rules $\mathcal{R}$, $\vdash^{\mathcal{R}}$ is the smallest consequence relation extending $\vdash$ for which $\Gamma \vdash \Delta$ holds for all Γ/Δ in $\mathcal{R}$. Similarly for single-conclusion rules and single-conclusion consequence relations. In case of a single rule R we write $\vdash^R$ for $\vdash^{\{R\}}$. We leave it to the reader to verify that given any $\vdash$ and $\mathcal{R}$ there indeed exists such a smallest consequence relation and that it is unique.

Given a consequence relation $\vdash$, a set of rules $\mathcal{R}$ is a *basis* for a consequence relation $\vdash' \supseteq \vdash$ if $\vdash' = \vdash^{\mathcal{R}}$. A rule $R = \Gamma/\Delta$ is *derivable* if $\Gamma \vdash \Delta$. It is *admissible*, written $\Gamma \mathrel{|\!\!\sim} \Delta$, if $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash^R)$, and $\mathsf{Th}(\vdash) = \mathsf{Th}(\vdash^R)$ in case $\vdash$ and R are single-conclusion. A set of rules is admissible if all of its members are.

A few remarks are in order. As can be seen from the definition, a rule is admissible when one can add it to the consequence relation without obtaining new theorems, just (possibly) new derivations. This shows that admissibility solely depends on the theorems of a consequence relation. Admissibility $\mathrel{|\!\!\sim}$ itself is a consequence relation, namely the largest consequence relation with the same theorems as $\vdash$. Therefore the admissible rules of $\vdash$ are all possible inference steps that could be added to $\vdash$ to derive the theorems of $\vdash$ and nothing more. Or to put it less formally, admissible rules tell us exactly what arguments are allowed to obtain the truths of a theory.

Naturally, this leads to questions such as: is it decidable whether a rule is admissible, is there a nice description of the rules that are admissible, and so on. Answers to these questions form the main part of the rest of this article.

Derivable rules are those admissible rules that are explicitly captured by the consequence relation, the reason they are often called trivial. Derivability depends on a design choice: one can have two consequence relations with the same theorems where a certain rule is nonderivable in one of them and derivable in the other. For example, if $\vdash$ has a nonderivable admissible rule R, then $\vdash$ and $\vdash^R$ have the same theorems, but R is not derivable in $\vdash$, while it is so in $\vdash^R$. As mentioned above, admissibility does not depend on a design choice, it is a fundamental notion depending on the theorems of a consequence relation (the logic) and on nothing else.

The following proposition provides the link between admissibility and unification. It applies to consequence relations that are *saturated*, meaning that

$$\vdash \Delta \Rightarrow \exists A \in \Delta \vdash A.$$

Every single-conclusion consequence relation $\vdash$ is clearly saturated, and so is its multi-conclusion variant $\vdash^m$. The notion is related to the disjunction property in that in every logic for which there is a consequence relation $\vdash$ such that $\vdash \Delta$ if and only if $\vdash \bigvee \Delta$, being saturated is equivalent to having the disjunction property.

Proposition 3.1 *For every saturated consequence relation* $\vdash$,

$$\Gamma \mathrel{|\!\!\sim} \Delta \;\Leftrightarrow\; \forall \sigma : \;(\forall A \in \Gamma \vdash \sigma A) \Rightarrow \;\exists B \in \Delta \vdash \sigma B.$$

Therefore every single-conclusion consequence relation satisfies

$$\Gamma \mathrel{|\!\!\sim} A \;\Leftrightarrow\; \forall \sigma : \;(\forall B \in \Gamma \vdash \sigma B) \Rightarrow \vdash \sigma A.$$

In the literature admissibility is often defined via the equivalence above.

Proposition 3.1 shows that via multi-conclusion consequence relations one can express the disjunction property. It implies that an intermediate logic has the disjunction property if and only if $\{p \vee q\}/\{p, q\}$ is admissible, and similarly for modal logic and the modal disjunction property $\{\Box p \vee \Box q\}/\{p, q\}$.

A single-conclusion consequence relation $\vdash$ is *structurally complete* if all proper extensions in the same language have new theorems, and *hereditarily structurally complete* if all extensions in the same language are structurally complete. It is not difficult to see that $\vdash$ is structurally complete if and only if it coincides with $\mid\!\sim$. For the latter means that every nonderivable rule R is not admissible, which is equivalent to $\vdash^R$ having new theorems for all nonderivable R, that is, to being structurally complete. Thus structural completeness means that there are no "hidden" principles of inference, no underivable admissible rules, all valid arguments are already captured by the consequence relation itself.

Structural completeness first appeared in the work of Pogorzelski in 1971 [35]. The name "admissible" for rules goes back to a paper by Lorenzen from 1955 [30]. One of the first appearances of the phenomenon of admissible rules, though not under this name, is in a paper by Johansson on minimal logic from 1937 [29], where it is observed that the rule $B \vee (A \wedge \neg A)/B$ is admissible but underivable in minimal logic.

We wish to associate with well-known logics L a particular single-conclusion consequence relation $\vdash_{\mathsf{L}}$ that we can refer to when considering the logic. As we will be concerned with admissibility rather than derivability, it is not important which particular consequence relation we choose, except that in the case conjunction and implication are present in the language we choose $\vdash_{\mathsf{L}}$ such that rules corresponding to provable implications are derivable, where $\mathsf{Th}(\mathsf{L})$ denotes the set of theorems of L:

$$\left(\bigwedge \Gamma \to A\right) \in \mathsf{Th}(\mathsf{L}) \Rightarrow \Gamma \vdash_{\mathsf{L}} A.$$

Once $\vdash_{\mathsf{L}}$ has been fixed we take $\vdash_{\mathsf{L}}^m$ for the multi-conclusion consequence relation corresponding to L.

4 Admissibility

One of the reasons that admissibility is a notion that has long been overlooked might well be that in classical propositional logic CPC all (single-conclusion) admissible rules are derivable, that is, CPC is structurally complete. Indeed, suppose that $\Gamma \mid\!\sim_{\mathsf{CPC}} A$. By Proposition 3.1 this means that for all substitutions σ that replace the propositional variables in Γ by $\top$ or $\bot$, $\vdash_{\mathsf{CPC}} \bigwedge \sigma\Gamma$ implies that $\vdash_{\mathsf{CPC}} \sigma A$. Note that this is precisely the definition of an implication $\bigwedge \Gamma \to A$ being valid in classical logic.

Once tertium non datur does not hold the picture changes drastically. Probably the most famous admissible rule that is not always derivable is the *Kreisel-Putnam Rule*:

$$\frac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)}\mathsf{KP}$$

Prucnal discovered the universal character of this rule, a result later strengthened by Minari and Wroński in [31].

Theorem 4.1 [38] KP *is admissible in any intermediate logic.*

That HR is not always derivable follows from its underivability in intuitionistic logic. To prove that it is admissible one can use what has become known as Prucnal's trick [37]. It is an instance of a more general phenomenon that we will explain first.

Given a formula A and a valuation v of the propositional variables in the classical sense (mapping them to 0 or 1), we define the corresponding substitution σ_v as

$$\sigma_v^A(p) \equiv_{def} \begin{cases} A \wedge p & \text{if } v(p) = 0 \\ A \to p & \text{if } v(p) = 1. \end{cases}$$

It is not difficult to see that for all subformulas B of A:

$$\vdash_{\mathsf{CPC}} \sigma_v^A(B) \leftrightarrow \begin{cases} A \wedge B & \text{if } v(B) = 0 \\ A \to B & \text{if } v(B) = 1. \end{cases}$$

As $\vdash_{\mathsf{CPC}} A \to A$, this gives $\vdash_{\mathsf{CPC}} \sigma_v^A(A)$ for all v that satisfy A.

To prove Theorem 4.1, assume $\vdash_{\mathsf{L}} \neg A \to B \vee C$ holds in L. Thus $\vdash_{\mathsf{L}} \neg\sigma A \to \sigma B \vee \sigma C$ for any substitution σ. If $\vdash_{\mathsf{L}} \neg\neg A$, then $\vdash_{\mathsf{L}} \neg A \to D$ holds for any D. Therefore assume this is not so, which by Glivenko's Theorem provides us with a valuation v such that $v(\neg A) = 1$. Let σ denote $\sigma_v^{\neg A}$. The discussion above shows that $\vdash_{\mathsf{CPC}} \neg\sigma A$, and again by Glivenko's Theorem $\vdash_{\mathsf{L}} \neg\sigma A$. Hence $\vdash_{\mathsf{L}} \sigma B \vee \sigma C$. Clearly, $\vdash_{\mathsf{CPC}} \neg A \to (\sigma D \leftrightarrow D)$ for all D, and thus $\vdash_{\mathsf{L}} (\neg A \to B) \vee (\neg A \to C)$, which is what we had to show.

Substitutions as the above play an important role in the study of admissibility as we will see in the context of unification. This is not the place for a technical discussion on the precise use of them, but we have included the brief argument above to give the reader a taste of their usefulness.

5 Decidability

Rybakov may well be called the father of admissible rules. His [42] contains numerous results on admissibility, most importantly the decidability of the admissibility relation of IPC, K4, GL, S4 and several other intermediate and modal logics. Thus answering Harvey Friedman's 1975 question about the decidability of admissibility in intuitionistic logic positively. The method Rybakov employs is semantical. Essentially, the admissibility of a rule is reduced to its validity in a certain characterizing model. Then it is shown that from a valuation that refutes the rule a definable valuation refuting the rule can be obtained, which then provides a substitution proving that a rule is not admissible. Decidability of admissibility then follows once one can show the finiteness of the objects involved.

This method is fruitful in that it can be adapted to many other logics, as has been done in [2, 33, 43, 44], where the decidability of admissibility in various temporal logics and minimal logic has been studied. A slight drawback of the method is that the algorithms that the method proves to exist are not easy to obtain from those proofs. In several cases there are, however, other ways to construct algorithms, as shown by Ghilardi in [11]. In [19, 20] Metcalfe and the author developed proof systems for admissibility for several well-known intermediate and modal logics, from which decision algorithms can be obtained as well.

How complex admissibility is in case it is decidable has been studied by Jeřábek who proved it to be coNEXP-complete in many natural modal and intermediate logics such as K4, S4, GL and IPC [23]. Thus checking admissibility in these logics is strictly more complex than checking derivability.

Of course, if derivability in a logic is undecidable, so is admissibility. The converse is not true: there are decidable logics for which admissibility is undecidable, as was first shown in [3], and later also in [54].

6 Bases

Knowing that admissibility in a logic is decidable, a next challenge is to give a description or axiomatization of its admissible rules, that is, a nice basis (Section 3). The word *nice* is used to stress that we are, of course, not interested, in the trivial basis consisting of *all* admissible rules, but rather in bases that tell us something about admissibility. Over the last twenty years many results that characterize admissible rules via a basis have been obtained, mainly for intermediate, modal and substructural logics and their fragments. The rules we discuss can best be expressed via sequents and we therefore in this section use a sequent-style formulation of the results.

Rybakov in [42] showed that in various modal and intermediate logics, including IPC and K4, there does not exist a finite basis for the admissible rules. This, of course, does not exclude that these logics have an infinite basis that still can be described in a compact way. As we will see, this often is the case.

There are three collections of rules that play a central role in admissibility. In intermediate logic these are the multi- and the single-conclusion *Visser Rules*, in which Γ is required to consist of implications only:

$$\frac{\Gamma \Rightarrow \Delta}{\{\Gamma \Rightarrow A \mid A \in \Gamma^a \cup \Delta\}}\overline{\mathsf{V}}$$

Γ^a consists of the antecedents of the implications that belong to Γ, which means $\Gamma^a \equiv_{def} \{A \mid (A \to B) \in \Gamma \text{ for some } B\}$. In modal logic these are the multi-conclusion *Modal Visser Rules*:

$$\frac{\Box\Gamma \Rightarrow \Box\Delta}{\{\boxdot\Gamma \Rightarrow A \mid A \in \Delta\}}\overline{\mathsf{V}}^{\bullet} \qquad \frac{\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta}{\{\boxdot\Gamma \Rightarrow A \mid A \in \Delta\}}\overline{\mathsf{V}}^{\circ}$$

Here $\boxdot\Gamma$ is short for $\{A \wedge \Box A \mid A \in \Gamma\}$ and $\Box\Gamma \equiv \Gamma$ for $\{\Box A \leftrightarrow A \mid A \in \Gamma\}$. The single-conclusion versions of $\overline{\mathsf{V}}$, $\overline{\mathsf{V}}^{\bullet}$ and $\overline{\mathsf{V}}^{\circ}$ are denoted respectively by V, $\mathsf{V}^{\bullet}$ and V°.

Roziére [40] was the first to provide a concrete basis for the admissible rules for a logic for which the problem is not trivial, by proving in 1992 that $\overline{\mathsf{V}}$ is a basis for the admissible rules of IPC. This result was not published and was independently but later obtained by the author, who strengthened it to Theorem 6.1. The Visser Rules also appeared in the work of Visser [49, 50] who proved that the admissible rules of IPC and Heyting Arithmetic are equal, and Skura [45], who used them in the context of refutation systems.

Observe that the Visser Rules are in general not derivable. That they are admissible can be shown in various ways. For IPC, for example, the following proof-theoretic argument could be given. Consider the sequent calculus G3i [48] and observe that a cut-free proof of $\Gamma \Rightarrow \bigvee \Delta$ in case Γ consists of implications only, must necessarily end in an inference which has $\Gamma \Rightarrow A$ as a premiss for some $A \in \Gamma^a \cup \Delta$. Similar reasoning can be used to prove the admissibility of the Modal Visser Rules. Admissibility of these rules can also be proved semantically by constructing on the basis of counter models to the conclusion of the rule a counter model to its premisses. To prove that the rules not only are admissible but form a basis is much harder, the main ingredients of the proof will be discussed in the section on unification. A full proof is outside the scope of this paper.

Theorem 6.1 [16, 17] *In every intermediate logic in which* V *is admissible it forms a basis for the single-conclusion admissible rules.*

This theorem has implications for several intermediate logics for which it is easy to see or well-known that the Visser rules are admissible. It implies that $\overline{\mathsf{V}}$ is a basis for the multi-conclusion admissible rules in IPC, and that V is a basis for the admissible rules in the logics of frames with exactly n maximal nodes. In particular, V is a basis in KC. It also follows that in logics with linear frames all admissible rules are derivable, as the Visser Rules are easily seen to be derivable in such logics. In particular this holds for the finite Gödel logics and LC.

Examples of intermediate logics in which not all Visser Rules are admissible are the Gabbay–de Jongh logics [13] and Medvedev logic, which is structurally complete [17, 39, 53]. Admissibility in the Gabbay–de Jongh logics neatly corresponds to the levels of the hierarchy of Visser Rules: in the n-th such logic the Visser Rules in which the size of Γ is at most $n + 1$ form a basis. Just as the union of these logics is IPC, so is the union of their bases a basis for the admissible rules of IPC. Another example of an intermediate logic with a basis for admissibility different from the Visser rules is the logic of frames of depth at most 2, as shown by Goudsmit [14]. Goudsmit also provides a nice relation between admissibility and refutation systems in [15].

Using similar techniques as in the intermediate case, Jeřábek proved the following modal analogue of Theorem 6.1 about multi-conclusion rules.

Theorem 6.2 [22] *In every transitive irreflexive modal logic in which* $\overline{\mathsf{V}}^\bullet$ *is admissible it forms a basis. The same holds for* $\overline{\mathsf{V}}^\circ$ *in transitive reflexive modal logics. If the logic is neither reflexive nor irreflexive and* $\overline{\mathsf{V}}^\bullet$ *and* $\overline{\mathsf{V}}^\circ$ *are admissible, they form a basis.*

It follows that $\overline{\mathsf{V}}^\bullet$ is a basis for the admissible rules in GL, and the same holds for $\overline{\mathsf{V}}^\circ$ in S4, and for $\overline{\mathsf{V}}^\bullet$ and $\overline{\mathsf{V}}^\circ$ in K4. In [24] other interesting bases for the admissible rules of these logics and IPC are provided. In contrast to the Visser Rules, these bases are independent. Interestingly, not in all logics admissible rules have independent bases [41]. For modal logics below K4 much less is known about admissibility. Some partial answers can be found in [28, 51, 52].

As one would expect, admissibility is very sensitive to the language one uses. It has long been known that the implicational fragment of IPC is hereditarily structurally complete [37] (the first occurrence of Prucnal's trick). The same holds for the implication–conjunction and some other fragments of IPC [32, 47]. In [32] Mints showed that any admissible underivable rule of IPC must contain both implication and disjunction.

Interestingly, the implication–negation fragment of IPC is not structurally complete, as was first observed by Wroński. In [5] Cintula and Metcalfe show that the following *Wroński Rules* are a basis for the multi-conclusion admissible rules of this fragment (they also provide a basis for the single-conclusion rules).

$$\frac{A_1 \to (A_2 \to (\cdots \to (A_n \to \bot)\dots)}{\{\neg\neg A_i \to A_i \mid 1 \leq i \leq n\}}\mathsf{W}$$

As often, to see that these rules are admissible is not hard, but to prove that they actually are a basis requires considerable effort. Cintula and Metcalfe make use of a technique by now well-known in the study of rules, and which will be explained in the section on unification. An nontrivial example of a logic for which the implication–negation fragment is structurally complete is relevant logic [46].

For substructural logics much less is known about bases of admissible rules, as many of the techniques used for modal and intermediate logics fail in this setting. Exceptions are results on structural completeness and Łukasiewicz logic. In [26] Jeřábek proves that admissibility in the latter is decidable in PSPACE, and an explicit basis is given in [27]. In [34] substructural logics without weakening are considered and many are proved to be hereditarily structurally complete. Due to lack of space and the fact that the author is not an expert in this area concrete results are not included, but see [4, 34].

7 Canonicity

Research on admissibility in modal and intermediate logics can roughly be divided in three groups, based on the techniques that are used. There is the approach developed by Rybakov that makes use of characterizing models, the approach initiated by Ghilardi that uses projectivity and finite frames, and the approach by Jeřábek that uses canonical rules. The first approach is discussed in Section 5, the second approach will be discussed in the next section, and the third approach will be discussed in this section.

Zakharyaschev, in a series of papers, showed that intermediate and normal transitive modal logics can be axiomatized by canonical formulas, which are formulas based on finite frames [57–60]. Jeřábek showed that one can prove a similar result for rules [25]. Namely, that every rule is equivalent (over IPC in the case of intermediate and over K4 in the case of modal logic) to a set of canonical rules, which are rules based on finite frames.

Canonical formulas can be a useful tool to establish that certain logics have certain properties, such as decidability and the finite model property. In [25] Jeřábek uses canonical rules to obtain several quite strong results on admissibility in intermediate and modal logics. He provides a semantical necessary and sufficient condition for a canonical rule to be admissible, and from this obtains many known results on decidability and bases for admissibility.

Jeřábek also proves that certain logics (IPC, K4, GL, S4 among them) have the *rule dichotomy property*. This is the property that every rule is equivalent to a set of rules that are either admissible in the logic or assumption-free. This means in particular that any intermediate logic L has a basis for its admissible rules that consists of rules that are admissible in IPC. Thus the basis consists of rules that are derivable (over IPC) from $\overline{\mathsf{V}}$. Observe that this improves Theorem 6.1, which only applies to logics in which the Visser rules are admissible. Similar observations can be made about modal logics.

All in all, it shows that the approach via canonicity is as powerful in the context of rules as it is in the context of formulas. And it leads to results about admissibility for which at the moment no other proofs are known.

8 Unification Theory

Proving that a certain set of rules is a basis for the admissible rules of a certain logic can be very hard. One of the reasons is the role played by substitutions, as witnessed by Proposition 3.1. The far reaching connection with unification theory was discovered by Ghilardi in a series of papers that form a cornerstone in the field [9–12]. We restrict ourselves to intermediate and modal propositional logics, which were the first logics for which this method was developed.

Central is the notion of a *projective unifier* of a formula A, which is a substitution σ that is a *unifier* of A, which means that $\vdash \sigma A$, and such that for every atom p:

$$A \vdash p \leftrightarrow \sigma p.$$

This implies that, given A, σ is the identity on all formulas B: $A \vdash B \leftrightarrow \sigma B$. A formula for which such a projective unifier exists is a *projective formula*.

In IPC, for example, the substitutions σ_v^A from Section 4 are a projective unifier of A if they are a unifier of A. Clearly, if $v(p) = 1$, then $\vdash_{\mathsf{IPC}} \sigma_v^p p$, thus proving that p is projective in IPC. For $\neg p$, any v with $v(p) = 0$ gives $\vdash_{\mathsf{IPC}} \neg\sigma_v^{\neg p} p$. Hence $\neg p$ is projective too. An example of a formula that is not projective is $p \vee \neg p$. That it is not projective can be seen using a semantical characterization of projectivity by Ghilardi [9, 10] that falls outside the scope of this paper. Similar reasoning as for IPC can be used in modal logics to show, for example, that $p \wedge \Box p$ and $\neg p \wedge \Box\neg p$ are projective formulas in K4 while $\Box p \vee \Box\neg p$ is not.

The importance of projective formulas in the context of admissible rules stems from the fact that for such formulas admissibility and derivability coincide.

Fact 8.1 If A is projective, then for all formulas B:

$$A \mathrel{|\!\sim} B \Leftrightarrow A \vdash B.$$

Projective formulas can also be used to approximate nonprojective formulas in the following sense. A *projective approximation* Π_A of a formula A is a finite set of projective formulas such that

$$\bigvee \Pi_A \vdash A \mid\!\sim \Pi_A.$$

This implies that $\bigvee \Pi_A \sim\!\mid\mid\!\sim A$. Therefore $\bigvee \Pi_A$ can be viewed as a normal form for A with respect to $\mid\!\sim$.

In IPC, for example, $\{p, \neg p\}$ is a projective approximation of $p \vee \neg p$. Above we saw that p and $\neg p$ are projective. And $\bigvee\{p, \neg p\} \vdash_{\mathsf{IPC}} p \vee \neg p \mid\!\sim \{p, \neg p\}$ holds because IPC has the disjunction property.

Projective approximations can be used to prove that a certain set of rules is a basis for the admissible rules of a logic in the following way.

Proposition 8.2 *Let $\mathcal{R}$ be a set of admissible rules such that every formula A has a projective approximation Π_A such that*

$$\bigvee \Pi_A \vdash_{\mathsf{L}} A \vdash_{\mathsf{L}}^{\mathcal{R}} \Pi_A. \tag{1}$$

Then $\mathcal{R}$ is a basis for the admissible rules of L.

Because of Lemma 8.2 one has but to find a way of constructing projective approximations using a certain set of rules in order to obtain a basis for the admissible rules of a logic. In practice this goal can be hard to reach. But at least for intermediate and transitive modal logics this method has been highly successful in providing ways to prove Theorems 6.1 and 6.2.

As to the constructivity of this method: [11, 19, 20] contain algorithms to obtain the projective approximation of a formula in several well-known logics and in [18, 21] constructive proofs of the projectivity of the formulas in the approximation are given, by providing explicit proofs of the formulas under their projective unifier.

Not only can projective approximations be used in the study of admissible rules, they also can be used to establish the unification type of a logic. In practice, these notions are often studied in parallel, which is the reason why we briefly discuss unification types here. First, we need some terminology.

Substitutions can be ordered according to their generality: $\tau \leqslant \sigma$ if and only if τ is an instance of σ:

$$\tau \leqslant \sigma \;\Leftrightarrow\; \exists \tau' \forall p : \vdash \tau(p) \leftrightarrow \tau'\sigma(p).$$

In this case τ is *less general* than σ. A substitution σ is a *maximal* unifier of A if among the unifiers of A it is maximal in the order $\leqslant$. It is a *most general unifier (mgu)* of A if all unifiers of A are less general than it.

Projective unifiers are most general unifiers because if σ is a projective unifier of A, and τA is derivable, then $\tau p \leftrightarrow \tau\sigma p$. Hence $\tau \leqslant \sigma$, proving that any unifier of A is less general than σ. The converse is not true: not every most general unifier is projective, but the only most general unifiers we will be concerned with are. A set of maximal unifiers of A is *complete* if every unifier of A is less general than a unifier in the set and if no two unifiers in the set are comparable.

The unification type of a logic describes the level of generality unifiers have in this logic. A logic L has unification type *unitary* if every unifiable formula has a mgu, and *finitary* if every unifiable formula has a finite complete set of maximal unifiers. There are other possible types, but we will not discuss them here.

In Section 4 we saw that in CPC, if v is a valuation that satisfies A, then σ_v^A is a unifier of A. Above we explained that σ_v^A is projective, and therefore it is a most general unifier of A. Thus proving that the unification type of classical logic is unitary. This no longer is the case for IPC: every unifier σ of $p \vee \neg p$ has to be a unifier of p or $\neg p$ by the disjunction property. Thus $\vdash_{\mathsf{IPC}} \sigma(p) \leftrightarrow \top$ or $\vdash_{\mathsf{IPC}} \sigma(p) \leftrightarrow \bot$. Therefore a most general unifier of $p \vee \neg p$ should be equivalent to $\top$ and more general than $\bot$ or vice versa, which cannot be. Using the modal disjunction property, a similar argument for $\Box p \vee \Box \neg p$ shows that many modal logics do not have unitary unification either.

Ghilardi introduced projective approximations to prove that the unification type of IPC and many transitive modal logics, $\mathsf{K4}$, GL and $\mathsf{S4}$ among them, is finitary. Namely, he observed that if every formula A has a projective approximation Π_A, then L has finitary or unitary unification, and then proved the existence of projective approximations in several intermediate and modal logics. The projective unifiers he uses are compositions of the substitutions σ_v^A that we encountered before.

Summarizing, a logic that satisfies the assumptions of Lemma 8.2, not only has $\mathcal{R}$ as a basis for its admissible rules but also has finitary unification type. This is a connection between unification and admissible rules that exists for many modal and intermediate logics and has lead to some of the deepest results in the area.

9 What to do

In the field of admissible rules there are many challenges, both technical and philosophical. Let me start with the first. The proof method using projective approximations that has been described in the last section does not seem to apply (easily or at all) to substructural logics and modal logics that are not transitive. And therein lies one of the major hurdles to be taken, hopefully, in the coming years: to find ways to treat admissibility in these logics for which known methods fail.

A topic that deserves more attention is admissibility in predicate logic. Pogorzelski and Prucnal have studied structural completeness in the context of classical predicate logic [36], and Dzik provided structurally complete intermediate predicate logics [7, 8]. Visser has proved that the propositional admissible rules of Heyting Arithmetic and IPC are equal [50], and that in Heyting Arithmetic the admissibility relation for predicate rules is Π_2-complete [49]. This does not exclude the possibility that the admissible rules have a decent description in some way or another, but it shows how complex the notion of admissibility in predicate logic is.

Then there is the philosophical issue regarding the reason for a rule to be admissble. For example, in intuitionistic logic the disjunction property is perfectly explainable from the constructive point of view. That it really holds still requires a proof, but the meaning of disjunction in intuitionistic logic foretold us that it would hold. For admissible rules one would like such an intuitive explanation as well, but it is conspicuously lacking. In the case of intuitionistic logic the type theoretic interpretation might lead the way, but how to understand admissibility in modal logic seems even less clear.

Another matter, both philosophical and technical in nature, concerns the framework of admissibility. In the literature one often presents it in the context of consequence relations, as is done in this paper. But there is a slight feeling of uneasiness in that this framework is not quite as flexible as one would like it to be. In practice, rules do not always consist of pairs of (sets of) formulas only but have side conditions as well (such as "x is not free in ..."). Consequence relations as presented here do not allow for that, which is why one might wish to look for alternatives.

These are some of the numerous questions regarding admissibility for which no solution is known. In this field answers are in general hard to obtain, but I think that the future has some beautiful results in store for us.

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